STUDIA MATHEMATICA 211 (1) (2012)

Shift-modulation invariant spaces on LCA groups

by

Carlos Cabrelli and Victoria Paternostro (Buenos Aires)

Abstract. A (K, Λ) shift-modulation invariant space is a subspace of $L^2(G)$ that is invariant under translations along elements in K and modulations by elements in Λ. Here G is a locally compact abelian group, and K and Λ are closed subgroups of G and the dual group $\hat{G}$, respectively.

We provide a characterization of shift-modulation invariant spaces when K and Λ are uniform lattices. This extends previous results known for $L^2(\mathbb{R}^d)$. We develop fiberization techniques and suitable range functions adapted to LCA groups needed to provide the desired characterization.

1. Introduction. Shift invariant spaces (SIS) play an important role in approximation theory, wavelets and frames. They have also proven to be useful as models in signal processing applications.

Shift-modulation invariant (SMI) spaces are shift invariant spaces that have the extra property of also being invariant under some group of modulations. Such spaces are usually known as Gabor or Weyl–Heisenberg spaces. They have been intensively studied in [Bow07], [CCJ01], [CC01], [Chr03], [Dau92], [GH04], [GH03], [Gro01].

A deep and detailed study of the structure of shift-modulation invariant spaces of $L^2(\mathbb{R}^d)$ was made by Bownik [Bow07], who provided a characterization of SMI spaces based on fiberization techniques and range functions.

In the case of SIS, the $L^2(\mathbb{R}^d)$ theory using range functions has been extended to the context of locally compact abelian (LCA) groups in [CP10] and [KR08]. This general framework allows for a more complete view, in which relationships among the groups involved and their properties are more transparent. Further, the LCA group setting includes the finite case. Having a valid theory for groups such as $\mathbb{Z}_n$ is important for applications.

Since modulations become translations in the Fourier domain, shift-modulations invariant spaces are spaces that are shift invariant in time and

2010 *Mathematics Subject Classification*: Primary 43A77; Secondary 43A15.
Key words and phrases: shift-modulation invariant space, LCA groups, range functions, fibers.

DOI: 10.4064/sm211-1-1

frequency. As a consequence, the techniques of shift-invariant spaces can be applied to study the structure of SMI spaces. Having at hand a theory of SIS on LCA groups, it is natural to ask whether a general theory of SMI spaces could be developed in this more general context.

In this article we define and study the structure of SMI spaces using range functions and fiberization techniques, in the context of LCA groups. First we introduce the notion of shift-modulation spaces where translations are on a closed subgroup of an LCA group G and modulations are on a closed subgroup of the dual group $\widehat{G}$. Next we focus our attention on the case where both translations and modulations are along uniform lattices of G and the dual group of G respectively, with some minor hypotheses. We prove a characterization of shift-modulation invariant spaces, extending the result obtained by Bownik [Bow07] for $L^2(\mathbb{R}^d)$ to the case of LCA groups. The LCA setting allows us to visualize the role played by the different components: the groups, their duals, the quotients, and the relationships between the different lattices involved and their annihilators. These roles are somehow hidden in the Euclidean case. For example in the classical case, the simple structure of the lattices allows one to reduce the study to the case when one of the lattices is $\mathbb{Z}^d$. Since the dual group of $\mathbb{R}^d$ is isomorphic to itself and the annihilator of $\mathbb{Z}^d$ is isomorphic to $\mathbb{Z}^d$, most of the analysis can be done in the original group $\mathbb{R}^d$. As in the general case we do not have these isomorphisms, it requires some effort to discover the role of each component. The lack of structure of the lattices involved creates additional difficulties in establishing the precise setting. This is particularly relevant in defining the different isomorphisms that lead to a suitable decomposition of $L^2(G)$ and consequently to a range function well adapted to spaces having this double invariance. A diversity of results are then required such as the existence of Borel sections and their relations (see for example (2.3) and (2.4)).

On the other hand, once the proper setting is obtained, there is a more clear picture of the general interrelationships, which has the additional advantage of simplifying some reasonings.

We have organized the article as follows. In Section 2 we review some of the basic facts on LCA groups. Then we develop the notion of shift-modulation invariant spaces in this context and we set our standing assumptions. In Section 3 we outline how the results on shift invariant spaces can be used for shift-modulation invariant spaces. Section 4 establishes the fiberization isometry and the concept of shift-modulation range functions. Section 5 contains the main result of the paper, that is, the characterization of shift-modulation invariant spaces under uniform lattices. Finally, we include some examples in Section 6.

2. Preliminaries. In this section we collect the known results and notation needed for this paper. We also state our standing assumptions that will be in force for the remainder of the article.

Let G be an arbitrary locally compact Hausdorff abelian group written additively. We will denote by m_G its Haar measure. The dual group of G, that is, the set of continuous characters on G, is denoted by Γ or $\widehat{G}$. The value of the character $\gamma \in \Gamma$ at the point $x \in G$ is written (x, γ).

When two LCA groups G_1 and G_2 are topologically isomorphic we will write $G_1 \approx G_2$. In particular, it is known that $\widehat{\Gamma} \approx G$, where $\widehat{\Gamma}$ is the dual of the dual group of G.

For an LCA group G and K a closed subgroup of G, the Haar measures m_G, m_K and $m_{G/K}$ can be so chosen that Weil's formula

$$\int_G f(x)\, dm_G(x) = \int_{G/K} \int_K f(x+k)\, dm_K(k)\, dm_{G/K}([x]) \tag{2.1}$$

holds for each $f \in L^1(G)$. Here $[x]$ denotes the coset of x in G/K. Given a subgroup K of an LCA group G we will indicate by $\Pi_{G/K} \subseteq G$ a section for the quotient G/K.

The Fourier transform of a Haar integrable function f on G is the function $\widehat{f}$ on Γ defined by

$$\widehat{f}(\gamma) = \int_G f(x)(x, -\gamma)\, dm_G(x), \qquad \gamma \in \Gamma.$$

When the Haar measures m_G and m_Γ are normalized so that the inversion formula holds (see [Rud62]), the Fourier transform on $L^1(G) \cap L^2(G)$ can be extended to a unitary operator from $L^2(G)$ onto $L^2(\Gamma)$, called the Plancherel transformation and also denoted by "$\widehat{\ }$".

If K is a subgroup of G, then the subgroup

$$K^* = \{\gamma \in \Gamma : (k, \gamma) = 1,\ \forall k \in K\}$$

of Γ is called the *annihilator* of K. Since every character in Γ is continuous, K^* is a closed subgroup of Γ. If K is a closed subgroup, we have $(K^*)^* \approx K$. For every closed subgroup K of G the following duality relationships are valid:

$$K^* \approx \widehat{G/K} \quad \text{and} \quad \Gamma/K^* \approx \widehat{K}. \tag{2.2}$$

DEFINITION 2.1. Given an LCA group G, a *uniform lattice in* G is a discrete subgroup K of G such that the quotient group G/K is compact.

For every uniform lattice K in G, there exists a measurable (Borel) section of G/K with finite m_G-measure (see [KK98] and [FG64]). Another important fact is that if K is a countable (finite or countably infinite) uniform lattice in G, then K^* is a countable uniform lattice in Γ.

Remark 2.2. If $K_1 \subsetneq K_2$ are lattices in G, then K_2/K_1 is finite. To see this, observe that since $K_2^* \subseteq K_1^*$, we have $\widehat{K_1^*}/\widehat{K_2^*} \approx K_2/K_1$ due to (2.2). Therefore, K_2/K_1 is both compact and discrete, hence finite.

Let G be an LCA group, K a countable uniform lattice in G, and $\Pi_{\Gamma/K^*} \subseteq \Gamma$ an m_Γ-measurable section of Γ/K^*. Throughout we will identify the space $L^p(\Pi_{\Gamma/K^*})$ with the set $\{\varphi \in L^p(\Gamma) : \varphi = 0 \text{ a.e. in } \Gamma \setminus \Pi_{\Gamma/K^*}\}$ for $p = 1$ and $p = 2$.

Proposition 2.3. *Let G be an LCA group and K a countable uniform lattice in G. Then $\{\eta_k\}_{k\in K}$ is an orthogonal basis for $L^2(\Pi_{\Gamma/K^*})$, where $\eta_k(\gamma) = (k, -\gamma)\chi_{\Pi_{\Gamma/K^*}}(\gamma)$. If moreover $m_\Gamma(\Pi_{\Gamma/K^*}) = 1$, then this basis is orthonormal.*

The following proposition will be needed. Its proof can be found in [CP10].

