

On a variant of the Hardy inequality between weighted Orlicz spaces

by

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Abstract. Let M be an N -function satisfying the Δ_2 -condition, and let ω, φ be two other functions, with $\omega \geq 0$. We study Hardy-type inequalities

$$\int_{\mathbb{R}_+} M(\omega(x)|u(x)) \exp(-\varphi(x)) dx \leq C \int_{\mathbb{R}_+} M(|u'(x)|) \exp(-\varphi(x)) dx,$$

where u belongs to some set \mathcal{R} of locally absolutely continuous functions containing $C_0^\infty(\mathbb{R}_+)$. We give sufficient conditions on the triple (ω, φ, M) for such inequalities to be valid for all u from a given set \mathcal{R} . The set \mathcal{R} may be smaller than the set of Hardy transforms. Bounds for constants are also given, yielding classical Hardy inequalities with best constants.

1. INTRODUCTION

General framework and classical approach. Hardy type inequalities have been the subject of intensive research, going back to Hardy, who in the early 1920's ([25, 26]) obtained inequalities of the form

$$(1.1) \quad \int_{\mathbb{R}_+} |u(t)|^p t^{\alpha-p} dt \leq C \int_{\mathbb{R}_+} |u'(t)|^p t^\alpha dt.$$

This inequality was generalized to

$$\left(\int_{\mathbb{R}_+} \left| \int_0^t f(\tau) d\tau \right|^q d\mu(t) \right)^{1/q} \leq C \left(\int_{\mathbb{R}_+} |f(t)|^p d\nu(t) \right)^{1/p},$$

with two nonnegative Radon measures μ, ν , and further to inequalities in the Orlicz-space setting:

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$$(1.2) \quad Q^{-1} \left(\int_{\mathbb{R}_+} Q(\omega(x)|Tf(x)|)r(x) dx \right) \leq P^{-1} \left(\int_{\mathbb{R}_+} P(C\rho(x)|f(x)|)v(x) dx \right).$$

Inequalities in L^p were studied by Muckenhoupt [46], Maz'ya and Rozin [44], Bradley [8], Kokilashvili [33], Sinnamoni [53], Sawyer [51], Bloom and Kerman [6], Stepanov [54], and many others (we refer to the monographs [38, 36, 37, 34, 45] and references therein). As to the Orlicz-space result (1.2), several authors contributed to a complete characterization of admissible weights ω, r, ρ, v and nondecreasing functions P, Q which allow for (1.2) with $Tf(x) = \int_0^x K(x, y)f(y) dy$ being the generalized Hardy operator with kernel K . To name just a few, we refer to the papers of Bloom and Kerman [5, 6], Lai [40, 41, 42, 43], Heinig and Maligranda [28], Heinig and Lai [27] and their references. For a more detailed account of such results, we refer to Sections 3.1 and 3.2.

Hardy-type inequalities are widely applicable in PDE theory and functional analysis. For example, one can derive various Sobolev embedding theorems in the L^p setting, which can then be used to prove the existence of solutions of the Cauchy problem for elliptic and parabolic PDEs (see e.g. [9, 22, 36, 15, 44, 55, 39]), to study the asymptotic behaviour of solutions [2, 57], as well as their stability [10, 11]. They are present in probability theory (see e.g. [17, 50, 19]). Hardy inequalities are also interesting in their own right (see e.g. [37, 38, 56]). For latest results, see the very recent papers [9, 13, 23, 47] and their references.

Investigation of weighted or nonweighted Orlicz–Sobolev spaces defined by an N -function different from λ^p is suggested by physical models (see e.g. [1, 3, 14, 48, 15]). Therefore it is worthwhile to examine Hardy-type inequalities in general Orlicz spaces as well.

One of the central problems around Hardy inequalities can be expressed as follows. Consider the sets \mathcal{H} of Hardy transforms and \mathcal{H}^* of conjugate Hardy transforms:

$$(1.3) \quad \begin{aligned} \mathcal{H} &= \left\{ u(t) = \int_0^t f(s) ds : \int_0^a |f(\tau)| d\tau < \infty \text{ for every } a > 0 \right\} \\ &= \{ u \in W_{\text{loc}}^{1,1}(\mathbb{R}_+) : \lim_{r \rightarrow 0} u(r) = 0 \}, \\ \mathcal{H}^* &= \left\{ u(t) = \int_t^\infty f(s) ds : \int_a^\infty |f(\tau)| d\tau < \infty \text{ for every } a > 0 \right\} \\ &= \{ u \in W_{\text{loc}}^{1,1}(\mathbb{R}_+) : \lim_{r \rightarrow \infty} u(r) = 0 \}. \end{aligned}$$

and let Tf be either the Hardy transform of f , or the conjugate Hardy transform of f . Then the following problem arises.

PROBLEM 1 (classical). Given two N -functions Q, P , describe all possible weights (ω, r, ϱ, v) for which inequality (1.2) holds with some constant C independent of f (with T the Hardy transform or the conjugate Hardy transform).

This problem has been completely solved for the Hardy transform by Bloom and Kerman in [6] for modular functions P and Q such that Q dominates P in some special sense (for the details see Section 3.2). A further generalization of (3.3), without the domination restriction, can be found in Lai's paper [43]. Therefore Problem 1 can be considered as settled.

Another approach and its motivation. We are concerned with another problem, which is expressed as follows.

PROBLEM 2 (general). Given two N -functions Q, P and weights (ω, r, ϱ, v) , find a possibly large set \mathcal{R} contained in the set of locally absolutely continuous functions for which the inequality (1.2) holds with some constant C independent of $u \in \mathcal{R}$.

Let us make some comments here.

It can happen that the given weights (ω, r, ϱ, v) obey the known requirements for the validity of (1.2), say for Tf being the Hardy transform of f , as described in Problem 1. In that case the set \mathcal{R} contains the full set \mathcal{H} of Hardy transforms. On the other hand, if this requirement is not satisfied, we cannot expect (1.2) to hold for every $u \in \mathcal{H}$. In such a case $\mathcal{R} \cap \mathcal{H}$ will be a proper subset of \mathcal{H} . The solution to Problem 2 would therefore lead to Hardy-type inequalities in a (possibly) narrower class of functions.

If we substitute $u = Tf$ in (1.2), where T is either the Hardy transform or the conjugate Hardy transform, then $u' = f$, so that (1.2) reduces to the special case of (1.2).

The reduced problem and partial answers. In this paper we deal with a special variant of Problem 2, which reads as follows.

PROBLEM 3 (reduced). Given an N -function M satisfying the Δ_2 -condition, and a pair of functions (ω, φ) where $\omega \geq 0$, describe a possibly large set \mathcal{R} contained in the set of locally absolutely continuous functions for which the inequality

$$(1.4) \quad \int_{\mathbb{R}_+} M(\omega(x)|u(x)|) \exp(-\varphi(x)) dx \leq C \int_{\mathbb{R}_+} M(|u'(x)|) \exp(-\varphi(x)) dx$$

holds with some constant C independent of $u \in \mathcal{R}$.

Variants of (1.4) with $M(\lambda) = \lambda^p$ and \mathcal{R} determined by constraints concerning M, ω, φ have been studied e.g. in [4, 16, 20, 21] (see also their references). To the best of our knowledge their extension to the Orlicz space setting has not been considered so far.

Our main result (Theorem 2.1) states that if the triple (ω, φ, M) satisfies certain simple compatibility conditions, then we can indicate a set \mathcal{R} such that (1.4) holds for every $u \in \mathcal{R}$. Moreover, we give some bounds on the constant C , which can be expressed in terms of the Simonenko lower and upper indices (see [52] and [24], [18] for interesting related results).

The condition $u \in \mathcal{R}$ depends on the behaviour of u near zero and near infinity, which is very natural in problems arising from PDEs: when analyzing a particular equation one can often say that its solution (i.e. our function u) has some “good” properties near the boundary, expressed in terms of its rate of decay near the boundary.

As an illustration we derive the classical Hardy inequalities (1.1) with best constants (see Section 3.1.1 for discussion).

We also obtain sufficient conditions for (1.4) to hold for every $u \in \mathcal{H}$ (see Proposition 5.2 in Section 5.2.1). These conditions can be easily implemented in practice, and since the verification of the classical Bloom–Kerman conditions (5.6) from [6] seems rather hard, by using our approach one can avoid that verification and quickly deduce that (1.4) is satisfied for every $u \in \mathcal{H}$.

Perhaps it is even more interesting to deal with the case when the Bloom–Kerman conditions (5.6) are not satisfied, so that inequality (1.4) is not valid for all $u \in \mathcal{H}$. Then we find a set \mathcal{R} such that $\mathcal{R} \cap \mathcal{H}$ is smaller than \mathcal{H} and (1.4) holds for all its elements (see Section 5.2.2).

As a particular type of inequalities like (1.4), we analyze those with $\omega = |\varphi'|$ (see Section 3.3), and in the class of admissible φ 's we obtain the inequality

$$(1.5) \quad \int_{\mathbb{R}_+} M(|\varphi'(x)|u(x)) \exp(-\varphi(x)) dx \leq C \int_{\mathbb{R}_+} M(|u'(x)|) \exp(-\varphi(x)) dx.$$

For $M(\lambda) = \lambda^p$ and $\psi(x) = \exp\{-\varphi(x)/p\}$ we get

$$\int_{\mathbb{R}_+} (|\psi'(x)u(x)|)^p dx \leq C \int_{\mathbb{R}_+} (|u'(x)\psi(x)|)^p dx,$$

which is nothing other than a particular case of the Caccioppoli inequality on \mathbb{R}^+ (see e.g. [12, 29]). Caccioppoli inequalities are commonly used in regularity theory, and so we believe that our variant (1.5) can be used in regularity theory as well.

