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Quasi-greedy bases and Lebesgue-type inequalities

by

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Abstract. We study Lebesgue-type inequalities for greedy approximation with respect to quasi-greedy bases. We mostly concentrate on the L_p spaces. The novelty of the paper is in obtaining better Lebesgue-type inequalities under extra assumptions on a quasi-greedy basis than known Lebesgue-type inequalities for quasi-greedy bases. We consider uniformly bounded quasi-greedy bases of L_p, $1 < p < \infty$, and prove that for such bases an extra multiplier in the Lebesgue-type inequality can be taken as $C(p)\ln(m+1)$. The known magnitude of the corresponding multiplier for general (no assumption of uniform boundedness) quasi-greedy bases is of order $m^{|1/2-1/p|}$, $p \neq 2$. For uniformly bounded orthonormal quasi-greedy bases we get further improvements replacing $\ln(m+1)$ by $(\ln(m+1))^{1/2}$.

1. Introduction. We study the efficiency of greedy algorithms for m-term nonlinear approximation with regard to bases. Our primary interest is in approximation in L_p with respect to quasi-greedy bases.

Let X be an infinite-dimensional separable Banach space with a norm $\|\cdot\| := \|\cdot\|_X$ and let $\Psi := \{\psi_k\}_{k=1}^{\infty}$ be a seminormalized basis for X ($0 < c_0 \le \|\psi_k\| \le C_0, k \in \mathbb{N}$). All bases considered in this paper are assumed to be seminormalized. For a given $f \in X$ we define the *best m-term* approximation with regard to Ψ as follows:

$$\sigma_m(f) := \sigma_m(f,\Psi)_X := \inf_{b_k,\Lambda} \Big\| f - \sum_{k\in\Lambda} b_k\psi_k \Big\|_X,$$

where the infimum is taken over coefficients b_k and sets Λ of indices with cardinality $|\Lambda| = m$. There is a natural algorithm of constructing an m-term approximant. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

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We call a permutation ρ, $\rho(j) = k_j$, $j = 1, 2, \dots$, of the positive integers *decreasing* and write $\rho \in D(f)$ if

$$|c_{k_1}(f)| \geq |c_{k_2}(f)| \geq \cdots .$$

In the case of strict inequalities here $D(f)$ consists of only one permutation. We define the *mth greedy approximant* of f with regard to the basis Ψ corresponding to a permutation $\rho \in D(f)$ by

$$G_m(f) := G_m(f, \Psi) := G_m(f, \Psi, \rho) := \sum_{j=1}^{m} c_{k_j}(f)\psi_{k_j}.$$

This algorithm is known in the theory of nonlinear approximation as the Thresholding Greedy Algorithm (TGA). The best we can achieve with the algorithm G_m is

$$\|f - G_m(f)\|_X = \sigma_m(f, \Psi)_X,$$

or a little weaker

$$\|f - G_m(f)\|_X \leq C\sigma_m(f, \Psi)_X$$

for all $f \in X$ with a constant C independent of f and m. The following concept of a greedy basis has been introduced in [9].

DEFINITION 1.1. We call a basis Ψ a *greedy basis* if for every $f \in X$ there exists a permutation $\rho \in D(f)$ such that

$$\|f - G_m(f, \Psi, \rho)\|_X \leq C\sigma_m(f, \Psi)_X$$

with a constant C independent of f and m.

We refer the reader to the survey [24] and the book [25] for further discussion of greedy type bases. In this paper we are interested in special inequalities—Lebesgue-type inequalities—for greedy approximation.

Lebesgue [12] proved the following inequality: for any 2π-periodic continuous function f we have

$$\|f - S_n(f)\|_\infty \leq \left(4 + \frac{4}{\pi^2}\ln n\right)E_n(f)_\infty, \tag{1.1}$$

where $S_n(f)$ is the nth partial sum of the Fourier series of f and $E_n(f)_\infty$ is the error of the best approximation of f by the trigonometric polynomials of order n in the uniform norm $\|\cdot\|_\infty$. The inequality (1.1) relates the error of a particular method (S_n) of approximation by the trigonometric polynomials of order n to the best-possible error $E_n(f)_\infty$ of approximation by the trigonometric polynomials of order n. By a *Lebesgue-type inequality* we mean an inequality that provides an upper estimate for the error of a particular method of approximation of f by elements of a special form, say $\mathcal{A}$, by the best-possible approximation of f by elements of the form $\mathcal{A}$. In the case of approximation with regard to bases (or minimal systems), Lebesgue-type inequalities are known both in linear and in nonlinear settings (see the surveys [10], [23], and [24]).

By Definition 1.1 greedy bases are those for which we have ideal (up to a multiplicative constant) Lebesgue-type inequalities for greedy approximation. In this paper we concentrate on a wider class of bases than greedy bases—quasi-greedy bases. The concept of quasi-greedy basis was introduced in [9].

DEFINITION 1.2. A basis Ψ is called *quasi-greedy* if there exists some constant C such that

$$\sup_m \|G_m(f,\Psi)\| \le C\|f\|.$$

Subsequently, Wojtaszczyk [28] proved that these are precisely the bases for which the TGA merely converges, i.e.,

$$\lim_{m\to\infty} G_m(f) = f.$$

The main result of [27] is the following Lebesgue-type inequality for greedy approximation with respect to a quasi-greedy basis in L_p.

THEOREM 1.3. *Let $1 < p < \infty$, $p \neq 2$, and let Ψ be a quasi-greedy basis of L_p. Then for each $f \in L_p$ we have*

$$\|f - G_m(f,\Psi)\|_{L_p} \le C(p,\Psi) m^{|1/2-1/p|} \sigma_m(f,\Psi)_{L_p}. \tag{1.2}$$

The above theorem does not cover the case $p = 2$. It is mentioned in [28] that for $p = 2$ one has

$$\|f - G_m(f,\Psi)\|_{L_2} \le C(\Psi)\ln(m+1)\sigma_m(f,\Psi)_{L_2}.$$

We do not know if the above inequality is sharp in the sense that an extra factor $\log m$ cannot be replaced by a slower growing factor. The reader can find a further discussion of this problem in [26].

We note that inequality (1.2) is known (see [28]) in the case of unconditional bases Ψ. It is proved in [22] that (1.2) holds for the trigonometric system $\Psi = \{e^{ikx}\}$ for all $1 \le p \le \infty$. It was also noticed in [22] that (1.2) holds for any uniformly bounded orthonormal bases of L_2. Thus, it was known that bases satisfying very different conditions—uniformly bounded orthonormal bases of L_2 or quasi-greedy bases of L_p—both guarantee that similar Lebesgue-type inequalities (1.2) hold for greedy approximation.

In this paper we continue to study Lebesgue-type inequalities for greedy approximation. We try to make a bridge between the above two conditions—uniformly bounded orthonormal bases of L_2 and quasi-greedy bases of L_p. We consider uniformly bounded quasi-greedy bases of L_q and study Lebesgue-type inequalities in L_p, $q \le p$. It turns out that even the question of existence of such bases is nontrivial. For instance, it is known (see [4]) that there is no uniformly bounded unconditional basis in L_p, $p \neq 2$.

Quasi-greedy bases are close to unconditional bases. However, surprisingly, it turns out that there exist uniformly bounded quasi-greedy bases in all L_q with $1 < q < \infty$. We discuss this issue in Section 3, where we present a construction of uniformly bounded quasi-greedy bases. In particular, we prove the following theorem there.

THEOREM 1.4. *There exists a uniformly bounded orthonormal quasi-greedy basis* $\Psi = \{\psi_j\}_{j=1}^{\infty}$ *in* L_p, $1 < p < \infty$, *that consists of trigonometric polynomials.*

We note that existence of uniformly bounded orthonormal quasi-greedy bases was proved by Nielsen [15]. The construction in [15] is a variation on a construction in [11]. The same type of construction was used in [28]. Our construction in Section 3 is a somewhat more general version of the known constructions. We include it in the paper without proofs for the sake of completeness and because some of our results in Section 4 rely on a particular choice of the parameters in this specific construction. Conceivably, too, this more general construction may have further applications. The construction in [15] is based on the Walsh system. We should emphasize that Theorem 1.4 could also be obtained from the arguments of [15] by replacing the Walsh system with the trigonometric system.

It is known from [1] that the space $C[0,1]$ does not have quasi-greedy bases and the space $L_1[0,1]$ has quasi-greedy bases. In Section 4 we prove, in particular, that $L_1[0,1]$ does not have a uniformly bounded quasi-greedy Markushevich basis. This result complements a theorem of Szarek [21] on the nonexistence of a uniformly bounded Schauder basis for $L_1[0,1]$. On the other hand, we show that the Hardy space $H_1(D)$ does have a uniformly bounded quasi-greedy basis of analytic polynomials.

In Section 5 we prove Lebesgue-type inequalities for greedy approximation in L_p, $2 \le p \le \infty$, under different assumptions on a basis Ψ. In that section we assume that Ψ is a uniformly bounded basis. In addition we assume that Ψ is a certain type basis (quasi-greedy basis, Riesz basis) in one of the spaces L_2, L_q, $1 < q < 2$, or L_q, $2 < q < \infty$. Here is a typical result from Section 5 (see Theorem 5.4). We will often use the notation $h(p) := |1/2 - 1/p|$. We also use the brief notation $\|\cdot\|_p := \|\cdot\|_{L_p}$.