Proposition 2.4. *Let G be an LCA group and K a countable uniform lattice on G. Fix a Borel section Π_{Γ/K^*} of Γ/K^* and choose m_{K^*} and m_{Γ/K^*} so that the inversion formula holds. Then*

$$\|a\|_{\ell^2(K)} = \frac{m_K(\{0\})^{1/2}}{m_\Gamma(\Pi_{\Gamma/K^*})^{1/2}} \Big\| \sum_{k\in K} a_k \eta_k \Big\|_{L^2(\Pi_{\Gamma/K^*})}$$

for each $a = \{a_k\}_{k\in K} \in \ell^2(K)$.

Definition 2.5. If $K \subseteq G$ and $\Lambda \subseteq \Gamma$ are closed subgroups, we will say that a closed subspace $V \subseteq L^2(G)$ is:

- *K-shift invariant* (or *shift invariant under K*) if
$$f \in V \ \Rightarrow\ T_k f \in V \ \forall k \in K, \quad \text{where} \quad T_k f(x) = f(x-k),$$
- *Λ-modulation invariant* (or *modulation invariant under Λ*) if
$$f \in V \ \Rightarrow\ M_\lambda f \in V \ \forall \lambda \in \Lambda, \quad \text{where} \quad M_\lambda f(x) = (x,\lambda) f(x),$$
- *(K,Λ)-invariant* (or *shift-modulation invariant under (K,Λ)*) if V is shift invariant under K and modulation invariant under Λ. In that case
$$f \in V \ \Rightarrow\ M_\lambda T_k f \in V \ \forall k \in K \text{ and } \lambda \in \Lambda.$$

For a subset $\mathcal{A} \subseteq L^2(G)$, define

$$E_{(K,\Lambda)}(\mathcal{A}) = \{M_\lambda T_k \varphi : \varphi \in \mathcal{A},\, k \in K,\, \lambda \in \Lambda\},$$
$$S_{(K,\Lambda)}(\mathcal{A}) = \overline{\operatorname{span}}\, E_{(K,\Lambda)}(\mathcal{A}).$$

A straightforward computation shows that the space $S_{(K,\Lambda)}(\mathcal{A})$ is shift-modulation invariant under (K,Λ). We call $S_{(K,\Lambda)}(\mathcal{A})$ the (K,Λ)-invariant space *generated by* $\mathcal{A}$. Note that for every (K,Λ)-invariant space V, there exists a countable set of generators $\mathcal{A} \subseteq L^2(G)$ such that $V = S_{(K,\Lambda)}(\mathcal{A})$.

Similarly, $S_K(\mathcal{A})$ will denote the closed subspace generated by translations along K of the elements of $\mathcal{A}$, and $S_\Lambda(\mathcal{A})$ the closed subspace generated by modulations from Λ of the elements of $\mathcal{A}$.

In this paper we characterize (F,Δ)-invariant spaces whenever F and Δ are uniform lattices in G and Γ respectively and $F\cap\Delta^*$ is a uniform lattice in G. The condition of $F\cap\Delta^*$ being a uniform lattice corresponds in the classical case $G=\mathbb{R}^d$ to rationally dependent lattices (see [Bow07]). Similar to the shift invariant case (see [CP10]), a characterization of shift-modulation invariant spaces will be established in terms of appropriate range functions using fiberization techniques.

We now set our standing assumptions which will be in force throughout.

Standing Assumptions 2.6.

- G is a second countable LCA group and Γ its dual group.
- F is a countable uniform lattice on G. (translations)
- Δ is a countable uniform lattice on Γ. (modulations)
- $E:=F\cap\Delta^*$ is a (countable) uniform lattice on G.

From our Standing Assumptions and Remark 2.2 we obtain:

(a) E^* is a uniform lattice in Γ and $\Delta\subseteq E^*$.
(b) E^*/Δ is finite.
(c) Δ^* is a uniform lattice in G.

Now we observe that if we fix a measurable section $\Pi_{\Gamma/E^*}\subseteq\Gamma$ for Γ/E^* and a finite section $\Pi_{E^*/\Delta^*}\subseteq E^*$ for E^*/Δ, then we can construct a measurable section $\Pi_{\Gamma/\Delta}$ for Γ/Δ as

$$\Pi_{\Gamma/\Delta}=\bigcup_{e\in\Pi_{E^*/\Delta}}\Pi_{\Gamma/E^*}+e. \tag{2.3}$$

Let $\Pi_{F/E}\subseteq F$ be a finite section for F/E. Note that, by the first isomorphism theorem for groups, F/E is isomorphic to $(F+\Delta^*)/\Delta^*$. Thus, $\Pi_{F/E}\subseteq F$ is also a section for $(F+\Delta^*)/\Delta^*$. Then, letting $\Pi_{G/(F+\Delta^*)}$ be a measurable section for $G/(F+\Delta^*)$, we see that

$$\Pi_{G/\Delta^*}=\bigcup_{d\in\Pi_{F/E}}\Pi_{G/(F+\Delta^*)}-d \tag{2.4}$$

is a section for G/Δ^*. The minus sign in (2.4) is just for notational convenience.

These sections will be used to define the fiberization isometry and the range function.

In order to avoid carrying over constants through the article, we fix the following normalization of the Haar measures considered. This particular choice of the Haar measures does not affect the generality of our results.

First, we choose m_{Δ^*} so that $m_{\Delta^*}(\{0\}) = 1$. Then we fix m_G and m_{G/Δ^*} such that Weil's formula holds for m_{Δ^*}, m_G and m_{G/Δ^*}. Furthermore, we choose m_{Γ/E^*}, m_{E^*} so as to get $m_{E^*}(\{0\})m_{\Gamma/E^*}(\Gamma/E^*) = 1/|\Pi_{E^*/\Delta^*}|$ where $|\Pi_{E^*/\Delta^*}|$ denotes the cardinality of Π_{E^*/Δ^*}. Then, we choose m_Γ so that Weil's formula holds for m_{Γ/E^*}, m_{E^*} and m_Γ.

If $\Pi_{\Gamma/\Delta}$ is given by (2.3), this normalization implies that $m_\Gamma(\Pi_{\Gamma/\Delta}) = 1$. This is due to the formula $m_\Gamma(\Pi_{\Gamma/E^*}) = m_{E^*}(\{0\})m_{\Gamma/E^*}(\Gamma/E^*)$ proved in [CP10, Lemma 2.10].

We will use different instances of the following space.

Definition 2.7. Let (X, μ) be a finite measure space and $\mathcal{H}$ a separable Hilbert space with inner product $\langle\cdot,\cdot\rangle_{\mathcal{H}}$. We define $L^2(X, \mathcal{H})$ as the space of all measurable functions $\Phi : X \to \mathcal{H}$ such that

$$\|\Phi\|_2^2 := \int_X \|\Phi(x)\|_{\mathcal{H}}^2 \, d\mu(x) < \infty,$$

where $\Phi : X \to \mathcal{H}$ is *measurable* if for each $v \in \mathcal{H}$, the function $x \mapsto \langle\Phi(x), v\rangle_{\mathcal{H}}$ from X to $\mathbb{C}$ is measurable in the usual sense. The space $L^2(X, \mathcal{H})$ with the inner product

$$\langle\Phi, \Psi\rangle := \int_X \langle\Phi(x), \Psi(x)\rangle_{\mathcal{H}} \, d\mu(x)$$

is a complex Hilbert space.

3. Heuristic.

3. Heuristic. Before we proceed to state the results and their proofs, we will give an informal discussion of the main ideas. We start by recalling some results from [CP10]. For further details we refer to [CP10].

Let G be an LCA group, H a uniform lattice in G, and Π_{Γ/H^*} a measurable section for Γ/H^*. Then the spaces $L^2(G)$ and $L^2(\Pi_{\Gamma/H^*}, \ell^2(H^*))$ are isometrically isomorphic via the isomorphism

$$(3.1) \qquad \mathcal{T}_H : L^2(G) \to L^2(\Pi_{\Gamma/H^*}, \ell^2(H^*)), \qquad \mathcal{T}_H f(\gamma) = \{\widehat{f}(\gamma + \delta)\}_{\delta \in H^*}.$$

The object $\{\widehat{f}(\gamma + \delta)\}_{\delta \in H^*}$ is called the H^*-*fiber* of $\widehat{f}$ at γ.