2. PRELIMINARIES AND STATEMENTS OF MAIN RESULTS

2.1. Preliminaries

Orlicz spaces. Let us recall some preliminary facts about Orlicz spaces, referring e.g. to [49] for details. Here we deal with Orlicz spaces of functions

defined on \mathbb{R}_+ . Suppose that μ is a positive Radon measure on \mathbb{R}_+ and let $M : [0, \infty) \rightarrow [0, \infty)$ be an N -function, i.e. a continuous convex function satisfying $\lim_{\lambda \rightarrow 0} M(\lambda)/\lambda = 0$ and $\lim_{\lambda \rightarrow \infty} M(\lambda)/\lambda = \infty$.

The weighted Orlicz space L_μ^M we deal with is

$$L_\mu^M := \left\{ f : \mathbb{R}_+ \rightarrow \mathbb{R} \text{ measurable :} \right. \\ \left. \int_{\mathbb{R}_+} M(|f(x)|/K) d\mu(x) \leq 1 \text{ for some } K > 0 \right\},$$

equipped with the Luxemburg norm

$$\|f\|_{L_\mu^M} = \inf \left\{ K > 0 : \int_{\mathbb{R}_+} M(|f(x)|/K) d\mu(x) \leq 1 \right\}.$$

This norm is complete and turns L_μ^M into a Banach space. When μ is the Lebesgue measure, it is dropped from the notation. For $M(\lambda) = \lambda^p$ with $p > 1$, the space L_μ^M coincides with the usual L_μ^p space (defined on \mathbb{R}_+).

The symbol M^* denotes the complementary function of an N -function M , i.e. its Legendre transform: $M^*(y) = \sup_{x>0} [xy - M(x)]$ for $y \geq 0$. It is again an N -function and from its definition we get the Young inequality

$$xy \leq M(x) + M^*(y) \quad \text{for } x, y \geq 0.$$

M is said to satisfy the Δ_2 -condition if, for some constant $c > 0$ and every $\lambda > 0$, we have

$$(2.1) \quad M(2\lambda) \leq cM(\lambda).$$

In the class of differentiable convex functions the Δ_2 -condition is equivalent to

$$\lambda M'(\lambda) \leq \tilde{c}M(\lambda) \quad \text{for all } \lambda > 0,$$

with the constant \tilde{c} independent of λ (see e.g. [35, Theorem 4.1]).

We will need the following property of modulars (see [35, formula (9.21)]):

$$(2.2) \quad \int_{\mathbb{R}_+} M\left(\frac{f(x)}{\|f\|_{L_\mu^M}}\right) d\mu(x) \leq 1.$$

If M satisfies the Δ_2 -condition, then (2.2) becomes an equality.

The function M_1 is said to *dominate* M_2 if there exist two positive constants K_1, K_2 such that $M_2(\lambda) \leq K_1 M_1(K_2 \lambda)$ for every $\lambda > 0$. In that case we have

$$(2.3) \quad \|\cdot\|_{L_\mu^{M_2}} \leq K \|\cdot\|_{L_\mu^{M_1}} \quad \text{with } K = K_2(K_1 + 1).$$

Functions M_1 and M_2 are called *equivalent* when M_2 dominates M_1 and M_1 dominates M_2 . In particular, equivalent N -functions give rise to equivalent Luxemburg norms.

On the set $\mathcal{L}_\mu^M = \{u \text{ measurable} : \int M(|u|) d\mu < \infty\}$, one introduces the so-called *dual norm*:

$$(2.4) \quad \|u\|_{L_\mu^{(M)}} = \sup \left\{ \int_{\mathbb{R}_+} u(x)v(x) d\mu(x) : \right. \\ \left. v \in L_\mu^{M^*}, \int_{\mathbb{R}_+} M^*(|v(x)|) d\mu(x) \leq 1 \right\}.$$

The advantage of this norm is the Hölder-type inequality:

$$(2.5) \quad \int_{\mathbb{R}_+} f \cdot g d\mu \leq \|f\|_{L_\mu^{(M)}} \|g\|_{L_\mu^{(M^*)}} \quad \text{for } f \in L_\mu^M, g \in L_\mu^{M^*}.$$

If M satisfies the Δ_2 -condition, then $\mathcal{L}_\mu^M = L_\mu^M$, and in general, the Orlicz space L_μ^M is the completion of \mathcal{L}_μ^M in the dual norm. The Luxemburg norm and the dual norm are equivalent:

$$\|u\|_{L_\mu^{(M)}} \leq \|u\|_{L_\mu^M} \leq 2\|u\|_{L_\mu^{(M)}}.$$

Assumptions. Throughout the paper we assume:

(M) $M : [0, \infty) \rightarrow [0, \infty)$ is a differentiable N -function, i.e. M is convex, $M(0) = M'_+(0) = 0$, $M(\lambda)/\lambda \rightarrow \infty$ as $\lambda \rightarrow \infty$, and moreover

$$(2.6) \quad d_M \frac{M(\lambda)}{\lambda} \leq M'(\lambda) \leq D_M \frac{M(\lambda)}{\lambda} \quad \text{for every } \lambda > 0,$$

where $D_M \geq d_M \geq 1$.

(μ) μ is a Radon measure on \mathbb{R}_+ , absolutely continuous with respect to the Lebesgue measure, and $\mu(dr) = e^{-\varphi(r)}$, where $\varphi \in C^2(\mathbb{R}_+)$ and φ' never vanishes,

(ω) $\omega : (0, \infty) \rightarrow [0, \infty)$ is a C^1 -function.

REMARK 2.1. The last inequality in (2.6) implies that M satisfies the Δ_2 -condition (see (2.1)). The condition $d_M > 1$ is equivalent to the Δ_2 -condition for M^* (see e.g. [35, Theorem 4.3] or [30, Proposition 4.1]). Moreover, for any N -function M , the left-hand inequality in (2.6) holds with $d_M = 1$.

If d_M and D_M are the best possible constants in (2.6) they obey the definition of the Simonenko lower and upper indices of M and are related to Boyd indices of $L^M(\mathbb{R}^n, \mu)$ (see [7], [52] for definitions, and [18], [24], [58] for discussion on those and other indices of Orlicz spaces).

Notation. Before we proceed, we need to introduce the following quantities. We set

$$(2.7) \quad \begin{aligned} \Omega &:= \{r \in \mathbb{R}_+ : \omega(r)u(r) \neq 0\}, \\ F &:= \{r \in \mathbb{R}_+ : \omega(r) \neq 0, \omega'(r)\varphi'(r) > 0\}, \\ G &:= \{r \in \mathbb{R}_+ : \omega(r) \neq 0, \omega'(r)\varphi'(r) < 0\}. \end{aligned}$$

Then we define

$$\begin{aligned}
 (2.8) \quad b_1(r, \omega, \varphi, M) &:= \left(1 + \frac{\varphi''(r)}{(\varphi'(r))^2} - \frac{\omega'(r)}{\omega(r)\varphi'(r)} [d_M \chi_G(r) + D_M \chi_F(r)] \right), \\
 b_1 &:= b_1(\omega, \varphi, M) := \inf \{ b_1(r, \omega, \varphi, M) : r \in \mathbb{R}_+ \}, \\
 b_2(r, \omega, \varphi, M) &:= \left(-1 - \frac{\varphi''(r)}{(\varphi'(r))^2} + \frac{\omega'(r)}{\omega(r)\varphi'(r)} [d_M \chi_F(r) + D_M \chi_G(r)] \right), \\
 b_2 &:= b_2(\omega, \varphi, M) := \inf \{ b_2(r, \omega, \varphi, M) : r \in \mathbb{R}_+ \}, \\
 (2.9) \quad L = L(\omega, \varphi) &:= \sup \left\{ \frac{\omega(r)}{|\varphi'(r)|} : r \in (0, \infty), \varphi'(r) \neq 0 \right\}.
 \end{aligned}$$

We use the conventions $\sup \emptyset = -\infty$, $\inf \emptyset = +\infty$, $c/\infty = 0$; and $f\chi_A$ is the function f extended by 0 outside A .

2.2. Main results. Our area of interest will be those triples (M, φ, ω) for which either

- (B1) $b_1 > 0$, $L < \infty$, or
- (B2) $b_2 > 0$, $L < \infty$.

We will deal with the following function:

$$(2.10) \quad h^u(r) = h^{(u, \omega, \varphi, M)}(r) := \frac{1}{\varphi'(r)} M(\omega(r)|u(r)|),$$

which is well defined since $\varphi'(r)$ is never zero.

