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Combinatorial inequalities and subspaces of L_1

by

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Abstract. Let M_1 and M_2 be N-functions. We establish some combinatorial inequalities and show that the product spaces $\ell^n_{M_1}(\ell^n_{M_2})$ are uniformly isomorphic to subspaces of L_1 if M_1 and M_2 are "separated" by a function t^r, $1 < r < 2$.

1. Introduction. The structure and variety of subspaces of L_1 is very rich. Over the years, tremendous effort has been put in characterizing subspaces of L_1. Although there are a number of sophisticated criteria at hand now, it might turn out to be nontrivial to decide for a specific Banach space whether it is isomorphic to a subspace of L_1.

Using the theorem of de Finetti it was shown in [3] that every Orlicz space with a 2-concave Orlicz function embeds into L_1. Consequently, all spaces whose norms are averages of 2-concave Orlicz norms embed into L_1. In fact, this characterizes all subspaces of L_1 with a symmetric basis. The corresponding finite-dimensional version of this result was proved in [7], using combinatorial and probabilistic tools.

Although this characterization gives a complete picture of which spaces with a symmetric basis embed into L_1, it might not be easy to apply. This becomes apparent when one considers Lorentz spaces [12] (see also [11]).

Here we study matrix subspaces of L_1, i.e., spaces $E(F)$ where E and F have a 1-symmetric basis $(e_i)_{i=1}^n$ and $(f_j)_{j=1}^n$, and where for all matrices $(x_{ij})_{i,j}$,

$$\|(x_{ij})_{i,j}\|_{E(F)} = \Big\|\sum_{i=1}^n \Big\|\sum_{j=1}^n x_{ij} f_j\Big\|_F e_i\Big\|_E.$$

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Our main result is the following:

THEOREM 1.1. *Let $1 < p < r < 2$ and M and N be N-functions with $M(t)/t^p$ pseudo-decreasing, $N(t)/t^r$ pseudo-increasing and $N(t)/t^2$ pseudo-decreasing. Then there is a constant $C > 0$ such that for all $n \in \mathbb{N}$ there is a subspace E of L_1 with* $\dim(E) = n^2$ *and*

$$d(E, \ell_M^n(\ell_N^n)) \leq C.$$

Here, d denotes the Banach–Mazur distance. A function $f : [0,\infty) \to [0,\infty)$ is *pseudo-increasing* if there is a constant $c > 0$ such that for all $s < t$ we have $f(s) \leq cf(t)$. A pseudo-decreasing function is also defined in this way.

As far as the hypothesis of Theorem 1.1 is concerned, by a regularization ([4, Theorem 1.6], [5] and [10]) we can pass to N-functions $\tilde{M}$ and $\tilde{N}$ such that $\tilde{M}(t)/t^p$ is decreasing, $\tilde{N}(t)/t^r$ increasing and $\tilde{N}(t)/t^2$ decreasing.

To prove Theorem 1.1 we first show that $\ell_M^n(\ell_r^n)$ are uniformly isomorphic to subspaces of L_1. To do this, we develop some technical combinatorial results related to Orlicz norms and use techniques first developed in [7] and [8]. These combinatorial inequalities, used to embed finite-dimensional Banach spaces into L_1, are interesting in themselves. Using a result of Bretagnolle and Dacunha-Castelle [3], that ℓ_N is a subspace of L_r if $N(t)/t^r$ is increasing and $N(t)/t^2$ decreasing, we obtain our main result.

In some sense the conditions that $M(t)/t^p$ is decreasing, $N(t)/t^r$ is increasing and $N(t)/t^2$ is decreasing, are sharp. This is a consequence of [8, Corollary 3.3]. Kwapień and Schütt proved that

$$\frac{1}{5\sqrt{2}}\|\mathrm{Id}\| \leq d\left(E(F), G\right),$$

where $\mathrm{Id} \in L(E,F)$ is the natural identity map, i.e., $\mathrm{Id}(\sum_{i=1}^n a_i e_i) = \sum_{j=1}^n a_j f_j$, and E, F are n-dimensional spaces with a 1-symmetric and 1-unconditional basis respectively. For $1 \leq p < r \leq 2$ they find that for any n^2-dimensional subspace G of L_1,

$$d(\ell_r^n(\ell_p^n), G) \geq \frac{1}{5\sqrt{2}} n^{1/p-1/r}.$$

Therefore, the conditions are sharp.

The technical difficulties that occur are that in general, Orlicz functions are not homogeneous for some p, i.e., $M(\lambda t) \neq \lambda^p M(t)$.

Furthermore, since our results are of a very technical nature in many places, we tried to make this paper as self-contained as possible and therefore easily accessible.

2. Preliminaries and combinatorial inequalities. A convex function $M : [0,\infty) \to [0,\infty)$ with $M(0) = 0$ and $M(t) > 0$ for $t > 0$ is called an

Orlicz function. An Orlicz function (as we define it) is bijective and continuous on $[0,\infty)$. We define the *Orlicz space* ℓ_M^n to be $\mathbb{R}^n$ equipped with the norm

$$\|x\|_M = \inf\Big\{\rho > 0 : \sum_{i=1}^n M(|x_i|/\rho) \le 1\Big\}.$$

Given an Orlicz function M, we define its *conjugate function* M^* by the Legendre transform, i.e.,

$$M^*(x) = \sup_{t\in[0,\infty)} (xt - M(t)).$$

An *N-function* M is an Orlicz function with

$$\lim_{t\to 0} \frac{M(t)}{t} = 0 \quad \text{and} \quad \lim_{t\to\infty} \frac{M(t)}{t} = \infty.$$

The conjugate function of an N-function is again an N-function. For all N-functions M and for all $0 \le t < \infty$ we have

$$t \le M^{-1}(t) M^{*-1}(t) \le 2t. \tag{2.1}$$

See [1] and [2, formula (6)]. We say that two Orlicz functions M and N are *equivalent* if there are positive constants a and b such that for all $t \ge 0$,

$$M(at) \le N(t) \le M(bt).$$

For N-functions M and N this is equivalent to

$$aN^{-1}(t) \le M^{-1}(t) \le bN^{-1}(t).$$

If two Orlicz functions are equivalent so are their norms. Notice that it is enough for the functions M and N to be equivalent in a neighborhood of 0 for the corresponding sequence spaces ℓ_M and ℓ_N to coincide [9].

Let X and Y be isomorphic Banach spaces. We say that they are C-*isomorphic* if there is an isomorphism $I : X \to Y$ with $\|I\|\,\|I^{-1}\| \le C$. We define the *Banach–Mazur distance* of X and Y by

$$d(X,Y) = \inf\left\{\|T\|\,\|T^{-1}\| : T \in L(X,Y) \text{ an isomorphism}\right\}.$$

Let $(X_n)_n$ be a sequence of n-dimensional normed spaces and let Z be also a normed space. If there exists a constant $C > 0$ such that for all $n \in \mathbb{N}$ there exists a normed space $Y_n \subseteq Z$ with $\dim(Y_n) = n$ and $d(X_n, Y_n) \le C$, then we say that $(X_n)_n$ *embeds uniformly into* Z or for short, X_n embeds into Z. For a detailed introduction to the concept of Banach–Mazur distance, see for example [13].

We will write $a \sim b$ to mean that there exist positive absolute constants c_1, c_2 such that $c_1 a \le b \le c_2 a$ and similarly use $a \lesssim b$ or $a \gtrsim b$.