The isometry $\mathcal{T}_H$ defined by (3.1) is used to characterize, by means of range functions, the subspaces of $L^2(G)$ that are shift invariant under H:

Definition 3.1. A *shift range function* with respect to the pair (G, H) is a mapping

$$J : \Pi_{\Gamma/H^*} \to \{\text{closed subspaces of } \ell^2(H^*)\}.$$

The space $J(\gamma)$ is called the *fiber space* associated to γ.

Then we have the following characterization:

Theorem 3.2 ([CP10, Theorem 3.10]). *Let V be a closed subspace of $L^2(G)$. Then V is H-shift invariant if and only if there exists a measurable* *shift range function J such that*

$$V = \{f \in L^2(G) : \mathcal{T}_H f(\gamma) \in J(\gamma) \text{ for a.e. } \gamma \in \Pi_{\Gamma/H^*}\}.$$

If $\mathcal{A}$ is a set of generators of V as an H-shift invariant space (i.e. $V = S_H(\mathcal{A})$), then for a.e. $\gamma \in \Pi_{\Gamma/H^*}$,

$$J(\gamma) = \overline{\operatorname{span}}\{\mathcal{T}_H \varphi(\gamma) : \varphi \in \mathcal{A}\}.$$

Basically the result states that a function $f \in L^2(G)$ belongs to V if and only if each fiber of the Fourier transform of f is in the corresponding fiber space.

From Theorem 3.2, we can easily derive a similar result for *modulation* invariant spaces. Namely, if Λ is a uniform lattice in Γ, and $W \subseteq L^2(G)$ is Λ-modulation invariant, then $\widehat{W}$, the image of W under the Fourier transform, is Λ-*shift* invariant in $L^2(\Gamma)$. Hence, fixing a section Π_{G/Λ^*} for G/Λ^* and using Theorem 3.2, we can derive the following characterization of modulation invariant spaces:

Corollary 3.3. *Let W be a closed subspace of $L^2(G)$. Then W is Λ-modulation invariant if and only if there exists a measurable shift range function J with respect to the pair (Γ, Λ) such that*

$$W = \{f \in L^2(G) : \widetilde{\mathcal{T}}_{\Lambda^*} f(x) \in J(x) \text{ for a.e. } x \in \Pi_{G/\Lambda^*}\},$$

where $\widetilde{\mathcal{T}}_{\Lambda^*}$ is the isometric isomorphism

$$\widetilde{\mathcal{T}}_{\Lambda^*} : L^2(G) \to L^2(\Pi_{G/\Lambda^*}, \ell^2(\Lambda^*)), \qquad \widetilde{\mathcal{T}}_{\Lambda^*} f(x) = \{f(x+h)\}_{h \in \Lambda^*}. \tag{3.2}$$

If $\mathcal{A}$ is a set of generators of W as a Λ-modulation invariant space (i.e. $V = S_\Lambda(\mathcal{A})$), then for a.e. $x \in \Pi_{G/\Lambda^*}$,

$$J(x) = \overline{\operatorname{span}}\{\widetilde{\mathcal{T}}_{\Lambda^*} \varphi(x) : \varphi \in \mathcal{A}\}.$$

That is, a function f belongs to W if and only if its fibers (and not the fibers of its Fourier transform) belong to the corresponding fiber space.

Now we describe how we will apply these results to shift-modulation invariant spaces.

Let G be an LCA group and Γ its dual, and let F and Δ be uniform lattices in G and Γ respectively satisfying Standing Assumptions 2.6.

Let V be an (F, Δ)-shift-modulation invariant space in $L^2(G)$ (see Definition 2.5). We now have two ways to characterize V: one using invariance under translations and the other using invariance under modulations. Assume we choose the characterization using the fact that our space is Δ-modulation invariant. Then, by Corollary 3.3, we have a range function defined on a section Π_{G/Δ^*}, with fiber spaces $J(x) \subseteq \ell^2(\Delta^*)$.

Next we will show that invariance under translations along the lattice $E = F \cap \Delta^*$ implies that the fiber spaces $J(x)$ are E-shift invariant in $\ell^2(\Delta^*)$. So using again Theorem 3.2, we obtain a range function for each of the fiber spaces $J(x)$. From this we will construct a *shift-modulation* range function that will produce the desired characterization.

Note that in this description we did not consider translations by elements of F that are not in E. As we will see later, the action of these elements produces some periodicity on the range function.

4. The fiberization isometry and range functions. The goal of this section is to define the fiberization isometry and a suitable range function required to achieve the characterization of (F,Δ)-invariant spaces. We start by defining the isometry that will produce a decomposition of the space $L^2(G)$, and we show its relation to the Zak transform in Section 4.1. In Section 4.2 we first introduce the concept of shift-modulation range function, to which we associate a shift-modulation invariant space; then we also construct a range function from a given shift-modulation invariant space.

4.1. The isometry. Let us fix $F \subseteq G$ and $\Delta \subseteq \Gamma$, countable uniform lattices satisfying Standing Assumptions 2.6.

In order to construct the fiberization isometry, we introduce the following isomorphisms.

Let $\widetilde{\mathcal{T}}_{\Delta^*} : L^2(G) \to L^2(\Pi_{G/\Delta^*}, \ell^2(\Delta^*))$ be the isometric isomorphism defined as in (3.2) for G and Δ^*, that is,

$$\widetilde{\mathcal{T}}_{\Delta^*} f(x) = \{f(x+h)\}_{h\in\Delta^*}. \tag{4.1}$$

On the other hand, consider $\mathcal{T}_E : \ell^2(\Delta^*) \to L^2(\Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta}))$ defined by

$$\mathcal{T}_E a(\xi) = \Big\{\sum_{h\in\Delta^*} a_h \eta_h(\xi+e)\Big\}_{e\in\Pi_{E^*/\Delta}}, \tag{4.2}$$

where the functions η_h are as in Proposition 2.3 and $a = \{a_h\}_{h\in\Delta^*}$.

Lemma 4.1. *The map $\mathcal{T}_E$ defined in* (4.2) *is an isometric isomorphism between $\ell^2(\Delta^*)$ and $L^2(\Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta}))$.*

Proof. Since $\Pi_{E^*/\Delta}$ is an index set, according to Definition 2.7 we have

$$\begin{aligned}\|\mathcal{T}_E a\|_2^2 &= \int_{\Pi_{\Gamma/E^*}} \sum_{e\in\Pi_{E^*/\Delta}} \Big|\sum_{h\in\Delta^*} a_h\eta_h(\xi+e)\Big|^2 \, dm_\Gamma(\xi)\\ &= \int_{\Pi_{\Gamma/\Delta}} \Big|\sum_{h\in\Delta^*} a_h\eta_h(\omega)\Big|^2 \, dm_\Gamma(\omega) = \Big\|\sum_{h\in\Delta^*} a_h\eta_h\Big\|^2_{L^2(\Pi_{\Gamma/\Delta})}.\end{aligned}$$

Now, applying Proposition 2.4 we obtain

$$\Big\|\sum_{h\in\Delta^*} a_h\eta_h\Big\|^2_{L^2(\Pi_{\Gamma/\Delta})} = \frac{m_\Gamma(\Pi_{\Gamma/\Delta})}{m_{\Delta^*}(\{0\})}\|a\|^2_{\ell^2(\Delta^*)}.$$

Hence, by our normalization of the Haar measures, $m_\Gamma(\Pi_{\Gamma/\Delta})/m_{\Delta^*}(\{0\}) = 1$ and so $\|\mathcal{T}_E a\|_2^2 = \|a\|^2_{\ell^2(\Delta^*)}$.

Let $\Phi \in L^2(\Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta}))$. Then Φ induces a function $\widetilde{\Phi} \in L^2(\Pi_{\Gamma/\Delta})$ given by

$$\widetilde{\Phi}(\omega) = (\Phi(\xi))_e,$$

where $\omega = \xi + e \in \Pi_{\Gamma/\Delta}$ with $\xi \in \Pi_{\Gamma/E^*}$ and $e \in \Pi_{E^*/\Delta}$. Here $(\Phi(\xi))_e$ denotes the value of the sequence $\Phi(\xi)$ at e. It is easy to check that $\|\Phi\|_2 = \|\widetilde{\Phi}\|_{L^2(\Pi_{\Gamma/\Delta})}$.