Let us introduce the following two classes of functions:

$$\begin{aligned}
 (2.11) \quad \mathcal{R}_{(\omega, \varphi, M)}^+ &:= \{ u \in W_{\text{loc}}^{1,1}(\mathbb{R}_+) : \\
 &\quad \lim_{n \rightarrow \infty} (h^u(R_n) e^{-\varphi(R_n)} - h^u(s_n) e^{-\varphi(s_n)}) \geq 0 \\
 &\quad \text{for some sequences } s_n \rightarrow 0, R_n \rightarrow \infty \},
 \end{aligned}$$

$$\begin{aligned}
 (2.12) \quad \mathcal{R}_{(\omega, \varphi, M)}^- &:= \{ u \in W_{\text{loc}}^{1,1}(\mathbb{R}_+) : \\
 &\quad \lim_{n \rightarrow \infty} (h^u(R_n) e^{-\varphi(R_n)} - h^u(s_n) e^{-\varphi(s_n)}) \leq 0 \\
 &\quad \text{for some sequences } s_n \rightarrow 0, R_n \rightarrow \infty \},
 \end{aligned}$$

where the limits may be infinite. For simplicity we will usually omit (ω, φ, M) from the notation.

Note that both sets \mathcal{R}^+ and \mathcal{R}^- contain the set of compactly supported $W_{\text{loc}}^{1,1}$ functions and that they sum up to the whole $W_{\text{loc}}^{1,1}(\mathbb{R}_+)$.

Our main result reads as follows.

THEOREM 2.1. *Suppose that M, φ, ω satisfy (M), (μ) , (ω) and (B1) (respectively (B2)). Then*

$$(2.13) \quad \int_{\mathbb{R}_+} M(\omega(r)|u(r)|) \mu(dr) \leq C \int_{\mathbb{R}_+} M(|u'(r)|) \mu(dr)$$

for every $u \in \mathcal{R}_{(\omega, \varphi, M)}^+$ (resp. $u \in \mathcal{R}_{(\omega, \varphi, M)}^-$), where $C = c(LD_M^2/b_1d_M)$ (resp. $C = c(LD_M^2/b_2d_M)$) with $c(x) = \max(x^{d_M}, x^{D_M})$.

As a direct consequence we also obtain the following theorem.

THEOREM 2.2. *Suppose that M, φ, ω satisfy (M), (μ) , (ω) and (B1) (respectively (B2)). Then*

$$\|\omega u\|_{L_\mu^M} \leq \tilde{C} \|u'\|_{L_\mu^M}$$

for every $u \in \mathcal{R}_{(\omega, \varphi, M)}^+$ (resp. $u \in \mathcal{R}_{(\omega, \varphi, M)}^-$) such that for every $\lambda > 0$, $\lambda u \in \mathcal{R}_{(\omega, \varphi, M)}^+$ (resp. $\lambda u \in \mathcal{R}_{(\omega, \varphi, M)}^-$) as well, and where $\tilde{C} = c(LD_M^2/b_1d_M) + 1$ (resp. $\tilde{C} = c(LD_M^2/b_2d_M) + 1$) with $c(x) = \max(x^{d_M}, x^{D_M})$.

3. PARTICULAR CASES

The main goal of this section is to illustrate Theorem 2.1 in various contexts. First we discuss the case when $M(\lambda) = \lambda^p$ (Subsection 3.1). Then we turn to general M (Subsection 3.2). Finally, in Subsection 3.3, we restrict ourselves to weights $\omega = |\varphi'|$.

3.1. Inequalities in the L^p setting. When $M(\lambda) = \lambda^p$ with $p > 1$, our conditions get simpler. In particular, $d_M = D_M = p$, and since φ' is assumed to be nonzero everywhere, we have

$$b_1(r, \varphi, \omega) = 1 + \frac{\varphi''(r)}{(\varphi'(r))^2} - p \frac{\omega'(r)}{\omega(r)\varphi'(r)} \chi_{\{\omega(r) \neq 0\}}(r) = -b_2(r, \varphi, \omega).$$

It follows that our theorem yields results when $L < \infty$ and either $\inf_{r>0} b_1(r, \varphi, \omega) > 0$, or $\sup_{r>0} b_1(r, \varphi, \omega) < 0$.

3.1.1. Classical Hardy inequalities. As the first example illustrating our methods, we get the classical Hardy inequality (see e.g. [26, Theorem 330] for the classical source, Theorem 5.2 in [36], or [38] for the statement, historical framework and discussion).

THEOREM 3.1. *Let $1 < p < \infty$ and $\alpha \neq p - 1$. Suppose that $u = u(t)$ is an absolutely continuous function in $(0, \infty)$ such that $\int_0^\infty |u'(t)|^p t^\alpha dt < \infty$, and let*

$$\begin{aligned} u^+(0) &:= \lim_{t \rightarrow 0} u(t) = 0 && \text{for } \alpha < p - 1, \\ u(\infty) &:= \lim_{t \rightarrow \infty} u(t) = 0 && \text{for } \alpha > p - 1. \end{aligned}$$

Then

$$(3.1) \quad \int_0^\infty |u(t)|^p t^{\alpha-p} dt \leq C \int_0^\infty |u'(t)|^p t^\alpha dt,$$

where $C = (p/|\alpha - p + 1|)^p$.

We consider the case $\alpha \neq 0$. Let us explain how this theorem follows from our results. Setting

$$M(r) = r^p, \quad \mu(dr) = r^\alpha dr = \exp(\alpha \ln r), \quad \omega(r) = 1/r,$$

we have

$$\begin{aligned} \varphi(r) &= -\alpha \ln r, & \varphi'(r) &= -\alpha/r, & \varphi''(r) &= \alpha/r^2, \\ \omega'(r) &= -1/r^2, & \omega'(r)\varphi'(r) &= \alpha/r^2 > 0, & d_M &= D_M = p. \end{aligned}$$

By a direct check we see that

$$b_1 = \frac{\alpha - (p-1)}{\alpha}, \quad b_2 = \frac{(p-1) - \alpha}{\alpha}, \quad L = \frac{1}{|\alpha|}.$$

Therefore for $\alpha > p-1$ and for $\alpha < 0$ we have $b_1 > 0$, while for $0 < \alpha < p-1$ we have $b_2 > 0$. In either case φ' never vanishes. The constant C in (2.13) is equal to $(p/|\alpha - (p-1)|)^p$, which coincides with the classical statement.

The only thing that remains to be checked is that any function u as in the statement of Theorem 3.1 for which the right hand side in (3.1) is finite belongs to \mathcal{R}^+ when $\alpha > p-1$ or $\alpha < 0$, and to \mathcal{R}^- when $0 < \alpha < p-1$. We will use standard arguments (see e.g. [36, proof of Theorem 5.2]).

First suppose that $\alpha > p-1$, and let u be as in the assumptions of Theorem 3.1. Then for any $t > 0$ we define $U(t) := \int_t^\infty |u(\tau)| d\tau$. One has

$$\begin{aligned} U(t) &= \int_t^\infty |u'(\tau)| d\tau \leq \int_t^\infty |u'(\tau)| \tau^{\alpha/p} \tau^{-\alpha/p} d\tau \\ &\leq \left(\int_t^\infty |u'(\tau)|^p \tau^\alpha d\tau \right)^{1/p} \left(\int_t^\infty \tau^{-\alpha/(p-1)} d\tau \right)^{(p-1)/p} \\ &= \left(\frac{p-1}{\alpha - (p-1)} \right)^{(p-1)/p} \left(\int_t^\infty |u'(\tau)|^p \tau^\alpha d\tau \right)^{1/p} t^{-(\alpha - (p-1))/p} < \infty. \end{aligned}$$

From this chain of inequalities and the condition $\int_0^\infty |u'(\tau)|^p \tau^\alpha d\tau < \infty$ we infer not only that $u \in W_{\text{loc}}^{1,1}(0, \infty)$ and that U is well defined, but also that $\lim_{R \rightarrow \infty} U(R)^p R^{\alpha - (p-1)} = 0$. Indeed, taking into account that $\lim_{t \rightarrow \infty} u(t) = 0$, we have

$$|u(t)| = \left| \int_t^\infty u'(\tau) d\tau \right| \leq \int_t^\infty |u'(\tau)| d\tau = U(t),$$

and therefore $\lim_{R \rightarrow \infty} |u(R)|^p R^{\alpha - (p-1)} = 0$ as well.

Recall now the formulas defining the class \mathcal{R}^+ . The function $h(r)e^{-\varphi(r)}$ appearing there is now equal to $-\alpha^{-1}|u(r)|^p r^{\alpha - (p-1)}$, vanishing for r tending to infinity. Therefore $u \in \mathcal{R}^+$.

When $\alpha < p-1$, we proceed similarly, but now we take $U(t) := \int_0^t |u'(\tau)| d\tau$. Again, U is well defined and $\lim_{r \rightarrow 0} U(r)^p r^{\alpha-(p-1)} = 0$, and since now $|u(r)| \leq U(r)$ as well, this yields $u \in \mathcal{R}^-$. We are done.

3.1.2. General approach within L^p -spaces. The following result is considered classical (see e.g. [44, Theorem 1 of Section 1.3.1]).

THEOREM 3.2. *Let μ, ν be nonnegative Borel measures on $(0, \infty)$, let ν^* be the absolutely continuous part of ν , and $1 \leq p \leq q \leq \infty$. Then*

$$(3.2) \quad \left(\int_{\mathbb{R}_+} \left| \int_0^t f(\tau) d\tau \right|^q d\mu(t) \right)^{1/q} \leq C \left(\int_{\mathbb{R}_+} |f(t)|^p d\nu(t) \right)^{1/p}$$

for all locally integrable functions f if and only if

$$B := \sup_{r>0} (\mu[r, \infty))^{1/q} \left(\int_0^r \left(\frac{d\nu^*}{d\tau} \right)^{-1/(p-1)} d\tau \right)^{(p-1)/p} < \infty.$$

The case $p = q \geq 1$ is due to Muckenhoupt [46]. Extensions to general p, q were proven by Mazya and Rozin ([44, Theorem 1 of Section 1.3.1]), Bradley [8] and Kokilashvili [33]. Some other generalizations (admitting also p, q below 1) were obtained by Sinnamon [53], Sawyer [51], Bloom and Kerman [6], Stepanov [54] and others.