We need the following two results by Kwapień and Schütt [7, 8].

Lemma 2.1 ([7, Lemma 2.1]). *Let $n, m \in \mathbb{N}$ with $n \le m$ and let $y \in \mathbb{R}^m$ with $y_1 \ge \cdots \ge y_m > 0$. Furthermore, let M be an N-function such that for* *all* $k=1,\dots,m$,

$$M^*\Big(\sum_{i=1}^{k} y_i\Big)=\frac{k}{m}. \tag{2.2}$$

Define $\|\cdot\|_y$ *by*

$$\|x\|_y=\max_{\sum_{i=1}^n k_i=m}\sum_{i=1}^{n}\Big(\sum_{j=1}^{k_i} y_j\Big)|x_i|.$$

Then, for all $x\in\mathbb{R}^n$,

$$\frac{1}{2}\|x\|_y\le\|x\|_M\le 2\|x\|_y.$$

Note that there is always an N-function M satisfying (2.2): We extend M^* affinely between the given values. Moreover, M^* is extended in a neighborhood of 0 and beyond the last point by a quadratic function. Then M^* is finite everywhere and takes the value 0 only at 0. So, M^* is an N-function. Its conjugate function is the desired M.

LEMMA 2.2 ([8, Lemma 2.5]). *Let* M *be an* N*-function. Then, for all* $x\in\mathbb{R}^n$,

$$\frac{1}{2}\Big(\frac{1}{2}-\frac{1}{n-1}\Big)\|x\|_M \le \frac{1}{n!}\sum_{\pi}\max_{1\le i\le n}\bigg|x_i n\cdot\bigg(M^{*-1}\Big(\frac{\pi(i)}{n}\Big)-M^{*-1}\Big(\frac{\pi(i)-1}{n}\Big)\bigg)\bigg|\le 2\|x\|_M,$$

where the sum is over all permutations $\pi\in S_n$.

LEMMA 2.3 ([8, Corollary 1.7]). *For all* $n\in\mathbb{N}$ *and all nonnegative numbers* $B(i,k,\ell)$, $1\le i,k,\ell\le n$,

$$\frac{1}{16n^2}\sum_{\alpha=1}^{n^2}s(\alpha)\le\frac{1}{(n!)^2}\sum_{\pi,\sigma\in S_n}\max_{1\le i\le n}B(i,\pi(i),\sigma(i))\le\frac{4}{n^2}\sum_{\alpha=1}^{n^2}s(\alpha),$$

where $s(1),\dots,s(n^3)$ *is the decreasing rearrangement of the numbers* $B(i,k,\ell)$, $1\le i,k,\ell\le n$.

From Lemmas 2.1, 2.2 and 2.3 we obtain the following result.

LEMMA 2.4. *Let* $a\in\mathbb{R}^n$ *with* $a_1\ge\dots\ge a_n>0$ *and let* M *be an* N*-function. Furthermore, let* N *be an* N*-function whose conjugate function* N^* *satisfies, for all* $\ell=1,\dots,n^2$,

$$N^{*-1}\Big(\frac{\ell}{n^2}\Big)=\frac{1}{n^2}\sum_{k=1}^{\ell}s(k),$$

where $s(1),\dots,s(n^2)$ is the decreasing rearrangement of

$$a_i n\left(M^{*-1}\left(\frac{j}{n}\right)-M^{*-1}\left(\frac{j-1}{n}\right)\right),\qquad i,j=1,\dots,n.$$

Then, for all $x\in\mathbb{R}^n$,

$$c\|x\|_N\le\frac{1}{n!}\sum_\pi\|(x_i a_{\pi(i)})_{i=1}^n\|_M\le 2\|x\|_N,$$

where $c>0$ is an absolute constant.

Furthermore, one can choose N so that N^{*-1} is an affine function between the values ℓ/n^2, $\ell=1,\dots,n^2$.

Proof. From Lemma 2.2, we know

$$c\|x\|_M\le\frac{1}{n!}\sum_\sigma\max_{1\le i\le n}\left|x_i n\left(M^{*-1}\left(\frac{\sigma(i)}{n}\right)-M^{*-1}\left(\frac{\sigma(i)-1}{n}\right)\right)\right|\le 2\|x\|_M.$$

Thus

$$\begin{aligned}
&c\frac{1}{n!}\sum_\pi\|(x_i a_{\pi(i)})_{i=1}^n\|_M\\
&\qquad\le\frac{1}{n!^2}\sum_{\sigma,\pi}\max_{1\le i\le n}\left|x_i a_{\pi(i)} n\left(M^{*-1}\left(\frac{\sigma(i)}{n}\right)-M^{*-1}\left(\frac{\sigma(i)-1}{n}\right)\right)\right|\\
&\qquad\le 2\frac{1}{n!}\sum_\pi\|(x_i a_{\pi(i)})_{i=1}^n\|_M.
\end{aligned}$$

Applying Lemmas 2.1 and 2.3 yields the desired result. ∎

Now we are able to develop the combinatorial ingredients that we need to prove Proposition 3.1. These results are extensions of the results proved in [7] and [8] respectively.

LEMMA 2.5. (i) *Let $1<r<\infty$ and $a_1\ge\dots\ge a_n>0$. Then there exists an N-function N whose conjugate function N^* satisfies, for all $\ell=1,\dots,n$,*

$$\begin{aligned}
N^{*-1}\left(\frac{\ell}{n}\right)&\le C_r\left(\frac{1}{n}\sum_{i=1}^{\ell}a_i+\left(\frac{\ell}{n}\right)^{1/r^*}\left(\frac{1}{n}\sum_{i=\ell+1}^{n}|a_i|^r\right)^{1/r}\right)\\
&\le 8N^{*-1}\left(\frac{\ell}{n}\right),
\end{aligned}\tag{2.3}$$

$$N^{*-1}\left(\frac{\ell}{n^2}\right)\le C_r\frac{1}{n}\left(\frac{\ell}{n}\right)^{1/r^*}\left(\sum_{i=1}^{\ell}|a_i|^r\right)^{1/r}\le 2N^{*-1}\left(\frac{\ell}{n^2}\right),\tag{2.4}$$

where $C_r=r^{1/r}(r^*)^{1/r^*}$. Furthermore, for all $x\in\mathbb{R}^n$,

$$c\|x\|_N\le\frac{1}{n!}\sum_\pi\left(\sum_{i=1}^n|x_i a_{\pi(i)}|^r\right)^{1/r}\le 2\|x\|_N.$$

(ii) *Let $1 < r < \infty$. There exists $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, all $a_1 \geq \cdots \geq a_n > 0$ and all Orlicz functions $\bar{N}$ satisfying, for all $\ell = 1, \dots, n$,*

$$\bar{N}^{*-1}\left(\frac{\ell}{n}\right) \leq C_r\left(\frac{1}{n}\sum_{i=1}^{\ell} a_i + \left(\frac{\ell}{n}\right)^{1/r^*}\left(\frac{1}{n}\sum_{i=\ell+1}^{n}|a_i|^r\right)^{1/r}\right) \leq 8\bar{N}^{*-1}\left(\frac{\ell}{n}\right) \tag{2.5}$$

and affine on the intervals $[\ell/n, (\ell+1)/n]$, $\ell = 0, \dots, n-1$, we have, for all $x \in \mathbb{R}^n$,

$$a_r\|x\|_{\bar{N}} \leq \frac{1}{n!}\sum_{\pi}\Big(\sum_{i=1}^{n}|x_i a_{\pi(i)}|^r\Big)^{1/r} \leq b_r\|x\|_{\bar{N}},$$

where a_r and b_r just depend on r and C_r as in (i).