THEOREM 1.5. *Assume that* Ψ *is a uniformly bounded quasi-greedy basis of* L_2. *Then for any* m*-term polynomial*

$$t_m = \sum_{k\in P} b_k \psi_k, \qquad |P| = m,$$

we have, for $2 \le p \le \infty$,

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + Cm^{h(p)} \ln(m+1) \|f - t_m\|_2.$$

In Section 6 we continue to prove Lebesgue-type inequalities for greedy approximation in L_p under different assumptions on the basis Ψ. We assume that Ψ is a seminormalized quasi-greedy basis for a pair of spaces: L_q, $1 < q < \infty$, and L_p, $q \le p$. It turns out that this assumption results in a dramatic improvement of the corresponding Lebesgue-type inequalities. This is demonstrated by the following result (see Theorem 6.1).

THEOREM 1.6. *Assume that Ψ is a seminormalized quasi-greedy basis for both L_q and L_p with $1 < q \le 2 \le p < \infty$. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \qquad |P| = m,$$

we have

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + C(p,q)\ln(m+1)\|f - t_m\|_q.$$

We now formulate some of the Lebesgue-type inequalities obtained in this paper. We already mentioned above (see Theorem 1.3) that Lebesgue-type inequalities in L_p, $1 < p < \infty$, under the assumption that Ψ is a quasi-greedy basis of L_p were obtained in [27]. First we give the definition of a democratic basis.

DEFINITION 1.7. We say that a basis $\Psi = \{\psi_k\}_{k=1}^{\infty}$ is a *democratic basis* for X if there exists a constant $D := D(X,\Psi)$ such that, for any two finite sets of indices P and Q with the same cardinality $|P| = |Q|$, we have

$$\Big\|\sum_{k\in P}\psi_k\Big\| \le D\Big\|\sum_{k\in Q}\psi_k\Big\|.$$

In Section 5 we prove that if Ψ is both quasi-greedy and democratic then, for any $f \in X$,

$$\|f - G_m(f,\Psi)\|_X \le C\ln(m+1)\sigma_m(f,\Psi)_X. \tag{1.3}$$

We note that it is proved in [2] that bases which are simultaneously quasi-greedy and democratic are exactly almost greedy bases. As a corollary of (1.3) we obtain a Lebesgue-type inequality for a uniformly bounded quasi-greedy basis of L_p, $1 < p < \infty$ (see Corollary 5.7):

$$\|f - G_m(f,\Psi)\|_p \le C(p)\ln(m+1)\sigma_m(f,\Psi)_p. \tag{1.4}$$

Here $\sigma_m(f,\Psi)_p := \sigma_m(f,\Psi)_{L_p}$. Comparing (1.4) with (1.2) we see that the extra assumption of uniform boundedness of the basis improves the Lebesgue-type inequalities dramatically.

In Section 6, making our assumptions on the basis even stronger, we improve (1.4) to

$$\|f - G_m(f,\Psi)\|_p \le C(p)(\ln(m+1))^{1/2}\sigma_m(f,\Psi)_p, \tag{1.5}$$

under the assumption that Ψ is a uniformly bounded orthonormal quasi-greedy basis of L_p, $2 \le p < \infty$.

In Section 5 we impose assumptions on the basis in L_q and obtain inequalities in L_p:

$$\|f - G_m(f,\Psi)\|_p \le C(p,q)m^{(1-q/p)/2} \ln(m+1)\sigma_m(f,\Psi)_p \tag{1.6}$$

under the assumption that Ψ is a uniformly bounded quasi-greedy basis of L_q, $1 < q < \infty$, and $q \le p \le \infty$. We note that in the case $p = q$ inequality (1.6) turns into (1.4).

We begin a systematic presentation with Section 2, where we list some properties of quasi-greedy bases that are used in this paper.

2. Properties of quasi-greedy bases

2.1. Quasi-greedy bases. The definition of a quasi-greedy basis is given in the Introduction (see Definition 1.2). We give here an equivalent definition (see [25, p. 34]). For a set Λ of indices we define the corresponding partial sum as follows:

$$S_\Lambda(f) := S_\Lambda(f,\Psi) := \sum_{k\in\Lambda} c_k(f)\psi_k.$$

DEFINITION 2.1. We say that a basis Ψ is *quasi-greedy* if there exists a constant C_Q such that, for any $f \in X$ and any finite set Λ of indices having the property

$$\min_{k\in\Lambda} |c_k(f)| \ge \max_{k\notin\Lambda} |c_k(f)|,$$

we have

$$\|S_\Lambda(f,\Psi)\| \le C_Q\|f\|.$$

First, we present some known useful properties of quasi-greedy bases. The reader can find the following two lemmas in [25, p. 37].

LEMMA 2.2. *Let Ψ be a quasi-greedy basis. Then, for any two finite sets $A \subseteq B$ of indices and coefficients $0 < t \le |c_j| \le 1$, $j \in B$, we have*

$$\Big\|\sum_{j\in A} c_j\psi_j\Big\| \le C(X,\Psi,t)\Big\|\sum_{j\in B} c_j\psi_j\Big\|.$$

It will be convenient to define the *quasi-greedy constant* K to be the least constant such that

$$\|G_m(f)\| \le K\|f\| \quad \text{and} \quad \|f - G_m(f)\| \le K\|f\|, \quad f \in X.$$

LEMMA 2.3. *Suppose Ψ is a quasi-greedy basis with quasi-greedy constant K. Then, for any real numbers c_j and any finite set P of indices, we have*

$$(4K^2)^{-1} \min_{j\in P} |c_j| \Big\|\sum_{j\in P} \psi_j\Big\| \le \Big\|\sum_{j\in P} c_j\psi_j\Big\| \le 2K \max_{j\in P} |c_j| \Big\|\sum_{j\in P} \psi_j\Big\|.$$

We present the following lemma from [1] with a proof for completeness.

LEMMA 2.4. *Let Ψ be a quasi-greedy basis of X. Then for any finite set Λ of indices we have, for all $f \in X$,*

$$\|S_\Lambda(f,\Psi)\| \le C\ln(|\Lambda|+1)\|f\|.$$

Proof. For a given element $f \in X$ we consider the expansion

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

Let a sequence k_j, $j = 1, 2, \dots$, of positive integers be such that

$$|c_{k_1}(f)| \ge |c_{k_2}(f)| \ge \cdots .$$

We will use the notation

$$a_n(f) := |c_{k_n}(f)|$$

for the decreasing rearrangement of the coefficients of f. Without loss of generality assume that f is normalized in such a way that guarantees that $|a_1(f)| \le 1$ and consider $m := |\Lambda| \ge 2$. Let, for integer $s \ge 0$,

$$\tau_s := \{k : 2^{-s} \le |c_k(f)| < 2^{1-s}\}.$$

Denote

$$\Lambda_s := \Lambda \cap \tau_s, \qquad \Lambda' := \Lambda \setminus \bigcup_{s\le\log_2 m} \Lambda_s.$$

The seminormalization property of the basis Ψ implies

$$\|S_{\Lambda'}(f)\| \le \frac{2}{m}|\Lambda'|C_0 \le 2C_0.$$

For $s \le \log_2 m$ we have

$$S_{\Lambda_s}(f) = S_{\Lambda_s}(S_{\tau_s}(f)).$$

By Lemma 2.2 we obtain

$$\|S_{\Lambda_s}(f)\| \le C\|S_{\tau_s}(f)\|.$$

Our assumption that Ψ is a quasi-greedy basis implies that, for all s,

$$\|S_{\tau_s}(f)\| \le C\|f\|.$$

Thus, for all $s \le \log_2 m$,

$$\|S_{\Lambda_s}(f)\| \le C\|f\|,$$

and therefore

$$\|S_\Lambda(f)\| \le C\ln(m+1)\|f\|. \ \blacksquare$$

Let

$$f = \sum_{k=1}^{\infty} c_k(f)\psi_k.$$

We define the following expansional best m-term approximation of f:

$$\tilde{\sigma}_m(f) := \tilde{\sigma}_m(f, \Psi) := \inf_{\Lambda,\, |\Lambda|=m} \Big\| f - \sum_{k\in\Lambda} c_k(f)\psi_k \Big\|.$$

It is clear that

$$\sigma_m(f, \Psi) \le \tilde{\sigma}_m(f, \Psi).$$

It is also clear that for an unconditional basis Ψ we have

$$\tilde{\sigma}_m(f, \Psi) \le C(X, \Psi)\sigma_m(f, \Psi).$$

Lemma 2.5 below is a new result that answers Question 2 from [7].

Lemma 2.5. *Let Ψ be a quasi-greedy basis of X. Then, for all $f \in X$,*

$$\tilde{\sigma}_m(f) \le C \ln(m+1)\sigma_m(f).$$

Proof. For a given $\epsilon > 0$ let p_m be an m-term polynomial

$$p_m := \sum_{k\in P} b_k\psi_k, \qquad |P| = m,$$

such that

$$\|f - p_m\| \le \sigma_m(f) + \epsilon.$$

Then by Lemma 2.4 we obtain

$$\tilde{\sigma}_m(f) \le \|f - S_P(f)\| = \|f - p_m - S_P(f - p_m)\| \le C \ln(m+1)(\sigma_m(f) + \epsilon).$$

This completes the proof of Lemma 2.5. ∎

Remark 2.6. After our paper was submitted we were informed that Lemma 2.5 was proved independently by Garrigós, Hernández, and Oikhberg.

We now formulate a result about quasi-greedy bases in L_p spaces. The following theorem is from [26]. We note that in the case $p = 2$ Theorem 2.7 was proved in [28].