Observe that inequality (2.13) corresponding to $\omega(r) = r$ is a particular case of (3.2) when one takes $\nu(dr) = e^{-\varphi(r)} dr$, $p = q$, $\mu(dr) = r^p e^{-\varphi(r)} dr$, but only for the representative $u(t) = \int_0^t u'(\tau) d\tau$. In particular, $u^+(0) = 0$, which we have not required (Theorem 3.1 shows that in general the condition $u^+(0) = 0$ may not hold). In this case (3.2) reads

$$\int_{\mathbb{R}_+} |u(\tau)\omega(\tau)|^p d\nu(\tau) \leq C \int_{\mathbb{R}_+} |u'(\tau)|^p d\nu(\tau),$$

and $u(t) = \int_0^t u'(s) ds$, u is an absolutely continuous function. The condition $B < \infty$ is equivalent to

$$\sup_{r>0} \left(\int_r^\infty x^p \exp(-\varphi(x)) dx \right) \left(\int_0^r \exp\left(\frac{\varphi(x)}{p-1}\right) dx \right)^{p-1} < \infty.$$

It is of different nature than our conditions (B1) and (B2) and is usually not easy to handle. But since $B < \infty$ is equivalent to the inequality (3.2) holding for all u in the set \mathcal{H} of Hardy transforms (see (1.3)), our assumptions can serve as a tool towards verifying $B < \infty$.

We may as well deal with the set \mathcal{H}^* of conjugate Hardy transforms $u = -\int_t^\infty f(\tau) d\tau$ (see (1.3)) instead of $u = \int_0^t f(\tau) d\tau \in \mathcal{H}$ as illustrated in Theorem 3.1, the case $\alpha > p-1$.

It may happen that inequalities (3.2) *do not hold* in general on the whole set \mathcal{H} , but they do hold on some smaller sets. These are the sets \mathcal{R}^+ , \mathcal{R}^- defined by (2.11) and (2.12).

3.2. Results in Orlicz spaces. The papers of Maz'ya, Bloom–Kerman and Lai [44, 6, 40, 41, 42, 43, 27] are concerned with inequalities of the form

$$(3.3) \quad Q^{-1}\left(\int_{\mathbb{R}_+} Q(\omega(x)|Tf(x)|)r(x) dx\right) \leq P^{-1}\left(\int_{\mathbb{R}_+} P(C\rho(x)|f(x)|)v(x) dx\right),$$

where Tf is the Hardy-type operator

$$Tf(x) = \int_0^x K(x, y)f(y) dy, \quad x > 0,$$

with a suitable kernel K . The case $K = 1$, corresponding to the classical Hardy operator, is also covered. P and Q are assumed to be nondecreasing functions on \mathbb{R}_+ satisfying

$$\lim_{t \rightarrow 0^+} P(t) = \lim_{t \rightarrow 0^+} Q(t) = 0, \quad \lim_{t \rightarrow \infty} P(t) = \lim_{t \rightarrow \infty} Q(t) = \infty.$$

Bloom and Kerman proved in [6] that within the class of modular functions P and Q satisfying the following domination property:

- there exists a constant $\eta > 0$ for which $\sum QP^{-1}(a_j) \leq QP^{-1}(\eta \sum a_j)$ whenever $\{a_j\}$ is any nonnegative sequence,

(3.3) is equivalent to the conditions

$$(3.4) \quad \begin{cases} \int_0^y P^*\left(\frac{G(\varepsilon, y)K(y, x)}{B\varepsilon v(x)\rho(x)}\right)v(x) dx \leq G(\varepsilon, y) < \infty, \\ \int_0^y P^*\left(\frac{H(\varepsilon, y)}{B\varepsilon v(x)\rho(x)}\right)v(x) dx \leq H(\varepsilon, y) < \infty, \end{cases}$$

holding for all $y > 0$ and $\varepsilon > 0$, where P^* is the Legendre transform of P ,

$$G(\varepsilon, y) = PQ^{-1}\left(\int_y^\infty Q(\varepsilon\omega(x))r(x) dx\right),$$

$$H(\varepsilon, y) = PQ^{-1}\left(\int_y^\infty Q(\varepsilon\omega(x)K(x, y))r(x) dx\right),$$

and $B > 0$ is a constant.

In our particular case: $P = Q (= M)$ with M satisfying the Δ_2 -condition, $r = v = \exp(-\varphi(x))$, $K \equiv 1$, $\rho \equiv 1$, inequality (3.3) reduces to

$$(3.5) \quad \int_{\mathbb{R}_+} M(\omega(x)Tf(x)) \exp(-\varphi(x)) dx \leq C \left(\int_{\mathbb{R}_+} M(f(x)) \exp(-\varphi(x)) dx \right),$$

which is an inequality of the type we are dealing with.

In this case conditions (3.4) simplify to

$$(3.6) \quad \int_0^y M^* \left(\frac{G(\varepsilon, y)}{B\varepsilon \exp(-\varphi(x))} \right) \exp(-\varphi(x)) dx \leq G(\varepsilon, y) < \infty$$

for all $y > 0$ and $\varepsilon > 0$, where

$$G(\varepsilon, y) = \int_y^\infty M(\varepsilon\omega(x)) \exp(-\varphi(x)) dx,$$

and $B > 0$ is a constant. A further generalization of (3.3), without the restriction $P \ll Q$, can be found in Lai's paper [43].

Condition (3.6) as well as Lai's condition are not easily implemented: in practice, for given ω, φ, M , it is usually hard to see whether (3.6) holds or not. Conditions (B1) and (B2) are much simpler. When they are satisfied, and when we know that the Hardy operator $Tf(x)$ (or the dual Hardy operator $T^*f(x) = \int_x^\infty f(\tau) d\tau$, see [43, last remark on page 671]) belongs to the set \mathcal{R}^- or \mathcal{R}^+ , then inequality (3.5) is just the statement of Theorem 2.1.

3.3. Special choice of weights. The case of $\omega = |\varphi'|$. Another case that substantially simplifies the approach is that of $\omega = |\varphi'|$ (in fact this was used in the proof of the classical Hardy inequality). Since we require φ to be C^1 with nonzero derivative, φ' is either always positive, or always negative.

This time, we have

$$b_1(r, |\varphi'|, \varphi, M) = \begin{cases} 1 + (1 - d_M) \frac{\varphi''(r)}{\varphi'(r)^2} & \text{if } \varphi''(r) \leq 0, \\ 1 + (1 - D_M) \frac{\varphi''(r)}{\varphi'(r)^2} & \text{if } \varphi''(r) \geq 0, \end{cases}$$

$$b_2(r, |\varphi'|, \varphi, M) = \begin{cases} -1 + (D_M - 1) \frac{\varphi''(r)}{(\varphi'(r))^2} & \text{if } \varphi''(r) \leq 0, \\ -1 + (d_M - 1) \frac{\varphi''(r)}{(\varphi'(r))^2} & \text{if } \varphi''(r) \geq 0, \end{cases}$$

$$L = 1.$$

In particular (as $D_M \geq d_M \geq 1$), we get

$$b_1 > 0 \quad \text{if and only if} \quad \sup_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} < \frac{1}{D_M - 1} =: \tilde{D}_M,$$

$$b_2 > 0 \quad \text{if and only if} \quad \inf_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} > \frac{1}{d_M - 1} =: \tilde{d}_M.$$

This leads to the following conclusion, which is of independent interest.

COROLLARY 3.1. *Assume that conditions (M) and (μ) are satisfied, and*

$$\text{either} \quad \sup_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} < \tilde{D}_M \quad \text{or} \quad \inf_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} > \tilde{d}_M.$$

Then

$$\int_{\mathbb{R}_+} M(|\varphi'(r)| |u(r)|) \exp(-\varphi(r)) \, dr \leq C \int_{\mathbb{R}_+} M(|u'(r)|) \exp(-\varphi(r)) \, dr$$

and

$$\|\varphi' u\|_{L_\mu^M} \leq \tilde{C} \|u'\|_{L_\mu^M}$$

for every $u \in \mathcal{R}_{(|\varphi'|, \varphi, M)}^+$ with $C = c(D_M^2/b_1 d_M)$, $\tilde{C} = c(D_M^2/b_1 d_M) + 1$ in the first case, and for every $u \in \mathcal{R}_{(|\varphi'|, \varphi, M)}^-$ with $C = c(D_M^2/b_2 d_M)$, $\tilde{C} = c(D_M^2/b_2 d_M) + 1$ in the other case, where $c(r) = \max(r^{d_M}, r^{D_M})$.