By Proposition 2.3, $\{\eta_h\}_{h\in\Delta^*}$ is an orthonormal basis for $L^2(\Pi_{\Gamma/\Delta})$. Thus, $\widetilde{\Phi} = \sum_{h\in\Delta^*} a_h\eta_h$ for some $a = \{a_h\}_{h\in\Delta^*} \in \ell^2(\Delta^*)$. Hence $\mathcal{T}_E a = \Phi$. Therefore, $\mathcal{T}_E$ is an isomorphism. ■

REMARK 4.2. Note that $E^{*(\Delta^*)}$, the annihilator of E as a subgroup of Δ^*, is topologically isomorphic to E^*/Δ. Hence, using the dual relationships (2.2), it follows that $\widehat{\Delta^*}/E^{*(\Delta^*)} \approx \Gamma/E^*$. This allows us to look at $\mathcal{T}_E$ as a particular case of the map defined in (3.1).

The isometric isomorphism $\mathcal{T}_E$ induces another isometric isomorphism

$$\Psi_1 : L^2(\Pi_{G/\Delta^*}, \ell^2(\Delta^*)) \to L^2(\Pi_{G/\Delta^*}, L^2(\Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta})))$$

defined by

$$\Psi_1(\phi)(x) = \mathcal{T}_E(\phi(x)).$$

We can also identify the Hilbert space $L^2(\Pi_{G/\Delta^*}, L^2(\Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta})))$ with $L^2(\Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta})))$ using the isometric isomorphism

$$\Psi_2 : L^2(\Pi_{G/\Delta^*}, L^2(\Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta}))) \to L^2(\Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta}))$$

given by

$$\Psi_2(\phi)(x,\xi) = \phi(x)(\xi).$$

DEFINITION 4.3. We define $\mathcal{T} : L^2(G) \to L^2(\Pi_{G/\Delta^*}\times\Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta}))$ as

$$\mathcal{T} = \Psi_2 \circ \Psi_1 \circ \widetilde{\mathcal{T}}_{\Delta^*}.$$

The mapping $\mathcal{T}$, which is actually an isometric isomorphism, and which we call *the fiberization isometry*, can be explicitly defined as

$$\mathcal{T}f(x,\xi) = \mathcal{T}_E(\widetilde{\mathcal{T}}_{\Delta^*} f(x))(\xi) = \Big\{\sum_{h\in\Delta^*} f(x-h)(h,\xi+e)\Big\}_{e\in\Pi_{E^*/\Delta}}. \tag{4.3}$$

4.1.1. *The isometry and the Zak transform.* As is well known, a natural tool to study shift-modulation invariant spaces is the Zak transform, first introduced in $\mathbb{R}$ by Gelfand [Gel50]. Weil [Wei64] extended this transform to general LCA groups, and independently Zak [Zak67] used it in physical problems. In what follows we show how the Zak transform is present in our analysis.

We recall the usual Zak transform $Z : L^2(G) \to \mathcal{Q}$ given by

$$Zf(x,\xi) = \sum_{h\in\Delta^*} f(x-h)(h,\xi),$$

where $\mathcal{Q}$ is the set of all measurable functions $F : G \times \Gamma \to \mathbb{C}$ satisfying

(a) $F(x+h,\xi) = (h,\xi)F(x,\xi)$ for all $h \in \Delta^*$,
(b) $F(x,\xi+\delta) = F(x,\xi)$ for all $\delta \in \Delta$ and
(c) $\|F\|^2 = \int_{\Pi_{\Gamma/\Delta}} \int_{\Pi_{G/\Delta^*}} |F(x,\xi)|^2 \, dm_G(x) \, dm_\Gamma(\xi) < \infty$.

Then it is clear that

$$\mathcal{T}f(x,\xi) = \{Zf(x,\xi+e)\}_{e\in\Pi_{E^*/\Delta}}.$$

The next lemma states an important property of $\mathcal{T}$, which is a straightforward consequence of properties (a)–(c) above.

Lemma 4.4. *For each $f \in L^2(G)$ the map $\mathcal{T}$ of Definition* 4.3 *satisfies* $\mathcal{T}(M_\delta T_y f)(x,\xi) = (x,\delta)(-z,\xi)\mathcal{T}(T_d f)(x,\xi)$ *for a.e.* $(x,\xi)\in\Pi_{G/\Delta^*}\times\Pi_{\Gamma/E^*}$, *where $\delta \in \Delta$, $y \in F$ and $y = z + d$ with $z \in E$ and $d \in \Pi_{F/E}$.*

4.2. Shift-modulation range functions. In this section we introduce the notion of shift-modulation range function adapted to the isometry defined above.

Definition 4.5. A *shift-modulation range function* with respect to the pair (F,Δ) is a mapping

$$J : \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*} \to \{\text{subspaces of } \ell^2(\Pi_{E^*/\Delta})\}$$

with the following periodicity property:

(4.4)
$$J(x,\xi) = J(x-d,\xi) \quad \forall d \in \Pi_{F/E} \text{ and a.e. } (x,\xi) \in \Pi_{G/(F+\Delta^*)} \times \Pi_{\Gamma/E^*}.$$

For a shift-modulation range function J, we associate to each $(x,\xi) \in \Pi_{G/\Delta^*}\times\Pi_{\Gamma/E^*}$ the orthogonal projection onto $J(x,\xi)$, $P_{(x,\xi)} : \ell^2(\Pi_{E^*/\Delta}) \to J(x,\xi)$.

We say that a shift-modulation range function J is *measurable* if the function $(x,\xi) \mapsto P_{(x,\xi)}$ from $\Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}$ to $\ell^2(\Pi_{E^*/\Delta})$ is measurable.

For a shift-modulation range function J (not necessarily measurable) we define

$$(4.5) \qquad M_J = \{\Psi \in L^2(\Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta})) : \\ \Psi(x,\xi) \in J(x,\xi) \text{ for a.e. } (x,\xi) \in \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}\}.$$

Remark 4.6. M_J is a closed subspace of $L^2(\Pi_{G/\Delta^*}\times\Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta}))$. For the proof see [CP10, Lemma 3.8].

4.2.1. *The shift-modulation invariant space associated to a range function.* The following proposition states that to any shift-modulation range function with respect to (F, Δ), we can associate an (F, Δ)-invariant space.

PROPOSITION 4.7. *Let J be a shift-modulation range function and define $V := \mathcal{T}^{-1} M_J$, where M_J is as in (4.5) and $\mathcal{T}$ is the fiberization isometry. Then V is an (F, Δ)-invariant space in $L^2(G)$.*

Proof. To begin with, observe that, since $\mathcal{T}$ is an isometry, $V \subseteq L^2(G)$ is a closed subspace, by Remark 4.6.

Let $f \in V$, $\delta \in \Delta$ and $y \in F$. We need to show that $M_\delta T_y f \in V$.

According to Lemma 4.4, we have

$$\mathcal{T}(M_\delta T_y f)(x, \xi) = (x, \delta)(-z, \xi)\mathcal{T}(T_d f)(x, \xi) \text{ for a.e. } (x, \xi) \in \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*},$$

where $y = z + d$ with $z \in E$ and $d \in \Pi_{F/E}$.

In particular, if $x \in \Pi_{G/(F+\Delta^*)}$ we can rewrite $\mathcal{T}(T_d f)(x, \xi)$ as $\mathcal{T} f(x - d, \xi)$. Then, since $\mathcal{T} f \in M_J$ and J satisfies (4.4), we have

$$\mathcal{T}(T_d f)(x, \xi) = \mathcal{T} f(x - d, \xi) \in J(x - d, \xi) = J(x, \xi)$$

for a.e. $(x, \xi) \in \Pi_{G/(F+\Delta^*)} \times \Pi_{\Gamma/E^*}$. Thus,

$$\mathcal{T}(M_\delta T_y f)(x, \xi) \in J(x, \xi) \quad \text{for a.e. } (x, \xi) \in \Pi_{G/(F+\Delta^*)} \times \Pi_{\Gamma/E^*}, \tag{4.6}$$

and this is valid for all $y \in F$ and $\delta \in \Delta$.

