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Orthogonally additive mappings on Hilbert modules

by

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Abstract. We study the representation of orthogonally additive mappings acting on Hilbert C^*-modules and Hilbert H^*-modules. One of our main results shows that every continuous orthogonally additive mapping f from a Hilbert module W over $\mathcal{K}(\mathcal{H})$ or $\mathcal{HS}(\mathcal{H})$ to a complex normed space is of the form $f(x) = T(x) + \Phi(\langle x, x\rangle)$ for all $x \in W$, where T is a continuous additive mapping, and Φ is a continuous linear mapping.

1. Introduction. Let $\mathcal{A}$ be a C^*-algebra or an H^*-algebra, $(W, \langle\cdot,\cdot\rangle)$ be a Hilbert module over $\mathcal{A}$, and G be a complex normed space. A continuous mapping $f : W \to G$ is said to be *orthogonally additive* if for all $x, y \in W$,

$$\langle x, y\rangle = 0 \;\Rightarrow\; f(x + y) = f(x) + f(y).$$

In this paper, we study the representation of orthogonally additive mappings. If $T : W \to G$ and $\Phi : \mathcal{A} \to G$ are continuous additive mappings, then clearly the mapping $f : W \to G$ defined by

$$f(x) = T(x) + \Phi(\langle x, x\rangle) \quad \text{for all } x \in W \tag{1.1}$$

is a continuous orthogonally additive mapping. One of our main goals is to show that the converse also holds true if $\mathcal{A}$ is a C^*-algebra of compact operators or an H^*-algebra. In particular, this answers [23, Problem 27] affirmatively, not only for Hilbert H^*-modules, but for Hilbert C^*-modules over a C^*-algebra of compact operators as well. Other related problems in [23] have been also solved in [11, 12, 13].

Orthogonally additive mappings have been extensively studied from many aspects. See the survey [17] and the references therein for the representation of orthogonally additive mappings on orthogonality spaces. Refer to [20, 21, 22] for the connection between the existence of even orthogonally ad-

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ditive mappings and inner product spaces. Recently, several mathematicians have obtained some interesting results on orthogonally additive polynomials. See, e.g., [7, 9, 10, 16].

The rest of this paper is organized as follows. In Section 2, we give some necessary background and set up some notation. Section 3 deals with $\perp$-additive mappings on abelian groups. The context there is very general, so the results there may be also useful in future. Applying the results of Section 3, we obtain in Section 4 the representation of orthogonally additive mappings in general Hilbert modules. In Section 5, we strengthen the results of Section 4 in the case when $\mathcal{A}$ is $\mathcal{K}(\mathcal{H})$ or $\mathcal{HS}(\mathcal{H})$. The main result is then generalized in Section 6 to any Hilbert module over a C^*-algebra of compact operators or H^*-algebra. In the last section, we obtain the representation of orthogonally additive mappings on $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

Let us end the introduction by the following remark: Although some results look very similar, there are many subtleties that make them conceptually different.

2. Preliminaries. In this section, we give some necessary background and set up our notation.

2.1. Hilbert C^*-modules and Hilbert H^*-modules. Hilbert modules arise as generalizations of a complex Hilbert space when the complex field is replaced by a C^*-algebra or an H^*-algebra. The idea of replacing the complex numbers by the elements of a C^*-algebra first appeared in the work of Kaplansky [14], and by the elements of a proper H^*-algebra in the work of Saworotnow [18].

A C^*-*algebra* is a complex Banach $*$-algebra $(\mathcal{A}, \|\cdot\|)$ such that $\|a^*a\| = \|a\|^2$ for all $a \in \mathcal{A}$. An H^*-*algebra* is a complex Banach $*$-algebra $(\mathcal{A}, \|\cdot\|)$ whose underlying Banach space is a Hilbert space with respect to the inner product $\langle \cdot, \cdot \rangle$ satisfying $\langle ab, c\rangle = \langle b, a^*c\rangle$ and $\langle ba, c\rangle = \langle b, ca^*\rangle$ for all $a, b, c \in \mathcal{A}$. The *trace-class* associated with an H^*-algebra $\mathcal{A}$ is defined as the set $\tau(\mathcal{A}) = \{ab : a, b \in \mathcal{A}\}$; it is a self-adjoint two-sided ideal of $\mathcal{A}$ which is dense in $\mathcal{A}$.

Some examples of C^*-algebras are $\mathcal{B}(\mathcal{H})$ and $\mathcal{K}(\mathcal{H})$, the algebras of all bounded linear operators, resp. all compact operators, on some complex Hilbert space $\mathcal{H}$. An example of an H^*-algebra is $\mathcal{HS}(\mathcal{H})$, the algebra of all Hilbert–Schmidt operators on $\mathcal{H}$.