Theorem 2.7. *Let $\Psi = \{\psi_k\}_{k=1}^\infty$ be a quasi-greedy basis of L_p, $1 < p < \infty$. Then for each $f \in X$ we have*

$$C_1(p) \sup_n n^{1/p} a_n(f) \le \|f\|_p \le C_2(p) \sum_{n=1}^\infty n^{-1/2} a_n(f), \qquad 2 \le p < \infty;$$

$$C_3(p) \sup_n n^{1/2} a_n(f) \le \|f\|_p \le C_4(p) \sum_{n=1}^\infty n^{1/p-1} a_n(f), \qquad 1 < p \le 2.$$

Remark 2.8. Theorem 2.7 was proved in [26] under the assumption that Ψ is a normalized basis. That proof works for a seminormalized basis as well.

Remark 2.9. The proof in [26] gives the following inequalities. Let $\Psi = \{\psi_k\}_{k=1}^\infty$ be a quasi-greedy basis of X. If for any set of indices A of cardinality m we have $\|\sum_{k\in A}\psi_k\|_X \le C'm^{1/2}$ then for each $f\in X$,

$$\|f\|_X \le C_1\sum_{n=1}^{\infty} n^{-1/2}a_n(f). \tag{2.1}$$

If for any set of indices A of cardinality m we have $\|\sum_{k\in A}\psi_k\|_X \ge c'm^{1/2}$ then for each $f\in X$,

$$\|f\|_X \ge c_1 \sup_n n^{1/2}a_n(f).$$

A general version of (2.1) was obtained in [7].

Now we define the *fundamental function* $\varphi(m) := \varphi(m,\Psi,X)$ of a basis Ψ in X by

$$\varphi(m,\Psi,X) := \sup_{|A|\le m}\Big\|\sum_{k\in A}\psi_k\Big\|.$$

LEMMA 2.10. *Let Ψ be a quasi-greedy basis of X. Then for each $f\in X$,*

$$\|f\| \le C\sum_{n=1}^{\infty} a_n(f)\varphi(n)\frac{1}{n}.$$

Proof. It is known (see [2, p. 581]) that $\varphi(n)/n$ is decreasing. Therefore, by Lemma 2.3 we obtain

$$\begin{aligned}\|f\| &\le \sum_{s=1}^{\infty}\Big\|\sum_{n=2^{s-1}}^{2^s-1} a_n(f)\psi_{k_n}\Big\| \\ &\le C\sum_{s=1}^{\infty} a_{2^{s-1}}(f)\varphi(2^{s-1}) \le C\sum_{n=1}^{\infty} a_n(f)\varphi(n)\frac{1}{n}. \ \blacksquare\end{aligned}$$

2.2. Uniformly bounded quasi-greedy bases. It is clear that any orthonormal basis of a Hilbert space H is an unconditional basis, and therefore a quasi-greedy basis of H. For example, the trigonometric basis is a uniformly bounded orthonormal basis of L_2. In L_p, $p\ne 2$, even the question of existence of a uniformly bounded quasi-greedy basis is nontrivial. It is known (see [4]) that there is no uniformly bounded unconditional basis in L_p, $p\ne 2$. As we already mentioned in the Introduction, there are uniformly bounded quasi-greedy bases in L_p, $1<p<\infty$ (see [15]). We sketch a construction of such bases in Section 3. Now we present some properties of these bases.

LEMMA 2.11. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_q, $1<q<\infty$. Then for any set Λ of indices we have, for $q<p\le\infty$,*

$$\|S_\Lambda(f)\|_p \le C|\Lambda|^{(1-q/p)/2}\|S_\Lambda(f)\|_q. \tag{2.2}$$

Moreover,

$$\|S_\Lambda(f)\|_q \le C\ln(|\Lambda|+1)\|f\|_q. \tag{2.3}$$

Proof. Inequality (2.3) follows from Lemma 2.4. We prove (2.2). We have

$$\|S_\Lambda(f)\|_\infty \le \sum_{k\in\Lambda} |c_k(f)|\,\|\psi_k\|_\infty \le C\sum_{n=1}^{m} a_n(S_\Lambda(f)).$$

By Proposition 2.18 below, we continue

$$\le C\sum_{n=1}^{m} n^{-1/2}\|S_\Lambda(f)\|_q \le Cm^{1/2}\|S_\Lambda(f)\|_q.$$

The above inequality combined with

$$\|g\|_p \le \|g\|_q^{q/p}\|g\|_\infty^{1-q/p}, \quad q\le p\le\infty, \tag{2.4}$$

gives (2.2).

We note that in the case $1<q\le 2$ we could use Theorem 2.7 instead of Proposition 2.18. ∎

Corollary 2.12. *Assume that Ψ is a uniformly bounded Riesz basis of L_2. Then for any set Λ of indices we have, for $2\le p\le\infty$,*

$$\|S_\Lambda(f)\|_p \le C|\Lambda|^{h(p)}\|f\|_2.$$

Proof. The case $p>2$ follows from Lemma 2.11 since Riesz bases are unconditional and hence quasi-greedy. The case $p=2$ follows from unconditionality. ∎

2.3. Uniformly bounded orthonormal quasi-greedy bases. We prove in Section 3 that there exist uniformly bounded orthonormal quasi-greedy bases in L_p, $1<p<\infty$. We also prove in Section 3 that if Ψ is a uniformly bounded orthonormal quasi-greedy basis in L_p, $2\le p<\infty$ then Ψ is a quasi-greedy basis of $L_{p'}$ ($p^{-1}+p'^{-1}=1$). Thus there are uniformly bounded bases which are quasi-greedy bases of two spaces L_p and $L_{p'}$, $2<p<\infty$. We now present some results in this direction.

Lemma 2.13. *Assume that Ψ is a seminormalized quasi-greedy basis for both L_q and L_p with $1<q\le 2\le p<\infty$. Then for any set Λ of indices,*

$$\|S_\Lambda(f)\|_p \le C\ln(|\Lambda|+1)\|f\|_q. \tag{2.5}$$

Proof. Using the notation $m:=|\Lambda|$ we obtain, by Theorem 2.7,

$$\|S_\Lambda(f)\|_p \le C_2(p)\sum_{n=1}^{m} n^{-1/2}a_n(S_\Lambda(f)) \le C(p)\sum_{n=1}^{m} n^{-1/2}a_n(f),$$

$$\le C(p)\sum_{n=1}^{m} n^{-1/2}C_3(q)^{-1}\|f\|_q n^{-1/2} \le C(p,q)\ln(m+1)\|f\|_q.$$

This proves (2.5). ∎

LEMMA 2.14. *Assume that Ψ is a uniformly bounded orthonormal quasi-greedy basis of L_p, $2 < p < \infty$. Then for any set Λ of indices we have*

$$\|S_\Lambda(f)\|_2 \le C(\ln(|\Lambda|+1))^{1/2}\|f\|_{p'}, \tag{2.6}$$

$$\|S_\Lambda(f)\|_p \le C(\ln(|\Lambda|+1))^{1/2}\|f\|_2. \tag{2.7}$$

Proof. Let $|\Lambda| = m$. By Theorem 2.16 below and Theorem 2.7 we have, in the case of (2.6),

$$\|S_\Lambda(f)\|_2 \le \Big(\sum_{n=1}^m a_n(f)^2\Big)^{1/2} \le C\Big(\sum_{n=1}^m n^{-1}\|f\|_{p'}^2\Big)^{1/2} \le C(\ln(m+1))^{1/2}\|f\|_{p'}.$$

In the case of (2.7) we obtain, by Theorem 2.7,

$$\begin{aligned}\|S_\Lambda(f)\|_p &\le C\sum_{n=1}^m n^{-1/2}a_n(f) \le C(\ln(m+1))^{1/2}\Big(\sum_{n=1}^m a_n(f)^2\Big)^{1/2}\\ &\le C(\ln(m+1))^{1/2}\|f\|_2. \ \blacksquare\end{aligned}$$

Let us discuss in more detail uniformly bounded orthonormal quasi-greedy bases. The existence of such bases was proved in [15]. We first recall the definition of bases which are called unconditional for constant coefficients (cf. [28]).

DEFINITION 2.15. A basis Ψ is called *unconditional for constant coefficients* (UCC) if there exist constants C_1 and C_2 such that for each finite subset $A \subset \mathbb{N}$ and for each choice of signs $\varepsilon_i = \pm 1$ we have

$$C_1\Big\|\sum_{i\in A}\psi_i\Big\| \le \Big\|\sum_{i\in A}\varepsilon_i\psi_i\Big\| \le C_2\Big\|\sum_{i\in A}\psi_i\Big\|.$$

It is known ([28]) that quasi-greedy bases are UCC bases. To formulate our results we need some of the basic concepts of the Banach space theory from [13, Definition 1.e.12]. First, let us recall the definition of type and cotype. Let $\{\varepsilon_i\}$ be a sequence of independent Rademacher variables. We say that a Banach space X has *type* p if there exists a universal constant C_3 such that, for $f_k \in, X$

$$\Big(\mathrm{Ave}_{\varepsilon_k=\pm1}\Big\|\sum_{k=1}^n \varepsilon_k f_k\Big\|^p\Big)^{1/p} \le C_3\Big(\sum_{k=1}^n \|f_k\|^p\Big)^{1/p},$$

and X is of *cotype* q if there exists a universal constant C_4 such that, for $f_k \in X$,

$$\Big(\mathrm{Ave}_{\varepsilon_k=\pm1}\Big\|\sum_{k=1}^n \varepsilon_k f_k\Big\|^q\Big)^{1/q} \ge C_4\Big(\sum_{k=1}^n \|f_k\|^q\Big)^{1/q}. \tag{2.8}$$

It is known that L_p, $2 \le p < \infty$, has type 2. Consider a uniformly bounded orthonormal quasi-greedy basis $\Psi = \{\psi_j\}_{j=1}^\infty$ in L_p, $2 < p < \infty$.