We now want to show that (4.6) holds on $\Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}$. Let $(x, \xi) \in \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}$. By (2.4) we can set $x = x' - d$ with $x' \in \Pi_{G/(F+\Delta^*)}$ and $d \in \Pi_{F/E}$. If we fix $\delta \in \Delta$ and $y \in F$, then $\mathcal{T}(M_\delta T_y f)(x, \xi) = \mathcal{T}(T_d M_\delta T_y f)(x', \xi)$. Since $M_\lambda T_k g = (k, \lambda) T_k M_\lambda g$ for all $g \in L^2(G)$, $\lambda \in \Delta$ and $k \in F$, we have

$$\mathcal{T}(T_d M_\delta T_y f)(x', \xi) = (-d, \delta)\mathcal{T}(M_\delta T_{d+y} f)(x', \xi) \in J(x', \xi) = J(x, \xi).$$

Thus, (4.6) holds on $\Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}$. Therefore, $M_\delta T_y f \in V$ for all $\delta \in \Delta$ and $y \in F$. ∎

4.2.2. *The range function associated to an (F, Δ)-invariant space.* Theorem 3.2 gives a specific way to describe the shift range function associated to each shift invariant space in terms of the fibers of its generators. The analogous description for modulation invariant spaces is given in Corollary 3.3. Now we will use these results to construct a shift-modulation range function from a given (F, Δ)-invariant space.

Assume that $V \subseteq L^2(G)$ is an (F, Δ)-invariant space and that $V = S_{(F,\Delta)}(\mathcal{A})$ for some countable set $\mathcal{A} \subseteq L^2(G)$. We will associate to V a shift-modulation range function.

Since V is Δ-modulation invariant, by Corollary 3.3 we have

$$V = \{f \in L^2(G) : \widetilde{\mathcal{T}}_{\Delta^*} f(x) \in J_{\Delta^*}(x) \text{ for a.e. } x \in \Pi_{G/\Delta^*}\}, \tag{4.7}$$

where $\widetilde{\mathcal{T}}_{\Delta^*}$ is the isometry defined in (4.1) and J_{Δ^*} is the shift range function associated to V given by

$$J_{\Delta^*} : \Pi_{G/\Delta^*} \to \{\text{closed subspaces of } \ell^2(\Delta^*)\},$$
$$J_{\Delta^*}(x) = \overline{\operatorname{span}}\{\widetilde{\mathcal{T}}_{\Delta^*}(T_y\varphi)(x) : y \in F,\, \varphi \in \mathcal{A}\}.$$

Now, let us see that $J_{\Delta^*}(x) \subseteq \ell^2(\Delta^*)$ is a shift invariant space under translations in E. Since $\Pi_{F/E} \subseteq F$ is a section for the quotient F/E, every $y \in F$ can be written in a unique way as $y = z + d$ with $z \in E$ and $d \in \Pi_{F/E}$. Then, using $\widetilde{\mathcal{T}}_{\Delta^*} T_z f = T_z \widetilde{\mathcal{T}}_{\Delta^*} f$ for all $z \in E$, we can rewrite $J_{\Delta^*}(x)$ as

$$J_{\Delta^*}(x) = \overline{\operatorname{span}}\{T_z \widetilde{\mathcal{T}}_{\Delta^*}(T_d\varphi)(x) : z \in E,\, d \in \Pi_{F/E},\, \varphi \in \mathcal{A}\}.$$

This description shows that $J_{\Delta^*}(x)$ is a shift invariant space under translations in E generated by the set $\{\widetilde{\mathcal{T}}_{\Delta^*}(T_d\varphi)(x) : d \in \Pi_{F/E},\, \varphi \in \mathcal{A}\}$.

Using Theorem 3.2, we can characterize $J_{\Delta^*}(x)$ for a.e. $x \in \Pi_{G/\Delta^*}$ as follows. For each $x \in \Pi_{G/\Delta^*} \setminus Z$, where Z is the exceptional zero m_G-measure set, there exists a range function $J_E^x : \Pi_{\Gamma/E^*} \to \{\text{subspaces of } \ell^2(\Pi_{E^*/\Delta})\}$ such that

$$J_{\Delta^*}(x) = \{a \in \ell^2(\Delta^*) : \mathcal{T}_E a(\xi) \in J_E^x(\xi) \text{ for a.e. } \xi \in \Pi_{\Gamma/E^*}\},$$

where $\mathcal{T}_E$ is the map given in (4.2). Moreover,

$$\begin{aligned} J_E^x(\xi) &= \overline{\operatorname{span}}\{\mathcal{T}_E(\widetilde{\mathcal{T}}_{\Delta^*} T_d\varphi(x))(\xi) : d \in \Pi_{F/E},\, \varphi \in \mathcal{A}\} \\ &= \overline{\operatorname{span}}\{\mathcal{T}(T_d\varphi)(x,\xi) : d \in \Pi_{F/E},\, \varphi \in \mathcal{A}\} \\ &= \operatorname{span}\{\mathcal{T}(T_d\varphi)(x,\xi) : d \in \Pi_{F/E},\, \varphi \in \mathcal{A}\}, \end{aligned}$$

where in the last equality we use the fact that $\dim(\ell^2(\Pi_{E^*/\Delta})) < \infty$.

This leads to the function $J : \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*} \to \{\text{subspaces of } \ell^2(\Pi_{E^*/\Delta})\}$ defined as

$$J(x,\xi) = \operatorname{span}\{\mathcal{T}(T_d\varphi)(x,\xi) : d \in \Pi_{F/E},\, \varphi \in \mathcal{A}\} \tag{4.8}$$

for a.e. $(x,\xi) \in \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}$.

Lemma 4.8. *Let $\mathcal{A} \subseteq L^2(G)$ be a countable set. Then the map defined in* (4.8) *is a shift-modulation range function.*

Proof. We need to show that J satisfies (4.4).

Let $d_0 \in \Pi_{F/E}$. For each $d \in \Pi_{F/E}$, we have

$$\mathcal{T}(T_d\varphi)(x - d_0, \xi) = \mathcal{T}(T_{d+d_0}\varphi)(x,\xi) \text{ for a.e. } (x,\xi) \in \Pi_{G/(F+\Delta^*)} \times \Pi_{\Gamma/E^*}.$$

Since $d + d_0 \in F$, we can write $d + d_0 = d' + z'$ with $d' \in \Pi_{F/E}$ and $z' \in E$. Then, according to Lemma 4.4, $\mathcal{T}(T_{d+d_0}\varphi)(x,\xi) = (z',\xi)\mathcal{T}(T_{d'}\varphi)(x,\xi)$. Thus, $\mathcal{T}(T_d\varphi)(x - d_0,\xi) \in J(x,\xi)$ because $\mathcal{T}(T_{d'}\varphi)(x,\xi) \in J(x,\xi)$. This shows that $J(x - d_0,\xi) \subseteq J(x,\xi)$ for a.e. $(x,\xi) \in \Pi_{G/(F+\Delta^*)} \times \Pi_{\Gamma/E^*}$ for each $d_0 \in \Pi_{F/E}$.

With an analogous argument, it can be proven that $J(x,\xi) \subseteq J(x-d_0,\xi)$ for a.e. $(x,\xi) \in \Pi_{G/(F+\Delta^*)} \times \Pi_{\Gamma/E^*}$ for each $d_0 \in \Pi_{F/E}$. ■

As we have seen in Proposition 4.7, each shift-modulation range function with respect to the pair (F,Δ) induces an (F,Δ)-invariant space. Furthermore, in Section 4.2.2 we associated to each shift-modulation invariant space V a shift-modulation range function from a system of generators of V. This leads to a natural question: If V is an (F,Δ)-invariant space and J the shift-modulation range function that V induces, what is the relationship between V and the (F,Δ)-invariant space induced from J?

This problem will be the topic of the following section.

5. The characterization of (F,Δ)-invariant spaces. We can now state our main result which characterizes (F,Δ)-invariant spaces in terms of the fiberization isometry and shift-modulation range functions.

THEOREM 5.1. *Let $V \subseteq L^2(G)$ be a closed subspace and $\mathcal{T}$ the fiberization isometry of Definition 4.3. Then V is an (F,Δ)-invariant space if and only if there exists a measurable shift-modulation range function $J : \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*} \to \{\text{subspaces of } \ell^2(\Pi_{E^*/\Delta})\}$ such that*

$$V = \{f \in L^2(G) : \mathcal{T}f(x,\xi) \in J(x,\xi) \text{ for a.e. } (x,\xi) \in \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}\}.$$

If we identify shift-modulation range functions which are equal almost everywhere, then the correspondence between (F,Δ)-invariant spaces and measurable shift-modulation range functions is one-to-one and onto.

