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Algebraic analysis in structures with the Kaplansky–Jacobson property

by

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Abstract. In 1950 N. Jacobson proved that if u is an element of a ring with unit such that u has more than one right inverse, then it has infinitely many right inverses. He also mentioned that I. Kaplansky proved this in another way. Recently, K. P. Shum and Y. Q. Gao gave a new (non-constructive) proof of the Kaplansky–Jacobson theorem for monoids admitting a ring structure. We generalize that theorem to monoids without any ring structure and we show the consequences of the generalized Kaplansky–Jacobson theorem for the theory of linear operators, and even for the classical Calculus. In order to do that, we recall some multiplicative systems, called pseudocategories, very useful in the algebraic theory of perturbations of linear operators. In the second part of the paper, basing on the Kaplansky–Jacobson theorem, we show how to use the above mentioned structures for building Algebraic Analysis of linear operators over a class of linear spaces. We also define (non-linear) logarithmic and antilogarithmic mappings on these structures.

In 1950 N. Jacobson proved the following theorem:

If u is an element of a ring with unit e such that u has more than one right inverse, then it has infinitely many right inverses (cf. [J1]).

Namely, all elements of the form $w_k = v_0 + (e - v_0 u)u^k$ $(k \in \mathbb{N})$ are also right inverses of u, for any fixed right inverse v_0.

In the same paper [J1] it is mentioned that I. Kaplansky proved this theorem in another way (private communication to Jacobson).

Recently, K. P. Shum and Y. Q. Gao gave a new (non-constructive) proof of the Kaplansky–Jacobson theorem for monoids admitting a ring structure (cf. [SG], also [B]). In this paper we generalize that theorem to monoids without any ring structure and we show the deep consequences of the gener-

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alized Kaplansky–Jacobson theorem for the theory of linear operators, and even for the classical Calculus.

In order to do that, we will recall some multiplicative systems, called pseudocategories, very useful in the algebraic theory of perturbations of linear operators. These were introduced in the monograph of the present author and S. Rolewicz ([PRR1]), where *pararings* and *paraalgebras* were defined for operators mapping a linear space into another. An axiomatics of pseudocategories was given in [PR1]. Next, in [PR2], in a rather complicated way, properties of pseudocategories were studied. It was shown that pseudocategories can be extended to concrete categories under some additional assumptions. Furthermore pararings and paraalgebras, ideals and radicals were defined by means of pseudocategories. Then, as in the above-mentioned book, the algebraic theory of perturbations of linear operators over a class of linear spaces was examined. This theory was simplified in an essential way by H. Lausch in 1985 (cf. [LPR]).

It should be pointed out that in the theory of pseudocategories the notion of objects does not play any role. The axiomatics contains only the notion of morphisms. The existence of a unit is also not necessary. This is natural in a sense. For instance, in a quotient paraalgebra of linear operators over a class of linear spaces, the spaces under consideration are not objects. One can introduce objects in this case, but in an artificial way.

In the second part of the present paper, due to the Kaplansky–Jacobson theorem, it will be shown how to use the above mentioned structures for building Algebraic Analysis of linear operators over a class of linear spaces. We shall also define (non-linear) logarithmic and antilogarithmic mappings on these structures.

1. Pseudocategories, pararings, paraalgebras and perturbations of linear operators. We recall some notions and properties (without proofs which can found in [LPR]) which will be used in what follows.

1a. *Pseudocategories.* The following definition is fundamental for all subsequent considerations:

DEFINITION 1a.1. A class P of morphisms is called a *pseudocategory* if for some ordered pairs (x, y) of morphisms $x, y \in P$ a product $z \in P$ is defined, denoted by $z = xy$, such that for all $a, b, c, d \in P$ the following axioms are satisfied:

$\mathbf{A}_1$: (ab, cd, cb exists) $\Rightarrow$ ad exists;

$\mathbf{A}_2$: (ab, bc exist) $\Rightarrow$ [$(ab)c$, $a(bc)$ exist and $(ab)c = a(bc)$]; call the last element abc;

A$_3$: $[\forall a\ (ab \text{ exists} \Rightarrow (ab)c \text{ exists})] \Rightarrow bc$ exists;
$[\forall b\ (bc \text{ exists} \Rightarrow a(bc) \text{ exists}) \Rightarrow ab$ exists ([1]);

A$_4$: $\{x \in P : ax, xb \text{ exist}\}$ is a set.

Definition 1a.2. If P is a pseudocategory and $a, b \in P$ then $\{x \in P : ax, xb \text{ exist}\}$ is called a *multiplicant* and is denoted by $M_{a|b}$.

Proposition 1a.1. *Let P be a pseudocategory.*

(i) *If $a, \widetilde{a}, b, \widetilde{b}, x, y \in P$ and $ax, \widetilde{a}x, yb, y\widetilde{b}$ exist then $M_{\widetilde{a}|\widetilde{b}} = M_{a|b}$;*
(ii) *if $c, d \in P$ and $M_{c|c} \cap M_{d|d} \neq \emptyset$ then $M_{c|c} = M_{d|d}$;*
(iii) *if $c, u, v \in P$ and uv exists then $M_{c|u} = M_{c|uv}$ and $M_{v|c} = M_{uv|c}$;*
(iv) *if $a, x, y, z \in P$ and $x, y \in M_{a|a}$ then the existence of zx implies the existence of zy and the existence of xz implies the existence of yz.*

Definition 1a.3. A subclass Q of a pseudocategory P is a *subpseudocategory* if whenever $x, y \in P$ and xy exists then $xy \in Q$.

Note that any subpseudocategory Q of a pseudocategory P, equipped with the multiplication of P, is itself a pseudocategory.

Proposition 1a.2. *Let P be a pseudocategory and let $a \in P$. Then both*

$$P_a = \{x \in P : M_{a|x} \neq \emptyset\} \quad \text{and} \quad P'_a = \{y \in P : M_{y|a} \neq \emptyset\}$$

are subpseudocategories of P.

Definition 1a.4. A pseudocategory P is *proper* if $M_{a|a} \neq \emptyset$ for all $a \in P$.

Theorem 1a.1. *If P is a proper pseudocategory then*

(i) *for all $a, b \in P$ either $M_{a|b} = \emptyset$ or there exists a $y \in P$ such that $M_{a|b} = M_{y|y}$;*
(ii) *$P = \bigcup_{y \in P} M_{y|y}$, i.e. the proper pseudocategory is the union of disjoint multiplicants of the form $M_{y|y}$, where $y \in P$.*

Proposition 1a.3. *Let P be a proper pseudocategory and let $a \in P$. Then $a \cdot a = a^2$ exists if and only if $a \in M_{a|a}$. If $a \in M_{a|a}$ then $M_{a|a}$ is a semigroup.*

Definition 1a.5. Let P be a pseudocategory and let $a \in P$ have the property that $a \in M_{a|a}$. A morphism $e_a \in M_{a|a}$ is called a *unit* in P if

$$\forall x \in P\ (xe_a \text{ exists} \Rightarrow xe_a = x \text{ and } e_a x \text{ exists} \Rightarrow e_a x = x).$$

We say that P *has units* if each multiplicant $M_{a|a}$ with $a \in M_{a|a}$ contains a unit.

([1]) This axiom is written here in a slightly modified form in comparison with [LPR].

Theorem 1a.2. *Let P be a proper pseudocategory. Then there exists a proper pseudocategory P_1 with units such that P is its subpseudocategory.*

Corollary 1a.1. *Every proper pseudocategory is a subpseudocategory of some category.*

Definition 1a.6. Two disjoint subclasses Q and R of a pseudocategory P are said to be *non-cooperating* if $y \in Q$, $z \in R$ implies that neither yz nor zy exist. A proper pseudocategory P is said to be *irreducible* if P has no two non-empty non-cooperating subpseudocategories (with respect to the same multiplication).