The orthonormality and UCC property imply that for any set of indices A of cardinality m we have

$$m^{1/2} = \Big\| \sum_{k\in A} \psi_k \Big\|_2 \le \Big\| \sum_{k\in A} \psi_k \Big\|_p \asymp \Big(\mathrm{Ave}_{\varepsilon_k=\pm1} \Big\| \sum_{k\in A} \varepsilon_k \psi_k \Big\|_p^p \Big)^{1/p} \asymp \Big\| \Big(\sum_{k\in A} |\psi_k|^2 \Big)^{1/2} \Big\|_p \le C(p) \Big(\sum_{k\in A} \|\psi_k\|_p^2 \Big)^{1/2} \asymp m^{1/2}. \tag{2.9}$$

This shows that for a uniformly bounded orthonormal quasi-greedy basis $\Psi = \{\psi_j\}_{j=1}^\infty$ in L_p, $2 < p < \infty$, we have $\varphi(m, \Psi, L_p) \asymp m^{1/2}$. In particular, (2.9) implies that Ψ is democratic. We consider along with the basis Ψ in L_p its dual basis Ψ^* in $L_{p'}$. By orthonormality of Ψ we get $\Psi^* = \Psi$. Properties of dual bases to quasi-greedy and almost greedy bases are discussed in detail in [2]. In particular, by Proposition 4.4 and Theorem 5.4 from [2] the relation $\varphi(m, \Psi, L_p) \asymp m^{1/2}$ implies that Ψ is also a quasi-greedy basis of $L_{p'}$. We formulate this conclusion as a theorem.

Theorem 2.16. *Let $\Psi = \{\psi_j\}_{j=1}^\infty$ be a uniformly bounded orthonormal quasi-greedy basis in L_p, $2 < p < \infty$. Then Ψ is a quasi-greedy basis of $L_{p'}$.*

Proposition 2.17. *Let $\Psi = \{\psi_j\}_{j=1}^\infty$ be a uniformly bounded quasi-greedy basis in L_q, $1 < q < \infty$. Then Ψ is democratic with fundamental function $\varphi(m, \Psi, L_q) \asymp m^{1/2}$.*

Proof. We give the proof only for $1 < q \le 2$, as the case $2 < q < \infty$ is similar. Using the UCC property of quasi-greedy bases and using the fact that L_q, $1 < q \le 2$, is of cotype 2 we obtain, as in (2.9),

$$\Big\| \sum_{k\in A} \psi_k \Big\|_q \asymp \Big(\mathrm{Ave}_{\varepsilon_k=\pm1} \Big\| \sum_{k\in A} \varepsilon_k \psi_k \Big\|_q^2 \Big)^{1/2} \ge C m^{1/2}.$$

Also

$$\Big\| \sum_{k\in A} \psi_k \Big\|_q \asymp \Big(\mathrm{Ave}_{\varepsilon_k=\pm1} \Big\| \sum_{k\in A} \varepsilon_k \psi_k \Big\|_q^2 \Big)^{1/2} \le \Big(\mathrm{Ave}_{\varepsilon_k=\pm1} \Big\| \sum_{k\in A} \varepsilon_k \psi_k \Big\|_2^2 \Big)^{1/2} \le C m^{1/2}. \ \blacksquare$$

Combination of Proposition 2.17 and Remark 2.9 gives the following inequalities which we will often use.

Proposition 2.18. *Let $\Psi = \{\psi_j\}_{j=1}^\infty$ be a uniformly bounded quasi-greedy basis in L_q, $1 < q < \infty$. Then for any $f \in L_q$ we have*

$$c_1(q) \sup_n n^{1/2} a_n(f) \le \|f\|_q \le C_1(q) \sum_{n=1}^\infty n^{-1/2} a_n(f). \tag{2.10}$$

This proposition implies the following analog of Lemma 2.13.

LEMMA 2.19. *Assume that $\Psi = \{\psi_j\}_{j=1}^{\infty}$ is a uniformly bounded quasi-greedy basis in L_q and L_p, $1 < q, p < \infty$. Then for any set Λ of indices we have*

$$\|S_\Lambda(f)\|_p \le C \ln(|\Lambda| + 1)\|f\|_q. \tag{2.11}$$

Proof. Let $|\Lambda| = m$. By Proposition 2.18 we obtain

$$\|S_\Lambda(f)\|_p \le C_1(p) \sum_{n=1}^{m} n^{-1/2} a_n(f) \le c_1(q)^{-1} C_1(p) \sum_{n=1}^{m} n^{-1}\|f\|_q$$
$$\le C \ln(|\Lambda| + 1)\|f\|_q. \ \blacksquare$$

3. Construction of quasi-greedy bases. In this section we describe without proofs a general scheme of construction of a quasi-greedy basis out of a given basis with special properties. This scheme generalizes the one used by Wojtaszczyk in [28]. Both schemes are based on the orthogonal Haar-type matrices, first used by Olevskiĭ to construct orthogonal systems (see [5, p. 120], [16], [17]). For the proofs of these results we refer the reader to [3].

3.1. Assumptions. Let X be a separable Banach space and $\Phi = \{\varphi_j\}_{j=1}^{\infty}$ be a seminormalized basis of X, $0 < c_0 \le \|\varphi_j\| \le C_0$. We assume that Φ is a Besselian basis of X: for any

$$f = \sum_{j=1}^{\infty} c_j(f)\varphi_j \tag{3.1}$$

we have

$$\Big(\sum_{j=1}^{\infty} |c_j(f)|^2\Big)^{1/2} \le C_1\|f\|. \tag{3.2}$$

Assume that Φ can be split into two systems $F = \{f_s\}_{s=1}^{\infty}$, $f_s = \varphi_{m(s)}$ and $E = \{e_j\}_{j=1}^{\infty}$, $e_j = \varphi_{n(j)}$ with increasing sequences $\{m(s)\}$ and $\{n(j)\}$ in such a way that E has the following special property: for any sequence $\{c_j\}$

$$\Big\|\sum_{j=1}^{\infty} c_j e_j\Big\| \le C_2\Big(\sum_{j=1}^{\infty} |c_j|^2\Big)^{1/2}. \tag{3.3}$$

In our construction of quasi-greedy bases we will use special matrices. Let a collection of matrices $\mathcal{A} = \{A(n)\}_{n=1}^{\infty}$, with $A(n)$ of size $n \times n$, have the following properties:

M1. The singular numbers of matrices $A(n)$ and their inverse $A(n)^{-1}$ are uniformly bounded:

$$s_j(A(n)) \le C_3, \quad s_j(A(n)^{-1}) \le C_3. \tag{3.4}$$

M2. For the elements of the first column of a matrix $A(n) = [a_{ij}(n)]$ we have

$$|a_{i1}(n)| \le C_4 n^{-1/2}. \tag{3.5}$$

3.2. Construction. Let $\{n_k\}_{k=0}^{\infty}$, $n_0 = 0$, be an increasing sequence of integers such that

$$n_{k+1} \ge n_k^2. \tag{3.6}$$

For a fixed natural number k we pick the basis elements

$$g_1^k := f_k, \quad g_i^k := e_{S_{k-1}+i-1}, \quad i = 2, \dots, n_k, \tag{3.7}$$

where $\{S_j\}$ is defined recursively as

$$S_j = S_{j-1} + n_j - 1, \quad j = 1, 2, \dots, \qquad S_0 = 0.$$

We build a new system of elements $\{\psi_i^k\}_{i=1}^{n_k}$ using a matrix $A(n_k)$ in the following way:

$$(\psi_1^k, \dots, \psi_{n_k}^k)^T = A(n_k)(g_1^k, \dots, g_{n_k}^k)^T. \tag{3.8}$$

In other words, for $i \in [1, n_k]$ we have

$$\psi_i^k = \sum_{j=1}^{n_k} a_{ij}(n_k) g_j^k.$$

We begin with a property of the system $G := \{g_i^k\}_{i=1,k=1}^{n_k,\infty} = \{g_{j(k,i)}\}$ ordered lexicographically: $j(k', i') > j(k, i)$ if either $k' > k$ or $k' = k$ and $i' > i$.

Proposition 3.1. *The system G is a Besselian basis of X.*

We define $\Psi = \{\psi_i^k\}_{i=1,k=1}^{n_k,\infty} = \{\psi_{j(k,i)}\}$ ordered lexicographically. The following theorem summarizes the properties of Ψ.

Theorem 3.2. *The basis Ψ is a Besselian quasi-greedy basis of X.*

3.3. Extra assumptions. First of all we note that if Φ forms an orthonormal basis of a Hilbert space H then G also forms an orthonormal basis in H. Second, if the matrices $A(n)$ are orthogonal then Ψ is an orthonormal basis of H.