By part (i) there is indeed an Orlicz function as specified in (ii): The N-function of (i) can be modified so that it is affine on the intervals $[\ell/n, (\ell+1)/n]$, $\ell = 0, \dots, n-1$.

Proof. (i) From Lemma 2.4 we obtain

$$c\|x\|_N \leq \frac{1}{n!}\sum_{\pi}\|(x(i)a_{\pi(i)})_{i=1}^n\|_M \leq 2\|x\|_N,$$

where $M(t) = t^r$,

$$N^*\left(\frac{1}{n^2}\sum_{k=1}^{\ell}s(k)\right) = \frac{\ell}{n^2}, \qquad \ell = 1, \dots, n^2,$$

and $s(1), \dots, s(n^2)$ is the decreasing rearrangement of the numbers

$$a_i n\left(M^{*-1}\left(\frac{j}{n}\right) - M^{*-1}\left(\frac{j-1}{n}\right)\right), \qquad 1 \leq i, j \leq n.$$

Obviously $M^*(s) = (1/r)^{1/r}(1/r^*)^{1/r^*}s^{r^*}$ and $M^{*-1}(t) = r^{1/r}(r^*)^{1/r^*}t^{1/r^*}$. We choose $C_r := r^{1/r}(r^*)^{1/r^*}$. For all $\ell \leq n^2$ we have

$$\begin{aligned}\frac{1}{n^2}\sum_{k=1}^{\ell}s(k) &= \max_{\substack{\sum_{i=1}^n \ell_i = \ell \\ \ell_i \leq n}} \frac{1}{n^2}\sum_{i=1}^{n}a_i\sum_{j=1}^{\ell_i}n\left(M^{*-1}\left(\frac{j}{n}\right) - M^{*-1}\left(\frac{j-1}{n}\right)\right) \\ &= \max_{\substack{\sum_{i=1}^n \ell_i = \ell \\ \ell_i \leq n}} \frac{1}{n^2}\sum_{i=1}^{n}a_i n M^{*-1}\left(\frac{\ell_i}{n}\right) \\ &= \max_{\substack{\sum_{i=1}^n \ell_i = \ell \\ \ell_i \leq n}} C_r\frac{1}{n}\sum_{i=1}^{n}a_i\left|\frac{\ell_i}{n}\right|^{1/r^*}.\end{aligned} \tag{2.6}$$

We now show the right inequality of (2.3). We consider the case $\ell = mn$, $1 \le m \le n$. Then

$$N^{*-1}\left(\frac{m}{n}\right) = \frac{1}{n^2}\sum_{k=1}^{nm} s(k),$$

and by (2.6),

$$N^{*-1}\left(\frac{m}{n}\right) = C_r \max_{\substack{\sum_{i=1}^n \ell_i = mn \\ \ell_i \le n}} \frac{1}{n}\sum_{i=1}^n a_i \left|\frac{\ell_i}{n}\right|^{1/r^*}.$$

For $m = 1$ we deduce from Lemma 2.1 that $N^{*-1}(1/n)$ is of the order $\|a\|_r$. Now we consider $m \ge 2$. We choose $\ell_1 = \cdots = \ell_m = n$ and $\ell_{m+1} = \cdots = \ell_n = 0$ to obtain

$$N^{*-1}\left(\frac{m}{n}\right) \ge C_r \frac{1}{n}\sum_{i=1}^m a_i. \tag{2.7}$$

We consider

$$y_j = M^{*-1}\left(\frac{j}{nm}\right) - M^{*-1}\left(\frac{j-1}{nm}\right), \qquad 1 \le j \le nm.$$

From Lemma 2.1 we get

$$\begin{aligned}\frac{1}{2}\|a\|_r \le \|a\|_y &= \max_{\sum_{i=1}^n \ell_i = mn} \sum_{i=1}^n a_i \Big(\sum_{j=1}^{l_i} y_j\Big)\\ &= C_r \max_{\sum_{i=1}^n \ell_i = mn} \sum_{i=1}^n a_i \left|\frac{\ell_i}{mn}\right|^{1/r^*}.\end{aligned}$$

This holds if and only if

$$\frac{1}{2} m^{1/r^*}\|a\|_r \le C_r \max_{\sum_{i=1}^n \ell_i = mn} \sum_{i=1}^n a_i \left|\frac{\ell_i}{n}\right|^{1/r^*}.$$

The inequality also holds for the modified vector $\tilde a$ with $\tilde a_1 = \cdots = \tilde a_m = a_m$ and $\tilde a_i = a_i$ for $i = m+1, \dots, n$, i.e.

$$\frac{1}{2} m^{1/r^*}\|\tilde a\|_r \le C_r \max_{\sum_{i=1}^n \ell_i = mn} \sum_{i=1}^n \tilde a_i \left|\frac{\ell_i}{n}\right|^{1/r^*}. \tag{2.8}$$

We show that without loss of generality, $\ell_i \le n$, $i \le n$. We have $\ell_1 \ge \cdots \ge \ell_m \ge \cdots \ge \ell_n \ge 0$. Obviously, $\ell_i \le n$ for all $i = m, \dots, n$, as otherwise $\sum_{i=1}^m \ell_i > mn$, which cannot occur. Therefore, it suffices to show that we can choose $\ell_1, \dots, \ell_m \le n$. To do this, we construct $\tilde\ell_i$, $i \le m$, such that $\tilde\ell_i \le n$, and such that the maximum in (2.8) is attained up to an absolute constant (we take $\tilde\ell_i = \ell_i$ for $i = m+1, \dots, n$). Now, let $\ell_1, \dots, \ell_n$ be such that the maximum in (2.8) is attained. Then we define, for $i \le m$,

$$\tilde{\ell}_i := \left\lfloor \frac{1}{m} \sum_{j=1}^{m} \ell_j \right\rfloor.$$

($\lfloor x \rfloor$ is the greatest integer smaller than x.) We may assume

$$\left\lfloor \frac{1}{m} \sum_{j=1}^{m} \ell_j \right\rfloor \ge 2,$$

because from $\lfloor m^{-1} \sum_{j=1}^{m} \ell_j \rfloor < 2$ we deduce immediately that $\ell_{m+1}, \dots, \ell_n \le 1$, and therefore $mn = \sum_{j=1}^{n} \ell_j < 2m + (n-m) = n+m$. Since $m \ge 2$ and we may assume that $n \ge 3$ we get a contradiction. Hence, for all $i \le m$,

$$\tilde{\ell}_i \ge \frac{1}{m} \sum_{j=1}^{m} \ell_j - 1 \ge \frac{1}{2} \frac{1}{m} \sum_{j=1}^{m} \ell_j.$$

Now we have

$$\begin{aligned} \sum_{i=1}^{m} \tilde{a}_i \left| \frac{\tilde{\ell}_i}{n} \right|^{1/r^*} &\ge \sum_{i=1}^{m} a_m \frac{1}{2^{1/r^*}} \left(\frac{1}{n} \frac{1}{m} \sum_{j=1}^{m} \ell_j \right)^{1/r^*} \\ &= a_m m^{1/r} \frac{1}{2^{1/r^*}} \left(\frac{1}{n} \sum_{j=1}^{m} \ell_j \right)^{1/r^*}. \end{aligned}$$