EXAMPLE 3.1 (classical inequalities). To illustrate this corollary we consider again $M(\lambda) = \lambda^p$, $p > 1$, $\varphi(\lambda) = -\alpha \ln r$ as in Section 3.1.1. In this case

$$d_M = D_M = p, \quad \frac{\varphi''}{(\varphi')^2} \equiv \frac{1}{\alpha}.$$

We have

$$\begin{aligned} \frac{1}{\alpha} &\equiv \sup_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} < \frac{1}{p-1} = \tilde{D}_M && \text{for } \alpha \in (-\infty, 0) \cup (p-1, \infty), \\ \frac{1}{\alpha} &\equiv \inf_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2} > \frac{1}{p-1} = \tilde{d}_M && \text{for } \alpha \in (0, p-1). \end{aligned}$$

Therefore the classical Hardy inequality follows from Corollary 3.1 as well.

EXAMPLE 3.2 (Hardy inequalities with logarithmic-type weights). As another illustration we show what inequality can be obtained for the measures $\mu(dr) = r^\alpha (\ln(1+r))^\beta dr$ with $\alpha, \beta > 0$.

In this case we have

$$\begin{aligned} \varphi(r) &=: \varphi_{\alpha, \beta}(r) = -\alpha \ln(r) - \beta \ln \ln(1+r), \\ \varphi'(r) &= -\frac{\alpha}{r} - \frac{\beta}{\ln(1+r)} \frac{1}{1+r}, \\ \varphi''(r) &= \frac{\alpha}{r^2} + \frac{\beta}{(1+r)^2} \frac{1}{\ln(1+r)} \left(1 + \frac{1}{\ln(1+r)} \right). \end{aligned}$$

Choose $\omega(r) = |\varphi'(r)|$; then

$$b_1(r) = 1 + (1 - D_M) \frac{\varphi''(r)}{(\varphi'(r))^2},$$

and so $b_1 = 1 - (D_M - 1) \sup_{r>0} \varphi''(r)/(\varphi'(r))^2$. As $\varphi'(r)$ is of order $1/r$ and $\varphi''(r)$ is of order $1/r^2$ on \mathbb{R}_+ , the supremum involved is finite. Set

$$(3.7) \quad s_{\alpha,\beta} = \sup_{r>0} \frac{\varphi''(r)}{(\varphi'(r))^2}.$$

Then

$$b_1 > 0 \Leftrightarrow D_M < 1 + \frac{1}{s_{\alpha,\beta}}.$$

As $1/r \sim \omega(r)$, we arrive at the following.

THEOREM 3.3. *Suppose $\alpha, \beta > 0$ and let M be an N -function satisfying (M) such that $D_M < 1 + 1/s_{\alpha,\beta}$, where $s_{\alpha,\beta}$ is given by (3.7). Then there exists a constant $C > 0$ such that*

$$\int_{\mathbb{R}_+} M\left(\frac{u(r)}{r}\right) r^\alpha (\ln(1+r))^\beta dr \leq C \int_{\mathbb{R}_+} M(|u'(r)|) r^\alpha (\ln(1+r))^\beta dr$$

for all $u \in \mathcal{R}^+(|\varphi'|, \varphi, M) \supset C_0^1(\mathbb{R}_+)$.

A similar analysis can be performed for negative α or β .

How to easily verify whether $u \in \mathcal{R}^+$ or $u \in \mathcal{R}^-$ will be shown in Subsection 5.1.

4. PROOFS OF THEOREMS 2.1 AND 2.2

Before we pass to the actual proofs, let us formulate two easy lemmas concerning Young functions. Although the lemmas may be known to specialists (see e.g. [24], [52] and [18] for related results), for the reader's convenience we supply the proofs.

LEMMA 4.1. *Let M be a differentiable N -function.*

(i) *Suppose that there exists a constant $D_M \geq 1$ such that*

$$(4.1) \quad M'(r) \leq D_M \frac{M(r)}{r} \quad \text{for every } r > 0.$$

Then for all $r > 0$ and $\lambda \geq 1$,

$$M(\lambda r) \leq \lambda^{D_M} M(r).$$

(ii) *Suppose that there exists a constant $d_M \geq 1$ such that*

$$d_M \frac{M(r)}{r} \leq M'(r) \quad \text{for every } r > 0.$$

Then for all $r > 0$ and $\lambda \leq 1$,

$$M(\lambda r) \leq \lambda^{d_M} M(r).$$

(iii) Suppose that there exist constants $1 \leq d_M \leq D_M$ such that

$$d_M \frac{M(r)}{r} \leq M'(r) \leq D_M \frac{M(r)}{r} \quad \text{for every } r > 0.$$

Then for all $r, \lambda > 0$,

$$(4.2) \quad M(\lambda r) \leq \max(\lambda^{d_M}, \lambda^{D_M}) M(r) =: c(\lambda) M(r).$$

We recall Remark 2.1 for interpretation of the constants d_M, D_M .

Proof. We only prove (i); (ii) is proven analogously, while (iii) is their direct consequence.

From (4.1) we get $M'(r)/M(r) \leq D_M/r$, and further, for any $r > 0$ and $\lambda > 1$,

$$\int_r^{\lambda r} \frac{M'(t)}{M(t)} dt \leq \int_r^{\lambda r} \frac{D_M}{t} dt,$$

which after integrating gives $[\ln M(t)]_r^{\lambda r} \leq [\ln t^{D_M}]_r^{\lambda r}$, and further $M(\lambda r) \leq \lambda^{D_M} M(r)$. ■

LEMMA 4.2. *Let M be a differentiable N -function, and suppose that $1 \leq d_M \leq D_M$ are two constants such that*

$$(4.3) \quad d_M \frac{M(r)}{r} \leq M'(r) \leq D_M \frac{M(r)}{r} \quad \text{for every } r > 0.$$

Then for all $r, s > 0$,

$$(4.4) \quad \frac{M(r)}{r} s \leq \frac{D_M - 1}{d_M} M(r) + \frac{1}{d_M} M(s).$$

Proof. Using the Young inequality $rs \leq M^*(r) + M(s)$ together with (4.3) we have

$$(4.5) \quad \frac{M(r)}{r} s \leq \frac{1}{d_M} M'(r) s \leq \frac{M^*(M'(r))}{d_M} + \frac{M(s)}{d_M}.$$

From the very definition of the conjugate function M^* we have

$$M(r) = rM'(r) - M^*(M'(r)),$$

and so

$$M^*(M'(r)) \leq D_M M(r) - M(r) = (D_M - 1)M(r).$$

Inserting this into (4.5) we get (4.4). ■

Proof of Theorem 2.1. Suppose $u \in \mathcal{R}_{(\omega, \varphi, M)}^+$ (resp. $u \in \mathcal{R}_{(\omega, \varphi, M)}^-$). Choose $s_n \rightarrow 0$ and $R_n \rightarrow \infty$ as in (2.11) (resp. (2.12)). To abbreviate,

we write

$$J := \int_0^{\infty} M(\omega(r))|u(r)|e^{-\varphi(r)} dr, \quad J_n := \int_{s_n}^{R_n} M(\omega(r))|u(r)|e^{-\varphi(r)} dr,$$

$$H := \int_0^{\infty} M(|u'(r)|)e^{-\varphi(r)} dr, \quad H_n := \int_{s_n}^{R_n} M(|u'(r)|)e^{-\varphi(r)} dr.$$

Let h^u be given by (2.10). Under our assumptions, it is well defined for every $r > 0$. Since $u \in W_{\text{loc}}^{1,1}(\mathbb{R}^n)$ and M is locally Lipschitz, we infer that $h^u \in W_{\text{loc}}^{1,1}(\mathbb{R}_+)$ and

$$(4.6) \quad (h^u)'(r) = \frac{d}{dr} \left(\frac{1}{\varphi'(r)} \right) M(\omega(r)|u(r)|) \\ + \frac{1}{\varphi'(r)} M'(\omega(r)|u(r)|)(\omega'(r)|u(r)| + \omega(r)u'(r) \operatorname{sgn} u(r)),$$

in the sense of distributions and almost everywhere, and h is absolutely continuous on each interval $[s, R] \subseteq (0, \infty)$ (see e.g. [44, Theorems 1 and 2, Sec. 1.1.3]). Moreover, for $0 < s < R < \infty$,

$$\int_s^R (h^u)'(r)e^{-\varphi(r)} dr = h^u(r)e^{-\varphi(r)} \Big|_s^R + \int_s^R M(\omega(r)|u(r)|)e^{-\varphi(r)} dr \\ =: \theta(R, s) + \int_s^R M(\omega(r)|u(r)|)e^{-\varphi(r)} dr,$$

and so we have

$$(4.7) \quad J_n = \int_{s_n}^{R_n} (h^u)'(r)e^{-\varphi(r)} dr - \theta_n,$$

where $\theta_n := \theta(R_n, s_n) \rightarrow \alpha \in [0, \infty]$ (resp. $[-\infty, 0]$).

Inserting (4.6) in (4.7) and using (2.7) yields, after some rearrangement,

$$(4.8) \quad J_n = \int_{s_n}^{R_n} \left(-\frac{\varphi''(r)}{(\varphi'(r))^2} \right) M(\omega(r)|u(r)|)e^{-\varphi(r)} dr \\ - \int_{s_n}^{R_n} \frac{1}{|\varphi'(r)|} M'(\omega(r)|u(r)|)|\omega'(r)||u(r)|\chi_G(r)e^{-\varphi(r)} dr \\ + \int_{s_n}^{R_n} \frac{1}{|\varphi'(r)|} M'(\omega(r)|u(r)|)(|\omega'(r)||u(r)|\chi_F(r) + \omega(r)u'(r)\operatorname{sgn} u(r))e^{-\varphi(r)} dr - \theta_n \\ =: I_n - II_n + III_n - \theta_n.$$

From now on, the proofs for the two cases: (B1) with $u \in \mathcal{R}^+$, and (B2) with $u \in \mathcal{R}^-$, differ slightly.