Proposition 1a.4. *Let P be an irreducible pseudocategory and let $a, b \in P$. Then $M_{a|b} \neq \emptyset$.*

Definition 1a.7. A pseudocategory P is said to be *indexed* if there exists a set A and a map $\mathsf{M} : \mathsf{A} \times \mathsf{A} \to \{M_{u|u} : u \in P\}$, $(\alpha, \beta) \mapsto M_{\alpha,\beta}$, such that

(i) $M_{\alpha,\beta} \cap M_{\alpha',\beta'} = \emptyset$ if $(\alpha, \beta) \neq (\alpha', \beta')$;
(ii) M is surjective;
(iii) $x \in M_{\alpha,\beta}$, $y \in M_{\beta,\gamma}$ $(\alpha, \beta, \gamma \in \mathsf{A})$ imply that xy exists and $xy \in M_{\alpha,\gamma}$.

M is called an *indexing* of P.

Theorem 1a.3. *If an irreducible pseudocategory P is a set then it is indexed.*

Definition 1a.8. Let $\mathfrak{M}$ be a class of sets and P be a class of mappings between sets of $\mathfrak{M}$. If P is a pseudocategory with respect to composition of mappings then P is called a *concrete* pseudocategory.

Theorem 1a.4. *Let P be a pseudocategory indexed by* $\mathsf{M} : \mathsf{A} \times \mathsf{A} \to \{M_{u|u} : u \in P\}$. *Then P is isomorphic* (*in the obvious sense*) *to a concrete pseudocategory with the same indexing* M.

A proper pseudocategory is said to be *commutative* if whenever ab exists then ba exists and $ba = ab$.

An irreducible pseudocategory which is a set is a semigroup if

$$ab \text{ exists} \Rightarrow ba \text{ exists.} \tag{1a.1}$$

Let P be a proper pseudocategory with units and let $x \in P$. If there exist $x^l \in P$ (respectively, $x^r \in P$) such that for some $a \in M_{a|a}$ we have $x^l x = e_a$ (respectively, $xx^r = e_a$) then x is said to be *left invertible* (respectively, *right invertible*). The morphisms x^l, x^r are called *left* and *right inverses* of x, respectively.

PROPOSITION 1a.5. *If an x belonging to a proper pseudocategory with units is simultaneously left and right invertible then its left inverse is equal to its right inverse.*

A morphism x satisfying the conditions of Proposition 1a.5 is said to be *invertible* and we write $x^{-1} = x^l = x^r$. The morphism x^{-1} is called an *inverse*. The inverse of every invertible morphism is, by definition, uniquely determined and $x^{-1} \in M_{x|x}$ (for both xx^{-1} and $x^{-1}x$ exist).

A proper pseudocategory P with units is said to be a *paragroup* if every $x \in P$ is invertible. An irreducible paragroup which is a set satisfying condition (1a.1) is a group.

It is easy to verify that if P is an irreducible pseudocategory then the class of all multiplicants contained in P with multiplication defined as follows is a *Brandt grupoid*:

(i) if $a \in M_{a|a}$ then $M_{a|a} \cdot M_{a|a} = M_{a|a}$;
(ii) if $a \notin M_{a|a}$ and $b \in M_{a|a}$ then $M_{a|a} \cdot M_{b|b} = M_{ba|ba}$; $M_{b|b} \cdot M_{a|a} = M_{ab|ab}$; $M_{a|a} \cdot M_{ab|ab} = M_{ba|ba} \cdot M_{a|a} = M_{a|a}$; $M_{ab|ab} \cdot M_{b|b} = M_{b|b} \cdot M_{ba|ba} = M_{b,b}$ (cf. H. Brandt [Bra], also R. H. Bruck [Bru]).

1b. *Pararings.* We start with

DEFINITION 1b.1. A *pararing* P is a proper pseudocategory such that for each $a \in P$ an addition $+$ is defined on $M_{a|a}$ which turns $M_{a|a}$ into an Abelian group and this addition satisfies

$\mathbf{A}_5$: if $x, y \in M_{a|a}$, $z \in P$, then

$$zx \text{ exists} \Rightarrow z(x+y) = zx + zy; \qquad xz \text{ exists} \Rightarrow (x+y)z = xz + yz.$$

Note that, by Proposition 1a.1(iv), the existence of zx (respectively, xz) implies the existence of zy and $z(x+y)$ (respectively yz and $(x+y)z$). Hence the right hand sides in $\mathbf{A}_5$ are always defined.

DEFINITION 1b.2. A subclass P_0 of a pararing P is said to be a *subpararing* of P if P_0 is a pararing with the addition and multiplication of P restricted to P_0. A subpararing J of a pararing P is an *ideal* of P if for all $x \in P$ and $z \in J$ the existence of xz implies that $xz \in J$ and the existence of zx implies that $zx \in J$. If $a \in P$ and $x \in M_{a|a}$ then $[x] = \{x + y : y \in M_{a|a} \cap J\}$ is called the *coset* (with respect to J) determined by x.

Note that $[x] \subseteq M_{a|a} \cap J$ for $x \in M_{a|a}$.

DEFINITION 1b.3. Let A, B be subsets of a pararing P. We write

$$A + B = \{x + y : x \in A,\ y \in B,\ x + y \text{ exists}\},$$
$$AB = \{xy : x \in A,\ y \in B,\ xy \text{ exists}\}.$$

THEOREM 1b.1. *Let J be an ideal in a pararing P and let $x_1, x_2, x_3 \in P$. Suppose that $x_1 + x_2$ and $x_1 x_3$ exist. Then $[x_1] + [x_2] = [x_1 + x_2]$ and $[x_1][x_3] \subseteq [x_1 x_3]$.*

DEFINITION 1b.4. Let J be an ideal of a pararing P. The *quotient pararing* P/J is the class of all cosets $[x]$ with respect to J, for $x \in P$, with addition and multiplication defined in the usual way.

Theorem 1b.1 tells us that P/J is a pararing and $[x][y]$ exists if and only if xy exists.

THEOREM 1b.2. *Suppose that P is a pararing and $a \in P$. Denote by 0_a the* neutral morphism *of the multiplicant $M_{a|a}$* ([1]) *and by $N(P)$ the class of neutral morphisms belonging to P. Then*

(i) *if $a \in M_{a|a}$ then $M_{a|a}$ is a ring*;
(ii) *if $a \in P$ then $M_{ax|ax}$ and $M_{xa|xa}$ are independent of the choice of $x \in M_{a|a}$*;
(iii) *if $a \in P$ and $x \in M_{a|a}$ then $M_{a|a}$ is a right (left) module* ([2]) *over the ring $M_{ax|ax}$ (respectively, $M_{xa|xa}$); moreover,*

$$0_{xa} x = 0_a, \qquad x 0_{ax} = 0_a.$$

A pararing P *has units* if P considered as a pseudocategory has units. A pararing with units is said to be a *parafield* if every non-neutral morphism is invertible. Evidently, an irreducible parafield which is a set is a field.

EXAMPLE 1b.1. Let X and Y be linear spaces over the same field of scalars. Denote by $L_0(X \to Y)$ the space of all linear operators defined on X and with range in Y. Write $L_0(X) = L_0(X \to X)$. The quadruple

$$L_0(X \rightleftharpoons Y) = \begin{pmatrix} & L_0(X) & \\ L_0(X \to Y) & & L_0(Y \to X) \\ & L_0(Y) & \end{pmatrix}$$

is a pararing which has been considered by the present author and S. Rolewicz in [PRR1]) (cf. also Example 1c.1).

One can prove that every pararing can be extended to an additive pseudocategory. However, this extension seems to be rather artificial (cf. [PR2]). On the other hand, there is a pararing which is not an additive category, for instance, the ring $\mathbb{Z}$ of all integers.

DEFINITION 1b.5. Let P be a pararing with units. The subclass

$$\mathbf{R}(P) = \{x \in P : \forall a, y, z \in P\ [a \in M_{a|a} \text{ and } yxz \in M_{a,a}] \Rightarrow e_a + yxz \text{ is invertible}\}$$

is called the *radical* of P.

([1]) That is, a morphism in $M_{a|a}$ such that $0_a x = x 0_a = 0_a$ for all $x \in P$.
([2]) See Jacobson [J2].

THEOREM 1b.3. *The radical* $\mathbf{R}(P)$ *of a pararing* P *with units is an ideal in* P.