From Hölder's inequality we get

$$\sum_{i=1}^{m} \tilde{a}_i \left| \frac{\ell_i}{n} \right|^{1/r^*} = a_m \sum_{i=1}^{m} \left| \frac{\ell_i}{n} \right|^{1/r^*} \le a_m m^{1/r} \left(\frac{1}{n} \sum_{i=1}^{m} \ell_i \right)^{1/r^*}.$$

Thus

$$\sum_{i=1}^{m} \tilde{a}_i \left| \frac{\tilde{\ell}_i}{n} \right|^{1/r^*} \ge \frac{1}{2^{1/r^*}} \sum_{i=1}^{m} \tilde{a}_i \left| \frac{\ell_i}{n} \right|^{1/r^*},$$

and therefore

$$\sum_{i=1}^{n} \tilde{a}_i \left| \frac{\tilde{\ell}_i}{n} \right|^{1/r^*} \ge \frac{1}{2^{1/r^*}} \sum_{i=1}^{n} \tilde{a}_i \left| \frac{\ell_i}{n} \right|^{1/r^*}.$$

Inequality (2.8) gives us

$$C_r \sum_{i=1}^{n} \tilde{a}_i \left| \frac{\tilde{\ell}_i}{n} \right|^{1/r^*} \ge \frac{1}{2^{1/r^*}} \frac{1}{2} m^{1/r^*} \|\tilde{a}\|_r.$$

So we have

$$\frac{1}{2^{1/r^*}} \frac{1}{2} m^{1/r^*} \|\tilde{a}\|_r \le C_r \max_{\substack{\sum_{i=1}^{n} \tilde{\ell}_i = mn \\ \tilde{\ell}_i \le n}} \sum_{i=1}^{n} \tilde{a}_i \left| \frac{\tilde{\ell}_i}{n} \right|^{1/r^*} = n N^{*-1} \left(\frac{m}{n} \right),$$

i.e.,

$$N^{*-1}\left(\frac{m}{n}\right) \geq \frac{1}{2^{1/r^*}}\,\frac{1}{2}\,\frac{m^{1/r^*}}{n}\|\tilde{a}\|_r,$$

and because of

$$\Big(\sum_{i=m+1}^{n} |a_i|^r\Big)^{1/r} \leq \|\tilde{a}\|_r$$

and (2.7) we obtain the right inequality of (2.3).

Now we estimate the left hand side of (2.3). By (2.6), for a suitable choice of ℓ_i,

$$\frac{1}{n^2}\sum_{k=1}^{mn} s(k) = C_r\frac{1}{n}\sum_{i=1}^{n} a_i\left|\frac{\ell_i}{n}\right|^{1/r^*}.$$

Since $\ell_i \leq n$, we obtain

$$\frac{1}{n^2}\sum_{k=1}^{mn} s(k) \leq C_r n^{-1/r^*-1}\Big(\sum_{i=1}^{m} a_i n^{1/r^*} + \sum_{i=m+1}^{n} a_i \ell_i^{1/r^*}\Big).$$

Hölder's inequality implies

$$\frac{1}{n^2}\sum_{k=1}^{mn} s(k) \leq C_r\frac{1}{n}\Big\{\sum_{i=1}^{m} a_i + n^{-1/r^*}\Big(\sum_{i=m+1}^{n} |a_i|^r\Big)^{1/r}\Big(\sum_{i=m+1}^{n} \ell_i\Big)^{1/r^*}\Big\},$$

and with $\sum_{i=1}^{n} \ell_i = mn$,

$$\frac{1}{n^2}\sum_{k=1}^{mn} s(k) \leq C_r\frac{1}{n}\Big(\sum_{i=1}^{m} a_i + m^{1/r^*}\Big(\sum_{i=m+1}^{n} |a_i|^r\Big)^{1/r}\Big).$$

Therefore, we obtain the left inequality of (2.3), i.e.

$$N^{*-1}\left(\frac{m}{n}\right) \leq C_r\left(\frac{1}{n}\sum_{i=1}^{m} a_i + \left(\frac{m}{n}\right)^{1/r^*}\left(\frac{1}{n}\sum_{i=m+1}^{n} |a_i|^r\right)^{1/r}\right).$$

Now we prove (2.4). Because $m \leq n$, from (2.6) we get

$$\begin{aligned}\frac{1}{n^2}\sum_{k=1}^{m} s(k) &= \max_{\sum_{i=1}^{n} \ell_i = m} C_r\frac{1}{n}\sum_{i=1}^{n} a_i\left|\frac{\ell_i}{n}\right|^{1/r^*} \\ &= C_r\frac{1}{n}\left(\frac{m}{n}\right)^{1/r^*} \max_{\sum_{i=1}^{m} \ell_i = m} \sum_{i=1}^{n} a_i\left|\frac{\ell_i}{m}\right|^{1/r^*}.\end{aligned}$$

For $m = 1, \ldots, n$, using Hölder's inequality, we get the left inequality of (2.4),

$$N^{*-1}\left(\frac{m}{n^2}\right) = \frac{1}{n^2}\sum_{k=1}^{m} s(k) \leq C_r\frac{1}{n}\left(\frac{m}{n}\right)^{1/r^*}\Big(\sum_{i=1}^{n} |a_i|^r\Big)^{1/r}.$$

From Lemma 2.1 we obtain for $m=1,\dots,n$ the right inequality of (2.4),

$$\frac{1}{n}\left(\frac{m}{n}\right)^{1/r^*}\left(\sum_{i=1}^{n}|a_i|^r\right)^{1/r}\leq\frac{2}{C_r}N^{*-1}\left(\frac{m}{n^2}\right).$$

(ii) Let N be an N-function as given by (i). We will show that for all t with $1/4^{r^*}n\leq t\leq 1$,

$$\frac{1}{8\cdot 4^{r^*}}N^{*-1}(t)\leq\bar{N}^{*-1}(t)\leq 32\cdot 4^{r^*}N^{*-1}(t). \tag{2.9}$$

From this it will follow that for all x,

$$\frac{1}{32\cdot 4^{r^*}}\|x\|_N\leq\|x\|_{\bar{N}}\leq(48\cdot 4^{r^*}+16)\|x\|_N. \tag{2.10}$$

We show (2.9) first for $1/n\leq t\leq 1$, and then for $1/4^{r^*}n\leq t\leq 1/n$.

For $1/n\leq t\leq 1$, we have

$$\frac{1}{16}N^{*-1}(t)\leq\bar{N}^{*-1}(t)\leq 16N^{*-1}(t). \tag{2.11}$$

Indeed, there exists an $\ell\in\{1,\dots,n-1\}$ such that $\ell/n\leq t\leq(\ell+1)/n$. By (2.3) and (2.5),

$$\begin{aligned}
N^{*-1}(t)&\leq N^{*-1}\left(\frac{\ell+1}{n}\right)\\
&\leq C_r\left\{\frac{1}{n}\sum_{i=1}^{\ell+1}a_i+\left(\frac{\ell+1}{n}\right)^{1/r^*}\left(\frac{1}{n}\sum_{i=\ell+2}^{n}|a_i|^r\right)^{1/r}\right\}\\
&\leq 2C_r\left\{\frac{1}{n}\sum_{i=1}^{\ell}a_i+\left(\frac{\ell}{n}\right)^{1/r^*}\left(\frac{1}{n}\sum_{i=\ell+1}^{n}|a_i|^r\right)^{1/r}\right\}\\
&\leq 16\bar{N}^{*-1}\left(\frac{\ell}{n}\right)\leq 16\bar{N}^{*-1}(t).
\end{aligned}$$

The inverse estimate is obtained in the same way.