Next, assume that Y is a subspace of X with a stronger norm: $\|f\|_X \le \|f\|_Y$. Assume that the basis Φ is from Y and $\|\varphi_j\|_Y \le B$, $j = 1, 2, \dots$. We impose an extra assumption on the matrices $A(n)$:

M3. Assume that for all n,

$$\sum_{j=1}^{n} |a_{ij}(n)| \le C_5. \tag{3.9}$$

Under condition **M3**, which is satisfied by the Haar matrices, we easily derive from the definition of Ψ that

$$\|\psi_i^k\|_Y \le C_5 B.$$

Let $X = L_p(0, 2\pi)$, $2 < p < \infty$, $Y = L_\infty(0, 2\pi)$. Consider $\Phi = \mathcal{T}$ to be the trigonometric system $\{e^{ikx}\}$. Define $E := \{e^{i2^j x}\}_{j=1}^\infty$. It is well known that (3.3) holds for this system. By Riesz' theorem $\mathcal{T}$ is a basis of L_p, $1 < p < \infty$. Trivially, $\mathcal{T}$ has the Besselian property in L_p, $2 < p < \infty$. Thus, applying the above construction for the case where the matrices $A(n)$ are orthogonal we obtain the following theorem.

THEOREM 3.3. *There exists a uniformly bounded orthonormal system $\Psi = \{\psi_j\}_{j=1}^\infty$ consisting of trigonometric polynomials which is a quasi-greedy basis for $L_p[0, 1]$ for all $1 < p < \infty$.*

Proof. The quasi-greedy property holds for all $p > 2$ by the preceding discussion. It also holds for $1 < p < 2$ by the duality result, Theorem 2.16. ∎

REMARK 3.4. The orthonormal system constructed in Theorem 3.3 will also be a quasi-greedy basis for other function spaces, including the Lorentz spaces $L_{p,q}[0, 1]$ for $1 < p < \infty$, $1 \le q < \infty$.

REMARK 3.5. Theorem 3.3 can also be obtained from the construction of [15] by replacing the Walsh system by the trigonometric system. We thank the referee for this observation.

4. Uniformly bounded quasi-greedy systems. The main result of this section is that there is no analog of Theorem 3.3 for $L_1[0, 1]$. It is known that $L_1[0, 1]$ has a quasi-greedy basis [1, Theorem 7.1] and, by a theorem of Szarek [21], that $L_1[0, 1]$ does not admit any uniformly integrable *Schauder* basis (see also [8]). On the other hand, the trigonometric system is a uniformly bounded Markushevich basis. Therefore, it is natural to ask whether $L_1[0, 1]$ admits a uniformly bounded (or uniformly integrable) quasi-greedy Markushevich basis Ψ. We answer this question negatively.

First, we recall the relevant definitions. Let X be a separable Banach space. Let $\Psi = \{\psi_j\}_{j=1}^\infty \subset X$ be a *seminormalized* fundamental system, i.e. there exist positive constants a and b such that

$$a \le \|\psi_j\| \le b \quad (j \ge 1), \tag{4.1}$$

with a biorthogonal sequence $\{\psi_j^*\}_{j=1}^\infty \subset X^*$. The system Ψ is said to be a *Markushevich basis* if the mapping $f \mapsto \{\psi_j^*(f)\}_{j=1}^\infty$ $(f \in X)$ is one-one. In other words, each $f \in X$ is uniquely determined by its coefficient sequence $\{\psi_j^*(f)\}_{j=1}^\infty$. We say that Ψ is *quasi-greedy* if there exists a constant C such that

$$\|G_m(f, \Psi)\| \le C\|f\| \quad (m \ge 1,\ f \in X). \tag{4.2}$$

Wojtaszczyk [28] proved that (4.2) is equivalent to the norm convergence of $\{G_m(f)\}$ to f for all $f \in X$.

It follows easily from (4.1) and (4.2) that $\{\psi_j^*\}_{j=1}^\infty$ is seminormalized in X^*. Indeed, for $f \in X$, we have

$$|\psi_j^*(f)| \le a_1(f) \le (1/a)\|G_1(f)\| \le (C/a)\|f\|,$$

and hence $\|\psi_j^*\| \le C/a$. On the other hand, since $\psi_j^*(\psi_j) = 1$, we also have $\|\psi_j^*\| \ge 1/\|\psi_j\| \ge 1/b$.

The following result was proved for quasi-greedy bases (actually for the larger class of *thresholding-bounded* bases) in [1, Lemma 8.2]. The proof easily carries over to quasi-greedy Markushevich bases (cf. also the proof of Lemma 2.4 above).

PROPOSITION 4.1. *Suppose that Ψ is a seminormalized quasi-greedy Markushevich basis for X. There exists a constant C such that for all finite sets $\Lambda \subset \mathbb{N}$ with $|\Lambda| = N \ge 2$, we have*

$$\max_{\pm} \Big\| \sum_{n\in\Lambda} \pm\psi_n^*(f)\psi_n \Big\| \le C(\ln N)\|f\| \quad (f \in X).$$

In particular,

$$\|S_\Lambda(f)\| \le C(\ln N)\|f\| \quad (f \in X).$$

Recall that a bounded operator $T\colon X \to Y$, where X and Y are Banach spaces, is *absolutely summing* if there exists a constant C such that, for all $n \ge 1$ and for all finite sequences $\{f_j\}_{j=1}^n \subset X$, we have

$$\sum_{j=1}^n \|T(f_j)\| \le C \max_{\pm} \Big\| \sum_{j=1}^n \pm f_j \Big\|.$$

The smallest such constant is denoted $\pi_1(T)$. A Banach space X is called a *GT space* [19] if every bounded operator $T\colon X \to \ell_2$ is absolutely summing. It is known that X is a GT space if and only if there exists a constant B such that $\pi_1(T) \le B\|T\|$ for all bounded $T\colon X \to \ell_2$. Grothendieck [6] proved that $L_1(\mu)$ spaces are GT spaces.

The proof of the following result is based on the methods used in [1, Section 8].

THEOREM 4.2. *Suppose that X is a GT space. Let Ψ be a seminormalized quasi-greedy Markushevich basis for X. Then Ψ is democratic and its fundamental function satisfies $\varphi(n) \asymp n$.*

For $1 \le p \le \infty$, recall that a Markushevich basis Ψ is said to be *p-Besselian* if there exists a constant C_p such that

$$\Big(\sum_{n=1}^\infty |\psi_n^*(f)|^p\Big)^{1/p} \le C_p\|f\| \quad (f \in X)$$

(with the obvious modification for $p=\infty$). Since ψ is quasi-greedy, $C_\infty = \sup_{n\geq 1}\|\psi_n^*\| < \infty$, so Ψ is ∞-Besselian.

We will derive Theorem 4.2 from the following theorem.

THEOREM 4.3. *Suppose that Ψ is a seminormalized quasi-greedy Markushevich basis for a GT space X. Then Ψ is r-Besselian for all $r>1$.*

We need the following key lemma.

LEMMA 4.4. *Suppose that Ψ is a seminormalized quasi-greedy Markushevich basis for a GT space X. If Ψ is p-Besselian for some $2\leq p\leq\infty$, then Ψ is r-Besselian for all r satisfying $1/r < 1/p + 1/2$.*

Proof. We shall give the proof for the case $2<p<\infty$, as the case $p=\infty$ requires only minor changes. Let $1/s = 1/p+1/2$. Suppose that $\Lambda\subset\mathbb{N}$, with $|\Lambda| = N$, and that $(\eta_n)_{n\in\Lambda}$ is any fixed choice of signs. Choose $f\in X$ with $\|f\|=1$ such that

$$\sum_{n\in\Lambda}\eta_n\psi_n^*(f) \geq \frac{1}{2}\Big\|\sum_{n\in\Lambda}\eta_n\psi_n^*\Big\|.$$

Next consider $T\colon X\to\ell_2(\Lambda)$ defined as follows:

$$T(g) = (\psi_n^*(g)|\psi_n^*(f)|^{s-1})_{n\in\Lambda}\quad (g\in X).$$

Then, applying Hölder's inequality and using the fact that Ψ is p-Besselian, we get

$$\begin{aligned}\|T(g)\| &= \Big(\sum_{n\in\Lambda}|\psi_n^*(f)|^{2s-2}|\psi_n^*(g)|^2\Big)^{1/2}\\ &\leq \Big(\sum_{n\in\Lambda}|\psi_n^*(f)|^{s}\Big)^{1-1/s}\Big(\sum_{n\in\Lambda}|\psi_n^*(g)|^p\Big)^{1/p}\\ &\leq C_p\Big(\sum_{n\in\Lambda}|\psi_n^*(f)|^{s}\Big)^{1-1/s}\|g\|.\end{aligned}$$

Hence $\|T\|\leq C_p(\sum_{n\in\Lambda}|\psi_n^*(f)|^s)^{1-1/s}$. Since X is a GT space, we have

$$\begin{aligned}\sum_{n\in\Lambda}|\psi_n^*(f)|^s &= \sum_{n\in\Lambda}|\psi_n^*(f)|\|T(\psi_n)\| \leq B\|T\|\sup_{\varepsilon_n=\pm1}\Big\|\sum_{n\in\Lambda}\varepsilon_n\psi_n^*(f)\psi_n\Big\|\\ &\leq BC_p\Big(\sum_{n\in\Lambda}|\psi_n^*(f)|^s\Big)^{1-1/s}\sup_{\varepsilon_n=\pm1}\Big\|\sum_{n\in\Lambda}\varepsilon_n\psi_n^*(f)\psi_n\Big\|.\end{aligned}$$

Thus,

$$\Big(\sum_{n\in\Lambda}|\psi_n^*(f)|^s\Big)^{1/s}\leq BC_p\sup_{\varepsilon_n=\pm1}\Big\|\sum_{n\in\Lambda}\varepsilon_n\psi_n^*(f)\psi_n\Big\|.$$

