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Composition operator and Sobolev–Lorentz spaces $WL^{n,q}$

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Abstract. Let $\Omega,\Omega'\subset\mathbb{R}^n$ be domains and let $f\colon\Omega\to\Omega'$ be a homeomorphism. We show that if the composition operator $T_f\colon u\mapsto u\circ f$ maps the Sobolev–Lorentz space $WL^{n,q}(\Omega')$ to $WL^{n,q}(\Omega)$ for some $q\neq n$ then f must be a locally bilipschitz mapping.

1. Introduction. Let $\Omega,\Omega'\subset\mathbb{R}^n$ be domains and let $f\colon\Omega\to\Omega'$ be a homeomorphism. Given a function space X we would like to characterize mappings f for which the composition operator $T_f\colon T_f(u)=u\circ f$ maps $X(\Omega')$ into $X(\Omega)$ continuously. This problem has been studied for many function spaces and one the most important is the following well-known result: The composition operator T_f maps $W^{1,n}_{\mathrm{loc}}(\Omega')$ into $W^{1,n}_{\mathrm{loc}}(\Omega)$ continuously if $f\colon\Omega\to\Omega'$ is a quasiconformal mapping ([N], [L], [VG], [VG2], [Re], [HKM], [K, Lemma 5.13]). Conversely, each homeomorphism f which maps $W^{1,n}_{\mathrm{loc}}(\Omega')$ into $W^{1,n}_{\mathrm{loc}}(\Omega)$ continuously is necessarily a quasiconformal mapping up to a reflection; this is a consequence of ring characterizations of quasiconformality ([N], [G1], [L], [V]). Similarly it is possible to characterize homeomorphisms for which the composition operator is continuous from $W^{1,p}_{\mathrm{loc}}$ to $W^{1,p}_{\mathrm{loc}}$ ([G2], [L], [M], [GRo], [GGR], [Kl1]). A homeomorphism $f\in W^{1,1}_{\mathrm{loc}}(\Omega,\mathbb{R}^n)$ is called a *quasiconformal mapping* if there is a constant $Q\geq 1$ such that

$$|Df(x)|^n\leq QJ_f(x)\quad\text{ for a.e. } x\in\Omega. \tag{1.1}$$

For the properties and further applications of quasiconformal mappings and their generalizations see e.g. [Re], [GR], [V], [Ri], [Vu], [IM], [AIM], [K].

The class of quasiconformal mappings serves as the best class of morphisms not only for $W^{1,n}_{\mathrm{loc}}$ functions but also for other function spaces that are "close" to $W^{1,n}_{\mathrm{loc}}$. Let us mention for example the stability under quasiconformal mappings for the BMO space (see [R]), fractional Sobolev spaces $\dot{M}^s_{n/s,q}$, $s\in(0,1]$ (see [KYZ, Theorem 1.3]; also [TV] and [HK]), absolutely contin-

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uous functions of several variables AC^n_λ (see [H]), exponential Orlicz space $\exp L(\Omega)$ in the plane (see [FG]) and Sobolev–Orlicz spaces like $WL^n \log^\alpha L$, $\alpha \in \mathbb{R}$ (see [HKl]).

We wanted to know if the similar result holds for the Sobolev–Lorentz space $WL^{n,1}$, which is important in many applications (see e.g. [S], [CP], [KKM]). Surprisingly we have found that it is false for any Sobolev–Lorentz space $WL^{n,q}$, $q \neq n$ (see Preliminaries for the definition). This is the first result that we are aware of that says that some reasonable and useful function space close to $W^{1,n}$ is not stable under quasiconformal mappings. The main reason behind this seems to be that quasiconformality is a geometrical notion while the use of the non-increasing rearrangement in the definition of Lorentz spaces erases the geometrical information about the function Du. In the meantime, a similar feature has been observed in the scale of homogeneous Besov spaces $\dot{B}^s_{n/s,q}$ with $sq \neq n$ (see [KKSS]).

It is easy to see that a composition of a bilipschitz mapping f and a function $u \in WL^{p,q}$ satisfies $u \circ f \in WL^{p,q}$ but the following results show that we cannot improve on this.