CASE 1. Assume that (B1) is satisfied and $u \in \mathcal{R}^+$. Since we have $-M'(\lambda) \leq -d_M M(\lambda)/\lambda$, it follows that

$$(4.9) \quad -II_n \leq - \int_{s_n}^{R_n} d_M M(\omega(r)|u(r)) \frac{|\omega'(r)|}{\omega(r)|\varphi'(r)} \chi_G(r) e^{-\varphi(r)} dr \\ = \int_{s_n}^{R_n} d_M M(\omega(r)|u(r)) \frac{\omega'(r)}{\omega(r)\varphi'(r)} \chi_G(r) e^{-\varphi(r)} dr =: IV_n.$$

For III_n , we first estimate $u'(r) \operatorname{sgn} u(r)$ by $|u'(r)|$, and then use the inequality $M'(\lambda) \leq D_M M(\lambda)/\lambda$. For every $r \in \Omega$ (see (2.7)) we have

$$(4.10) \quad M'(\omega(r)|u(r))(|\omega'(r)||u(r)|)\chi_F(r) + \omega(r)u'(r) \operatorname{sgn} u'(r) \\ \leq D_M \left[M(\omega(r)|u(r)) \frac{|\omega'(r)|}{\omega(r)} \chi_F(r) + M(\omega(r)|u(r)) \frac{|u'(r)|}{|u(r)|} \right] \\ =: D_M [A_1(r) + A_2(r)].$$

This implies

$$(4.11) \quad III_n \leq D_M \int_{\Omega \cap [s_n, R_n]} \frac{1}{|\varphi'(r)|} [A_1(r) + A_2(r)] e^{-\varphi(r)} dr =: V_n + VI_n,$$

where

$$V_n = \int_{\Omega \cap [s_n, R_n]} D_M \frac{|\omega'(r)|}{\omega(r)|\varphi'(r)} \chi_F(r) M(\omega(r)|u(r)) e^{-\varphi(r)} dr \\ = \int_{\Omega \cap [s_n, R_n]} D_M \frac{\omega'(r)}{\omega(r)\varphi'(r)} \chi_F(r) M(\omega(r)|u(r)) e^{-\varphi(r)} dr, \\ VI_n = \int_{\Omega \cap [s_n, R_n]} D_M \frac{|u'(r)|}{|u(r)||\varphi'(r)|} M(\omega(r)|u(r)) e^{-\varphi(r)} dr.$$

The definition (2.8) of the constant b_1 yields

$$b_1 J_n \leq J_n - I_n - IV_n - V_n,$$

and (4.8), (4.9), (4.11) give

$$J_n = I_n - II_n + III_n - \theta_n \leq I_n + IV_n + V_n + VI_n - \theta_n.$$

Combining the two we get

$$b_1 J_n \leq VI_n - \theta_n.$$

Consequently, since b_1 is assumed to be positive,

$$(4.12) \quad J_n \leq \frac{D_M}{b_1} \int_{\Omega \cap [s_n, R_n]} \frac{A_2(r)}{|\varphi'(r)|} e^{-\varphi(r)} dr - \frac{\theta_n}{b_1}.$$

Now we use the estimates from Lemmas 4.1 and 4.2. For any $0 < \delta \leq 1$ (or any $\delta > 0$ when $M(\lambda) = \lambda^p$) and $r \in \Omega$,

$$\begin{aligned} A_2(r) &= \delta \omega(r) \frac{M(\omega(r)|u(r)|)}{\omega(r)|u(r)|} \frac{|u'(r)|}{\delta} \\ &\leq \delta \omega(r) \left[\frac{D_M - 1}{d_M} M(\omega(r)|u(r)|) + \frac{1}{d_M} M\left(\frac{|u'(r)|}{\delta}\right) \right] \\ &\leq \delta \omega(r) \left[\frac{D_M - 1}{d_M} M(\omega(r)|u(r)|) + \frac{1}{d_M} c\left(\frac{1}{\delta}\right) M(|u'(r)|) \right] \end{aligned}$$

(Lemma 4.1 was used in the very last line).

Using now this estimate on $A_2(r)$, we obtain from (4.12)

$$\begin{aligned} J_n &\leq \frac{D_M \delta}{b_1} \left(\frac{D_M - 1}{d_M} \int_{s_n}^{R_n} \frac{\omega(r)}{|\varphi'(r)|} M(\omega(r)|u(r)|) e^{-\varphi(r)} dr \right. \\ &\quad \left. + \frac{1}{d_M} c\left(\frac{1}{\delta}\right) \int_{s_n}^{R_n} \frac{\omega(r)}{|\varphi'(r)|} M(|u'(r)|) e^{-\varphi(r)} dr \right) - \frac{\theta_n}{b_1} \\ &\leq \frac{D_M L \delta}{b_1} \left(\frac{D_M - 1}{d_M} J_n + \frac{1}{d_M} c\left(\frac{1}{\delta}\right) H_n \right) - \frac{\theta_n}{b_1}, \end{aligned}$$

or, after rearranging,

$$\frac{\theta_n}{b_1} + \left(1 - \frac{D_M L \delta}{b_1} \frac{D_M - 1}{d_M} \right) J_n \leq \frac{D_M L \delta}{b_1 d_M} c\left(\frac{1}{\delta}\right) H_n.$$

Choose now $\delta_0 = \frac{b_1 d_M}{L D_M^2}$ to obtain

$$\frac{D_M \theta_n}{b_1} + J_n \leq c\left(\frac{1}{\delta_0}\right) H_n.$$

Only now do we let $n \rightarrow \infty$. As the limits $\lim_{n \rightarrow \infty} J_n = J$, $\lim_{n \rightarrow \infty} H_n = H$ and $\lim_{n \rightarrow \infty} \theta_n$ are all well defined and nonnegative (finite or not), this implies

$$J \leq c\left(\frac{L D_M^2}{b_1 d_M}\right) H$$

and finishes the proof.

CASE 2. We now prove the statement under assumption (B2), for $u \in \mathcal{R}^-$. We start with an expression similar to (4.8), but the integrals are rearranged somewhat differently.

This time we write

$$\begin{aligned}
 (4.13) \quad J_n &= \int_{s_n}^{R_n} \left(-\frac{\varphi''(r)}{(\varphi'(r))^2} \right) M(\omega(r)|u(r)|) e^{-\varphi(r)} dr \\
 &\quad + \int_{s_n}^{R_n} \frac{1}{|\varphi'(r)|} M'(\omega(r)|u(r)|) |\omega'(r)| |u(r)| \chi_F(r) e^{-\varphi(r)} dr \\
 &\quad - \int_{s_n}^{R_n} \frac{1}{|\varphi'(r)|} M'(\omega(r)|u(r)|) (|\omega'(r)| |u(r)| \chi_G(r) + \omega(r) u'(r) \operatorname{sgn} u(r)) e^{-\varphi(r)} dr - \theta_n \\
 &=: I_n + II'_n - III'_n - \theta_n.
 \end{aligned}$$

As before, since $M'(\lambda) \geq d_M M(\lambda)/\lambda$, we get

$$III'_n \geq \int_{s_n}^{R_n} d_M M(\omega(r)|u(r)|) \frac{\omega'(r)}{\omega(r)\varphi'(r)} \chi_F(r) e^{-\varphi(r)} dr =: IV'_n$$

(one can omit the absolute values because $r \in F$). Furthermore, since $-M'(\lambda) \geq -D_M M(\lambda)/\lambda$, for every $r \in \Omega$ we have

$$\begin{aligned}
 &-M'(\omega(r)|u(r)|) (|\omega'(r)| |u(r)| \chi_G(r) + \omega(r) |u'(r)|) \\
 &\geq -D_M \left[M(\omega(r)|u(r)|) \frac{|\omega'(r)|}{\omega(r)} \chi_G(r) + M(\omega(r)|u(r)|) \frac{|u'(r)|}{|u(r)|} \right] \\
 &=: -D_M [B(r) + A_2(r)],
 \end{aligned}$$

where $A_2(r)$ is as in (4.10). This last estimate combined with (4.13) implies

$$-III'_n \geq -D_M \int_{\Omega \cap [s_n, R_n]} \frac{1}{|\varphi'(r)|} [B(r) + A_2(r)] e^{-\varphi(r)} dr =: V'_n + VI'_n.$$

On the other hand,

$$\begin{aligned}
 b_2 J_n &\leq I_n + IV'_n + V'_n - J_n, \\
 J_n &\geq I_n + II'_n - III'_n - \theta_n \geq I_n + IV'_n + V'_n + VI'_n - \theta_n.
 \end{aligned}$$

We get $b_2 J_n \leq -VI'_n + \theta_n$, which leads to

$$J_n \leq \frac{D_M}{b_2} \int_{\Omega \cap [s_n, R_n]} \frac{A_2(r)}{|\varphi'(r)|} e^{-\varphi(r)} dr + \frac{\theta_n}{b_2}.$$

Now the proof follows the lines of the first case, starting from (4.12), with b_2 replacing b_1 and θ_n replacing $-\theta_n$. ■

REMARK 4.1. Observe that for $M(\lambda) = \lambda^p$ we have $d_M = D_M = p$ and $c(x) = x^p$. Therefore under the assumptions of Theorem 2.1 the constant C equals either $(Lp/b_1)^p$ if $b_1 > 0$, or $(Lp/b_2)^p$ if $b_2 > 0$.