Now, we show that for all t with $1/4^{r^*}n\leq t\leq 1/n$,

$$\frac{1}{8\cdot 4^{r^*}}N^{*-1}(t)\leq\bar{N}^{*-1}(t)\leq 32\cdot 4^{r^*}N^{*-1}(t). \tag{2.12}$$

By (2.3) for $\ell=1$,

$$N^{*-1}\left(\frac{1}{n}\right)\leq C_r\left(\frac{a_1}{n}+\left(\frac{1}{n}\right)^{1/r^*}\left(\frac{1}{n}\sum_{i=2}^{n}|a_i|^r\right)^{1/r}\right).$$

For n with $n \geq 2 \cdot 4^{r^*}$ we have $2 \cdot 4^{r^*}[n/4^{r^*}] \geq n$. By Hölder's inequality,

$$N^{*-1}\left(\frac{1}{n}\right) \leq 2^{1/r^*} C_r \frac{1}{n}\Big(\sum_{i=1}^{n} |a_i|^r\Big)^{1/r} \tag{2.13}$$
$$\leq C_r 2^{1/r^*} 2 \cdot 4^{r^*} \frac{1}{n}\Big(\sum_{i=1}^{[n/4^{r^*}]} |a_i|^r\Big)^{1/r}$$
$$\leq C_r 2^{1/r^*} 2 \cdot 4^{r^*} N^{*-1}\left(\frac{[n/4^{r^*}]}{n^2}\right)$$
$$\leq C_r 2^{1/r^*} 2 \cdot 4^{r^*} N^{*-1}\left(\frac{1}{4^{r^*} n}\right)$$
$$\leq C_r 2^{1/r^*} 2 \cdot 4^{r^*} N^{*-1}(t).$$

The function $\bar{N}^{*-1}$ takes the values $\bar{N}^{*-1}(t) = tn\bar{N}^{*-1}(1/n)$ on the interval $[0, 1/n]$. Hence, for all t with $1/4^{r^*}n \leq t \leq 1/n$,

$$\bar{N}^{*-1}\left(\frac{1}{4^{r^*} n}\right) \leq \bar{N}^{*-1}(t) = tn\bar{N}^{*-1}\left(\frac{1}{n}\right)$$
$$\leq \bar{N}^{*-1}\left(\frac{1}{n}\right) = 4^{r^*} \bar{N}^{*-1}\left(\frac{1}{4^{r^*} n}\right).$$

Thus, we have

$$N^{*-1}(t) \leq N^{*-1}\left(\frac{1}{n}\right) \leq C_r\left(a_1/n + \left(\frac{1}{n}\right)^{1/r^*}\left(\frac{1}{n}\sum_{i=2}^{n} |a_i|^r\right)^{1/r}\right)$$
$$\leq 8\bar{N}^{*-1}\left(\frac{1}{n}\right) \leq 8 \cdot 4^{r^*} \bar{N}^{*-1}(t)$$

and

$$\bar{N}^{*-1}(t) \leq \bar{N}^{*-1}\left(\frac{1}{n}\right) \leq C_r\left(\frac{a_1}{n} + \left(\frac{1}{n}\right)^{1/r^*}\left(\frac{1}{n}\sum_{i=2}^{n} |a_i|^r\right)^{1/r}\right)$$
$$\leq 8N^{*-1}\left(\frac{1}{n}\right) \leq C_r 2^{1/r^*} 16 \cdot 4^{r^*} N^{*-1}(t).$$

Hence, (2.12) follows.

Furthermore, we have

$$N^{*-1}\left(\frac{1}{4^{r^*} n}\right) \leq \frac{2}{3} N^{*-1}\left(\frac{1}{n}\right) \quad \text{and} \quad \bar{N}^{*-1}\left(\frac{1}{4^{r^*} n}\right) = \frac{1}{4^{r^*}} \bar{N}^{*-1}\left(\frac{1}{n}\right). \tag{2.14}$$

Indeed, the equality is obvious. We show the inequality. By (2.4) we get, for $\ell = [n/4^{r^*}]$,

$$\frac{1}{C_r}N^{*-1}\Big(\frac{1}{4^{r^*}n}\Big) \le \frac{1}{C_r}N^{*-1}\Big(\frac{[n/4^{r^*}]+1}{n^2}\Big)$$
$$\le \frac{1}{n}\Big(\frac{[n/4^{r^*}]+1}{n}\Big)^{1/r^*}\Big(\sum_{i=1}^{[n/4^{r^*}]+1}|a_i|^r\Big)^{1/r}.$$

By (2.4), for $\ell = n$ and sufficiently large n we get

$$\frac{1}{C_r}N^{*-1}\Big(\frac{1}{4^{r^*}n}\Big) \le \frac{1}{3n}\Big(\sum_{i=1}^{n}|a_i|^r\Big)^{1/r} \le \frac{2}{3C_r}N^{*-1}\Big(\frac{1}{n}\Big).$$

Now we show (2.10). Let $x \in \mathbb{R}^n$ with $\|x\|_{N^*} = 1$ and $x_1 \ge \cdots \ge x_n \ge 0$. Furthermore, let $t \in \mathbb{R}^n$ be such that $x_i = N^{*-1}(t_i)$. Let i_0 be such that

$$t_1 \ge \cdots \ge t_{i_0} \ge \frac{1}{4^{r^*}n} > t_{i_0+1} \ge \cdots \ge t_n.$$

Then $\sum_{i=1}^n t_i = 1$. We choose

$$\tilde{t} = (t_1,\dots,t_{i_0},0,\dots,0), \qquad \tilde{\tilde{t}} = (0,\dots,0,t_{i_0+1},\dots,t_n),$$
$$\tilde{x} = (x_1,\dots,x_{i_0},0,\dots,0), \quad \tilde{\tilde{x}} = (0,\dots,0,x_{i_0+1},\dots,x_n).$$

We will prove that $\|\tilde{\tilde{x}}\|_{N^*} \le 2/3$ and

$$\frac{1}{32\cdot 4^{r^*}}\|x\|_{N^*} \le \|\tilde{x}\|_{\bar{N}^*} \le 48\cdot 4^{r^*}\|\tilde{x}\|_{N^*}. \tag{2.15}$$

Indeed, we have

$$\|\tilde{\tilde{x}}\|_{N^*} = \inf\Big\{\rho > 0 \;\Big|\; \sum_{i=i_0+1}^{n} N^*\Big(\frac{N^{*-1}(t_i)}{\rho}\Big) \le 1\Big\}.$$

By (2.14),

$$\sum_{i=i_0+1}^{n} N^*\Big(\frac{N^{*-1}(t_i)}{\rho}\Big) \le nN^*\Big(\frac{N^{*-1}(1/4^{r^*}n)}{\rho}\Big) \le nN^*\Big(\frac{2N^{*-1}(1/n)}{3\rho}\Big),$$

and thus

$$\|\tilde{\tilde{x}}\|_{N^*} \le 2/3.$$

Therefore, $\|\tilde{x}\|_{N^*} \ge 1/3$. From (2.12) it follows that

$$\sum_{i=1}^{i_0}\bar{N}^*\Big(\frac{N^{*-1}(t_i)}{\rho}\Big) \le \sum_{i=1}^{i_0}\bar{N}^*\Big(\frac{16\cdot 4^{r^*}\bar{N}^{*-1}(t_i)}{\rho}\Big).$$

Thus, we have

$$\|\tilde{x}\|_{\bar{N}^*} \le 16\cdot 4^{r^*}.$$

Using this and $\|\tilde{x}\|_{N^*} \ge 1/3$ we obtain

$$\|\tilde{x}\|_{\bar{N}^*} \le 48\cdot 4^{r^*}\|\tilde{x}\|_{N^*}.$$

Hence, the right inequality of (2.15) is proved.