Theorem 1.1. *Let $1 \leq q \leq \infty$, $q \neq n$. Let $f\colon \Omega \to \Omega'$ be a Sobolev homeomorphism. Assume that T_f maps $W_0L^{n,q}(\Omega')$ to $WL^{n,q}(\Omega)$. Then f is a locally bilipschitz mapping.*

As one of the steps of the proof we need the following result which is of independent interest for $p \neq n$. Note that especially for $q = p$ we obtain a new proof of the characterization of homeomorphisms for which the composition operator is continuous from $W^{1,p}_{\mathrm{loc}}$ to $W^{1,p}_{\mathrm{loc}}$. Our proof uses a different idea and in contrast to the original article [GGR] (see also [Kl1]) we do not need to use the volume derivative instead of J_f, or the technical assumption that f is differentiable a.e. (See also [Kl2] for a more general version of the following theorem.)

Theorem 1.2. *Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$. Let $f\colon \Omega \to \Omega'$ be a Sobolev homeomorphism. Suppose that T_f maps $W_0L^{p,q}(\Omega')$ into $WL^{p,q}(\Omega)$. Then there exists $Q \in \mathbb{R}$ such that*

$$|Df(x)|^p \leq Q|J_f(x)| \qquad \text{for a.e. } x \in \Omega.$$

2. Preliminaries. Throughout this paper, Ω and Ω' are open subsets of $\mathbb{R}^n$. We use the symbol $|A|$ for the Lebesgue measure of a measurable set $A \subset \mathbb{R}^n$. We say that a function $f\colon \Omega \to \mathbb{R}^n$ satisfies the *Luzin* (N) *condition* on $A \subset \Omega$ if $|f(S)| = 0$ for every measurable set $S \subset A$ with $|S| = 0$. If we do not specify where the Luzin (N) condition is satisfied, we mean the case $A = \Omega$.

The function $f : \Omega \to \mathbb{R}$ is called *essentially unbounded* if $|\{x : f(x) > t\}| > 0$ for every $t \in \mathbb{R}$. The support of the function f is denoted by $\operatorname{supp} f = \overline{\{x : f(x) \neq 0\}}$.

The term *Sobolev function* is used for a function of class $W^{1,1}_{\mathrm{loc}}(\Omega)$. Similarly we define Sobolev mappings. A Sobolev mapping $f\colon \Omega \to \Omega'$ is a *Sobolev homeomorphism* if f is homeomorphic and onto Ω'.

It is well-known that the *area formula*

$$\int_A J_f(x)\,dx = |f(A)| \tag{2.1}$$

holds for every quasiconformal mapping f and measurable set $A \subset \mathbb{R}^n$ (see e.g. [IM, Theorem 16.13.4] or [K, Remark 6.1]). Moreover, it is valid for every Sobolev homeomorphism on every set A on which the Luzin (N) condition holds. It is well-known that for each Sobolev mapping $f\colon \Omega \to \mathbb{R}^n$ we can find a set N such that $|N| = 0$ and f satisfies the Luzin (N) condition on $\Omega \setminus N$ (see e.g. [Ha]).

We denote by C a generic real constant which can change at each occurrence. Similarly we use c for a constant > 0.

2.1. Lorentz spaces. Let $1 \leq p < \infty$ and $1 \leq q \leq \infty$. The Lorentz space $L^{p,q}(\Omega)$ is defined as the class of all measurable functions (modulo equality a.e.) $g\colon \Omega \to \mathbb{R}^m$ for which the following "norm" is finite:

$$\|g\|_{L^{p,q}(\Omega)} = \begin{cases} \Big(\int_0^\infty s^{q-1}|\{x \in \Omega\colon |g(x)| > s\}|^{q/p}\,ds\Big)^{1/q} & \text{for } q < \infty,\\ \sup_{s>0} s|\{x \in \Omega\colon |g(x)| > s\}|^{1/p} & \text{for } q = \infty. \end{cases}$$

In fact, $\|\cdot\|_{L^{p,q}(\Omega)}$ is a genuine norm only for $1 \leq q \leq p$, but for $1 < p < q$ it still remains to be a quantity equivalent to a norm. It is well-known that for each $1 \leq p < \infty$ and $1 < q < \infty$ and for each function $g\colon \Omega \to \mathbb{R}$ we have

$$c\|g\|_{L^{p,\infty}(\Omega)} \leq \|g\|_{L^{p,q}(\Omega)} \leq C\|g\|_{L^{p,1}(\Omega)}. \tag{2.2}$$

For an introduction to Lorentz spaces see e.g. [SW].