Moreover, if $V = S_{(F,\Delta)}(\mathcal{A}) \subseteq L^2(G)$ for some countable subset $\mathcal{A}$ of $L^2(G)$, then the measurable shift-modulation range function J associated to V is given by

$$J(x,\xi) = \operatorname{span}\{\mathcal{T}T_d\varphi(x,\xi) : d \in \Pi_{F/E},\ \varphi \in \mathcal{A}\}$$

for a.e. $(x,\xi) \in \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}$.

For the proof of Theorem 5.1 we need the following lemma, which is an adaptation of [CP10, Lemma 3.11]. For its proof see [CP10].

LEMMA 5.2. *If J and J' are measurable shift-modulation range functions such that $M_J = M_{J'}$, where M_J and $M_{J'}$ are given by (4.5), then $J(x,\xi) = J'(x,\xi)$ for a.e. $(x,\xi) \in \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}$. That is, J and J' are equal almost everywhere.*

Proof of Theorem 5.1. If V is an (F,Δ)-invariant space, then, since $L^2(G)$ is separable, it follows that $V = S_{(F,\Delta)}(\mathcal{A})$ for some countable subset $\mathcal{A}$ of $L^2(G)$.

Let us consider the function J defined as

$$J(x,\xi) = \operatorname{span}\{\mathcal{T}(T_d\varphi)(x,\xi) : d \in \Pi_{F/E},\ \varphi \in \mathcal{A}\}$$

defined on $\Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}$ and taking values in $\{\text{subspaces of } \ell^2(\Pi_{E^*/\Delta})\}$.

By Lemma 4.8, J is a shift-modulation range function. We must prove that $\mathcal{T}V = M_J$ where M_J is as in (4.5), and that J is measurable.

We first show $\mathcal{T}V = M_J$.

Take $\delta \in \Delta$, $y \in F$ written as $y = z + d$ with $z \in E$ and $d \in \Pi_{F/E}$, and $\varphi \in \mathcal{A}$. Then, by Lemma 4.4,

$$\mathcal{T}(M_\delta T_y \varphi)(x,\xi) = (x,\delta)(-z,\xi)\mathcal{T}(T_d\varphi)(x,\xi) \text{ for a.e. } (x,\xi) \in \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}.$$

Thus, since $\mathcal{T}(T_d\varphi)(x,\xi) \in J(x,\xi)$, we have $\mathcal{T}(M_\delta T_y\varphi)(x,\xi) \in J(x,\xi)$ for a.e. $(x,\xi) \in \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}$. Therefore,

$$\mathcal{T}(\operatorname{span}\{M_\delta T_y \varphi : \varphi \in \mathcal{A},\, y \in F,\, \delta \in \Delta\}) \subseteq M_J.$$

Using the fact that $\mathcal{T}$ is a continuous function and Remark 4.6, we can compute

$$\begin{aligned}\mathcal{T}V &= \mathcal{T}(\overline{\operatorname{span}}\{M_\delta T_y \varphi : \varphi \in \mathcal{A},\, y \in F,\, \delta \in \Delta\}) \\ &\subseteq \overline{\mathcal{T}(\operatorname{span}\{M_\delta T_y \varphi : \varphi \in \mathcal{A},\, y \in F,\, \delta \in \Delta\})} \\ &\subseteq \overline{M_J} = M_J.\end{aligned}$$

Let us suppose that $\mathcal{T}V \subsetneq M_J$. Then there exists $\Psi \in M_J \setminus \{0\}$ orthogonal to $\mathcal{T}V$. In particular, $\langle \Psi, \mathcal{T}(M_\delta T_y\varphi)\rangle = 0$ for all $\varphi \in \mathcal{A}$, $y \in F$ and $\delta \in \Delta$. Hence, if we write $y = z + d$ with $z \in E$ and $d \in \Pi_{F/E}$, by Lemma 4.4 we obtain

$$\begin{aligned}0 &= \int_{\Pi_{G/\Delta^*}} \int_{\Pi_{\Gamma/\Delta}} \langle \Psi(x,\xi), \mathcal{T}(M_\delta T_y\varphi)(x,\xi)\rangle \, dm_\Gamma(\xi)\, dm_G(x) \\ &= \int_{\Pi_{G/\Delta^*}} \int_{\Pi_{\Gamma/\Delta}} (x,\delta)(-z,\xi)\langle \Psi(x,\xi), \mathcal{T}(T_d\varphi)(x,\xi)\rangle \, dm_\Gamma(\xi)\, dm_G(x) \\ &= \int_{\Pi_{G/\Delta^*}} \int_{\Pi_{\Gamma/\Delta}} \eta_\delta(x)\eta_{-z}(\xi)\langle \Psi(x,\xi), \mathcal{T}(T_d\varphi)(x,\xi)\rangle \, dm_\Gamma(\xi)\, dm_G(x),\end{aligned}$$

where η_δ and η_{-z} are as in Proposition 2.3.

If we define $\nu_{(\delta,z)}(x,\xi) := \eta_\delta(x)\eta_{-z}(\xi)$, then, using Proposition 2.3, it can be seen that $\{\nu_{(\delta,z)}\}_{(\delta,z)\in\Delta\times E}$ is an orthogonal basis for $L^2(\Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*})$. Therefore, $\langle \Psi(x,\xi), \mathcal{T}(T_d\varphi)(x,\xi)\rangle = 0$ for a.e. $(x,\xi) \in \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}$ and all $d \in \Pi_{F/E}$.

This shows that $\Psi(x,\xi) \in J(x,\xi)^\perp$ for a.e. $(x,\xi) \in \Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}$, and since $\Psi \in M_J$, we must have $\Psi = 0$, which is a contradiction. Thus $\mathcal{T}V = M_J$.

Let us now prove that J is measurable. If $\mathcal{P}$ is the orthogonal projection on M_J, $\mathcal{I}$ is the identity mapping in $L^2(\Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta}))$, and if $\Psi \in L^2(\Pi_{G/\Delta^*} \times \Pi_{\Gamma/E^*}, \ell^2(\Pi_{E^*/\Delta}))$, we deduce that $(\mathcal{P} - \mathcal{I})\Psi$ is orthogonal to M_J. Then, by the above reasoning, $(\mathcal{P} - \mathcal{I})\Psi(x,\xi) \in J(x,\xi)^\perp$ for a.e.

$(x,\xi) \in \varPi_{G/\varDelta^*} \times \varPi_{\varGamma/E^*}$. Thus,

$$P_{(x,\xi)}((\mathcal{P}-\mathcal{I})\varPsi(x,\xi)) = 0 \quad \text{for a.e. } (x,\xi) \in \varPi_{G/\varDelta^*} \times \varPi_{\varGamma/E^*},$$

and therefore $\mathcal{P}\varPsi(x,\xi) = P_{(x,\xi)}(\varPsi(x,\xi))$ for a.e. $(x,\xi) \in \varPi_{G/\varDelta^*} \times \varPi_{\varGamma/E^*}$. If in particular $\varPsi(x,\xi) = a$ for all $(x,\xi) \in \varPi_{G/\varDelta^*} \times \varPi_{\varGamma/E^*}$, we find that $\mathcal{P}a(x,\xi) = P_{(x,\xi)}(a)$. Therefore, since $(x,\xi) \mapsto \mathcal{P}a(x,\xi)$ is measurable, $(x,\xi) \mapsto P_{(x,\xi)}a$ is measurable as well.

Conversely, if J is a shift-modulation range function, then by Proposition 4.7, $V := \mathcal{T}^{-1}M_J$ is an $(F,\varDelta)$-invariant space. Hence $V = S_{(F,\varDelta)}(\mathcal{A})$ for some countable subset $\mathcal{A}$ of $L^2(G)$, and by Lemma 4.8 we can define the shift-modulation range function J' as

$$J'(x,\xi) = \operatorname{span}\{\mathcal{T}(T_d\varphi)(x,\xi) : d \in \varPi_{F/E},\ \varphi \in \mathcal{A}\}$$

for a.e. $(x,\xi) \in \varPi_{G/\varDelta^*} \times \varPi_{\varGamma/E^*}$. Thus, as we have shown, J' is measurable and $M_{J'} = \mathcal{T}V = M_J$. Then Lemma 5.2 gives us $J = J'$ a.e.