Since $|\Lambda| = N$, Proposition 4.1 yields

$$\sup_{\varepsilon_n = \pm 1} \Big\| \sum_{n\in\Lambda} \varepsilon_n \psi_n^*(f)\psi_n \Big\| \le C'(\ln N),$$

where C' is independent of N. Hence

$$\Big(\sum_{n\in\Lambda} |\psi_n^*(f)|^s\Big)^{1/s} \le BC'C_p(\ln N).$$

Thus,

$$\begin{aligned}\Big\|\sum_{n\in\Lambda}\eta_n\psi_n^*\Big\| &\le 2\sum_{n\in\Lambda}\eta_n\psi_n^*(f) \le 2\Big(\sum_{n\in\Lambda}|\psi_n^*(f)|^s\Big)^{1/s} N^{1-1/s}\\ &\le BC'C_p(\ln N)N^{1-1/s}.\end{aligned}$$

Now suppose that $g \in X$ with $\|g\| = 1$. For $a > 0$, let

$$\Lambda(a) = \{n \in \mathbb{N} \colon |\psi_n^*(g)| \ge a\} \quad \text{and} \quad N(a) = |\Lambda(a)|.$$

Then, for some choice of signs (η_n), we have

$$aN(a) \le \sum_{n\in\Lambda(a)} \eta_n\psi_n^*(g) \le \Big\|\sum_{n\in\Lambda(a)}\eta_n\psi_n^*\Big\| \le BC'C_p(\ln N(a))N(a)^{1-1/s}.$$

Thus, for some constant C'', we have $N(a) \le C''a^{-t}$ provided t satisfies

$$\frac{1}{r} < \frac{1}{t} < \frac{1}{s}.$$

Note that

$$\sup_{n\ge 1}|\psi_n^*(g)| \le \sup_{n\ge 1}\|\psi_n^*\|_\infty = C_\infty.$$

Therefore

$$\sum_{n=1}^\infty |\psi_n^*(g)|^r \le \sum_{n=0}^\infty N(2^{-n}C_\infty)(2^{1-n}C_\infty)^r \le 2^r C'' \sum_{n=0}^\infty (2^{-n}C_\infty)^{r-t} < \infty.$$

Hence Ψ is r-Besselian. ∎

Applying the lemma twice, starting with $p = \infty$, we conclude that Ψ is r-Besselian for all $r > 1$, which proves Theorem 4.3.

Proof of Theorem 4.2. By Theorem 4.2, Ψ is 2-Besselian with constant $C_2 < \infty$. Hence, for every finite $\Lambda \subset \mathbb{N}$, the mapping $T \colon X \to \ell_2(\Lambda)$ given by $f \mapsto (\psi_n^*(f))_{n\in\Lambda}$ satisfies $\|T\| \le C_2$. Since X is a GT space, the absolutely summing norm of T satisfies $\pi_1(T) \le BC_2$. Thus,

$$|\Lambda| = \sum_{n\in\Lambda}\|T(\psi_n)\|_2 \le BC_2 \max_{\pm}\Big\|\sum_{n\in\Lambda}\pm\psi_n\Big\|.$$

Since Ψ is quasi-greedy, and hence unconditional for constant coefficients, it follows that $\varphi(n) \asymp n$. ∎

From [2, Lemma 3.2] and Theorem 4.2 we obtain the following result.

COROLLARY 4.5. *Suppose that Ψ is a seminormalized quasi-greedy Markushevich basis for a GT space X. There exists a constant C such that for all $g \in X$ we have*

$$a_n(g) \leq Cn^{-1}\|g\|.$$

Recall that a system $\{f_j\} \subset L_1[0,1]$ is *uniformly integrable* if, given $\varepsilon > 0$, there exists $\delta > 0$ such that if $\lambda(A) < \delta$, where λ denotes Lebesgue measure, then $\int_A |f_j|\, d\lambda < \varepsilon$ for all $j \geq 1$. Clearly, uniformly bounded systems are uniformly integrable.

THEOREM 4.6. *Let Ψ be a seminormalized quasi-greedy Markushevich basis for $L_1[0,1]$. Then no subsequence of Ψ is uniformly integrable. Hence every subsequence of Ψ contains a further subsequence equivalent to the unit vector basis of ℓ_1.*

Proof. Let $\{f_j\} \subset L_1[0,1]$ be any uniformly integrable system. Given $\varepsilon > 0$, choose $M > 0$ such that $\|f_j \chi_{\{|f_j|>M\}}\|_1 < \varepsilon$ for all j. Then

$$\operatorname{Ave}_{\pm} \Big\| \sum_{j=1}^{n} \pm f_j \Big\|_1 \leq n\varepsilon + \operatorname{Ave}_{\pm} \Big\| \sum_{j=1}^{n} \pm f_j \chi_{\{|f_j| \leq M\}} \Big\|_2 \leq n\varepsilon + M\sqrt{n}.$$

Hence $\operatorname{Ave}_{\pm} \|\sum_{j=1}^{n} \pm f_j\|_1 = o(n)$. Since $L_1[0,1]$ is a GT space, Theorem 4.2 implies that $\{f_j\}$ is not a subsequence of any quasi-greedy Markushevich basis. Finally, it is well known that seminormalized sequences in $L_1[0,1]$ are either uniformly integrable or contain a subsequence equivalent to the unit vector basis of ℓ_1. ∎

REMARK 4.7. Complemented subspaces of L_1 spaces are GT spaces. Hence the previous theorem extends to quasi-greedy Markushevich bases of complemented (infinite-dimensional) subspaces of $L_1[0,1]$. A related result of Popov [20] asserts that complemented subspaces of $L_1[0,1]$ do not admit a uniformly integrable Schauder basis.

Next we consider the Hardy spaces $H_p(D)$ $(1 \leq p < \infty)$ of analytic functions on the disk $D := \{z \in \mathbb{C} \colon |z| < 1\}$ equipped with the norm

$$\|f\|_p = \sup_{0<r<1} \left(\frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p \, d\theta \right)^{1/p}.$$

Using the system $\{z^n\}_{n=0}^{\infty}$ instead of $\mathcal{T}$ in the proof of Theorem 3.3 yields the following result.

THEOREM 4.8. *There exists an orthonormal system of uniformly bounded analytic polynomials which is a quasi-greedy basis for $H_p(D)$ for $1 < p < \infty$.*

Using some deep results from Banach space theory we can extend the latter result also to the case $p = 1$.

THEOREM 4.9. *$H_1(D)$ admits a seminormalized uniformly bounded quasi-greedy basis consisting of analytic polynomials.*

Proof. By Paley's inequality [18],

$$\Big(\sum_{n=1}^{\infty} |\hat{f}(2^n)|^2\Big)^{1/2} \asymp \Big\| \sum_{n=1}^{\infty} \hat{f}(2^n) z^{2^n} \Big\|_1 \le 2\|f\|_1.$$

Hence $P(f) = \sum_{n=1}^{\infty} \hat{f}(2^n) z^{2^n}$ is a bounded projection on $H_1(D)$. Let $X := \ker P$ and let $H := [e_j]_{j=1}^{\infty}$, where $e_j := z^{2^j}$. Then $H_1(D)$ is linearly isomorphic to $X \oplus H$ (equipped with the sum norm), which in turn is isomorphic to $X \oplus \ell_2$. Since X contains a complemented subspace isomorphic to ℓ_2 (e.g., the subspace spanned by $(z^{3^j})_{j=1}^{\infty}$), it follows that X is also isomorphic to $X \oplus \ell_2$ and thus also isomorphic to $H_1(D)$. Hence, by a theorem of Maurey [14, p. 79], X has a normalized unconditional basis $(f_j)_{j=1}^{\infty}$. The intersection of X with the linear space of analytic polynomials is dense in X. Hence we may assume that each f_j is an analytic polynomial. Since $H_1(D)$ has cotype 2 (see (2.8)), it follows that $\{f_j\}_{j=1}^{\infty}$ is Besselian. Then $\Phi = \{f_j\}_{j=1}^{\infty} \cup \{e_j\}_{j=1}^{\infty}$ is a basis for $H_1(D)$ satisfying (3.2) and (3.3). Assume the matrices $\{A(n)\}$ satisfy **M1**–**M3** and, in addition, that $n_k \ge \|f_k\|_{\infty}^2$. By the construction, the system Ψ is a seminormalized quasi-greedy basis. Moreover, by **M2** and **M3**,

$$\sup_{j\ge 1} \|\psi_j\|_{\infty} \le \sup_{k\ge 1} \frac{\|f_k\|_{\infty}}{\sqrt{n_k}} + C_5 \sup_{k\ge 1} \|e_k\| \le 1 + C_5.$$

Thus, Ψ is uniformly bounded. ∎

Finally, let us mention that Theorem 3.3 may be generalized to certain closed subspaces of $L_p[0,1]$, for $p > 2$, including those spanned by any subsequence of the trigonometric system. Recall that $\{\psi_j\} \subset L_2[0,1]$ is a *Riesz sequence* if $\|\sum c_j\psi_j\|_2 \asymp (\sum |c_j|^2)^{1/2}$ for all scalars $\{c_j\}$.