OPEN QUESTIONS. Does every non-commutative pararing without divisors of neutral morphisms have an extension to a parafield? Is every ideal in a pararing with units contained in a maximal ideal? Is the radical of a pararing with units an intersection of maximal ideals?

1c. *Paraalgebras and perturbations of linear operators*

DEFINITION 1c.1. A pararing P with units is said to be a *paraalgebra* if

(i) every multiplicant $M_{a|a}$ is a linear space over a field $\mathbb{F}$ of scalars;
(ii) whenever xy exists then $t(xy) = (tx)y = x(ty)$ for $t \in \mathbb{F}$.

EXAMPLE 1c.1. Suppose that a class $\{X_\alpha\}_{\alpha\in\mathbf{A}}$ of linear spaces (over a field $\mathbb{F}$ of scalars) is given. It is easy to verify that

$$L_0(X_\alpha \rightleftharpoons X_\beta; \alpha, \beta \in \mathbf{A}) = \bigcup_{\alpha,\beta\in\mathbf{A}} L_0(X_\alpha \to X_\beta)$$

is a paraalgebra. If $\mathbf{A}$ is a set then this paraalgebra is indexed by $\mathbf{A}$. The spaces $L_0(X_\alpha \to X_\beta)$ are multiplicants, $L_0(X_\alpha \to X_\alpha)$ are multiplicants such that $T^2 = T \cdot T \in L_0(X_\alpha \to X_\alpha)$ whenever $T \in L_0(X_\alpha \to X_\alpha)$. The paraalgebra $L_0(X_\alpha \rightleftharpoons X_\beta; \alpha, \beta \in \mathbf{A})$ has units, namely the identity operators I_α of the spaces X_α (cf. Example 1b.1).

EXAMPLE 1c.2. Let $\mathcal{L}_T$ be the class of **all** linear topological spaces (over $\mathbb{C}$ or $\mathbb{R}$). The class $LT(X \rightleftharpoons Y; X, Y \in \mathcal{L}_T)$ of all linear continuous operators defined on spaces $X \in \mathcal{L}_T$ and mapping X into spaces belonging to $\mathcal{L}_T$ is a paraalgebra which is not a set.

Note that Banach paraalgebras have been considered by K. H. Förster (cf. [F]).

EXAMPLE 1c.3. Let $\mathcal{E}(\mathbb{E}^m \rightleftharpoons \mathbb{E}^n; m, n \in \mathbb{N})$ be the paraalgebra of all $m \times n$ matrices with coefficients from a field $\mathbb{E}$ of scalars. If $m \neq n$ then a matrix $U \in \mathcal{E}(\mathbb{E}^m \rightleftharpoons \mathbb{E}^n; m, n \in \mathbb{N})$ is not invertible. This implies the following fact: if $L_0(X_\alpha \rightleftharpoons X_\beta; \alpha, \beta \in \mathbf{A})$ is a paraalgebra defined in Example 1c.1 and $\dim X_\alpha < \infty$ for all $\alpha \in \mathbf{A}$ then an operator $T \in L_0(X_\alpha \rightleftharpoons X_\beta; \alpha, \beta \in \mathbf{A})$ is invertible if and only if $T \in L_0(X_\alpha \to X_\alpha)$.

Let J be an ideal in a paraalgebra P and let $a \in P$. An element r is called a *left* (*right*) *regularizer* to the ideal J if ra (respectively, ar) exists and, for some $c \in P$ such that $c \in M_{c|c}$, $ra \in M_{c|c}$ (respectively, $ar \in M_{c|c}$) and $ra = e_c + b$ (respectively, $ar = e_c + b$), where $b \in J \cap M_{c|c}$. This means that in the quotient paraalgebra P/J the corresponding coset is left (right) invertible. If r is simultaneously a left and right regularizer then r is called a *simple regularizer* and the corresponding coset is invertible.

Let A be a linear operator defined on a linear subset $\operatorname{dom} A$ of a linear space X, called the *domain* of A, and mapping $\operatorname{dom} A$ into a linear space Y (both over the same field $\mathbb{F}$ of scalars). The set of all such operators will be denoted by $L(X \to Y)$. Write

$$\alpha_A = \dim\ker A = \dim\{x \in \operatorname{dom} A : Ax = 0\},\ \beta_A = \operatorname{codim} E_A = \dim Y/E_A,$$

where $E_A = A(\operatorname{dom} A) \subset Y$ is the *range* of A; and

$$\kappa_A = \begin{cases} -\infty & \text{if } \alpha_A = \infty,\ \beta_A < \infty; \\ \infty & \text{if } \alpha_A < \infty,\ \beta_A = \infty; \\ \beta_A - \alpha_A & \text{if } \alpha_A,\ \beta_A < \infty. \end{cases}$$

The numbers $\alpha_A, \beta_A, \kappa_A$ are called the *nullity, deficiency* and *index* of the operator $A \in L(X \to Y)$, respectively. The ordered pair (α_A, β_A) is called the *d-characteristic* (*dimensional characteristic*) of A (cf. [PRR1]). If both α_A and β_A are finite then the d-characteristic is said to be *finite*.

Suppose that $\mathcal{X}$ is a class of linear spaces (over the same field of scalars). Let $L(X \rightleftharpoons Y; \mathcal{X})$ be the paraalgebra of linear operators mapping spaces belonging to $\mathcal{X}$ into spaces from $\mathcal{X}$. This paraalgebra has units, namely, the identity operators I_X of spaces $X \in \mathcal{X}$.

If every operator $A \in L(X \rightleftharpoons Y; \mathcal{X})$ with a finite d-characteristic has a simple regularizer to an ideal $J \subset L(X \rightleftharpoons Y; \mathcal{X})$ then we say that $L(X \rightleftharpoons Y; \mathcal{X})$ is *regularizable* to the ideal J. The paraalgebra $L_0(X_\alpha \rightleftharpoons X_\beta; \alpha, \beta \in \mathbf{A})$ is regularizable to the ideal of all finite-dimensional operators contained in this paraalgebra.

An ideal J in a paraalgebra $L(X \rightleftharpoons Y; \mathcal{X})$ is said to be a *quasi Fredholm ideal* if for all $X \in \mathcal{X}$ and for all $T \in J \cap L_0(X \to X)$ the operator $I_X + T$ has a finite d-characteristic. If, moreover, $\kappa_{I_X+T} = 0$ then J is said to be a *Fredholm ideal*.

THEOREM 1c.1. *Suppose that a paraalgebra $\mathcal{L}_{\mathcal{X}} = L(X \rightleftharpoons Y; \mathcal{X})$ is regularizable to a quasi Fredholm ideal in $J \subset \mathcal{L}_{\mathcal{X}}$. Denote by $\mathbf{R}(\mathcal{L}_{\mathcal{X}}/J)$ the radical of the quotient paraalgebra $\mathcal{L}_{\mathcal{X}}/J$. Then the ideal $J_0 = \{U \in \mathcal{L}_{\mathcal{X}} : [U] \in \mathbf{R}(\mathcal{L}_{\mathcal{X}}/J)\}$ is the maximal quasi Fredholm ideal, i.e. $J \subset J_0$.*

An ideal J in a paraalgebra $\mathcal{L}_{\mathcal{X}} = L(X \rightleftharpoons Y; \mathcal{X})$ is called a *positive* (*negative*) *semi-Fredholm ideal* if for all $X \in \mathcal{X}$ and $T \in J \cap L_0(X \to X)$ the operator $I_X + T$ has a finite nullity (deficiency). A paraalgebra $\mathcal{L}_{\mathcal{X}} = L(X \rightleftharpoons Y; \mathcal{X})$ is called *left* (*right*) *regularizable* to an ideal $J \subset \mathcal{L}_{\mathcal{X}}$ if every operator $\in \mathcal{L}_{\mathcal{X}}$ with a finite nullity (deficiency) has a left (right) regularizer to the ideal J.

Let $\mathfrak{A}$ be a class (not necessarily linear) of linear operators. A linear operator B is called a *perturbation of an operator* $A \in \mathfrak{A}$ if $A + B \in \mathfrak{A}$. B is said to be a *perturbation of the class* $\mathfrak{A}$ if $A+B \in \mathfrak{A}$ for all $A \in \mathfrak{A}$ such that the sum $A+B$ is well defined.