The *Sobolev–Lorentz space* $WL^{p,q}(\Omega)$ is defined as the class of all Sobolev functions $f\colon \Omega \to \mathbb{R}$ such that $f \in L^{p,q}(\Omega)$ and $|Df| \in L^{p,q}(\Omega)$.

3. p-quasiconformality

Lemma 3.1. *Let $E \subset \mathbb{R}^n$ be a measurable set and $g\colon E \to [0,\infty]$ be an essentially unbounded measurable function. Then there exists an infinite sequence $\{U_k\}_{k\in\mathbb{N}}$ of pairwise disjoint open sets in $\mathbb{R}^n$ such that g is essentially unbounded on each $E \cap U_k$.*

Proof. Using the fact that g is essentially unbounded, it is not difficult to find $z \in \mathbb{R}^n$ and $r_j \searrow 0$ such that

$$|\{x \in E \cap B(z, r_j) \setminus B(z, r_{j+1}) : |g(x)| \geq j\}| > 0, \qquad j = 1, 2, \ldots.$$

Then we write

$$A_j = B(z, r_j) \setminus \overline{B}(z, r_{j+1})$$

and set

$$U_{k+1} = A_{2^k} \cup A_{3 \cdot 2^k} \cup A_{5 \cdot 2^k} \cup \cdots, \qquad k = 0, 1, 2, \ldots. \blacksquare$$

LEMMA 3.2. *Let $f \colon \Omega \to \Omega'$ be a Sobolev homeomorphism. Let $E \subset \Omega$ be a Borel measurable set of finite positive measure and $t > |f(E)|$. Then there exist a compact set $K \subset E$, an open set $G \subset \Omega'$ and a function $u \in \mathcal{C}_c^\infty(\Omega')$ such that*

$$G \supset f(E), \tag{3.1}$$

$$|K| \geq \frac{1}{2n+1}|E|, \tag{3.2}$$

$$|G| \leq t, \tag{3.3}$$

$$\operatorname{supp} u \subset G, \tag{3.4}$$

$$|\nabla u| \leq 2 \quad \text{in } G, \tag{3.5}$$

$$|D\ (u \circ f)| \geq \frac{1}{2n}|Df| \qquad \text{in } K. \tag{3.6}$$

Proof. Consider the sets

$$E_i = \left\{x \in E\colon |Df_i(x)| \geq \frac{1}{n}|Df(x)|\right\}$$

and take $i \in \{1, \ldots, n\}$ such that

$$|E_i| \geq \frac{1}{n}|E|.$$

Find a compact set $K_i \subset E_i$ such that

$$|K_i| > \frac{2}{2n+1}|E|$$

and an open set $G \subset \Omega'$ such that $f(E) \subset G$ and $|G| \leq t$. Let $\eta \in \mathcal{C}_c^\infty(\Omega')$ be a cut-off function such that $\operatorname{supp} \eta \subset G$, $0 \leq \eta \leq 1$ and $\eta = 1$ on $f(K_i)$. Find $m \in \mathbb{N}$ such that $\|D\eta\|_\infty \leq m$. Choose K to be one of the sets

$$K^{\sin} = \{x \in K_i : \cos^2(mf_i(x)) \geq 1/2\},$$
$$K^{\cos} = \{x \in K_i : \sin^2(mf_i(x)) \geq 1/2\}$$

such that

$$|K| \geq \frac{1}{2}|K_i| \geq \frac{1}{2n+1}|E|$$

and set

$$u(y)=\begin{cases}\frac{1}{m}\eta(y)\sin(my_i) & \text{if } K=K^{\sin},\\ \frac{1}{m}\eta(y)\cos(my_i) & \text{if } K=K^{\cos}.\end{cases}$$