An element a in a C^*-algebra $\mathcal{A}$ is called *positive* ($a \geq 0$) if it is self-adjoint and has nonnegative spectrum. An element a in an H^*-algebra $\mathcal{A}$ is called *positive* ($a \geq 0$) if $\langle ax, x\rangle \geq 0$ for all $x \in \mathcal{A}$. If $\mathcal{A}$ is a C^*-algebra (resp. an H^*-algebra) then every positive $a \in \mathcal{A}$ (resp. $a \in \tau(A)$) can be written as $a = b^*b$ for some $b \in \mathcal{A}$. Every $a \in \mathcal{A}$ can be written as a linear combination of four positive elements, in both structures. A projection (i.e., self-adjoint idempotent) $e \in \mathcal{A}$ is called *minimal* if $e\mathcal{A}e = \mathbb{C}e$.

Let $\mathcal{A}$ be a C^*-algebra or an H^*-algebra. Let W be an algebraic right $\mathcal{A}$-module which is a complex linear space with a compatible scalar multiplication, i.e. $\lambda(xa) = (\lambda x)a = x(\lambda a)$ for all $x \in W$, $a \in \mathcal{A}$, $\lambda \in \mathbb{C}$. The space W is called a (*right*) *inner product* $\mathcal{A}$*-module* if there exists a *generalized inner product*, that is, a mapping $\langle \cdot, \cdot \rangle$ from $W \times W$ to $\mathcal{A}$ if $\mathcal{A}$ is a C^*-algebra, and to $\tau(A)$ if $\mathcal{A}$ is an H^*-algebra, having the following properties:

(i) $\langle x, y+z \rangle = \langle x, y \rangle + \langle x, z \rangle$ for all $x, y, z \in W$,
(ii) $\langle x, ya \rangle = \langle x, y \rangle a$ for all $x, y \in W$ and $a \in \mathcal{A}$,
(iii) $\langle x, y \rangle^* = \langle y, x \rangle$ for all $x, y \in W$,
(iv) $\langle x, x \rangle \geq 0$ for all $x \in W$, and $\langle x, x \rangle = 0 \Leftrightarrow x = 0$.

If W is an inner product module over $(\mathcal{A}, \|\cdot\|)$, then for $x \in W$ we write $\|x\|_W = \|\langle x, x \rangle\|^{1/2}$. If W is complete with respect to this norm, then it is called a *Hilbert* $\mathcal{A}$*-module*, or a *Hilbert* C^**-module* (resp. H^**-module*) over the C^*-algebra (resp. H^*-algebra) $\mathcal{A}$.

We shall use the symbol $\langle W, W \rangle$ for the linear span of all inner products $\langle x, y \rangle$, $x, y \in W$. A Hilbert $\mathcal{A}$-module W is *full* if $\langle W, W \rangle$ is dense in $\mathcal{A}$. Notice that $\mathcal{A}$ is a (full) Hilbert $\mathcal{A}$-module via $\langle x, y \rangle = x^* y$ for all $x, y \in \mathcal{A}$.

Let $w \in W$. If $\langle w, w \rangle = e$ is a projection in $\mathcal{A}$, then $we = w$. Indeed,

$$\langle w - we, w - we \rangle = \langle w, w \rangle - \langle w, w \rangle e - e\langle w, w \rangle + e\langle w, w \rangle e = 0$$

(see the paragraph before Lemma 1 in [6]). This property will be used frequently later.

The main difference between Hilbert C^*-modules and Hilbert H^*-modules is the fact that Hilbert H^*-modules can be equipped with the structure of a complex Hilbert space. Although both structures obey the same axioms as ordinary Hilbert spaces (except that the inner product takes values in a more general structure than the field of complex numbers), there are some properties that differentiate Hilbert C^*-modules from Hilbert spaces. For example, a closed submodule V of a Hilbert C^*-module W need not be (orthogonally) complemented, that is, $V \oplus V^\perp \neq W$ in general, where $V^\perp$ denotes $\{x \in W : \langle x, y \rangle = 0 \text{ for all } y \in V\}$; more details can be found in e.g. [15]. However, Hilbert C^*-modules over compact operators share many nice properties with Hilbert spaces; in particular all closed submodules of such modules are complemented.