THEOREM 1c.2. *Suppose that a paraalgebra* $\mathcal{L}_{\mathcal{X}} = L(X \rightleftharpoons Y; \mathcal{X})$ *is left* (*right*) *regularizable to a positive* (*negative*) *semi-Fredholm ideal* $J \subset \mathcal{L}_{\mathcal{X}}$. *Then all operators belonging to a positive* (*negative*) *semi-Fredholm ideal* $J_1 \subset \mathcal{L}_{\mathcal{X}}$ *are perturbations of the class of all operators with a finite nullity* (*deficiency*) *belonging to* $\mathcal{L}_{\mathcal{X}}$.

COROLLARY 1c.1. *If a paraalgebra* $\mathcal{L}_{\mathcal{X}} = L(X \rightleftharpoons Y; \mathcal{X})$ *is regularizable to a quasi Fredholm ideal* $J \subset \mathcal{L}_{\mathcal{X}}$ *then every* $T \in J$ *is a perturbation of the class of all operators with a finite d-characteristic. If* J *is a Fredholm ideal then this perturbation* preserves the index, *i.e. for all* $A \in \mathcal{L}_{\mathcal{X}}$ *with a finite d-characteristic and for all* $T \in J$ *such the sum* $A+T$ *is well defined*, *we have* $\kappa_{A+T} = \kappa_A$.

Regularizers of integral and singular integral operators have been in common use for several years (cf. for instance Nguyen Van Mau [N1], [N2]).

2. Pararings with the Kaplansky–Jacobson property. We start with

THEOREM 2.1. *Let* P *be a pararing with units. Let* $x \in P$ *be right invertible and let* x^r *be its right inverse. Then there is an* $a \in P$ *such that the morphism* $p_x = e_a - x^r x$ *is well defined and* $p_x \in M_{a|a}$. *Moreover*, p_x *is an idempotent such that* $p_x x^r = 0$ *and* $x p_x = 0$ ([1]).

Proof. By definition, there is an $a \in P$ such that $a \in M_{a|a}$ and $xx^r = e_a$. By Proposition 1a.1(iv), $x^r x$ exists and belongs to $M_{a|a}$. Thus the morphism $p_x = e_a - x^r x$ is well defined and belongs to $M_{a|a}$. By Proposition 1a.3, p_x^2 exists, belongs to $M_{a|a}$ and

$$\begin{aligned} p_x^2 &= (e_a - x^r x)^2 = (e_a - x^r x)(e_a - x^r x) \\ &= e_a^2 - (x^r x)e_a - e_a(x^r x) + (x^r x)(x^r x) \\ &= e_a - (e_a x^r)x - x^r(xe_a) + x^r(xx^r)x \\ &= e_a - x^r x - x^r x + x^r e_a x = e_a - x^r x = p_x, \end{aligned}$$

i.e. p_x is an idempotent. Moreover, $xp_x = x(e_a - x^r x) = xe_a - (xx^r)x = x - e_a x = 0$ and $p_x x^r = (e_a - x^r x)x^r = e_a x^r - x^r(xx^r) = x^r - x^r e_a = 0$. ■

COROLLARY 2.1. *Let* P *be a pararing with units. Let* $x \in P$ *be right invertible and let* x^r *be its right inverse*, *i.e. there is an* $a \in P$ *such that the idempotent* $p_x = e_a - x^r x \in M_{a|a}$. *Then every right inverse of* x *is of the*

([1]) Here and in what follows we denote all neutral morphisms 0_a in P by 0, since this does not lead to any misunderstanding.

form $x_1^r = x^r + p_x y$, *where* $p_x y$ *exists and belongs to* $M_{a|a}$. *In particular,* $x_k^r = x^r + p_x x^k$ *for* $k \in \mathbb{N}$ *are right inverses of* x (1).

Proof. By Theorem 2.1, if $p_x y$ exists and belongs to $M_{a|a}$ then $xx_1^r = x(x^r + p_x y) = xx^r + (xp_x)y = e_a + 0 = e_a$, i.e. x_1^r is a right inverse of x. A similar proof for x_k^r, since $x^r x$ and x^k exist. ∎

THEOREM 2.2. *Let* P *be a pararing with units. Let* $x \in P$ *be right invertible and let* $p_x = e_a - x^r x \in M_{a|a}$ *be an idempotent corresponding to a right inverse* x^r *of* x $(a \in M_{a|a} \subset P)$. *Whenever* yx *exists and belongs to* $M_{a|a}$, *the morphism* $\widetilde{p}_x = p_x(e_a - yx)$ *is an idempotent corresponding to a right inverse* $\widetilde{x}^r = x^r - \widetilde{p}_x x^r$ *of* x *such that* $x\widetilde{p}_x = 0$ *and* $\widetilde{p}_x \widetilde{x}^r = 0$.

Proof. By definitions,

$$\widetilde{p}_x \widetilde{x}^r = \widetilde{p}_x(x^r - \widetilde{p}_x x^r) = \widetilde{p}_x x^r - \widetilde{p}_x^2 x^r = 0,$$
$$x\widetilde{p}_x = xp_x(e_a - yx) = xp_x - (xp_x)yx = 0,$$
$$x\widetilde{x}_r = x(x^r - \widetilde{p}_x x^r) = xx^r - (x\widetilde{p}_x)x^r = e_a - 0 = e_a,$$

i.e. $\widetilde{x}^r$ is a right inverse of x. The morphism $\widetilde{p}_x$ is an idempotent. Indeed,

$$\widetilde{p}_x^2 = [p_x(e_a - yx)]^2 = (p_x - p_x yx)^2 = p_x^2 - p_x^2 yx - p_x yxp_x + (p_x yx)(p_x yx)$$
$$= p_x - (p_x y)x - (p_x y)(xp_x) + (p_x y)(xp_x yx) = p_x. \ \blacksquare$$

DEFINITION 2.1. A pararing P has the *Kaplansky–Jacobson property* if P has units and the existence of a right inverse of a morphism $x \in P$ implies the existence of infinitely many right inverses. A pararing with this property will be called briefly a *K-J-pararing*. A *K-J-paraalgebra* is a paraalgebra which, as a pararing, is a K-J-pararing.

This definition and Corollary 2.1 immediately imply

COROLLARY 2.2. *Every pararing with units has the Kaplansky–Jacobson property.*

COROLLARY 2.3. *Every irreducible pararing with units is a ring with the Kaplansky–Jacobson property.*

COROLLARY 2.4. *Every paraalgebra has the Kaplansky–Jacobson property.*

In a similar manner we obtain *dual* results for left invertible operators:

THEOREM 2.3. *Let* P *be a pararing with units. Let* $x \in P$ *be left invertible and let* x^l *be its left inverse. Then there is an* $a \in P$ *such that the morphism* $q_x = e_a - xx^l$ *is well defined and* $q_x \in M_{a|a}$. *Moreover,* q_x *is an idempotent such that* $q_x x = 0$ *and* $x^l q_x = 0$.

(1) Cf. Jacobson [J1].

Corollary 2.5. *Let P be a pararing with units. Let $x \in P$ be left invertible and let x^l be its left inverse, i.e. there is an $a \in P$ such that the idempotent $q_x = e_a - xx^l \in M_{a|a}$. Then every left inverse of x is of the form $x_1^l = x^l + yq_x$, where yq_x exists and belongs to $M_{a|a}$. In particular, $x_k^l = x^k qp_x$ for $k \in \mathbb{N}$ are right inverses of x.*

Theorem 2.4. *Let P be a pararing with units. Let $x \in P$ be left invertible and let $q_x = e_a - xx^l \in M_{a|a}$ be an idempotent corresponding to a left inverse x^l of x $(a \in M_{a|a} \subset P)$. Whenever xy exists and belongs to $M_{a|a}$, the morphism $\widetilde{q}_x = (e_a - xy)q_x$ is an idempotent corresponding to a left inverse $\widetilde{x}^l = x^l - x^l\widetilde{q}_x$ of x such that $\widetilde{x}^l\widetilde{q}_x = 0$ and $\widetilde{q}_x x = 0$.*

Definition 2.2. A pararing P has the *dual Kaplansky–Jacobson property* if P has units and the existence of a left inverse of a morphism $x \in P$ implies the existence of infinitely many left inverses. A pararing with this property will be called briefly a *dual K-J-pararing*. A *dual K-J-paraalgebra* is a paraalgebra which, as a pararing, is a dual K-J-pararing.