This also proves that the correspondence between $(F,\varDelta)$-invariant spaces and shift-modulation measurable range functions is one-to-one and onto. ∎

REMARK 5.3. All the results of this paper are valid for uniform lattices satisfying Standing Assumptions 2.6. However, for shift-modulation invariant spaces where the translations (modulations) are along the whole group (dual group) we can still give a characterization as a corollary of Wiener's theorem (see [Hel64], [Sri64], [Rud87] and [HS64]).

We say that a set $B \subseteq G$ is *K-translation invariant* if $B + k = B$ for all $k \in K$.

PROPOSITION 5.4. *Let $V \subseteq L^2(G)$ be a closed subspace and $\varLambda \subseteq \varGamma$ be a closed subgroup. Then V is $(G,\varLambda)$-invariant if and only if there exists an $m_\varGamma$-measurable set $B \subseteq \varGamma$ which is $\varLambda$-translation invariant and such that*

$$V = \{f \in L^2(G) : \operatorname{supp}(\widehat{f}) \subseteq B\}.$$

Proof. By Wiener's theorem, there exists an $m_\varGamma$-measurable set $B \subseteq \varGamma$ satisfying $V = \{f \in L^2(G) : \operatorname{supp}(\widehat{f}) \subseteq B\}$. Since V is $\varLambda$-modulation invariant, it follows that B is $\varLambda$-translation invariant. ∎

The following proposition is analogous to the previous one for the case when the subspace is invariant along every modulation.

PROPOSITION 5.5. *Let $V \subseteq L^2(G)$ and $T \subseteq G$ be a closed subgroup. Then V is $(T,\varGamma)$-invariant if and only if there exists an m_G-measurable set $A \subseteq G$ which is T-translation invariant and such that*

$$V = \{f \in L^2(G) : \operatorname{supp}(f) \subseteq A\}.$$

Finally, we have the following corollary.

Corollary 5.6. *Let $V \subseteq L^2(G)$ be a non-zero closed subspace. If V is (G,Γ)-invariant, then $V = L^2(G)$.*

6. Examples. In order to illustrate the constructions of the previous sections we now present some examples.

Example 6.1. Let $G = \mathbb{R}$. Then $\Gamma = \mathbb{R}$. Now fix $F = \frac{2}{3}\mathbb{Z}$ as the lattice for translations and $\Delta = \mathbb{Z}$ as the lattice for modulations. Since $\Delta^* = \mathbb{Z}$, the lattice $E = F \cap \Delta^*$ is $2\mathbb{Z}$ and $F + \Delta^* = \frac{2}{3}\mathbb{Z} + \mathbb{Z} = \frac{1}{3}\mathbb{Z}$. Thus, $E^* = \frac{1}{2}\mathbb{Z}$ and $E^*/\Delta \approx \mathbb{Z}_2$. Hence, we can fix $\Pi_{\Gamma/E^*} = [0, 1/2)$ and $\Pi_{E^*/\Delta} = \{0, 1/2\}$.

On the other hand, we set $\Pi_{F/E} = \{0, -2/3, -4/3\}$ and $\Pi_{G/(F+\Delta^*)} = [0, 1/3)$. Then, by (2.4), $\Pi_{G/\Delta^*} = [0, 1/3) \cup [2/3, 1) \cup [4/3, 5/3)$ is a section for G/Δ^*.

Therefore, the fundamental isometry of Definition 4.3 applied to $f \in L^2(\mathbb{R})$ is given by the formula

$$\mathcal{T} f(x, \xi) = (Zf(x, \xi), Zf(x, \xi + 1/2)),$$

where $x \in [0, 1/3) \cup [2/3, 1) \cup [4/3, 5/3)$, $\xi \in [0, 1/2)$ and Z is the Zak transform in $\mathbb{R}$ given by $Zf(x, \xi) = \sum_{k\in\mathbb{Z}} f(x - k)e^{2\pi i k\xi}$.

Let $\varphi \in L^2(\mathbb{R})$ and let $S_{F,\Delta}(\varphi)$ be the (F,Δ)-invariant space generated by $\{\varphi\}$. If J is the shift-modulation range function associated to $S_{F,\Delta}(\varphi)$ through Theorem 5.1, then

$$J(x, \xi) = \operatorname{span}\{\mathcal{T}\varphi(x, \xi), \mathcal{T}\varphi(x + 2/3, \xi), \mathcal{T}\varphi(x + 4/3, \xi)\},$$

where $x \in [0, 1/3)$ and $\xi \in [0, 1/2)$. ■

In the next example we change the lattice of translations. We will see that the fiberization isometry in Example 6.2 has the same formula as in Example 6.1.

Example 6.2. Consider now $G = \mathbb{R}$, $F = \frac{2}{5}\mathbb{Z}$ and $\Delta = \mathbb{Z}$. Then $E = 2\mathbb{Z}$. By the same reasoning as in Example 6.1 we find that $\Pi_{G/\Delta^*} = [0, 1/5) \cup [2/5, 3/5) \cup [4/5, 1) \cup [6/5, 7/5) \cup [8/5, 9/5)$ and $\Pi_{\Gamma/E^*} = [0, 1/2)$. Hence, if $\varphi \in L^2(\mathbb{R})$ the fiberization isometry is

$$\mathcal{T}\varphi(x, \xi) = (Z\varphi(x, \xi), Z\varphi(x, \xi + 1/2)),$$

where $x \in [0, 1/5) \cup [2/5, 3/5) \cup [4/5, 1) \cup [6/5, 7/5) \cup [8/5, 9/5)$, $\xi \in [0, 1/2)$ and Z is the usual Zak transform in $\mathbb{R}$.

In this case, since $\Pi_{F/E} = \{0, -2/5, -4/5, -6/5, -8/5\}$, the shift-modulation range function associated to $S_{F,\Delta}(\varphi)$ is

$$J(x, \xi) = \operatorname{span}\{\mathcal{T}\varphi(x, \xi), \mathcal{T}\varphi(x+2/5, \xi), \mathcal{T}\varphi(x+4/5, \xi), \mathcal{T}\varphi(x+6/5, \xi), \mathcal{T}\varphi(x+8/5, \xi)\},$$

where $x \in [0, 1/5)$ and $\xi \in [0, 1/2)$. ■

REMARK 6.3. Note that in Examples 6.1 and 6.2, the fiberization isometries have the same formula but different domains. This is due to the fact that the lattice E is the same in both cases. The difference here appears in the translations that are outside of E and they are mainly reflected in the shift-modulation range function.

Our last example is for $G = \mathbb{T}$.

EXAMPLE 6.4. Let $G = \mathbb{T}$. Then $\Gamma = \mathbb{Z}$. Let us fix $m, n \in \mathbb{N}$. We consider

$$F = \frac{1}{m}\mathbb{Z}_m = \left\{0, \frac{1}{m}, \frac{2}{m}, \dots, \frac{m-1}{m}\right\} \subseteq \mathbb{T}$$

as the lattice for translations and $\Delta = n\mathbb{Z}$ as the lattice for modulations. Since $\Delta^* = 1/n\mathbb{Z}_n = \{0, 1/n, 2/n, \dots, (n-1)/n\}$, we see that $E = F \cap \Delta^* = \frac{1}{(n:m)}\mathbb{Z}_{(n:m)}$ where $(n : m)$ is the greatest common divisor of n and m, and that $F + \Delta^* = \frac{1}{[n:m]}\mathbb{Z}_{[n:m]}$ where $[n : m]$ is the least common multiple of n and m. The construction of the fiberization isometry and the shift-modulation range function can be done for general n and m. For simplicity we fix $m = 15$ and $n = 12$.