Proof of Theorem 2.2. Without loss of generality we can assume that $A = \|u'\|_{L_\mu^M}$ is finite. Let us substitute $u_A := u/A$ in (2.13). Then we get

$$\int_{\mathbb{R}_+} \frac{M}{C} \left(\frac{\omega|u|}{A} \right) \exp(-\varphi) \leq 1,$$

which implies $\|\omega u\|_{L_\mu^{M/C}} \leq A$. As the N -functions M/C and M are equivalent, we see by (2.3) that

$$\|\omega \mu\|_{L_\mu^M} \leq (C+1)\|\omega u\|_{L_\mu^{M/C}} \leq (C+1)A = (C+1)\|u'\|_{L_\mu^M}.$$

Therefore the result follows. ■

REMARK 4.2. One can compare our results with those recently proven in [9]. Namely, in Theorem 3.1, p. 416 there, the authors obtain the following inequality for the Gaussian measure, for $M(\lambda) = \lambda^p$:

$$\int_{\mathbb{R}_+} |\omega u|^p \exp\left(-\frac{x^2}{2}\right) dx \leq \left(\frac{p}{p-1}\right)^p \int_{\mathbb{R}_+} |u'|^p \exp\left(-\frac{x^2}{2}\right) dx,$$

where

$$\omega(x) = \frac{\exp(x^2/2(p-1))}{\int_0^x \exp(\sigma^2/2(p-1)) d\sigma},$$

holding for every $u \in W_0^{1,p}(\mathbb{R}_+, d\mu)$ (the completion of $C_0^\infty(\mathbb{R}_+)$ in the weighted Sobolev space $W^{1,p}(\mathbb{R}_+, d\mu)$, where $\mu(x) = \exp(-x^2/2) dx$ is the Gaussian measure). In this case we have $\varphi(x) = x^2/2$ and the quantity $\omega(x)/\varphi'(x)$ is not bounded as required by our Theorem 3.1. Instead, the weight ω obeys a different requirement, namely the ODE $x\omega - (p-1)\omega' = (p-1)\omega^2$.

5. ANALYSIS OF SETS \mathcal{R}

5.1. Verification of the condition $u \in \mathcal{R}$. The purpose of this subsection is two-fold. First we give an easy practical method to verify whether $u \in \mathcal{R}$ (here \mathcal{R} stands for \mathcal{R}^+ or \mathcal{R}^-). Further, we discuss when the condition

$$(5.1) \quad \int_{\mathbb{R}_+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty$$

together with $u \in \mathcal{H}$ (respectively $u \in \mathcal{H}^*$) implies that $u \in \mathcal{R}$.

We start with the following result.

PROPOSITION 5.1. *Suppose M, φ satisfy the conditions (M), (μ) , (ω) , $\varphi \in C^2(\mathbb{R}_+)$, φ' never vanishes and $L < \infty$ (see (2.9)). Then the following statements hold true.*

- (i) Assume that $\varphi'(R) \rightarrow 0$ as $R \rightarrow \infty$, and $u \in W_{\text{loc}}^{1,1}(\mathbb{R}_+)$ is such that $u(R)$ and $u(R)e^{-\varphi(R)}$ are bounded for $R \rightarrow \infty$. Then for $\varphi' < 0$ we have $u \in \mathcal{R}_{(\omega, \varphi, M)}^+$, while for $\varphi' > 0$ we have $u \in \mathcal{R}_{(\omega, \varphi, M)}^-$.
- (ii) Assume that $\varphi'(r) \rightarrow 0$ as $r \rightarrow 0$, and $u \in W_{\text{loc}}^{1,1}(\mathbb{R}_+)$ is such that $u(r)$ and $u(r)e^{-\varphi(r)}$ are bounded for $r \rightarrow 0$. Then for $\varphi' > 0$ we have $u \in \mathcal{R}_{(\omega, \varphi, M)}^+$, while for $\varphi' < 0$ we have $u \in \mathcal{R}_{(\omega, \varphi, M)}^-$.

Proof. (i) We have (h^u was defined in (2.10))

$$\begin{aligned} |h^u(R)| \exp(-\varphi(R)) &= \frac{M(\omega(R)|u(R)|)}{|\varphi'(R)|} \exp(-\varphi(R)) \\ &= \frac{M(|\varphi'(R)||u(R)|\frac{\omega(R)}{|\varphi'(R)|})}{|\varphi'(R)|} \exp(-\varphi(R)) \\ &\leq c(L) \frac{M(|\varphi'(R)||u(R)|)}{|\varphi'(R)|} \exp(-\varphi(R)) \chi_{\{u(R) \neq 0\}}(R), \end{aligned}$$

where $c(\cdot)$ is defined in (4.2). The property $M(r)/r \rightarrow 0$ as $r \rightarrow 0$ and our assumptions imply

$$\frac{M(|\varphi'(R)||u(R)|)}{|\varphi'(R)||u(R)|} \chi_{\{u(R) \neq 0\}}(R) \rightarrow 0 \quad \text{as } R \rightarrow \infty$$

and so $c(L)|u(R)|\exp(-\varphi(R))$ is bounded for $R \rightarrow \infty$. Therefore $|h^u(R)| \times \exp(-\varphi(R)) \rightarrow 0$ as $R \rightarrow \infty$ and the statement now follows from the definition of \mathcal{R}^+ and \mathcal{R}^- .

(ii) As in the proof of (i), we check that $|h^u(r)| \exp(-\varphi(r)) \rightarrow 0$ as $r \rightarrow 0$. ■

In the remaining part of this subsection we examine the property (5.1). Let us set some additional notation. First, recalling $c(\cdot)$ from (4.2), define, for $r > 0$,

$$(5.2) \quad \begin{aligned} f_\varphi(r) &= c^{-1}(e^{-\varphi(r)}), \\ A_\varphi(r) &= \|1/f_\varphi\|_{L^{(M^*)}(0,r)}, \quad B_\varphi(r) = \|1/f_\varphi\|_{L^{(M^*)}(r,\infty)}. \end{aligned}$$

The norms here are the dual norms defined by (2.4).

We will distinguish the cases when A_φ is well defined, and when B_φ is well defined.

CASE 1. Assume that $A_\varphi(r)$ is well defined for small r 's. We now analyze when $u \in \mathcal{H}$ which satisfies (5.1) belongs to \mathcal{R} .

We start with the following lemma.

LEMMA 5.1. Assume that M, φ, ω satisfy (M), (φ) , (ω) , and the function

$$K(r) = \frac{M(\omega(r)A_\varphi(r))}{|\varphi'(r)|} e^{-\varphi(r)}$$

is bounded for $r \rightarrow 0$. Then for every $u \in W_{\text{loc}}^{1,1}(\mathbb{R}_+)$ such that

$$\int_0^\infty M(|u'(r)|)e^{-\varphi(r)} dr < \infty \quad \text{and} \quad \lim_{r \rightarrow 0} u(r) = 0$$

the function h^u defined by (2.10) satisfies

$$(5.3) \quad \lim_{r \rightarrow 0} h^u(r)e^{-\varphi(r)} = 0.$$

Proof. Set

$$U(r) = \int_0^r |u'(\varrho)| d\varrho.$$

From inequality (2.5) we have

$$(5.4) \quad \begin{aligned} U(r) &= \int_0^r |u'(\varrho)| d\varrho = \int_0^r |u'(\varrho)f_\varphi(\varrho)| \cdot \frac{1}{|f_\varphi(\varrho)|} d\varrho \\ &\leq \|u' \cdot f_\varphi\|_{L^{(M)}(0,r)} \cdot A_\varphi(r). \end{aligned}$$

We have

$$(5.5) \quad \|u' f_\varphi\|_{L^M(0,r)} \leq \|u'\|_{L_\mu^M(0,r)}.$$

Indeed, if $A = \|u' f_\varphi\|_{L^M(0,r)}$ then the definition (5.2) of f_φ and the property (4.2) yield

$$1 = \int_{(0,r)} M\left(\frac{|u'|f_\varphi}{A}\right) dx \leq \int_{(0,r)} c(f_\varphi)M\left(\frac{|u'|}{A}\right) dx = \int_{(0,r)} M\left(\frac{|u'|}{A}\right) \exp(-\varphi) dx.$$

Therefore $A \leq \|u'\|_{L_\mu^M(0,r)}$, which is (5.5).

As the dual and Luxemburg norms are equivalent, we get

$$\|u' f_\varphi\|_{L^{(M)}(0,r)} \leq A \|u'\|_{L_\mu^{(M)}(0,r)} \leq A \|u'\|_{L_\mu^{(M)}(0,\infty)} < \infty,$$

where A is some universal constant. Therefore U is well defined.

From the assumption $\lim_{r \rightarrow 0} u(r) = 0$ we get $|u(r)| \leq U(r)$, and the estimate (5.4) holds true for u in place of U as well.