This definition and Corollary 2.5 immediately imply

Corollary 2.6. *Every pararing with units has the dual Kaplansky–Jacobson property.*

Corollary 2.7. *Every irreducible pararing with units is a ring with the dual Kaplansky–Jacobson property.*

Corollary 2.8. *Every paraalgebra has the dual Kaplansky–Jacobson property.*

If the projectors p_x (respectively, q_x) are well defined for some $x \in P$ then they induce partitions of units. Namely, every morphism of the form $e_a - p_x = x^r x$ (respectively, $e_a - q_x = xx^l$) is also an idempotent and

xy exists $\Rightarrow y = p_x y + (e_a - p_x)y$ (resp. zx exists $\Rightarrow z = zq_x + z(e_a - q_x)$).

Theorem 2.5. *Let P be a K-J pararing. Let $x \in P$ be right (left) invertible and let $p_x = e_a - x^r x \in M_{a|a}$ (respectively, $q_x = e_a - xx^l$) be an idempotent corresponding to a right inverse x^r (respectively, a left inverse x^l) of x $(a \in M_{a|a} \subset P)$. Whenever $x + y$ exists and belongs to $M_{a|a}$, the morphism y is called an x-perturbation if $x + y$ is again right (left) invertible. Then all morphisms of the form $y = x'p_x$ (respectively, $y = q_x x'$, if they exist) are x-perturbations.*

Proof. By Theorem 2.4, $(x+y)x^r = xx^r + (x'p_x)x^r = e_a + x'(p_x x^r) = e_a$. A similar proof for left invertible morphisms. ∎

3. Algebraic Analysis in Kaplansky–Jacobson paraalgebras. Let X be a linear space over a field $\mathbb{F}$ of scalars (of characteristic zero). Suppose that $D \in R(X)$, i.e. D is a *right invertible operator* with $\operatorname{dom} D \subset X$ and range also in X. (For all notions and properties connected with right invertible operators cf. [PR3].) Note that, by definition, D is a surjective mapping.

Write $\mathbb{N}_0 = \{0\} \cup \mathbb{N}$, $\mathbb{N}_\infty = \{0, 1, 2, \ldots, +\infty\}$ and

$$D_0 = X, \quad D_k = \operatorname{dom} D^k \quad \text{for } k \in \mathbb{N}, \quad D_\infty = \bigcap_{k \in \mathbb{N}_0} D_k. \tag{3.1}$$

Clearly,

$$D_\infty \subset \cdots \subset D_k = \operatorname{dom} D^k \subset \cdots \subset D_1 = \operatorname{dom} D \subset D_0 = X$$

and $D_j \neq D_k$ if $\operatorname{dom} D \neq X$ $(j, k \in \mathbb{N}_\infty)$.

For a given $D \in R(X)$ we denote by I the identity operator and

- $\ker D = \{z \in \operatorname{dom} D : Dz = 0\}$, the *kernel* of D, also called the *space of constants*;
- $\mathcal{R}_D = \{R \in L_0(X) : DR = I\}$, the set of *right inverses* of D;
- $\mathcal{F}_D = \{F \in L_0(X) : F^2 = F,\ FX = \ker D,\ \exists_{R \in \mathcal{R}_D}\ FR = 0\}$, the set of *initial operators* for D.

We have $\operatorname{dom} D = RX \oplus \ker D$ for all $R \in \mathcal{R}_D$.

F is an initial operator for $D \in R(X)$ corresponding to an $R \in \mathcal{R}_D$, i.e., $F \in \mathcal{F}_D$, if and only if

$$F = I - RD \quad \text{on } \operatorname{dom} D. \tag{3.2}$$

Even more, any projection F' *onto* $\ker D$ is an initial operator for D corresponding to a right inverse $R' = R - FR'$, for any $R \in \mathcal{R}_D$.

If we know at least one right inverse R then we also have the general forms of the sets of right inverses and initial operators for a right invertible operator D. Namely,

$$\mathcal{R}_D = \{R + FA : A \in L_0(X)\}, \quad \mathcal{F}_D = \{F(I - AD) : A \in L_0(X)\},$$

where F is any initial operator for D corresponding to R (cf. Theorem 2.1 and Corollary 2.1).

If two right inverses (respectively, initial operators) commute with each other, then they are equal.

Write $\mathcal{R}_D = \{R_\gamma\}_{\gamma \in \Gamma}$. Then, by (3.2), $\mathcal{R}_D$ induces the family $\mathcal{F}_D = \{F_\gamma\}_{\gamma \in \Gamma}$ of initial operators defined by

$$F_\gamma = I - R_\gamma D \quad \text{on } \operatorname{dom} D \quad (\gamma \in \Gamma). \tag{3.3}$$

Formula (3.3) yields (by a two-lines induction) the *Taylor–Gontcharov Formula*, which plays a fundamental role in our theory. Namely, let $\{\gamma_n\} \subset \Gamma$ be an arbitrary sequence. Then for all positive integers N the following identity holds:

$$(3.4)\qquad I=F_{\gamma_0}+\sum_{k=0}^{N-1} R_{\gamma_0}\cdots R_{\gamma_{k-1}}F_{\gamma_k}D^k+R_{\gamma_0}\cdots R_{\gamma_{N-1}}D^N \quad \text{on } \operatorname{dom} D^N\ (N\in\mathbb{N}).$$

Let F be an initial operator for D corresponding to an $R\in\mathcal{R}_D$. Putting in (3.4) $R_{\gamma_n}=R$ and $F_{\gamma_n}=F$ $(n\in\mathbb{N})$, we obtain the *Taylor Formula*:

$$(3.5)\qquad I=\sum_{k=0}^{n-1} R^kFD^k+R^nD^n \quad \text{on } \operatorname{dom} D^n\ (n\in\mathbb{N}).$$

Formula (3.5) gives partitions of units, since the operators R^kFD^k $(k=0,1,\dots,n-1)$, R^nD^n are projections onto the kernel of D^n. Thus it is easy to see the following

THEOREM 3.1. *The paraalgebra*

$$\mathcal{D}(D_k;\mathbb{N}_\infty)=L_0(D_k\rightleftharpoons D_j;\ j,k\in\mathbb{N}_\infty)$$

has the Kaplansky–Jacobson property (cf. Corollary 2.4).

Theorem 2.5 immediately implies

THEOREM 3.2. *Let $A\in\mathcal{D}(D_k;\mathbb{N}_\infty)$. Then A is a D-perturbation if $A=TF$, where* $\operatorname{dom} T\supset\ker D$ *and F is an initial operator for D corresponding to a right inverse R.*

COROLLARY 3.1. *Suppose that J is a quasi Fredholm ideal in the paraalgebra $\mathcal{D}(D_k;\mathbb{N}_\infty)$. Then*

$$J\mathcal{F}_D=\{TF\in\mathcal{F}_D: T\in J,\ \operatorname{dom} D\supset\ker D\}\subset J,$$

i.e. if $A=TF$ is a D-perturbation with $T\in J$ then $TF\in J$ and the operator $I+TF$ has a finite d-characteristic.

The following question arises: Are all D-perturbations of the form TF?

Suppose that $D_1,\dots,D_m\in R(X)$ and that the superposition $D=D_1\cdots D_m$ is well defined. Let F_j be an initial operator for D_j corresponding to an $R_j\in\mathcal{R}_{D_j}$ $(j=1,\dots,m)$. Write

$$R=R_m\cdots R_1,\qquad F=F_m+R_mF_{m-1}D_m+\dots+R_m\cdots R_2F_1D_2\cdots D_m.$$

Then $D\in R(X)$, $R\in\mathcal{R}_D$ and F is an initial operator for D corresponding to R.