Now we show the left one. By (2.9),

$$\sum_{i=1}^{i_0} \bar{N}^*\left(\frac{N^{*-1}(t_i)}{\rho}\right) \ge \sum_{i=1}^{i_0} \bar{N}^*\left(\frac{\bar{N}^{*-1}(t_i)}{32\cdot 4^{r^*}\rho}\right).$$

Thus $\|\tilde{x}\|_{\bar{N}^*} \ge 1/32\cdot 4^{r^*}$. Using $\|x\|_{N^*}=1$, we obtain the left inequality of (2.15):

$$\|\tilde{x}\|_{\bar{N}^*} \ge \frac{1}{32\cdot 4^{r^*}}\|x\|_{N^*}.$$

Now, the left inequality of (2.15) implies the left inequality of (2.10). The right inequality of (2.15) implies

$$\|x\|_{\bar{N}^*} \le \|\tilde{x}\|_{\bar{N}^*} + \|\tilde{\tilde{x}}\|_{\bar{N}^*} \le 48\cdot 4^{r^*}\|\tilde{x}\|_{N^*} + \|\tilde{\tilde{x}}\|_{\bar{N}^*}.$$

It is left to estimate the second summand. By (2.11),

$$\sum_{i=i_0+1}^{n} \bar{N}^*\left(\frac{N^{*-1}(t_i)}{\rho}\right) \le n\bar{N}^*\left(\frac{N^{*-1}(1/n)}{\rho}\right) \le n\bar{N}^*\left(\frac{16\bar{N}^{*-1}(1/n)}{\rho}\right).$$

Hence, $\|\tilde{\tilde{x}}\|_{\bar{N}^*} \le 16$ and

$$\|x\|_{\bar{N}^*} \le (48\cdot 4^{r^*}+16)\|x\|_{N^*}. \ ■$$

LEMMA 2.6. *Let $1\le p<r<\infty$ and $a\in\mathbb{R}^n$ with $a_1\ge\cdots\ge a_n>0$. Then there exists an N-function N whose conjugate function N^* satisfies, for all $\ell=1,\dots,n$,*

$$\begin{aligned}(2.16)\qquad &\frac{1}{2}N^{*-1}\left(\frac{\ell}{n}\right)\\ &\le C_r\left\{\left(\frac{\ell}{n}\right)^{1/p^*}\left(\frac{1}{n}\sum_{i=1}^{\ell}|a_i|^p\right)^{1/p} + \left(\frac{\ell}{n}\right)^{1/r^*}\left(\frac{1}{n}\sum_{i=\ell+1}^{n}|a_i|^r\right)^{1/r}\right\}\\ &\le 2^{-1/p}8N^{*-1}\left(\frac{\ell}{n}\right),\end{aligned}$$

and for all $x\in\mathbb{R}^n$ we have

$$\alpha_{r,p}\|x\|_N \le \left(\frac{1}{n!}\sum_{\pi}\left(\sum_{i=1}^{n}(|x(i)a_{\pi(i)}|^r\right)^{p/r}\right)^{1/p} \le \beta_{r,p}\|x\|_N,$$

where $\alpha_{r,p}$ and $\beta_{r,p}$ are constants, just depending on r and p.

Proof. For all $x\in\mathbb{R}^n$,

$$\frac{1}{n!}\sum_{\pi}\|(x(i)a_{\pi(i)})_{i=1}^n\|_r^p = \frac{1}{n!}\sum_{\pi}\left(\sum_{i=1}^{n}(|x(i)|^p|a_{\pi(i)}|^p)^{r/p}\right)^{p/r}.$$

Using Lemma 2.5, we get the existence of an N-function M with

$$a_{r/p}\|(|x(i)|^p)_{i=1}^n\|_M \le \frac{1}{n!}\sum_{\pi}\Big(\sum_{i=1}^n (|x(i)|^p|a_{\pi(i)}|^p)^{r/p}\Big)^{p/r} \le b_{r/p}\|(|x(i)|^p)_{i=1}^n\|_M$$

and

$$M^{*-1}\Big(\frac{\ell}{n}\Big) \le C_r\Big\{\frac{1}{n}\sum_{i=1}^{\ell}|a_i|^p + \Big(\frac{\ell}{n}\Big)^{1-p/r}\Big(\frac{1}{n}\sum_{i=\ell+1}^{n}|a_i|^r\Big)^{p/r}\Big\} \le 4M^{*-1}\Big(\frac{\ell}{n}\Big). \tag{2.17}$$

It follows that

$$(a_{r/p})^{1/p}\|(|x(i)|^p)_{i=1}^n\|_M^{1/p} \le \Bigg(\frac{1}{n!}\sum_{\pi}\Big(\sum_{i=1}^n (|x(i)|^p|a_{\pi(i)}|^p)^{r/p}\Big)^{p/r}\Bigg)^{1/p} \le (b_{r/p})^{1/p}\|(|x(i)|^p)_{i=1}^n\|_M^{1/p}.$$

Furthermore, we have

$$\|(|x(i)|^p)_{i=1}^n\|_M^{1/p} = \|x\|_{M\circ t^p},$$

since

$$\|(|x(i)|^p)_{i=1}^n\|_M^{1/p} = \Big\{\rho^{1/p} > 0 \,\Big|\, \sum_{i=1}^n M\Big(\frac{|x(i)|^p}{\rho}\Big) \le 1\Big\} = \Big\{\eta > 0 \,\Big|\, \sum_{i=1}^n M\Big(\Big|\frac{x(i)}{\eta}\Big|^p\Big) \le 1\Big\}.$$

We choose $N = M\circ t^p$. Then

$$(a_{r/p})^{1/p}\|x\|_N \le \Bigg(\frac{1}{n!}\sum_{\pi}\Big(\sum_{i=1}^n (|x(i)|^p|a_{\pi(i)}|^p)^{r/p}\Big)^{p/r}\Bigg)^{1/p} \le (b_{r/p})^{1/p}\|x\|_N.$$

Inequality (2.1) gives, for all $u \ge 0$,

$$u \le M^{-1}(u)M^{*-1}(u) \le 2u.$$

Hence

$$N^{*-1}(t) \ge \frac{t}{N^{-1}(t)} = \frac{t}{(M^{-1}(t))^{1/p}} \ge 2^{-1/p}t^{1-1/p}(M^{*-1}(t))^{1/p},$$

$$\frac{1}{2}N^{*-1}(t) \le \frac{t}{N^{-1}(t)} = \frac{t}{(M^{-1}(t))^{1/p}} \le t^{1-1/p}(M^{*-1}(t))^{1/p}.$$

Using (2.17), we get

$$\frac{1}{2}N^{*-1}\Big(\frac{\ell}{n}\Big) \le C_r\Big\{\Big(\frac{\ell}{n}\Big)^{1/p^*}\Big(\frac{1}{n}\sum_{i=1}^{\ell}|a_i|^p\Big)^{1/p} + \Big(\frac{\ell}{n}\Big)^{1/r^*}\Big(\frac{1}{n}\sum_{i=\ell+1}^{n}|a_i|^r\Big)^{1/r}\Big\}$$
$$\le 2^{-1/p}N^{*-1}\Big(\frac{\ell}{n}\Big). \ \blacksquare$$

The vector $(a_i)_{i=1}^n = (n/i)^{1/p}$ $(1 < p < 2)$ generates the ℓ_p-norm, i.e., for all $x \in \mathbb{R}^n$ we have

$$c\|x\|_p \le \operatorname*{Ave}_{\pi}\Big(\sum_{i=1}^{n}|x_i a_{\pi(i)}|^2\Big)^{1/2} \le C\|x\|_p, \tag{2.18}$$

where $c, C > 0$ are absolute constants just depending on p. This follows from Lemma 2.6.