Since $(12 : 15) = 3$ and $[12 : 15] = 60$, we have $E = \{0, 1/3, 2/3\}$ and $F + \Delta^* = \frac{1}{60}\mathbb{Z}_{60}$. Then $E^* = 3\mathbb{Z}$ and we can fix $\Pi_{E^*/\Delta}$ as $\{1, 3, 6, 9\}$ and Π_{Γ/E^*} as $\{0, 1, 2\}$. Now, in order to construct Π_{G/Δ^*} following (2.4) we choose

$$\Pi_{G/F+\Delta^*} = \left[0, \frac{1}{60}\right) \quad \text{and} \quad \Pi_{F/E} = \left\{0, \frac{3}{15}, \frac{6}{15}, \frac{9}{15}, \frac{12}{15}\right\}.$$

Then the section Π_{G/Δ^*} is

$$\Pi_{G/\Delta^*} = \left[0, \frac{1}{60}\right) \cup \left[\frac{12}{60}, \frac{13}{60}\right) \cup \left[\frac{24}{60}, \frac{25}{60}\right) \cup \left[\frac{36}{60}, \frac{37}{60}\right) \cup \left[\frac{48}{60}, \frac{49}{60}\right).$$

For $\varphi \in L^2(\mathbb{T})$ the fiberization isometry applied to φ is

$$\mathcal{T}\varphi(x, \xi) = (Z\varphi(x, \xi), Z\varphi(x, \xi + 3), Z\varphi(x, \xi + 6), Z\varphi(x, \xi + 9))$$

where $x \in \Pi_{G/\Delta^*}$, $\xi \in \Pi_{\Gamma/E^*} = \{0, 1, 2\}$ and the Zak transform is given by $Z\varphi(x, \xi) = \sum_{j=0}^{11} \varphi(x - j/12)e^{2\varphi ii \frac{j}{12}\xi}$.

We now focus on the particular case when $\varphi = \chi_{[0,1/60)}$. For all $\xi \in \{0, 1, 2\}$, $Z\varphi(x, \xi) = 1$ if $x \in [0, 1/60)$ and $Z\varphi(x, \xi) = 0$ if $x \notin [0, 1/60)$. Moreover, it can be proven that for each $r \in \{0, 3, 6, 9, 12\}$,

$$Z(T_{r/15}\varphi)(x, \xi) = \begin{cases} 1 & \text{if } x \in \left[\frac{4r}{60}, \frac{4r+1}{60}\right), \\ 0 & \text{if } x \in \Pi_{G/\Delta^*} \setminus \left[\frac{4r}{60}, \frac{4r+1}{60}\right). \end{cases}$$

Then, for all $\xi \in \{0,1,2\}$ and for each $r \in \{0,3,6,9,12\}$,

$$\mathcal{T}(T_{\frac{r}{15}}\varphi)(x,\xi) = \begin{cases} (1,1,1,1) & \text{if } x \in \left[\frac{4r}{60}, \frac{4r+1}{60}\right), \\ (0,0,0,0) & \text{if } x \in \Pi_{G/\Delta^*} \setminus \left[\frac{4r}{60}, \frac{4r+1}{60}\right). \end{cases} \tag{6.1}$$

The shift-modulation range function associated to $S_{(F,\Delta)}(\varphi)$ is

$$J(x,\xi) = \operatorname{span}\big\{\mathcal{T}(T_{r/15}\varphi)(x,\xi) : r \in \{0,3,6,9,12\}\big\},$$

and, using (6.1) we obtain $J(x,\xi) = \operatorname{span}\{(1,1,1,1)\}$ for $(x,\xi) \in [0,1/60) \times \{0,1,2\}$.

Now consider the translation of $\varphi = \chi_{[0,1/60)}$ by $1/2$. We will show that $T_{1/2}\varphi \notin S_{(F,\Delta)}(\varphi)$. With similar computations to the above, it can be shown that, for all $\xi \in \{0,1,2\}$,

$$\mathcal{T}(T_{1/2}\varphi)(x,\xi) = \begin{cases} (1+e^{\pi i\xi}, 1-e^{\pi i\xi}, 1+e^{\pi i\xi}, 1-e^{\pi i\xi}) & \text{if } x \in [0,1/60), \\ (0,0,0,0) & \text{if } x \in \Pi_{G/\Delta^*} \setminus [0,1/60). \end{cases}$$

From this, we deduce that $\mathcal{T}(T_{1/2}\varphi)(x,1) = (0,2,0,2) \notin J(x,1)$ for $x \in [0,1/60)$. Using Theorem 5.1 we conclude that $T_{1/2}\varphi \notin S_{(F,\Delta)}(\varphi)$. ∎

Acknowledgements. We thank the referees for their suggestions that led to a great improvement of the manuscript. We also thank Magalí Anastasio for fruitful discussions.

The research of the authors was partially supported by Grants: CONICET, PIP 398, UBACyT X638 and X502.

References

[Bow07] M. Bownik, *The structure of shift-modulation invariant spaces: the rational case*, J. Funct. Anal. 244 (2007), 172–219.

[CP10] C. Cabrelli and V. Paternostro, *Shift-invariant spaces on LCA groups*, J. Funct. Anal. 258 (2010), 2034–2059.

[CC01] P. G. Casazza and O. Christensen, *Weyl–Heisenberg frames for subspaces of* $L^2(\mathbb{R})$, Proc. Amer. Math. Soc. 129 (2001), 145–154.

[CCJ01] P. G. Casazza, O. Christensen, Ole and A. J. E. M. Janssen, *Weyl–Heisenberg frames, translation invariant systems and the Walnut representation*, J. Funct. Anal. 180 (2001), 85–147.

[Chr03] O. Christensen, *An Introduction to Frames and Riesz Bases*, Appl. Numer. Harmonic Anal., Birkhäuser Boston, Boston, MA, 2003.

[Dau92] I. Daubechies, *Ten Lectures on Wavelets*, CBMS-NSF Reg. Conf. Ser. Appl. Math. 61, SIAM, Philadelphia, PA, 1992.

[FG64] J. Feldman and F. P. Greenleaf, *Existence of Borel transversals in groups*, Pacific J. Math. 25 (1968), 455–461.

[GH03] J. Gabardo and D. Han, *Aspects of Gabor analysis and operator algebras*, in: Advances in Gabor Analysis, Appl. Numer. Harmonic Anal., Birkhäuser Boston, Boston, MA, 2003, 129–152.

[GH04] J. Gabardo and D. Han, *Balian–Low phenomenon for subspace Gabor frames*, J. Math. Phys. 45 (2004), 3362–3378.

[Gel50] I. M. Gel'fand, *Expansion in characteristic functions of an equation with periodic coefficients*, Dokl. Akad. Nauk SSSR (N.S.) 73 (1950), 1117–1120.

[Gro01] K. Gröchenig, *Foundations of Time-Frequency Analysis*, Appl. Numer. Harmonic Anal., Birkhäuser Boston, Boston, MA, 2001.

[HS64] M. Hasumi and T. P. Srinivasan, *Doubly invariant subspaces. II*, Pacific J. Math. 14 (1964), 525–535.

[Hel64] H. Helson, *Lectures on Invariant Subspaces*, Academic Press, New York, 1964.

[KR08] R. A. Kamyabi Gol and R. R. Tousi, *The structure of shift invariant spaces on a locally compact abelian group*, J. Math. Anal. Appl. 340 (2008), 219–225.

[KK98] E. Kaniuth and G. Kutyniok, *Zeros of the Zak transform on locally compact abelian groups*, Proc. Amer. Math. Soc. 126 (1998), 3561–3569.

[Rud87] W. Rudin, *Real and Complex Analysis*, 3rd ed., McGraw-Hill, New York, 1987.

[Rud62] W. Rudin, *Fourier Analysis on Groups*, Wiley Classics Library, reprint of the 1962 original, Wiley, New York, 1990.

[Sri64] T. P. Srinivasan, *Doubly invariant subspaces*, Pacific J. Math. 14 (1964), 701–707.

[Wei64] A. Weil, *Sur certains groupes d'opérateurs unitaires*, Acta Math. 111 (1964), 143–211.

[Zak67] J. Zak, *Finite translations in solid state physics*, Phys. Rev. Lett. 19 (1967), 1385–1397.

Carlos Cabrelli, Victoria Paternostro
Departamento de Matemática
Facultad de Ciencias Exactas y Naturales
Universidad de Buenos Aires
Ciudad Universitaria, Pabellón I
1428 Buenos Aires, Argentina
and
IMAS-CONICET
Consejo Nacional de Investigaciones Científicas y Técnicas, Argentina
E-mail: cabrelli@dm.uba.ar
vpater@dm.uba.ar

Received September 2, 2011
Revised version June 5, 2012 (7291)