PROPOSITION 4.10. *Let X be a closed subspace of $L_p[0,1]$ for $2 \le p < \infty$. Suppose that X has a uniformly bounded Schauder basis $\{\psi_j\}$ which is a Riesz sequence in $L_2[0,1]$. Then X admits a uniformly bounded quasi-greedy basis.*

Proof. Since $L_p[0,1]$ has an unconditional basis and $\{\psi_j\}$ is weakly null, it follows by a standard "gliding hump" argument that some subsequence $\{\psi_j\}_{j\in A}$ is unconditional. Since $L_p[0,1]$ has type 2 (for the upper estimate) and $\{\psi_j\}$ is a Riesz sequence in $L_2[0,1]$ (for the lower estimate), it follows that $\|\sum_{j\in A} c_j\psi_j\|_p \asymp (\sum_{j\in A} |c_j|^2)^{1/2}$ for all scalars $\{c_j\}$, i.e., $\{\psi_j\}_{j\in A}$ is a sequence in X that is equivalent to the unit vector basis of ℓ_2. Since $\{\psi_j\}$

is a Riesz sequence in $L_2[0,1]$, we have, for all $f \in X$,

$$\Big\| \sum c_j(f)\psi_j \Big\|_p \ge \Big\| \sum c_j(f)\psi_j \Big\|_2 \ge k_1 \Big(\sum_{j\in A} |c_j(f)|^2 \Big)^{1/2}$$
$$\ge k_2 \Big\| \sum_{j\in A} c_j(f)\psi_j \Big\|_p,$$

where k_1 and k_2 are constants. Hence the projection $Pf = \sum_{j\in A} c_j(f)\psi_j$ is bounded on X, which implies that X is linearly isomorphic to $[\psi_j]_{j\notin A} \oplus [\psi_j]_{j\in A}$. The fact that $\{\psi_j\}$ is a Riesz sequence in $L_2[0,1]$ implies that $\{\psi_j\}$ is a (uniformly bounded) Besselian basis for X. The proof is completed as in the discussion preceding Theorem 3.3. ∎

5. Lebesgue-type inequalities I. Our main interest in this section is to prove Lebesgue-type inequalities for greedy approximation in L_p, $2 \le p \le \infty$, under different assumptions on a basis Ψ. In this section we assume that Ψ is a uniformly bounded basis. In addition we assume that Ψ is a certain type basis (quasi-greedy basis, Riesz basis) in one of the spaces L_2, L_q, $1 < q < 2$, or L_q, $2 < q < \infty$. We will often use the following lemma.

LEMMA 5.1. *Suppose that $X \subset Y$ are two Banach spaces such that $\|\cdot\|_Y \le \|\cdot\|_X$. Assume that a basis Ψ of X has the following property: For any set Λ of indices,*

$$\|S_\Lambda(f)\|_X \le w(|\Lambda|)\|f\|_Y.$$

Then for each $f \in X$ and any m-term polynomial

$$p_m = \sum_{k\in P} b_k\psi_k, \qquad |P| = m,$$

we have

$$\|f - S_P(f)\|_X \le \|f - p_m\|_X + w(m)\|f - p_m\|_Y.$$

Proof. It is a simple two-line proof:

$$\|f - S_P(f)\|_X = \|f - p_m(f) - S_P(f - p_m(f)\|_X$$
$$\le \|f - p_m\|_X + w(m)\|f - p_m\|_Y. \ ∎$$

We now proceed to a systematic presentation of new results.

THEOREM 5.2. *Assume that Ψ is a uniformly bounded Riesz basis of L_2. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \qquad |P| = m,$$

we have, for $2 \le p \le \infty$,

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + Cm^{h(p)}\|f - t_m\|_2.$$

COROLLARY 5.3. *Assume that Ψ is a uniformly bounded Riesz basis of L_2. Then for $2 \le p \le \infty$ we have*

$$\|f - G_m(f,\Psi)\|_p \le Cm^{h(p)}\sigma_m(f,\Psi)_p.$$

Proof of Theorem 5.2. Denote by Q the set of indices picked by the greedy algorithm after m iterations,

$$G_m(f) := G_m(f,\Psi) := \sum_{k\in Q} c_k(f)\psi_k.$$

We use the representation

$$f - G_m(f) = f - S_Q(f) = f - S_P(f) + S_P(f) - S_Q(f). \tag{5.1}$$

First, we bound $\|f - S_P(f)\|_p$. By Lemmas 5.1 and 2.12 we get

$$\|f - S_P(f)\|_p \le \|f - t_m\|_p + Cm^{h(p)}\|f - t_m\|_2. \tag{5.2}$$

Second, we write

$$\begin{aligned} \|S_P(f) - S_Q(f)\|_p &= \|S_{P\setminus Q}(f) - S_{Q\setminus P}(f)\|_p \\ &\le \|S_{P\setminus Q}(f)\|_p + \|S_{Q\setminus P}(f)\|_p. \end{aligned} \tag{5.3}$$

Using Lemma 2.12 again, we obtain

$$\|S_P(f) - S_Q(f)\|_p \le Cm^{h(p)}(\|S_{P\setminus Q}(f)\|_2 + \|S_{Q\setminus P}(f)\|_2). \tag{5.4}$$

The definition of Q implies

$$\begin{aligned} \|S_{P\setminus Q}(f)\|_2 &\le C\Big(\sum_{k\in P\setminus Q} |c_k(f)|^2\Big)^{1/2} \\ &\le C\Big(\sum_{k\in Q\setminus P} |c_k(f)|^2\Big)^{1/2} \le C\|S_{Q\setminus P}(f)\|_2. \end{aligned} \tag{5.5}$$

Next,

$$\|S_{Q\setminus P}(f)\|_2 = \|S_{Q\setminus P}(f - t_m)\|_2 \le C\|f - t_m\|_2. \tag{5.6}$$

Combining (5.1)–(5.6) completes the proof. ∎

We now impose a slightly weaker assumption on a basis Ψ than the one in Theorem 5.2.

THEOREM 5.4. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_2. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \qquad |P| = m,$$

we have, for $2 \le p \le \infty$,

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + Cm^{h(p)}\ln(m+1)\|f - t_m\|_2.$$

COROLLARY 5.5. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_2. Then for $2 \le p \le \infty$,*

$$\|f - G_m(f,\Psi)\|_p \le Cm^{h(p)} \ln(m+1)\sigma_m(f,\Psi)_p.$$

Proof of Theorem 5.4. This proof goes along the lines of the proof of Theorem 5.2. However, the details are different because we need to use properties of quasi-greedy bases instead of properties of Riesz bases. We use notations from the proof of Theorem 5.2 and the representation (5.1). By Lemmas 5.1 and 2.11 we get, for $\|f - S_P(f)\|_p$,

$$\|f - S_P(f)\|_p \le \|f - t_m\|_p + Cm^{h(p)} \ln(m+1)\|f - t_m\|_2. \tag{5.7}$$

Using Lemma 2.11 again, from (5.3) we obtain

$$\|S_P(f) - S_Q(f)\|_p \le Cm^{h(p)}(\|S_{P\setminus Q}(f)\|_2 + \|S_{Q\setminus P}(f)\|_2). \tag{5.8}$$

Next, by Theorem 2.7 we have

$$\begin{aligned} &\|S_{Q\setminus P}(f)\|_2 \\ &\quad = \|S_{Q\setminus P}(f - t_m)\|_2 \le C_2(2) \sum_{n=1}^{m} n^{-1/2} a_n(S_{Q\setminus P}(f - t_m)) \\ &\quad \le C \sum_{n=1}^{m} n^{-1/2} a_n(f - t_m) = C \sum_{n=1}^{m} n^{-1}(n^{1/2} a_n(f - t_m)) \\ &\quad \le C \ln(m+1) \sup_n n^{1/2} a_n(f - t_m) \le C \ln(m+1)\|f - t_m\|_2. \end{aligned} \tag{5.9}$$

For $S_{P\setminus Q}(f)$ we have

$$\begin{aligned} \|S_{P\setminus Q}(f)\|_2 &\le C_2(2) \sum_{n=1}^{m} n^{-1/2} a_n(S_{P\setminus Q}(f)) \\ &\le C_2(2) \sum_{n=1}^{m} n^{-1/2} a_n(S_{Q\setminus P}(f)) \\ &= C_2(2) \sum_{n=1}^{m} n^{-1/2} a_n(S_{Q\setminus P}(f - t_m)) \end{aligned}$$

which has been estimated in (5.9) as

$$\le C \ln(m+1)\|f - t_m\|_2. \tag{5.10}$$

Combining (5.7)–(5.10) completes the proof. ∎

THEOREM 5.6. *Assume that Ψ is a democratic quasi-greedy basis of X. Then, for any $f \in X$,*

$$\|f - G_m(f,\Psi)\|_X \le C \ln(m+1)\sigma_m(f,\Psi)_X.$$

COROLLARY 5.7. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_p, $1 < p < \infty$. Then*

$$\|f - G_m(f,\Psi)\|_p \le C(p)\ln(m+1)\sigma_m(f,\Psi)_p.$$

Proof of Theorem 5.6 and Corollary 5.7. It is known (see [2]) that a democratic and quasi-greedy basis is an almost greedy basis. Therefore, the inequality

$$\|f - G_m(f,\Psi)\|_X \le C\tilde{\sigma}_m(f,\Psi)_X$$

holds for any $f \in X$. It remains to apply Lemma 2.5 to complete the proof the theorem. The corollary follows from the theorem and from Proposition 2.17. ∎

THEOREM 5.8. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_q, $1 < q < \infty$. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \qquad |P| = m,$$

we have, for $q \le p \le \infty$,

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + C(p,q)m^{(1-q/p)/2}\ln(m+1)\|f - t_m\|_q.$$