If X is an algebra, $D\in R(X)$ and $xy,yx\in\operatorname{dom} D$ whenever $x,y\in\operatorname{dom} D$ then we write $D\in\mathbf{A}(X)$. If X is a commutative algebra and $D\in\mathbf{A}(X)$ then we write $D\in\mathsf{A}(X)$. If $D\in\mathbf{A}(X)$ then we may write

$$D(xy)=c_D[xDy+(Dx)y]+f_D(x,y)\quad \text{for } x,y\in\operatorname{dom} D,$$

where c_D is a scalar depending on D only, f_D is called the *non-Leibniz component* and f_D (by its definition) is a bilinear mapping of $\operatorname{dom} D \times \operatorname{dom} D$ into X. In commutative algebras the mapping f_D is *symmetric*. If $c_D = 1$ and $f_D = 0$ then X is said to be a *Leibniz algebra*, because

$$D(xy) = xDy + (Dx)y \quad \text{for } x, y \in \operatorname{dom} D.$$

In commutative Leibniz algebras the *generalized Leibniz formula* holds:

$$D^k(xy) = c_D^k(xD^k y + yD^k x) + f_D^k(x,y) \quad \text{for } x, y \in D_k \ (k \in \mathbb{N}), \tag{3.6}$$

where

$$f_D^1 = f_D = 0, \qquad f_D^k(x,y) = \sum_{j=1}^{k-1} \binom{k}{j} (D^{k-j}x)(D^j y) \quad \text{for } k \geq 2.$$

Definition 3.1. Suppose that $D \in R(X)$. The paraalgebra $\mathcal{D}(D_k; \mathbb{N}_\infty) = L_0(D_k \rightleftharpoons D_j;\ j, k \in \mathbb{N}_\infty)$ is said to be a *D-paraalgebra* if $D \in \mathbf{A}(X)$. The paraalgebra $\mathcal{D}(D_k; \mathbb{N}_\infty)$ is a *Leibniz paraalgebra* if X is a Leibniz algebra.

Clearly, if $\operatorname{dom} D = X$ then $\mathcal{D}(D_k; \mathbb{N}_\infty)$ is an algebra, since

$$\mathcal{D}(D_k; \mathbb{N}_\infty) = L_0(D_k \rightleftharpoons D_j;\ j, k \in \mathbb{N}_\infty) = L_0(X \to X) = L_0(X).$$

Note that the multiplication in X is not necessarily commutative.

Some results can also be proved for left invertible operators. Write

- $\Lambda(X)$ is the set of all *left invertible* operators belonging to $L_0(X)$ (we assume that $\operatorname{dom} T = X$ for $T \in \Lambda(X)$);
- $\mathcal{L}_T = \{S \in L(X) : ST = I\}$ is the set of all *left inverses* to $T \in \Lambda(X)$;
- $\mathcal{G}_T = \{G \in L(X) : G^2 = G,\ \ker G = TX,\ \exists_{S \in \mathcal{L}_T} SQ = 0\}$ is the set of all *co-initial* operators for $T \in \Lambda(X)$ (cf. Theorem 2.4 and Corollary 2.8).
- $\mathcal{I}(X) = R(X) \cap \Lambda(X)$ is the set of invertible operators ([1]).

Clearly, if $\ker D \neq \{0\}$, then the operator D is right invertible, but not invertible. Here the invertibility of an operator $A \in L(X)$ means that the equation $Ax = y$ has a unique solution for every $y \in X$. If $D \in \mathcal{I}(X)$ then $\mathcal{F}_D = \mathcal{G}_D = \{0\}$ and $\mathcal{R}_D = \mathcal{L}_D = \{D^{-1}\}$.

The sets defined above are, in a sense, dual. Namely, if $D \in R(X)$ and $F \in \mathcal{F}_D$ corresponds to an $R \in \mathcal{R}_D$, i.e. $FR = 0$, then $R \in \Lambda(X)$ and $F \in \mathcal{G}_D$ corresponds to $D \in \mathcal{L}_D$.

With these facts one can obtain Calculus and solutions to linear equations (under appropriate assumptions on the equations considered). If the

([1]) Note that the domain of a left invertible operator is the whole space X, so, in this case, instead of a paraalgebra we have an algebra.

field $\mathbb{F}$ is algebraically closed then solutions of linear equations with scalar coefficients can be calculated by a decomposition of a rational function into vulgar fractions (as in Operational Calculus).

Recall that any solution of the equation

$$Dx = y, \quad y \in X,\ D \in R(X),$$

is of the form $x = Ry + z$, where $R \in \mathcal{R}_D$ and $z \in \ker D$ is arbitrary. This form is *independent* of the choice of R. Indeed, if $R' \in \mathcal{R}_D$ and $R' \neq R$ then for all $x \in X$ we have $R'x - Rx \in \ker D$. So a change of R implies only a change of the constant z.

The equation

$$Tx = y, \quad y \in TX,\ T \in \Lambda(X),$$

has a unique solution $x = Sy$ which is independent of the choice of $S \in \mathcal{L}_T$. Indeed, by definition, $x = STx = Sy$ and $\ker T = \{0\}$. Write

$$Q(D) = \sum_{k=0}^{N} Q_k D^k, \quad \text{where } D \in R(X),\ Q_0, \ldots, Q_{N-1} \in L(X),\ Q_N = I.$$

If there is an $R \in \mathcal{R}_D$ such that the operator

$$Q(I, R) = \sum_{k=0}^{N} Q_k R^{N-k}$$

is invertible, then $Q(D) \in R(X)$ and $R^N[Q(I,R)]^{-1} \in \mathcal{R}_{Q(D)}$. Thus an *initial value problem* for $Q(D)$, that is, the problem of finding solutions of the equation $Q(D)x = y$, $y \in X$, with the initial conditions

$$FD^k x = y_k, \quad \text{where } y_k \in \ker D \text{ are given } (k = 0, 1, \ldots, N-1),$$

has for every $y, y_0, \ldots, y_{N-1}$ a unique solution of the form

$$x = R^N[Q(I,R)]^{-1}\Big[y - \sum_{m=0}^{N-1}\Big(\sum_{k=0}^{m} Q_k R^{m-k}\Big) y_m\Big] + \sum_{k=0}^{N-1} R^k y_k,$$

i.e. this problem is well-posed.

An $A \in L_0(X)$ is said to be a *Volterra operator* if the operators $I - \lambda A$ are invertible for every $\lambda \in \mathbb{F}$, i.e. if for every $\lambda \in \mathbb{F} \setminus \{0\}$ the number $1/\lambda$ is a regular value of A. The set of all Volterra operators belonging to $L(X)$ will be denoted by $V(X)$.

Main advantages of Algebraic Analysis are: (i) *simplifications of proofs* due to an algebraic description of problems under consideration; (ii) *algorithms* for solving "similar" problems, although these similarities could be rather far from one another and very formal, and (iii) several new results even in the classical case of the operator d/dt, which was, indeed, unexpected (for various extensions of the above results, examples and applications cf. [PR3]–[PR13], [PRW1], [PRW2], [M1], [M2], [N2], [T], [V], and others).

There are several applications to ordinary and partial differential equations with scalar and variable coefficients, functional-differential equations and discrete analogues of these equations, for instance, difference equations. There are also some results for non-linear equations.

It should be pointed out that in Algebraic Analysis a notion of *convolution* is not necessary. Also there is no need to have a structure of the Mikusiński field. This, together with the noncommutativity of right inverses and initial operators, shows the essential distinction of Algebraic Analysis from Operational Calculus.

EXAMPLE 3.1. Let $X = C[0,1]$ over $\mathbb{F}$, where $\mathbb{F} = \mathbb{R}$ or $\mathbb{C}$, and let $D = d/dt$. Then $\ker D$ consists of all constant functions. The operators $R_a = \int_a^t$, $a \in [0,1]$, are Volterra right inverses of D. Observe that $\dim \ker D = 1$. The family of initial operators induced by R_a is defined as follows: $(F_a x)(t) = x(a)$ for $x \in X$ and $a \in [0,1]$. Consider operators of the form

$$(\widetilde{F}_a x)(t) = \frac{\int_a^1 m(s)x(s)\,ds}{\int_a^1 m(s)\,ds} \quad \text{for } x \in X,\ a \in [0,1],\ m \in X,$$

$$\lambda F_a + (1-\lambda)\widetilde{F}_b, \quad \lambda F_a + (1-\lambda)F_b, \quad \text{for } b \in [0,1],\ \lambda \in \mathbb{F}.$$

These are also initial operators, because they are projections *onto* the space of constants. In several particular cases the corresponding right inverses have eigenvalues.