We shall deal with Hilbert C^*-modules over C^*-algebras of compact operators, and Hilbert H^*-modules. These structures have orthonormal bases. More precisely, if W is a Hilbert $\mathcal{A}$-module, where $\mathcal{A}$ is a C^*-algebra of compact operators or an H^*-algebra, then there exists a net $\{w_i : i \in I\}$ generating a dense submodule of W, such that $\langle w_i, w_i \rangle$ is a minimal projection in $\mathcal{A}$, and $\langle w_i, w_j \rangle = 0$ for $i \neq j$. All these orthonormal bases have the same cardinal number, which is called the *orthogonal dimension* of W, denoted by $\dim_{\mathcal{A}} W$. More details on orthonormal bases for Hilbert C^*-modules over C^*-algebras of compact operators can be found in [6], and for Hilbert H^*-modules in [8].

2.2. Notation and conventions. Let W be a Hilbert C^*-module (resp. H^*-module) over a C^*-algebra (resp. an H^*-algebra) $\mathcal{A}$. We simply use "Hilbert $\mathcal{A}$-module" or "Hilbert module over $\mathcal{A}$" to refer to either of them.

If $(\mathcal{H}, (\cdot,\cdot))$ is a Hilbert space and $\xi, \eta \in \mathcal{H}$, then we write $\xi \otimes \eta$ for the rank one operator defined by $(\xi \otimes \eta)(\nu) = (\nu, \eta)\xi$ for all $\nu \in \mathcal{H}$.

All spaces are assumed to be over complex numbers.

If G is an abelian group, we always use "+" as its group operation.

"Orthogonally additive mapping(s)" are abbreviated as "o.a.m.".

Finally, if W is a group (resp. normed space, Hilbert module), then by $W_0 \leq W$ we mean that W_0 is a subgroup (resp. subspace, submodule) of W.

3. $\perp$-additive mappings on abelian groups. Let W and G be abelian groups. Suppose that $\perp$ is a binary relation on W. We say that a mapping $f : W \to G$ is *$\perp$-additive* if for all $x, y \in W$,

$$x \perp y \;\Rightarrow\; f(x+y) = f(x) + f(y),$$

and that a mapping $F : W \times W \to G$ is *$\perp$-preserving* if for all $x, y \in W$,

$$x \perp y \;\Rightarrow\; F(x,y) = 0.$$

Let us recall that a mapping $T : W \to G$ is called *additive* if $T(x+y) = T(x) + T(y)$ for all $x, y \in W$, a mapping $B : W \times W \to G$ is called *biadditive* if it is additive in both variables, a mapping $Q : W \to G$ is called *quadratic* if $Q(x+y) + Q(x-y) = 2Q(x) + 2Q(y)$ for all $x, y \in W$, and if W is a complex vector space then a mapping $S : W \times W \to G$ is called *sesquilinear* if it is linear in the first variable and conjugate linear in the second.

LEMMA 3.1. *Let W be an abelian group with a binary relation $\perp$, and V, G be uniquely 2-divisible abelian groups. Suppose that there exist additive mappings $\varphi, \psi : V \to W$ with the following properties:*

$$\varphi(V) \perp \psi(V) \quad \textit{and} \quad (\varphi + \psi)(V) \perp (\varphi - \psi)(V). \tag{3.1}$$

Let $W_0 := \phi(V) + \psi(V) \leq W$. If $f : W \to G$ is a $\perp$-additive mapping, then:

(i) *If f is odd (resp. even), then f is additive (resp. quadratic) on W_0.*
(ii) *If $x \perp y$ implies $(-x) \perp (-y)$, then there exist mappings $T : W \to G$ and $B : W \times W \to G$ such that T is additive on W_0, B is $\perp$-preserving symmetric biadditive on $W_0 \times W_0$, and*

$$f(x) = T(x) + B(x,x) \quad \textit{for all } x \in W_0.$$

Proof. (i) Since φ and ψ are additive, clearly they are odd. Now using the $\perp$-additivity of f, additivity of φ and ψ, as well as the properties given in (3.1), we obtain

$$\begin{aligned}(3.2)\qquad & f(\varphi(x)+\varphi(y))+f(\psi(x)-\psi(y))\\ &= f(\varphi(x+y))+f(\psi(x-y)) = f(\varphi(x+y)+\psi(x-y))\\ &= f\big(\varphi(x)+\varphi(y)+\psi(x)-\psi(y)\big) = f\big((\varphi+\psi)(x)+(\varphi-\psi)(y)\big)\\ &= f((\varphi+\psi)(x))+f((\varphi-\psi)(y)) = f(\varphi(x)+\psi(x))+f(\varphi(y)-\psi(y))\\ &= f(\varphi(x))+f(\psi(x))+f(\varphi(y))+f(-\psi(y))\end{aligned}$$

for all $x, y \in V$.