COROLLARY 5.9. *Assume that Ψ is a uniformly bounded quasi-greedy basis of L_q, $1 < q < \infty$. Then, for $q \le p \le \infty$,*

$$\|f - G_m(f,\Psi)\|_p \le C(p,q)m^{(1-q/p)/2}\ln(m+1)\sigma_m(f,\Psi)_p.$$

Proof of Theorem 5.8. This proof goes along the lines of the proof of Theorem 5.4. We use the notation from the proof of Theorem 5.4 and the representation (5.1). By Lemmas 5.1 and 2.11 we get

$$\|f - S_P(f)\|_p \le \|f - t_m\|_p + C(p,q)m^{(1-q/p)/2}\ln(m+1)\|f - t_m\|_q. \tag{5.11}$$

Using Lemma 2.11 again, from (5.3) we obtain

$$\|S_P(f) - S_Q(f)\|_p \le Cm^{(1-q/p)/2}(\|S_{P\setminus Q}(f)\|_q + \|S_{Q\setminus P}(f)\|_q). \tag{5.12}$$

By Lemma 2.4 we estimate

$$\|S_{Q\setminus P}(f)\|_q = \|S_{Q\setminus P}(f - t_m)\|_q \le C\ln(m+1)\|f - t_m\|_q.$$

We give another proof of this bound because it will be used in estimating

$\|S_{P\setminus Q}(f)\|_q$. By Proposition 2.18 we have

$$\begin{aligned}(5.13)\qquad \|S_{Q\setminus P}(f)\|_q &= \|S_{Q\setminus P}(f-t_m)\|_q \le C(q)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f-t_m))\\ &\le C(q)\sum_{n=1}^{m} n^{-1/2}a_n(f-t_m) = C(q)\sum_{n=1}^{m} n^{-1}(n^{1/2}a_n(f-t_m))\\ &\le C(q)\ln(m+1)\sup_n n^{1/2}a_n(f-t_m) \le C(q)\ln(m+1)\|f-t_m\|_q.\end{aligned}$$

For $S_{P\setminus Q}(f)$ we have

$$\begin{aligned}\|S_{P\setminus Q}(f)\|_q &\le C(q)\sum_{n=1}^{m} n^{-1/2}a_n(S_{P\setminus Q}(f))\\ &\le C(q)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f))\\ &= C(q)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f-t_m)),\end{aligned}$$

which has been estimated in (5.13) as

$$(5.14)\qquad \le C(q)\ln(m+1)\|f-t_m\|_q.$$

Combining (5.11)–(5.14) completes the proof. ■

6. Lebesgue-type inequalities II. In this section we continue to prove Lebesgue-type inequalities for greedy approximation in L_p under different assumptions on the basis Ψ. In this section we assume that Ψ is a quasi-greedy basis for a pair of spaces: L_q, $1 < q < \infty$, and L_p, $q \le p$.

THEOREM 6.1. *Assume that Ψ is a seminormalized quasi-greedy basis for both L_q and L_p with $1 < q \le 2 \le p < \infty$. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \qquad |P| = m,$$

we have

$$\|f - G_m(f,\Psi)\|_p \le \|f-t_m\|_p + C(p,q)\ln(m+1)\|f-t_m\|_q.$$

COROLLARY 6.2. *Assume that Ψ is a seminormalized quasi-greedy basis for both L_q and L_p with $1 < q \le 2 \le p < \infty$. Then*

$$\|f - G_m(f,\Psi)\|_p \le C(p,q)\ln(m+1)\sigma_m(f,\Psi)_p.$$

Proof of Theorem 6.1. This proof goes along the lines of the proof of Theorem 5.4. We use the notation from the proof of Theorem 5.2 and the representation (5.1). By Lemmas 5.1 and 2.13 we get, for $\|f - S_P(f)\|_p$,

$$\|f - S_P(f)\|_p \le \|f - t_m\|_p + C(p,q)\ln(m+1)\|f - t_m\|_q. \tag{6.1}$$

From (5.3) we obtain

$$\|S_P(f) - S_Q(f)\|_p \le \|S_{P\setminus Q}(f)\|_p + \|S_{Q\setminus P}(f)\|_p. \tag{6.2}$$

Next, by Theorem 2.7 we have

$$\begin{aligned}\|S_{Q\setminus P}(f)\|_p &= \|S_{Q\setminus P}(f - t_m)\|_p \le C_2(p)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f - t_m)) \\ &\le C(p)\sum_{n=1}^{m} n^{-1/2}a_n(f - t_m) = C(p)\sum_{n=1}^{m} n^{-1}(n^{1/2}a_n(f - t_m)) \\ &\le C(p)\ln(m+1)\sup_n n^{1/2}a_n(f - t_m) \le C(p,q)\ln(m+1)\|f - t_m\|_q.\end{aligned} \tag{6.3}$$

For $S_{P\setminus Q}(f)$ we have, by Theorem 2.7,

$$\begin{aligned}\|S_{P\setminus Q}(f)\|_p &\le C_2(p)\sum_{n=1}^{m} n^{-1/2}a_n(S_{P\setminus Q}(f)) \\ &\le C_2(p)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f)) \\ &= C_2(p)\sum_{n=1}^{m} n^{-1/2}a_n(S_{Q\setminus P}(f - t_m))\end{aligned}$$

which has been estimated in (6.3) as

$$\le C(p,q)\ln(m+1)\|f - t_m\|_q. \tag{6.4}$$

Combining (6.1)–(6.4) completes the proof. ∎

Remark 6.3. The statement of Corollary 6.2 holds even if we drop the assumption that Ψ is a quasi-greedy basis of L_q.

Proof. The assumption that Ψ is seminormalized for both L_q and L_p, $q \le 2 \le p$, implies that it is seminormalized in L_2. Then as in Proposition 2.17 we can prove that Ψ is democratic with $\varphi(m) \asymp m^{1/2}$. It remains to apply Theorem 5.6. ∎

Now we prove sharper results for a uniformly bounded orthonormal quasi-greedy basis.

Theorem 6.4. *Assume that Ψ is a uniformly bounded orthonormal quasi-greedy basis for L_p, $2 \le p < \infty$. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \qquad |P| = m,$$

we have

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + C(p)\ln(m+1)\|f - t_m\|_{p'}, \tag{6.5}$$

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + C(p)(\ln(m+1))^{1/2}\|f - t_m\|_2. \tag{6.6}$$

COROLLARY 6.5. *Assume that Ψ is a uniformly bounded orthonormal quasi-greedy basis for L_p, $2 \le p < \infty$. Then*

$$\|f - G_m(f,\Psi)\|_p \le C(p)(\ln(m+1))^{1/2}\sigma_m(f,\Psi)_p.$$

Proof of Theorem 6.4. By Theorem 2.16, Ψ is a quasi-greedy basis of $L_{p'}$. Thus, (6.5) follows from Theorem 6.1 with $q = p'$. We now prove (6.6). As in the proof of Theorem 6.1 we obtain, by Lemmas 5.1 and 2.14,

$$\|f - S_P(f)\|_p \le \|f - t_m\|_p + C(p)(\ln(m+1))^{1/2}\|f - t_m\|_2. \tag{6.7}$$

By Theorem 2.7 we find that

$$\begin{aligned} \|S_{Q\setminus P}(f)\|_p &= \|S_{Q\setminus P}(f - t_m)\|_p \\ &\le C(p)\sum_{n=1}^{m} n^{-1/2}a_n(f - t_m) \\ &\le C(p)\Big(\sum_{n=1}^{m} n^{-1}\Big)^{1/2}\Big(\sum_{n=1}^{m} a_n(f - t_m)^2\Big)^{1/2} \\ &\le C(p)(\ln(m+1))^{1/2}\|f - t_m\|_2. \end{aligned} \tag{6.8}$$

Again as in the proof of Theorem 6.1 we get

$$\|S_{P\setminus Q}(f)\|_p \le C(p)\sum_{n=1}^{m} n^{-1/2}a_n(f - t_m),$$

and by the intermediate step in (6.8),

$$\le C(p)(\ln(m+1))^{1/2}\|f - t_m\|_2.$$

It remains to use the representation (5.1) and inequality (6.2). ∎

If Ψ is assumed to be uniformly bounded, then the Lebesgue-type inequality of Theorem 6.1 holds whenever $q \le p$.

THEOREM 6.6. *Assume that Ψ is a uniformly bounded quasi-greedy basis for both L_q and L_p with $1 < q \le p < \infty$. Then for any m-term polynomial*

$$t_m = \sum_{k\in P} b_k\psi_k, \qquad |P| = m,$$

we have

$$\|f - G_m(f,\Psi)\|_p \le \|f - t_m\|_p + C(p,q)\ln(m+1)\|f - t_m\|_q.$$

Proof. As in the proof of Theorem 6.1 we obtain, by Lemmas 5.1 and 2.19,

$$\|f - S_P(f)\|_p \le \|f - t_m\|_p + C(p,q)\ln(m+1)\|f - t_m\|_q. \tag{6.9}$$

By Proposition 2.18 we deduce that

$$\|S_{Q\setminus P}(f)\|_p = \|S_{Q\setminus P}(f-t_m)\|_p \le C(p,q)\sum_{n=1}^{m} n^{-1/2}a_n(f-t_m) \le C(p,q)\sum_{n=1}^{m} n^{-1}\|f-t_m\|_q \le C(p,q)\ln(m+1)\|f-t_m\|_q. \tag{6.10}$$

As in the proof of Theorem 6.1 we get

$$\|S_{P\setminus Q}(f)\|_p \le C(p,q)\sum_{n=1}^{m} n^{-1/2}a_n(f-t_m),$$

and by the intermediate step in (6.10),

$$\le C(p,q)\ln(m+1))\|f-t_m\|_n q.$$

It remains to use representation (5.1) and inequality (6.2). ∎

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