If we consider the space $C[0,1]$ over $\mathbb{C}$ then the only continuous Volterra right inverses are $\int_a^t$, $a \in [0,1]$. However, if $\mathbb{F} = \mathbb{R}$, then one can find a Volterra right inverse which is not of that form (cf. [PRR2], [PRR3]).

The Taylor formula applied to a function $x \in \operatorname{dom} D^N = C^N[0,1]$ gives a classical Taylor formula with remainder in integral form. In order to obtain remainders either in the Lagrange form or in the Cauchy form it is enough to apply the Darboux property of continuous functions satisfying two different estimates, without any intermediate value theorem, like the Rolle theorem or Lagrange theorem.

Observe that $X = C[0,1]$ is a commutative algebra (a linear ring) with respect to the usual pointwise multiplication. The operator D satisfies the so-called *Leibniz condition*:

$$D(xy) = xDy + yDx \quad \text{for } x, y \in \operatorname{dom} D = C^1[0,1].$$

By this condition, $\mathcal{D}(C^k[0,1];\mathbb{N}_\infty)$ is a Leibniz paraalgebra.

We point out that the restriction to the interval $[0,1]$ is not essential. The same considerations are valid also for any interval $[a,b]$ and for a half-axis or the real line.

EXAMPLE 3.2. Let $X = C[0,T]$ with multiplication defined by convolution $*$ and let $D = d/dt$. Then X is not a Leibniz algebra, for the following Duhamel condition holds:

$$\frac{d}{dt}(x*y)(t) = \left(x * \frac{d}{dt}\right)(t) + x(t)y(0) - x(0)y(t), \qquad x \in C[0,T),\ y \in C^1(0,T). \tag{3.7}$$

If we write a similar formula for $y * x$ then, by commutativity, we find

$$x * y = \frac{1}{2}(x*y + y*x) = \frac{1}{2}\left(x * \frac{d}{dt} + \frac{d}{dt}x * y\right),$$

which implies that the non-Leibniz component f_D is 0, but $c_D = \frac{1}{2}$. Thus $\mathcal{D}(C^k[0,T];\ k \in \mathbb{N}_\infty)$ is a paraalgebra, but it is not a Leibniz paraalgebra.

Note that in both examples we have $\operatorname{dom} D \subset X$ but $\operatorname{dom} D \neq X$. We should point out that in order to solve some problems, it is necessary to assume that the field $\mathbb{F}$ of scalars under consideration is *algebraically closed*. Clearly, if $\mathbb{F} = \mathbb{R}$, we may consider a natural extension to $\mathbb{C}$.

4. Logarithms and antilogarithms in paraalgebras with the Kaplansky–Jacobson property. Let X be an algebra (over a field of scalars $\mathbb{F}$ with characteristic zero) and let $D \in \mathbf{A}(X)$. As in the previous section, consider the paraalgebra

$$\mathcal{D}(D_k;\mathbb{N}_\infty) = L_0(D_k \rightleftharpoons D_j;\ j,k \in \mathbb{N}_\infty),$$

which, by Corollary 2.4, has the Kaplansky–Jacobson property. Write $I(X)$ for the set of all *invertible* elements in X.

LEMMA 4.1. *Suppose that* $a \in \mathcal{D}(D_k;\mathbb{N}_\infty)$, $a \in M_{a|a}$ *and* $u, Dx \in M_{a|a} \cap \operatorname{dom} D$. *Then* uDx, $(Dx)u$ *exist and belong to* $M_{a|a}$.

Proof. By the axiom $\mathbf{A}_1$, the existence of $u = ue_a$, $e_a = e_a^2$, $e_a Dx$ implies the existence of uDx. Moreover, $a(uDx)a = (au)[(Dx)]a$, i.e. $uDx \in M_{a|a}$. A similar proof for $(Dx)u$. ■

DEFINITION 4.1. Suppose that $a \in \mathcal{D}(D_k;\mathbb{N}_\infty)$ and $u, Dx \in M_{a|a} \cap \operatorname{dom} D$ for all a such that $a \in M_{a|a}$. Let $\Omega_r, \Omega_l : \operatorname{dom} D \to 2^{\operatorname{dom} D}$ be multifunctions defined as follows: for $u \in \operatorname{dom} D$,

$$\Omega_r u = \{x \in \operatorname{dom} D : uDx \text{ exists} \Rightarrow Du = uDx\}, \tag{4.1}$$

$$\Omega_l u = \{x \in \operatorname{dom} D : (Dx)u \text{ exists} \Rightarrow Du = (Dx)u\}. \tag{4.2}$$

The equations

$$\begin{aligned} Du &= uDx &&\text{for } (u,x) \in \operatorname{graph} \Omega_r, \\ Du &= (Dx)u &&\text{for } (u,x) \in \operatorname{graph} \Omega_l \end{aligned} \tag{4.3}$$

are said to be the *right* and *left basic equations*, respectively. Clearly,

$$\Omega_r^{-1}x = \{u \in \operatorname{dom} D : uDx \text{ exists} \Rightarrow Du = uDx\}, \tag{4.4}$$

$$\Omega_l^{-1}x = \{u \in \operatorname{dom} D : (Dx)u \text{ exists} \Rightarrow Du = (Dx)u\} \tag{4.5}$$

for $x \in \operatorname{dom} D$.

If $D \in \mathsf{A}(X)$ then $\Omega_r = \Omega_l$ and we write $\Omega_r = \Omega_l = \Omega$. Clearly, in this case we have $\operatorname{dom} \Omega_r = \operatorname{dom} \Omega_l = \operatorname{dom} \Omega$.

The multifunctions Ω_r, Ω_l, Ω are well defined. If D is right (left) invertible or invertible, their domains are non-empty (cf. [PR4] for the case of algebras). By definition, if $x, y \in \Omega_r$ (respectively, Ω_l, Ω) then $x-y \in \ker D$, i.e. it is a *constant*. Moreover, these multifunctions map $\ker D$ into itself.

Suppose that $(u_r, x_r) \in \operatorname{graph} \Omega_r$, $(u_l, x_l) \in \operatorname{graph} \Omega_l$, L_r, L_l are selectors of Ω_r, Ω_l, respectively, and E_r, E_l are selectors of Ω_r^{-1}, Ω_l^{-1}, respectively. By definition, $L_r u_r \in \operatorname{dom} \Omega_r^{-1}$, $E_r x_r \in \operatorname{dom} \Omega_r$, $L_l u_l \in \operatorname{dom} \Omega_l^{-1}$, $E_l x_l \in \operatorname{dom} \Omega_l$ and the following equations are satisfied:

$$Du_r = u_r D L_r u_r, \qquad D E_r x_r = (E_r x_r) D x_r;$$
$$Du_l = (D L_l u_l) u_l, \qquad D E_l x_l = (D x_l) E_l x_l.$$

Let F be an initial operator for D corresponding to an $R \in \mathcal{R}_D$ and let $u_r, u_l \in I(X)$. Then these equations can be written in equivalent forms

$$L_r u_r = R[u_r^{-1} D u_r] + F L_r u_r, \qquad E_r x_r - R[(E_r x_r)(D x_r)] = F E_r x_r;$$
$$L_l u_l = R[(D u_l) u_l^{-1}] + F L_l u_l, \qquad [I - R(D x_l)] E_l x_l = F E_l x_l.$$

Definition 4.2. Any invertible selector L_r of Ω_r is said to be a *right logarithmic mapping* and its inverse $E_r = L_r^{-1}$ is said to be a *right antilogarithmic mapping*. If $(u_r, x_r) \in \operatorname{graph} \Omega_r$ and L_r is an invertible selector of Ω_r then the element $L_r u_r$ is said to be a *right logarithm* of u_r and $E_r x_r$ is said to be a *right antilogarithm* of x_r. By $G[\Omega_r]$ we denote the set of all pairs (L_r, E_r), where L_r is an invertible selector of Ω_r and $E_r = L_r^{-1}$. Similarly, any invertible selector L_l of Ω_l is said to be a *left logarithmic mapping* and its inverse $E_l = L_l^{-1}$ is said to be a *left antilogarithmic mapping*. If $(u_l, x_l) \in \operatorname{graph} \Omega_l$ and L_l is an invertible selector of Ω_l then the element $L_l u$ is said to be a *left logarithm* of u_l and $E_l x_l$ is said to be a *left antilogarithm* of x_l. By $G[\Omega_l]$ we denote the set of all pairs (L_l, E_l), where L_l is an invertible selector of Ω_l and $E_l = L_l^{-1}$.