Then the properties (3.1)–(3.4) follow directly from the construction. Inequality (3.5) follows easily by the product rule as $\|D\sin(my_i)\|_\infty\le m$ and $\|D\eta\|_\infty\le m$. For (3.6) we assume e.g. that $K=K^{\sin}$. Since $K\subset E_i$, for $x\in K$ we have

$$|D(u\circ f)(x)|=|\cos(mf_i(x))|\,|Df_i(x)|\ge\frac{1}{\sqrt{2}}\frac{1}{n}|Df(x)|\ge\frac{|Df(x)|}{2n}.\ \blacksquare$$

THEOREM 3.3. *Let $1\le p<\infty$ and $1\le q\le\infty$. Let $f\colon\Omega\to\Omega'$ be a Sobolev homeomorphism. Suppose that there is a constant C_f such that*

$$\|D(u\circ f)\|_{L^{p,q}(\Omega)}\le C_f\|Du\|_{L^{p,q}(\Omega')}\tag{3.7}$$

for every $u\in\mathcal{C}_c^\infty(\Omega')$. Then there exists $Q\in\mathbb{R}$ such that

$$|Df(x)|^p\le Q|J_f(x)|\qquad\textit{for a.e. } x\in\Omega.\tag{3.8}$$

Proof. Fix a point $z\in\Omega$ of approximate differentiability of f and of approximate continuity of Df. Moreover, we assume that $Df(z)\neq 0$, otherwise there is nothing to prove. Choose $\varepsilon>0$ and find $\delta>0$ with $B(z,\delta)\subset\Omega$ such that for each $0<r<\delta$ we have

$$\big|\big\{x\in B(z,r):|Df(x)|<\tfrac12|Df(z)|\text{ or }|J_f(x)|>|J_f(z)|+\varepsilon\big\}\big|<\tfrac12|B(z,r)|.$$

Consider $r\in(0,\delta)$ and set

$$\tilde E=\big\{x\in B(z,r):|Df(x)|\ge\tfrac12|Df(z)|\text{ and }|J_f(x)|\le|J_f(z)|+\varepsilon\big\}.\tag{3.9}$$

Then we pass to a Borel measurable subset $E\subset\tilde E$ of full measure in $\tilde E$ such that f fulfills the Luzin (N) condition on E. Using Lemma 3.2 we find a compact set $K\subset E$, an open set $G\subset\Omega$ and a test function $u\in\mathcal{C}_c^\infty(\Omega')$ such that the properties (3.1)–(3.6) are satisfied with $t=|f(E)|+\varepsilon|E|$. Then $u=0$ on $\Omega'\setminus G$, $|Du|\le 2$ a.e., and thus (with the aid of (2.2))

$$\|Du\|_{L^{p,q}}\le C\|Du\|_{L^{p,1}}\le C\int_0^2|G|^{1/p}\,ds\le C|G|^{1/p}\le Ct^{1/p}.$$

Using (2.2) and the morphism property (3.7), we obtain

$$\|D(u\circ f)\|^p_{L^{p,\infty}}\le C\|D(u\circ f)\|^p_{L^{p,q}}\le CC_f^p\|Du\|^p_{L^{p,q}}\le CC_f^p t.\tag{3.10}$$

This means that for each $s>0$ we have

$$s^p\big|\big\{x\in\Omega:|D(u\circ f)(x)|>s\big\}\big|\le Ct.\tag{3.11}$$

(Here and below we do not indicate the dependence of the generic constant C on C_f.) In particular, it is useful to put $s:=\frac{1}{4n}|Df(z)|$. Indeed, by (3.6),

$$|D(u\circ f)(x)|\ge\frac{1}{4n}|Df(z)|=s,\qquad x\in K.\tag{3.12}$$

From (3.11) and (3.12) we obtain

$$|Df(z)|^p|K| \le Ct = C(|f(E)| + \varepsilon|E|). \tag{3.13}$$

Now, in view of the condition (N) on E we can use the area formula (2.1) and (3.9) to obtain

$$|f(E)| = \int_E |J_f(x)|\,dx \le |E|(|J_f(z)| + \varepsilon). \tag{3.14}$$

From (3.13), (3.14) and (3.2) we easily infer that

$$|Df(z)|^p \le C\frac{|E|(|J_f(z)| + 2\varepsilon)}{|K|} \le C(|J_f(z)| + 2\varepsilon).$$