First assume that f is odd. Switching x and y in (3.2) gives

$$\begin{aligned}(3.3)\qquad & f(\varphi(x)+\varphi(y))+f(\psi(y)-\psi(x))\\ &\qquad\qquad = f(\varphi(y))+f(\psi(y))+f(\varphi(x))+f(-\psi(x)).\end{aligned}$$

Add (3.2) and (3.3) and use the fact that f is odd to get

$$2f(\varphi(x)+\varphi(y)) = 2f(\varphi(x))+2f(\varphi(y)),$$

or equivalently,

$$f(\varphi(x)+\varphi(y)) = f(\varphi(x))+f(\varphi(y))$$

as G is uniquely 2-divisible. Thus f is additive on $\varphi(V)$. Then (3.2) reduces to

$$f(\psi(x)-\psi(y)) = f(\psi(x))+f(-\psi(y)),$$

that is,

$$f(\psi(x)+\psi(y)) = f(\psi(x))+f(\psi(y))$$

as ψ is odd. Hence, f is additive on $\psi(V)$ as well. It is now easy to verify that f is additive on W_0.

Now assume that f is even. Put $y = x$ in (3.2) to get

$$(3.4)\qquad f(2\varphi(x))+f(0) = 2f(\varphi(x))+2f(\psi(x)),$$

then put $y = -x$ in (3.2) to get

$$(3.5)\qquad f(0)+f(2\psi(x)) = 2f(\varphi(x))+2f(\psi(x)).$$

Comparing (3.4) and (3.5) yields $f(2\varphi(x)) = f(2\psi(x))$, that is, $f(\varphi(2x)) = f(\psi(2x))$ for all $x \in V$. Since V is uniquely 2-divisible, we have

$$(3.6)\qquad f(\varphi(x)) = f(\psi(x)) \quad \text{for all } x \in V.$$

Then $f(\psi(x)-\psi(y)) = f(\varphi(x)-\varphi(y))$ for all $x, y \in V$. This together with (3.2) implies

$$f(\varphi(x)+\varphi(y))+f(\varphi(x)-\varphi(y)) = 2f(\varphi(x))+2f(\varphi(y)),$$

so f is quadratic on $\varphi(V)$. Hence it follows from (3.6) that f is also quadratic on $\psi(V)$. Therefore f is quadratic on W_0.

(ii) Set

$$T(x)=\tfrac{1}{2}(f(x)-f(-x)),\quad F(x)=\tfrac{1}{2}(f(x)+f(-x)),\qquad \text{for all } x\in W.$$

Then T is odd and $\perp$-additive. By (i), T is additive on W_0. Furthermore, F is even and $\perp$-additive. Again by (i), F is quadratic on W_0. Then $F(0)=0$, so

$$F(x+x)+F(x-x)=2F(x)+2F(x)$$

yields $F(2x)=4F(x)$ for all $x\in W_0$.

Define

$$B(x,y)=\tfrac{1}{4}(F(x+y)-F(x-y))\qquad \text{for all } x,y\in W.$$

Since F is even, B is symmetric. It is well-known that B is biadditive (on W_0), but below we prove this fact for the reader's convenience. Obviously, $B(x,0)=B(0,x)=0$ and $B(x,x)=\frac{1}{4}(F(2x)-F(0))=F(x)$ for all $x\in W_0$. Since F is quadratic on W_0, for all $x,y,u\in W_0$ we have

$$\begin{aligned}4B(x+y,2u)&=F(x+y+2u)-F(x+y-2u)\\&=F((x+u)+(y+u))+F((x+u)-(y+u))\\&\quad-F((x-u)-(y-u))-F((x-u)+(y-u))\\&=2(F(x+u)+F(y+u))-2(F(x-u)+F(y-u))\\&=8B(x,u)+8B(y,u).\end{aligned}$$

Since G is uniquely 2-divisible, this implies

$$B(x+y,2u)=2B(x,u)+2B(y,u). \tag{3.7}$$

Inserting $y=0$ and $x=z$ yields

$$B(z,2u)=2B(z,u).$$

If we put $x+y$ instead of z, using (3.7) we get

$$B(x+y,u)=B(x,u)+B(y,u).$$

Hence, B is biadditive on $W_0\times W_0$. Finally,

$$f(x)=T(x)+F(x)=T(x)+B(x,x)$$

for all $x\in W_0$. Notice that, for all $x,y\in W_0$, $x\perp y$ implies

$$\begin{aligned}2B(x,y)&=B(x,y)+B(y,x)=B(x+y,x+y)-B(x,x)-B(y,y)\\&=(f(x+y)-T(x+y))-(f(x)-T(x))-(f(y)-T(y))\\&=\big(f(x+y)-f(x)-f(y)\big)-\big(T(x+y)-T(x)-T(y)\big)=0.\end{aligned}$$