3. Embedding of $\ell_M^n(\ell_N^n)$ into L_1. To embed $\ell_M^n(\ell_r^n)$ into L_1 we have to extend the combinatorial expressions by another average over permutations. We use the term

$$\operatorname*{Ave}_{\pi,\sigma,\eta}\Big(\sum_{i,j=1}^{n}|a_{ij}x_{\pi(i)}y_{\sigma(j)}z_{\eta(j)}|^2\Big)^{1/2}, \tag{3.1}$$

which is, as we will show, under appropriate choices of $x, y, z \in \mathbb{R}^n$, equivalent to

$$\big\|(\|(a_{ij})_{i=1}^n\|_r)_{j=1}^n\big\|_M.$$

Since (3.1) is equivalent to the L_1-norm, we obtain an embedding into L_1. Using $z = ((n/j)^{1/p})_{j=1}^n$, $1 < p < r < 2$, we "pass through" an ℓ_p space to obtain the result.

PROPOSITION 3.1. *Let $1 < p < r < 2$. Let $y \in \mathbb{R}^n \setminus \{0\}$ with $y_1 \ge \cdots \ge y_n > 0$, $(x_i)_{i=1}^n = ((n/i)^{1/r})_{i=1}^n$ and $(z_j)_{j=1}^n = ((n/j)^{1/p})_{j=1}^n$. Then, for all matrices $a = (a_{ij})_{i,j=1}^n$,*

$$a_{r,p}\big\|(\|(a_{ij})_{i=1}^n\|_r)_{j=1}^n\big\|_{M_y} \le \frac{1}{(n!)^3}\sum_{\pi,\sigma,\eta}\Big(\sum_{i,j=1}^{n}|a_{ij}x_{\pi(i)}y_{\sigma(j)}z_{\eta(j)}|^2\Big)^{1/2} \tag{3.2}$$
$$\le b_{r,p}\big\|(\|(a_{ij})_{i=1}^n\|_r)_{j=1}^n\big\|_{M_y},$$

where

$$M_y\Big(\frac{\ell}{n}\Big) \sim \frac{1}{n}\sum_{i=1}^{\ell}y_i + \Big(\frac{\ell}{n}\Big)^{1/p^*}\Big(\frac{1}{n}\sum_{i=\ell+1}^{n}|y_i|^p\Big)^{1/p}.$$

In particular, $\ell_{M_y}^n(\ell_r^n)$ is isomorphic to a subspace of L_1.

Proof. We start with the upper bound. By (2.18), z generates the ℓ_p-norm. Thus

(3.3)
$$\operatorname*{Ave}_{\pi,\sigma,\eta}\Big(\sum_{i,j=1}^n |a_{ij}x_{\pi(i)}y_{\sigma(j)}z_{\eta(j)}|^2\Big)^{1/2} \sim \operatorname*{Ave}_{\pi,\sigma}\Big(\sum_{j=1}^n y_{\sigma(j)}^p\Big(\sum_{i=1}^n|a_{ij}x_{\pi(i)}|^2\Big)^{p/2}\Big)^{1/p}.$$

By Jensen's inequality,

$$\operatorname*{Ave}_{\pi,\sigma}\Big(\sum_{j=1}^n y_{\sigma(j)}^p\Big(\sum_{i=1}^n|a_{ij}x_{\pi(i)}|^2\Big)^{p/2}\Big)^{1/p} \le \operatorname*{Ave}_{\sigma}\Big(\sum_{j=1}^n y_{\sigma(j)}^p\operatorname*{Ave}_{\pi}\Big(\sum_{i=1}^n|a_{ij}x_{\pi(i)}|^2\Big)^{p/2}\Big)^{1/p}.$$

By Lemma 2.6, for all $j \le n$,

$$\Big(\operatorname*{Ave}_{\pi}\Big(\sum_{i=1}^n|a_{ij}x_{\pi(i)}|^2\Big)^{p/2}\Big)^{1/p} \sim \|(a_{ij})_{i=1}^n\|_N,$$

where

$$N^{*-1}\Big(\frac{\ell}{n}\Big) \sim \Big(\frac{\ell}{n}\Big)^{1/p^*}\Big(\frac{1}{n}\sum_{i=1}^{\ell}|x_i|^p\Big)^{1/p} + \Big(\frac{\ell}{n}\Big)^{1/2}\Big(\frac{1}{n}\sum_{i=\ell+1}^{n}|x_i|^2\Big)^{1/2}$$
$$\sim \Big(\frac{\ell}{n}\Big)^{1/p^*}\Big(\frac{1}{n}\sum_{i=1}^{\ell}\Big(\frac{n}{i}\Big)^{p/r}\Big)^{1/p} + \Big(\frac{\ell}{n}\Big)^{1/2}\Big(\frac{1}{n}\sum_{i=\ell+1}^{n}\Big(\frac{n}{i}\Big)^{2/r}\Big)^{1/2}.$$

Since $p < r < 2$,
$$N^{*-1}(\ell/n) \sim (\ell/n)^{1/r^*},$$

which means that the N-norm is equivalent to the ℓ_r-norm. Hence, we have shown the upper estimate of (3.2), where M_y is the N-function as specified in Lemma 2.6.

For the lower bound, we obtain

$$\operatorname*{Ave}_{\pi,\sigma,\eta}\Big(\sum_{i,j=1}^n |a_{ij}x_{\pi(i)}y_{\sigma(j)}z_{\eta(j)}|^2\Big)^{1/2} \sim \operatorname*{Ave}_{\pi,\sigma}\Big(\sum_{j=1}^n y_{\sigma(j)}^p\Big(\sum_{i=1}^n|a_{ij}x_{\pi(i)}|^2\Big)^{p/2}\Big)^{1/p}.$$

Now we use the triangle inequality to get

$$\operatorname*{Ave}_{\pi,\sigma}\Big(\sum_{j=1}^n y_{\sigma(j)}^p\Big(\sum_{i=1}^n|a_{ij}x_{\pi(i)}|^2\Big)^{p/2}\Big)^{1/p} \ge \operatorname*{Ave}_{\sigma}\Big(\sum_{j=1}^n y_{\sigma(j)}^p\Big|\operatorname*{Ave}_{\pi}\Big(\sum_{i=1}^n|a_{ij}x_{\pi(i)}|^2\Big)^{1/2}\Big|^p\Big)^{1/p}.$$