Letting $\varepsilon \to 0$ we conclude that

$$|Df(z)|^p \le C|J_f(z)|. \ \blacksquare$$

Proof of Theorem 1.2. We assume that f is not p-quasiconformal, i.e. does not satisfy (3.8). Our aim is to construct $u \in W_0L^{p,q}(\Omega')$ such that $u \circ f \notin WL^{p,q}(\Omega)$. Set

$$g(x) = \begin{cases} |Df(x)|^p/|J_f(x)|, & J_f(x) \ne 0, \\ \infty, & J_f(x) = 0,\ Df(x) \ne 0, \\ 1, & Df(x) = 0. \end{cases}$$

Then the function g is essentially unbounded. By Lemma 3.1, there exist pairwise disjoint open sets $U_k \subset \Omega$, $k = 1, 2, \ldots$, such that g is essentially unbounded on each U_k. We know that f is not p-quasiconformal on U_k and hence the assumptions of Theorem 3.3 cannot be satisfied there. It follows that we can construct $u_k \in \mathcal{C}_c^\infty(f(U_k))$ such that

$$\|Du_k\|_{L^{p,q}(f(U_k))} \le 2^{-k} \quad \text{and} \quad \|D(u_k \circ f)\|_{L^{p,q}(U_k)} \ge 2^k.$$

We extend the domain of the functions u_k by putting $u_k = 0$ on $\Omega' \setminus U_k$. Set

$$u = \sum_{k=1}^{\infty} u_k.$$

Then the sum converges in the norm of $W_0L^{p,q}(\Omega')$. Now, assume that the function $u \circ f$ is a Sobolev function on Ω, otherwise there is nothing to prove. Then $D(u \circ f) = D(u_k \circ f)$ a.e. in U_k and it easily follows that $D(u \circ f) \notin L^{p,q}(\Omega)$. $\blacksquare$

4. Bilipschitz property

THEOREM 4.1. *Let $1 \le q \le \infty$, $q \ne n$. Let $f\colon \Omega \to \Omega'$ be a Sobolev homeomorphism. Assume that*

$$\|D(u \circ f)\|_{L^{n,q}(\Omega)} \le C\|Du\|_{L^{n,q}(\Omega')} \tag{4.1}$$

for every $u \in \mathcal{C}_c^\infty(\Omega')$. Then f is a locally bilipschitz mapping.

Proof. We already know from Theorem 3.3 that there exists a constant Q such that

$$|Df(x)|^n \le Q|J_f(x)|. \tag{4.2}$$

The reverse inequality

$$|J_f(x)| \le |Df(x)|^n \tag{4.3}$$

holds always. Thus, f is quasiconformal or antiquasiconformal. We may assume that f is quasiconformal; then it follows that $J_f > 0$ a.e. If there are $0 < c < C$ such that $c \le J_f \le C$ a.e., then we may use (4.3) and (4.2) to show that

$$c \le |Df|^n \le QC \quad \text{a.e.}$$

Since f is quasiconformal, f satisfies the (N) condition and thus for a.e. $y = f(x) \in \Omega'$ we have

$$|Df^{-1}(y)| = |(Df(x))^{-1}| \le \frac{|Df(x)|^{n-1}}{J_f(x)} \le \frac{(QC)^{(n-1)/n}}{c}.$$

This is enough to conclude that f is locally bilipschitz.