Hence $B(x,y)=0$, that is, B is $\perp$-preserving on $W_0\times W_0$. ■

Remark 3.2. Examining the definitions of T and B, it is easy to see that they are uniquely determined by f. Actually,

$$T(x) = \tfrac{1}{2}(f(x) - f(-x)),$$
$$B(x,y) = \tfrac{1}{8}\big(f(x+y) + f(-x-y) - f(x-y) - f(-x+y)\big)$$

for all $x, y \in W$. The reason why we only have

$$f(x) = T(x) + B(x,x)$$

for all $x \in W_0$, instead of W, is because it is only known that $F(x) = B(x,x)$ for all $x \in W_0$.

Lemma 3.3. *Let W, V, G be normed spaces, and $\perp$ be a binary relation on W such that $x \perp y$ implies $(-x) \perp (-y)$. Suppose that there are continuous linear mappings $\varphi, \psi : V \to W$ with the following properties:*

$$\varphi(V) \perp \psi(V) \quad \text{and} \quad (\varphi + \lambda\psi)(V) \perp (\varphi - \lambda\psi)(V) \quad \text{for } \lambda \in \{1, i\}. \tag{3.8}$$

Let $W_0 := \varphi(V) + \psi(V) \leq W$.

If $f : W \to G$ is a continuous $\perp$-additive mapping, then there exist continuous mappings $T : W \to G$ and $S : W \times W \to G$ such that T is additive on W_0, S is sesquilinear on $W_0 \times W_0$ with the property that for all $x, y \in W_0$, $x \perp y$ implies $S(x,y) + S(y,x) = 0$, and

$$f(x) = T(x) + S(x,x) \quad \text{for all } x \in W_0.$$

Furthermore, if we also assume that $x \perp y$ implies $x \perp iy$, then S is $\perp$-preserving on $W_0 \times W_0$.

Proof. By Lemma 3.1(ii), there exist mappings $T : W \to G$ and $B : W \times W \to G$ such that T is additive on W_0, B is $\perp$-preserving symmetric biadditive on $W_0 \times W_0$, and

$$f(x) = T(x) + B(x,x) \quad \text{for all } x \in W_0.$$

Since f is continuous, clearly so are T and B (see Remark 3.2).

Since ψ is linear and B is $\perp$-preserving on $W_0 \times W_0$, it follows from (3.8) that for all $x, y \in V$ and $\lambda \in \{1, i\}$ we have

$$\begin{aligned} 0 &= B\big((\varphi+\lambda\psi)(x), (\varphi-\lambda\psi)(y)\big) = B\big(\varphi(x)+\lambda\psi(x), \varphi(y)-\lambda\psi(y)\big) \\ &= B(\varphi(x),\varphi(y)) + B(\psi(\lambda x),\varphi(y)) - B(\varphi(x),\psi(\lambda y)) - B(\lambda\psi(x),\lambda\psi(y)) \\ &= B(\varphi(x),\varphi(y)) - B(\lambda\psi(x),\lambda\psi(y)). \end{aligned}$$

This implies

$$\begin{aligned} B(i\psi(x), i\psi(y)) &= B(\varphi(x),\varphi(y)) = B(\psi(x),\psi(y)), \\ B(i\varphi(x), i\varphi(y)) &= B(\varphi(ix),\varphi(iy)) = B(\psi(ix),\psi(iy)) \\ &= B(i\psi(x), i\psi(y)) = B(\varphi(x),\varphi(y)). \end{aligned}$$

Hence,

$$B(ix, iy) = B(x, y)$$

for all $x, y \in \varphi(V) + \psi(V) = W_0$.

Since B is biadditive and continuous on $W_0 \times W_0$, it is also $\mathbb{R}$-bilinear on $W_0 \times W_0$. Define $S : W \times W \to G$ by

$$S(x, y) = B(x, y) + iB(x, iy).$$

Then, for all $x, y \in W_0$,

$$\begin{aligned} S(ix, y) &= B(ix, y) + iB(ix, iy) = B(ix, y) + iB(x, y) \\ &= i(B(x, y) - iB(ix, y)) = i(B(x, y) + iB(x, iy)) = iS(x, y), \end{aligned}$$

and analogously

$$S(x, iy) = -iS(x, y).$$

Since the mapping B is continuous $\mathbb{R}$-bilinear on $W_0 \times W_0$, the mapping S is continuous $\mathbb{R}$-bilinear on $W_0 \times W_0$ as well. However, from the above we conclude that S is continuous sesquilinear on $W_0 \times W_0$.