If $D \in \mathsf{A}(X)$ then $\Omega_l = \Omega_r = \Omega$ and we write

$$L_r = L_l = L, \qquad E_r = E_l = E, \qquad (L, E) \in G[\Omega].$$

Selectors L, E of Ω are said to be *logarithmic* and *antilogarithmic* mappings, respectively. For any $(u, x) \in \operatorname{graph} \Omega$, $(L, E) \in G[\Omega]$ the elements Lu, Ex are said to be the *logarithm* of u and the *antilogarithm* of x, respectively.

Note that in the paraalgebra defined in Example 3.2 invertible selectors of the multifunction Ω, hence also logarithms, do not exist.

Clearly, by definition, for all $(L_r, E_r) \in G[\Omega_r]$, $(u_r, x_r) \in \text{graph } \Omega_r$, $(L_l, E_l) \in G[\Omega_l]$, $(u_l, x_l) \in \text{graph } \Omega_l$ we have

$$E_r L_r u_r = u_r, \quad L_r E_r x_r = x_r; \quad E_l L_l u_l = u_l, \quad L_l E_l x_l = x_l; \tag{4.6}$$

$$\begin{gathered} D E_r x_r = (E_r x_r) D x_r, \qquad D u_r = u_r D L_r u_r; \\ D E_l x_l = (D x_l)(E_l x_l), \qquad D u_l = (D L_l u_l) u_l. \end{gathered} \tag{4.7}$$

Moreover,

$$u D L_r u - (D L_l u) u = 0 \quad \text{for all } u \in \text{dom}\, \Omega_r \cap \text{dom}\, \Omega_l. \tag{4.8}$$

If $D \in R(X)$ (respectively, $\Lambda(X)$, $I(X)$) then the right (left) logarithm of zero is not defined. On the other hand, $E_r(0), E_l(0), E(0) \in \ker D$.

Definition 4.3. A right logarithmic mapping L_r (left logarithmic mapping L_l, logarithmic mapping L, respectively) is said to be of the *exponential type* if whenever uv exists, then

$$\begin{aligned} L_r(uv) &= L_r u + L_r v \quad && \text{for } u, v \in \text{dom}\, \Omega_r; \\ L_l(uv) &= L_l u + L_l v \quad && \text{for } u, v \in \text{dom}\, \Omega_l; \\ L(uv) &= Lu + Lv \quad && \text{for } u, v \in \text{dom}\, \Omega, \end{aligned}$$

respectively.

Theorem 4.1. *Let* $D \in R(X)$.

(i) *If* $D \in \mathbf{A}(X)$, $(L_r, E_r) \in G[\Omega_r]$, $(L_l, E_l) \in G[\Omega_l]$ *and* L_r (L_l, *respectively*) *is of the exponential type then*

$$E_r(x+y) = (E_r x)(E_r y), \quad E_l(x+y) = (E_l x)(E_l y),$$

whenever $x, y \in \text{dom}\, \Omega_r^{-1}$ ($x, y \in \text{dom}\, \Omega_l^{-1}$, *respectively*);

(ii) *if* $D \in \mathsf{A}(X)$, $(L, E) \in G[\Omega]$ *and* L *is of the exponential type then*

$$E(x+y) = (Ex)(Ey) \quad \textit{for } \ x, y \in \text{dom}\, \Omega^{-1};$$

(iii) *if* $L_r \in M(X)$ ([1]) *then* $E_r \in M(X)$, *if* $L_l \in M(X)$ *then* $E_l \in M(X)$, *if* $L \in M(X)$ *then* $E \in M(X)$;

(iv) *if* $D \in \mathsf{A}(X) \cap ML(X)$ *then* $L, E \in M(X)$.

By Theorem 4.1(ii), if left (right) logarithmic mappings are of the exponential type, then the corresponding left (right) antilogarithmic mappings

([1]) By $M(X)$ we denote the class of all *multiplicative* mappings of X into itself, i.e. mappings A such that $A(xy) = (Ax)(Ay)$ whenever xy and $(Ax)(Ay)$ exist.

satisfy the same functional equation as the usual exponential functions $ce^{\alpha t}$, $c, \alpha \in \mathbb{R}$ (cf. M. Kuczma [K])). A similar conclusion for logarithmic mappings.

Definition 4.4. Let $D \in \mathbf{A}(X)$. By $\mathbf{Lg}_r(D)$ ($\mathbf{Lg}_l(D)$, respectively) we denote the class of those D-paraalgebras with units $e_a \in \operatorname{dom} \Omega_r$ ($e \in \operatorname{dom} \Omega_l$, respectively), $a \in M_{a|a}$, for which there exist invertible selectors of Ω_r (Ω_l, respectively), i.e. there exist $(L_r, E_r) \in G[\Omega_r]$ ($(L_l, E_l) \in G[\Omega_l]$, respectively). If $D \in \mathsf{A}(X)$ then $\mathbf{Lg}_r(D) = \mathbf{Lg}_l(D)$. This class is denoted by $\mathbf{Lg}(D)$.

Theorem 4.2. *Let either $X \in \mathbf{Lg}_r(D)$ or $X \in \mathbf{Lg}_l(D)$. Let $R \in \mathcal{R}_D$ and $g = Re$. Then $g \in \operatorname{dom} \Omega_r$ ($g \in \operatorname{dom} \Omega_l$, $g \in \operatorname{dom} \Omega$, respectively) if $g \in I(X)$. In other words: a right logarithm (left logarithm, logarithm) of $g = Re$ exists if g is invertible.*

Theorem 4.3. *Let $X \in \mathbf{Lg}_r(D)$ ($X \in \mathbf{Lg}_l(D)$, $X \in \mathbf{Lg}(D)$, respectively). Then right (left) logarithms and antilogarithms (logarithms and antilogarithms, respectively) are uniquely determined up to a constant. These constants are additive for right (left) logarithms and logarithms and multiplicative for right (left) antilogarithms and antilogarithms.*

Theorem 4.4. *Suppose that $D \in \mathsf{A}(X)$, $X \in \mathbf{Lg}(D)$ and $(L, E) \in G[\Omega]$. Then L is of the exponential type if and only if X is a Leibniz algebra.*

In other words: *In commutative algebras with unit the Leibniz condition is a necessary and sufficient condition for logarithms to be of the exponential type. Thus, by Proposition* 2.6, *the Leibniz condition is a necessary and sufficient condition for the corresponding antilogarithms to satisfy the classical functional equation for exponential functions: $E(x + y) = (Ex)(Ey)$ whenever $x, y \in \operatorname{dom} \Omega^{-1}$* (cf. Theorem 4.2).

Theorem 4.4 motivates the use of the name *antilogarithmic* mapping for the mapping inverse to a logarithmic mapping, since, in general, antilogarithmic mappings are not exponentials.

Theorem 4.5. *Suppose that $D \in R(X)$ and $X \in \mathbf{Lg}(D)$ is a Leibniz algebra with unit e. Let $u \in \operatorname{dom} D$. Then $u \in I(X)$ if and only if $u \in \operatorname{dom} \Omega$.*

In other words: *An essential property of Leibniz commutative algebras with right invertible operators is that their elements have logarithms if and only if they are invertible.* Note that in noncommutative algebras (hence also paraalgebras) Theorem 4.5 does not hold, as shown by a counterexample of A. Di Bucchianico (cf. [DB1], [DB2], also [PR4]).

Proofs are similar to those of [PR4] and following papers (cf. [PR7]–[PR13]).

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