Now, assume for a contradiction that J_f is not essentially bounded by C from above or by c from below. Choose $k \in \mathbb{N}$. Then there exist $0 < a_1 < a_2 < \cdots < a_k$ and measurable sets A_j, $j = 1, \ldots, k$, such that $|A_j| > 0$ and

$$2^n a_{j-1} < a_j, \qquad j = 2, \ldots, k, \tag{4.4}$$

$$a_j \le J_f \le 2a_j \quad \text{on } A_j, \qquad j = 1, \ldots, k. \tag{4.5}$$

Choose positive constants $\mu_j < |A_j|$ and λ_j, $j = 1, \ldots, k$, to be specified later. Find compact sets $E_j \subset A_j$ such that

$$\mu_j = |E_j|. \tag{4.6}$$

Since E_j are pairwise disjoint, there exist pairwise disjoint "separating" open sets $W_j \subset \Omega$ such that $E_j \subset W_j$. Since

$$|f(E_j)| = \int_{E_j} |J_f(x)|\, dx > 0,$$

we can set

$$t_j = 2|f(E_j)| > |f(E_j)|.$$

According to Lemma 3.2 we construct compact sets $K_j \subset E_j$, open sets $G_j \subset f(W_j)$ and functions $u_j \in \mathcal{C}_c^\infty(f(\Omega))$ such that for $j = 1, \ldots, k$ we have

$$G_j \supset f(E_j), \tag{4.7}$$

$$|K_j| \ge \frac{1}{2n+1}|E_j|, \tag{4.8}$$

$$|G_j| \le t_j, \tag{4.9}$$

$$\operatorname{supp} u_j \subset G_j, \tag{4.10}$$

$$|Du_j| \le 2 \quad \text{in } G, \tag{4.11}$$

$$|D(u_j \circ f)| \ge \frac{1}{2n}|Df| \quad \text{in } K_j. \tag{4.12}$$

Set

$$u = \sum_{j=1}^{k} \lambda_j u_j.$$

Now we distinguish three cases.

Case A: $1 \le q < n$. We specify

$$\lambda_j = \lambda, \quad \mu_j = \frac{1}{a_j \lambda^n}, \qquad j = 1, \ldots, k, \tag{4.13}$$

where $\lambda > 0$ is chosen so large that $\mu_j < |A_j|$ for each j. From (4.9), (4.6) and (4.5) we obtain

$$|G_j| \le 2|f(E_j)| = 2 \int_{E_j} J_f(x)\, dx \le 4a_j \mu_j = 4\lambda^{-n}.$$

Hence by (4.10), (4.13) and (4.11),

$$|\{|Du| > s\}| \le \begin{cases} \sum_{j=1}^{k} |G_j| \le 4k\lambda^{-n}, & 0 < s < 2\lambda, \\ 0, & s \ge 2\lambda. \end{cases}$$

It follows that

$$\|Du\|^q_{L^{n,q}} \le \int_0^{2\lambda} s^{q-1} (4k\lambda^{-n})^{q/n}\, ds \le Ck^{q/n}. \tag{4.14}$$

On the other hand, from $|Df|^n \ge J_f$, (4.12) and (4.5), on K_j we have

$$|D(u \circ f)|^n = \lambda^n |D(u_j \circ f)|^n \ge \lambda^n \frac{1}{(2n)^n} |Df|^n \ge \frac{\lambda^n a_j}{(2n)^n},$$

so that

$$\big|\{|D(u \circ f)| > s\}\big| \ge |K_j|, \qquad 0 \le s < s_j := \frac{\lambda}{2n} a_j^{1/n}.$$

By (4.8), (4.6) and (4.13),

$$|K_j| \ge \frac{1}{2n+1}|E_j| = \frac{1}{(2n+1)a_j \lambda^n} \ge cs_j^{-n}. \tag{4.15}$$

With the convention $s_0 = 0$, from (4.4) we infer that

$$s_{j-1} \le \tfrac{1}{2} s_j, \qquad j = 1, \dots, k.$$

From (4.15) it follows that

$$\|D(u \circ f)\|^q_{L^{n,q}} \ge \sum_{j=1}^{k} \int_{s_{j-1}}^{s_j} s^{q-1} |K_j|^{q/n}\, ds \ge c \sum_{j=1}^{k} s_j^q |K_j|^{q/n} \ge ck. \tag{4.16}$$

Comparing (4.14) and (4.16) we observe that (4.1) leads to $k \le Ck^{q/n}$ with a constant independent of k. Since we can construct a corresponding function u for an arbitrary $k \in \mathbb{N}$, this is a contradiction.