We know that for all $j \le n$,

$$\operatorname*{Ave}_{\pi}\Big(\sum_{i=1}^{n}|a_{ij}x_{\pi(i)}|^2\Big)^{1/2} \sim \|(a_{ij})_{i=1}^n\|_r,$$

since $(x_i)_{i=1}^n = ((n/i)^{1/r})_{i=1}^n$. Hence, by Lemma 2.6 we get the lower estimate of (3.2),

$$\operatorname*{Ave}_{\pi,\sigma,\eta}\Big(\sum_{i,j=1}^{n}|a_{ij}x_{\pi(i)}y_{\sigma(j)}z_{\eta(j)}|^2\Big)^{1/2} \gtrsim \big\|(\|(a_{ij})_{i=1}^n\|_r)_{j=1}^n\big\|_{M_y}.$$

Let now ϵ and δ denote sequences of signs ± 1. Using (3.2) and Khinchin's inequality one can easily show that

$$\Psi_n : \ell_M^n(\ell_r^n) \to L_1^{n!^3 2^{2n}}, \qquad (a_{ij})_{i,j=1}^n \mapsto \Big(\sum_{i,j=1}^{n} a_{ij}x_{\pi(i)}y_{\sigma(j)}z_{\eta(j)}\varepsilon_i\delta_j\Big)_{\pi,\sigma,\eta,\delta,\varepsilon},$$

embeds $\ell_M^n(\ell_r^n)$ into L_1. ∎

COROLLARY 3.2. *Let $1 < r < 2$ and $1 < p < r$. Furthermore, let M be an α-convex N-function with $1 < \alpha < p$. Define $(x_i)_{i=1}^n = ((n/i)^{1/r})_{i=1}^n$, $(y_j)_{j=1}^n = (1/M^{-1}(j/n))_{j=1}^n$ and $(z_j)_{j=1}^n = ((n/j)^{1/p})_{j=1}^n$. Then, for all matrices $a = (a_{ij})_{i,j=1}^n$,*

$$\operatorname*{Ave}_{\pi,\sigma,\eta}\Big(\sum_{i,j=1}^{n}|a_{ij}x_{\pi(i)}y_{\sigma(j)}z_{\eta(j)}|^2\Big)^{1/2} \sim \big\|(\|(a_{ij})_{i=1}^n\|_r)_{j=1}^n\big\|_{M}.$$

In particular, $\ell_M^n(\ell_r^n)$ is isomorphic to a subspace of L_1.

Proof. We apply Proposition 3.1. We have to verify that the N-function M_y of Proposition 3.1 is equivalent to M. We have, for all $\ell \le n$,

$$(3.4)\qquad \frac{1}{n}\sum_{i=1}^{\ell}\frac{1}{M^{-1}(i/n)} \lesssim \frac{\ell}{n}\,\frac{1}{M^{-1}(\ell/n)} \overset{(2.1)}{\sim} M^{*-1}\Big(\frac{\ell}{n}\Big)$$

and

$$(3.5)\qquad \Big(\frac{\ell}{n}\Big)^{1/p^*}\Big(\frac{1}{n}\sum_{i=\ell+1}^{n}\Big|\frac{1}{M^{-1}(i/n)}\Big|^p\Big)^{1/p} \overset{(2.1)}{\lesssim} M^{*-1}\Big(\frac{\ell}{n}\Big),$$

since M is α-convex and therefore $(M^{-1})^\alpha$ is concave, i.e., for all $\ell \le n$,

$$M_y^{*-1}\Big(\frac{\ell}{n}\Big) \sim \frac{1}{n}\sum_{i=1}^{\ell} y_i + \Big(\frac{\ell}{n}\Big)^{1/p^*}\Big(\frac{1}{n}\sum_{i=\ell+1}^{n}|y_i|^p\Big)^{1/p} \lesssim M^{*-1}\Big(\frac{\ell}{n}\Big).$$

The lower bound is trivial, since M^{*-1} is an increasing function. ∎

We will now prove Theorem 1.1.

Proof of Theorem 1.1. It is enough to show the case $N(t)=t^r$. Indeed, by [3], ℓ_N is a subspace of L_r if $N(t)/t^r$ is increasing and $N(t)/t^2$ is decreasing.

We apply Proposition 3.1. We choose y_i, $i=1,\dots,n$, such that

$$M^{*-1}\left(\frac{\ell}{n}\right)=\frac{1}{n}\sum_{i=1}^{\ell} y_i.$$

We will show that M^* and M_y^{*-1} of Proposition 3.1 are equivalent. For all ℓ, we have

$$M^{*-1}\left(\frac{\ell}{n}\right)\geq \frac{\ell}{n}y_\ell.$$

Since

$$t\leq M^{-1}(t)M^{*-1}(t)\leq 2t,$$

we get

$$y_\ell\leq \frac{M^{*-1}(\ell/n)}{\ell/n}\leq \frac{2}{M^{-1}(\ell/n)}.$$

Therefore,

$$\frac{1}{n}\sum_{i=\ell+1}^{n}|y_i|^p\leq 2^p\frac{1}{n}\sum_{i=\ell+1}^{n}\frac{1}{|M^{-1}(i/n)|^p}=2^p\frac{1}{n}\sum_{i=\ell+1}^{n}\frac{|i/n|^{p/r}}{|M^{-1}(i/n)|^p}\left|\frac{n}{i}\right|^{p/r}.$$

Since $M(t)/t^r$ is decreasing, $s/|M^{-1}(s)|^r$ is decreasing. Therefore, since $r<p$,

$$\left|\frac{t}{|M^{-1}(t)|^r}\right|^{p/r}=\frac{t^{p/r}}{|M^{-1}(t)|^p}$$

is also decreasing. Thus

$$\frac{1}{n}\sum_{i=\ell+1}^{n}|y_i|^p\leq 2^p\frac{|\ell/n|^{p/r}}{|M^{-1}(\ell/n)|^p}\frac{1}{n}\sum_{i=\ell+1}^{n}\left|\frac{n}{i}\right|^{p/r}\leq 2^p\frac{|\ell|^{p/r}}{|M^{-1}(\ell/n)|^p}\frac{1}{n}\sum_{i=\ell+1}^{n}i^{-p/r}$$

$$\sim 2^p\frac{|\ell|^{p/r}}{|M^{-1}(\ell/n)|^p}\frac{1}{n}\ell^{1-p/r}=2^p\frac{\ell}{n}\frac{1}{|M^{-1}(\ell/n)|^p}.$$

Altogether,

$$\begin{aligned}M^{*-1}\left(\frac{\ell}{n}\right)&\leq \frac{1}{n}\sum_{i=1}^{\ell}y_i+\left(\frac{\ell}{n}\right)^{1/p^*}\left(\frac{1}{n}\sum_{i=\ell+1}^{n}|y_i|^p\right)^{1/p}\\&\leq M^{*-1}\left(\frac{\ell}{n}\right)+\left(\frac{\ell}{n}\right)^{1/p^*}\left(2^p\frac{\ell}{n}\frac{1}{|M^{-1}(\ell/n)|^p}\right)^{1/p}\\&=M^{*-1}\left(\frac{\ell}{n}\right)+2\frac{\ell}{n}\frac{1}{|M^{-1}(\ell/n)|}\leq 3M^{*-1}\left(\frac{\ell}{n}\right).\ \blacksquare\end{aligned}$$

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