Also, notice that

$$\begin{aligned} S(x, y) + S(y, x) &= B(x, y) + iB(x, iy) + B(y, x) + iB(y, ix) \\ &= 2B(x, y) + iB(x, iy) + iB(ix, y) = 2B(x, y) \end{aligned}$$

for all $x, y \in W_0$. In particular, we get

$$S(x, x) = B(x, x) \quad \text{for all } x \in W_0.$$

Then S is a continuous sesquilinear mapping on $W_0 \times W_0$ with the properties that for all $x, y \in W_0$,

$$x \perp y \;\Rightarrow\; S(x, y) + S(y, x) = 0$$

and

$$f(x) = T(x) + S(x, x) \quad \text{for all } x \in W_0.$$

Furthermore, if $x \perp y$ implies $x \perp iy$, then for all $x, y \in W_0$ satisfying $x \perp y$ we have

$$S(x, y) + S(y, x) = 0 \quad \text{and} \quad -iS(x, y) + iS(y, x) = 0.$$

Hence $S(x, y) = 0$, that is, S is $\perp$-preserving on $W_0 \times W_0$. ∎

By the definition of S and Remark 3.2, we conclude that S is also uniquely determined by f and

$$\begin{aligned} S(x, y) = \tfrac{1}{8}\big(&f(x + y) + if(x + iy) - f(x - y) - if(x - iy) \\ &+ f(-x - y) + if(-x - iy) - f(-x + y) - if(-x + iy)\big) \end{aligned}$$

for all $x, y \in W$.

It should also be mentioned that the mapping T is $\mathbb{R}$-linear on W_0 since it is continuous and additive on W_0, but it is not $\mathbb{C}$-linear in general.

4. O.a.m. on Hilbert modules. Let $(W, \langle\cdot,\cdot\rangle)$ be a Hilbert $\mathcal{A}$-module and let G be an abelian group. We shall study $\perp$-additive mappings and $\perp$-preserving mappings for the binary relation $\perp$ on W given by

$$x \perp y \;\Leftrightarrow\; \langle x, y\rangle = 0.$$

In this setting we shall use the term *orthogonally additive* instead of $\perp$-additive, and the term *orthogonality preserving* instead of $\perp$-preserving. More precisely, we shall say that a mapping $f : W \to G$ is orthogonally additive if

$$\langle x, y\rangle = 0 \;\Rightarrow\; f(x+y) = f(x) + f(y),$$

and that a mapping $B : W \times W \to G$ is orthogonality preserving if

$$\langle x, y\rangle = 0 \;\Rightarrow\; B(x, y) = 0.$$

A *morphism* between Hilbert $\mathcal{A}$-modules V and W is a mapping $\varphi : V \to W$ satisfying $\langle \varphi(x), \varphi(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$. It is clear that morphisms are continuous mappings, and it is not difficult to verify that they are also $\mathcal{A}$-linear mappings, that is, linear mappings satisfying $\varphi(xa) = \varphi(x)a$ for all $x \in V$ and $a \in \mathcal{A}$.

THEOREM 4.1. *Let W be a Hilbert $\mathcal{A}$-module, let V be a submodule of W, and let $\varphi : V \to W$ be a morphism such that $\varphi(V) \subseteq V^{\perp}$. Let $W_0 := V \oplus \varphi(V) \leq W$. Suppose that G is a uniquely 2-divisible abelian group and that $f : W \to G$ is an o.a.m. Then:*

(i) *There exist mappings $T : W \to G$ and $B : W \times W \to G$ such that T is additive on W_0, B is symmetric biadditive orthogonality preserving on $W_0 \times W_0$, and*

$$f(x) = T(x) + B(x, x) \quad \textit{for all } x \in W_0.$$

(ii) *If G is a normed space and f is continuous, then there exist continuous mappings $T : W \to G$ and $S : W \times W \to G$ such that T is additive on W_0, S is sesquilinear orthogonality preserving on $W_0 \times W_0$, and*

$$f(x) = T(x) + S(x, x) \quad \textit{for all } x \in W_0.$$

Proof. Set $x \perp y$ if and only if $\langle x, y\rangle = 0$. Let $\mathrm{id} : V \to V$ be the identity mappping. Since $\varphi(V) \subseteq V^{\perp}$, we have $\langle \varphi(V), \mathrm{id}(V)\rangle = 0$. Furthermore, for all $x, y \in V$, and $\lambda \in \{1, i\}$,

$$\langle (\varphi + \lambda\cdot\mathrm{id})(x), (\varphi - \lambda\cdot\mathrm{id})(y)\rangle = \langle \varphi(x), \varphi(y)\rangle - \langle x, y\rangle = 0.$$