Now,

$$\begin{aligned} |h^u(r)e^{-\varphi(r)}| &= \frac{M(\omega(r)|u(r))}{|\varphi'(r)|} e^{-\varphi(r)} \leq \frac{M(\omega(r)\|u'\|_{L_\mu^{(M)}(0,r)} A_\varphi(r))}{|\varphi'(r)|} e^{-\varphi(r)} \\ &\leq \frac{M(\omega(r)A_\varphi(r))}{|\varphi'(r)|} e^{-\varphi(r)} c(\|u'\|_{L_\mu^{(M)}(0,r)}) = K(r)c(\|u'\|_{L_\mu^{(M)}(0,r)}). \end{aligned}$$

But since $c(x) \rightarrow 0$ as $x \rightarrow 0$, and $\|u'\|_{L_\mu^{(M)}(0,r)} \rightarrow 0$ as $r \rightarrow 0$, the assertion (5.3) follows from the boundedness of $K(r)$ for small r 's. ■

As a corollary, we obtain straight from the definition of $\mathcal{R}^+, \mathcal{R}^-$:

COROLLARY 5.1. *Suppose that the assumptions of Lemma 5.1 are satisfied.*

(i) *If $\varphi' > 0$, then*

$$\left\{ u \in \mathcal{H} : \int_{\mathbb{R}_+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty \right\} \subset \mathcal{R}^+.$$

(ii) *If $\varphi' < 0$, then*

$$\left\{ u \in \mathcal{H} : \int_{\mathbb{R}_+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty \right\} \subset \mathcal{R}^-.$$

To illustrate the statements above we now discuss the following example.

EXAMPLE 5.1. Let $\varphi(r) = -\alpha \ln r$, $\alpha < p-1$, $M(\lambda) = \lambda^p$ and $\omega(r) = 1/r$ as in Theorem 3.1. Then $M^*(\lambda) = c_p \lambda^{p/(p-1)}$ and, in this range of α 's,

$$f_\varphi(r) = r^{\alpha/p}, \quad M^*(f_\varphi(r)) = c_p r^{\alpha/(p-1)}, \quad A_\varphi(r) = a_p r^{(p-1-\alpha)/p}.$$

Thus $K(r)$ is just a constant. Moreover, every $u \in \mathcal{H}$ with $\int_0^\infty |u'|^p x^\alpha dx < \infty$ belongs to $\mathcal{R}_{(1/x, x^\alpha, \lambda^p)}^-$ for $\alpha < 0$, and to $\mathcal{R}_{(1/x, x^\alpha, \lambda^p)}^+$ when $\alpha > 0$.

CASE 2. We now assume that $B_\varphi(R)$ is well defined for large R 's. In this case we have the following dual statements dealing with the question whether $u \in \mathcal{H}^*$ which satisfies (5.1) belongs to \mathcal{R} .

LEMMA 5.2. *Assume that M, φ, ω satisfy (M), (φ) , (ω) , and the function*

$$L(R) = \frac{M(\omega(R)B_\varphi(R))}{|\varphi'(R)|} e^{-\varphi(R)}$$

is bounded for $R \rightarrow \infty$. Then for every $u \in \mathcal{H}^$ with $\int_0^\infty M(|u'(r)|)e^{-\varphi(r)} dr < \infty$ the function h^u defined by (2.10) satisfies*

$$\lim_{R \rightarrow \infty} h^u(R) e^{-\varphi(R)} = 0.$$

Proof. It is almost identical to that of Lemma 5.1: just replace $U(r) = \int_0^r |u'(\varrho)| d\varrho$ with $U^*(R) = \int_R^\infty |u'(\varrho)| d\varrho$ and proceed as before. ■

As a counterpart of Corollary 5.1 we have the following.

COROLLARY 5.2. *Suppose that the assumptions of Lemma 5.2 are satisfied.*

(i) *If $\varphi' > 0$, then*

$$\left\{ u \in \mathcal{H}^* : \int_{\mathbb{R}_+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty \right\} \subset \mathcal{R}^-.$$

(ii) *If $\varphi' < 0$, then*

$$\left\{ u \in \mathcal{H}^* : \int_{\mathbb{R}_+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty \right\} \subset \mathcal{R}^+.$$

This result is illustrated by the following example.

EXAMPLE 5.2. We again consider the case $\varphi(r) = -\alpha \ln r$, $M(\lambda) = \lambda^p$ and $\omega(r) = 1/r$ as in Theorem 3.1, but now with $\alpha > p - 1$. We have $B_\varphi(r) = b_p r^{(-\alpha+p-1)/p}$ and $L(r)$ is a constant. Therefore every $u \in \mathcal{H}^*$ which satisfies $\int_0^\infty |u'|^p x^\alpha dx < \infty$ belongs to $\mathcal{R}_{(1/x, x^\alpha, \lambda^p)}^+$.

REMARK 5.1. Suppose that the assumptions of Theorem 2.1 are satisfied and put $\mathcal{R} = \mathcal{R}^+$ in the case of (B1) and $\mathcal{R} = \mathcal{R}^-$ in the case of (B2). One could ask whether the spaces

$$\left\{ u \in \mathcal{R} : \int_{\mathbb{R}_+} M(|u'(r)|) \exp(-\varphi(r)) dr < \infty \right\},$$

playing the crucial role in the inequality (2.13), can possibly be nonlinear. We do not know the answer to this question.

5.2. Relation to Bloom–Kerman results. We now compare our results with those of Bloom and Kerman [6].

5.2.1. *When our conditions imply Bloom and Kerman's.* Here we give an example where our assumptions yield inequality (3.5) for all functions $Tf = \int_0^t f(\tau) d\tau \in \mathcal{H}$, so the Bloom–Kerman condition is satisfied.

Corollary 5.1 yields the following proposition.

PROPOSITION 5.2. *Assume that (M, φ, ω) satisfy the assumptions of Lemma 5.1 and either $(\varphi' > 0, b_1 > 0, L < \infty)$ or $(\varphi' < 0, b_2 > 0, L < \infty)$. Then:*

(i) *There exists a constant $C > 0$ such that*

$$\int_0^\infty M(\omega(x)|u(x)|) \exp(-\varphi(x)) dx \leq C \int_0^\infty M(|u'(x)|) \exp(-\varphi(x)) dx$$

for every $u \in \mathcal{H}$.

(ii) *The triple (ω, φ, M) satisfies the Bloom–Kerman condition:*

$$(5.6) \quad \int_0^y M^* \left(\frac{G(\varepsilon, y)}{B\varepsilon \exp(-\varphi(x))} \right) \exp(-\varphi(x)) dx \leq G(\varepsilon, y) < \infty,$$

for all $y > 0$ and $\varepsilon > 0$, where

$$G(\varepsilon, y) = \int_y^\infty M(\varepsilon\omega(x)) \exp(-\varphi(x)) dx$$

and $B > 0$ is a constant.

Proof. (i) This is just a combination of Corollary 5.1 and Theorem 2.1.

(ii) Let $u(x) = \int_0^x f(\tau) d\tau = (Tf)(x) \in \mathcal{H}$ be the Hardy transform of f . Then (i) implies

$$\int_0^\infty M(\omega(x)Tf(x)) \exp(-\varphi(x)) dx \leq C \int_0^\infty M(f(x)) \exp(-\varphi(x)) dx,$$

which is equivalent to (5.6) (see Theorem 1.7 in [6] and our comments in Subsection 3.2). ■

5.2.2. *When Bloom and Kerman conditions are not satisfied.* It may happen that our conditions are satisfied and the Bloom–Kerman conditions are not. In that case inequalities (2.13) cannot hold for every Hardy transform $u \in \mathcal{H}$ (see (1.3)) but they hold on proper subsets in the set of Hardy transforms. This is illustrated by the following result.

PROPOSITION 5.3. *There exists a triple (ω, φ, M) such that conditions (M), (μ) , (ω) are satisfied and:*

(i) (ω, φ, M) satisfies (B1), in particular

$$(5.7) \quad \int_{\mathbb{R}_+} M(\omega(r))|u(r)| \mu(dr) \leq C \int_{\mathbb{R}_+} M(|u'(r)|) \mu(dr)$$

for all $u \in \mathcal{R}_{(\omega, \varphi, M)}^+$.

(ii) (ω, φ, M) does not satisfy the Bloom–Kerman condition (5.6).

(iii) The set

$$\mathcal{R}_{(\omega, \varphi, M)}^{(0,+)} := \mathcal{R}_{(\omega, \varphi, M)}^+ \cap \mathcal{H} \subseteq \mathcal{H}$$

is a proper subset of \mathcal{H} . Moreover, there is no constant C such that inequality (5.7) is satisfied for every $u \in \mathcal{H}$.

Proof. Let $p > 1$ and

$$M(\lambda) = \lambda^p, \quad \varphi(x) = -\frac{1}{2}x^2, \quad \omega(x) = x.$$

In particular, conditions (M), (μ) , (ω) are satisfied, $b_1 = 1 + \frac{1}{x^2}(p-1) > 0$, $L = 1$ and condition (B1) is also satisfied. Therefore (i) follows by Theorem 2.1.

(ii) The Bloom–Kerman condition does not hold: one has

$$G(\varepsilon, y) = \int_y^\infty (\varepsilon x)^p e^{x^2/2} dx = \infty,$$

and so (5.6) is violated.

(iii) The Laplace function

$$u(r) = \int_0^r \exp(-\tau^2) d\tau$$

belongs to $\mathcal{H} \setminus \mathcal{R}_{(\omega, \varphi, M)}^{0,+}$ and does not satisfy (5.7). ■

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