Case B: $n < q < \infty$. We specify

$$\lambda_j = (\mu a_j)^{-1/n}, \quad \mu_j = \mu, \qquad j = 1, \dots, k, \tag{4.17}$$

where $\mu > 0$ is chosen so small that $\mu < |A_j|$ for each j. From (4.9), (4.5) and (4.6) we obtain

$$|G_j| \le 2|f(E_j)| = 2 \int_{E_j} J_f(x)\, dx \le 4 a_j \mu.$$

By (4.4), $a_i \le 2^{n(j-i)} a_j$, $i = 1, \dots, j$, and thus

$$\big|\{|Du| > s\}\big| \le \begin{cases} \displaystyle\sum_{i=1}^{j} |G_i| \le C \sum_{i=1}^{j} \mu a_i \le C\mu a_j, & 0 < s < 2\lambda_j, \\ 0, & s \ge 2\lambda_1. \end{cases} \tag{4.18}$$

From (4.17) it follows that

$$\|Du\|^q_{L^{n,q}} \le C \sum_{j=1}^{k} \int_0^{2\lambda_j} s^{q-1} (\mu a_j)^{q/n}\, ds \le C \sum_{j=1}^{k} \lambda_j^q (\mu a_j)^{q/n} \le Ck. \tag{4.19}$$

On the other hand, as $|Df|^n \ge J_f$, by (4.12), (4.5) and (4.17), on K_j we have

$$|D(u \circ f)|^n = \lambda_j^n |D(u_j \circ f)|^n \ge \frac{\lambda_j^n}{(2n)^n} |Df|^n \ge \frac{\lambda_j^n a_j}{(2n)^n} = \frac{1}{(2n)^n \mu}.$$

By (4.8) and (4.6),

$$|K_j| \ge \frac{1}{2n+1} |E_j| \ge c\mu.$$

Consequently,

$$\big|\{|D(u \circ f)| > s\}\big| \ge \sum_{j=1}^{k} |K_j| \ge ck\mu, \qquad 0 \le s < \bar{s} := \frac{1}{2n\mu^{1/n}}. \tag{4.20}$$

Hence

$$\|D(u\circ f)\|^q_{L^{n,q}} \ge c\int_0^{\bar s} s^{q-1}(k\mu)^{q/n}\,ds \ge c\bar s^q(k\mu)^{q/n} \ge ck^{q/n}. \tag{4.21}$$

Comparing (4.19) and (4.21) we observe that (4.1) leads to $k^{q/n} \le Ck$, which is again a contradiction.

Case C: $q = \infty$. Again, we specify the choice of λ_j and μ_j by (4.17). From (4.18) we obtain

$$s\big|\{|Du| > s\}\big|^{1/n} \le \begin{cases} C\lambda_j(\mu a_j)^{1/n} \le C, & 0 < s \le 2\lambda_j,\ j = 1,\dots,k,\\ 0, & s \ge 2\lambda_1,\end{cases}$$

and thus

$$\|Du\|_{L^{n,\infty}} \le C. \tag{4.22}$$

From (4.20) we see that

$$\bar s\big|\big\{|D(u\circ f)| > \tfrac12\bar s\big\}\big|^{1/n} \ge ck^{1/n}.$$

Hence

$$\|D(u\circ f)\|_{L^{n,\infty}} \ge Ck^{1/n}. \tag{4.23}$$

Once more, comparing (4.22) and (4.23) we arrive at a contradiction. ■

Proof of Theorem 1.1. We already know from Theorem 1.2 that $|Df(x)|^n \le Q|J_f(x)|$. Thus, f is quasiconformal or antiquasiconformal. Hence it is enough to show that J_f is bounded from above by some c and from below by some $1/c$. Now, we proceed similarly to the proof of Theorem 1.2 with $p = n$ but instead of Theorem 3.3 we use Theorem 4.1. ■

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