Then (i) and (ii) follow from Lemma 3.1(ii) and Lemma 3.3, respectively. ∎

We write $U \sim V$ if U and V are unitarily equivalent Hilbert C^*-modules over a C^*-algebra $\mathcal{A}$, that is, if there exists a mapping $u : U \to V$ such that there is a mapping $u^* : V \to U$ satisfying $\langle u(x), y\rangle = \langle x, u^*(y)\rangle$ for all $x \in U$, $y \in V$, and

$$u^*u = \mathrm{id}_U, \qquad uu^* = \mathrm{id}_V .$$

It is clear that u is bijective and $\langle u(x), u(y)\rangle = \langle x, y\rangle$ for all $x, y \in U$.

A closed submodule V of a Hilbert C^*-module W is said to be *complemented* if $W = V \oplus V^\perp$, and *fully complemented* if V is complemented and $V^\perp \sim W$.

COROLLARY 4.2. *Let V be a fully complemented submodule of a Hilbert C^*-module W, G be a uniquely 2-divisible abelian group, and $f : W \to G$ be an o.a.m. Then there exist mappings $T : W \to G$ and $B : W \times W \to G$ such that T is additive on V, B is symmetric biadditive orthogonality preserving on $V \times V$, and*

$$f(x) = T(x) + B(x, x) \quad \text{for all } x \in V.$$

Furthermore, if G is a normed space and f is continuous then there exist a continuous mapping $T : W \to G$ which is additive on V, and a continuous mapping $S : W \times W \to G$ which is sesquilinear and orthogonality preserving on $V \times V$, such that

$$f(x) = T(x) + S(x, x) \quad \text{for all } x \in V.$$

Proof. Since V is a fully complemented submodule of W, there exists a linear operator $u : W \to V^\perp$ such that $\langle u(x), u(y)\rangle = \langle x, y\rangle$ for all $x, y \in W$. Set $\varphi = u|_V$. Then $\varphi : V \to V^\perp$ satisfies $\langle \varphi(x), \varphi(y)\rangle = \langle x, y\rangle$ for all $x, y \in V$. So it remains to apply Theorem 4.1. ∎

5. O.a.m. on Hilbert $\mathcal{K}(\mathcal{H})$-modules and Hilbert $\mathcal{HS}(\mathcal{H})$-modules. In this section, *$\mathcal{A}$ always denotes $\mathcal{K}(\mathcal{H})$ or $\mathcal{HS}(\mathcal{H})$*. Let W be a Hilbert $\mathcal{A}$-module and $e \in \mathcal{A}$ be a rank one projection in $\mathcal{A}$. Then there exists an orthonormal basis $\{w_i : i \in I\}$ for W such that $\langle w_i, w_i\rangle = e$ for all $i \in I$ (see [6, Remark 4(d)] for Hilbert $\mathcal{K}(\mathcal{H})$-modules, and [5, Proposition 1.5] for Hilbert $\mathcal{HS}(\mathcal{H})$-modules). The following lemma will allow us to deal with yet another suitable orthonormal basis for W.

LEMMA 5.1. *Let W be a Hilbert $\mathcal{A}$-module with $\dim_{\mathcal{A}} W \le \dim \mathcal{H}$, and $\{\xi_i : i \in I\}$ be an orthonormal basis for $\mathcal{H}$. Then there exists an orthonormal basis $\{w_i : i \in J \subseteq I\}$ for W such that $\langle w_i, w_i\rangle = \xi_i \otimes \xi_i$ for all $i \in J$.*

Proof. Let us fix an arbitrary $j_0 \in I$. Then there exists an orthonormal basis $\{g_i : i \in J\}$ for W such that $\langle g_i, g_i\rangle = \xi_{j_0} \otimes \xi_{j_0}$ for all $i \in J$. Since $\dim_{\mathcal{A}} W \le \dim \mathcal{H}$, we assume $J \subseteq I$. We define, for all $i \in J$,

$$w_i = g_i(\xi_{j_0} \otimes \xi_i).$$

Then $\langle w_i, w_j\rangle = 0$ if $i \neq j$, and

$$\begin{aligned}\langle w_i, w_i\rangle &= \langle g_i(\xi_{j_0} \otimes \xi_i), g_i(\xi_{j_0} \otimes \xi_i)\rangle = (\xi_i \otimes \xi_{j_0})\langle g_i, g_i\rangle(\xi_{j_0} \otimes \xi_i)\\ &= (\xi_i \otimes \xi_{j_0})(\xi_{j_0} \otimes \xi_{j_0})(\xi_{j_0} \otimes \xi_i) = \xi_i \otimes \xi_i\end{aligned}$$

for all $i \in J$. Furthermore, for all $x \in W$,

$$\begin{aligned}x &= \sum_{i\in J} g_i\langle g_i, x\rangle = \sum_{i\in J} g_i\langle g_i(\xi_{j_0} \otimes \xi_{j_0}), x\rangle = \sum_{i\in J} g_i(\xi_{j_0} \otimes \xi_{j_0})\langle g_i, x\rangle\\ &= \sum_{i\in J} g_i(\xi_{j_0} \otimes \xi_i)(\xi_i \otimes \xi_{j_0})\langle g_i, x\rangle\\ &= \sum_{i\in J} g_i(\xi_{j_0} \otimes \xi_i)\langle g_i(\xi_{j_0} \otimes \xi_i), x\rangle = \sum_{i\in J} w_i\langle w_i, x\rangle.\end{aligned}$$

By [6, Theorem 1], $\{w_i : i \in J\}$ is an orthonormal basis for W. ∎

REMARK 5.2. Let $\mathcal{H}$ be a Hilbert space with $\dim \mathcal{H} = \aleph_0$, W be a Hilbert $\mathcal{A}$-module such that $\dim_{\mathcal{A}} W = \aleph_0$, and $\{\xi_i : i \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{H}$. By Lemma 5.1 there exists an orthonormal basis $\{w_i : i \in \mathbb{N}\}$ for W such that $\langle w_i, w_i\rangle = \xi_i \otimes \xi_i$ for all $i \in \mathbb{N}$. Let $a \in \mathcal{A}$. Since

$$\begin{aligned}\Big\|\sum_{i=m}^{n} w_i a\Big\|_W^2 &= \Big\|\Big\langle \sum_{i=m}^{n} w_i a, \sum_{i=m}^{n} w_i a\Big\rangle\Big\|\\ &= \Big\|\sum_{i=m}^{n} a^*\langle w_i, w_i\rangle a\Big\| = \Big\|\sum_{i=m}^{n} a^*(\xi_i \otimes \xi_i)a\Big\|\end{aligned}$$

and $\sum_{i=1}^{\infty} a^*(\xi_i \otimes \xi_i)a = a^*a$, the sequence $(\sum_{i=1}^{n} w_i a)_{n=1}^{\infty}$ is a Cauchy sequence in W, so it converges. Hence, for all $a \in \mathcal{A}$, $\sum_{i=1}^{\infty} w_i a \in W$ and $\langle \sum_{i=1}^{\infty} w_i a, \sum_{i=1}^{\infty} w_i a\rangle = a^*a$.

Before giving the main result of this section, we provide a representation result of sesquilinear orthogonality preserving mappings $S : W \times W \to G$. This result may be of independent interest.

PROPOSITION 5.3. *Let W be a Hilbert $\mathcal{A}$-module and G be a normed space. If $S : W \times W \to G$ is a continuous sesquilinear orthogonality preserving mapping, then there is a unique linear mapping $\Phi : \langle W, W\rangle \to G$ such that*

$$S(x, y) = \Phi(\langle y, x\rangle) \quad \textit{for all } x, y \in W.$$

Furthermore, if $\mathcal{H}$ is finite-dimensional or $\dim \mathcal{H} = \dim_{\mathcal{A}} W = \aleph_0$, then Φ can be extended to a continuous linear mapping on $\mathcal{A}$.

Proof. Let $\{\xi_i : i \in I\}$ be an orthonormal basis for $\mathcal{H}$ and let $e_i = \xi_i \otimes \xi_i$ for all $i \in I$. Fix an arbitrary $i_0 \in I$. Let $\{w_j : j \in J\}$ be an orthonormal basis for W such that $\langle w_j, w_j\rangle = e_{i_0}$ for all $j \in J$. Then for all $j, k \in J$,

$j \neq k$, and all $a, b \in \mathcal{A}$ we have

$$\langle w_j a - w_k a, w_j b + w_k b\rangle = a^*\langle w_j, w_j\rangle b - a^*\langle w_k, w_k\rangle b = a^* e_{i_0} b - a^* e_{i_0} b = 0,$$

hence

$$0 = S(w_j a - w_k a, w_j b + w_k b) = S(w_j a, w_j b) - S(w_k a, w_k b).$$

In particular, for $b = e_{i_0}$, one obtains

$$S(w_j a, w_j) = S(w_j a, w_j e_{i_0}) = S(w_k a, w_k e_{i_0}) = S(w_k a, w_k)$$

for all $j, k \in J$. Hence the mapping $\Phi_{i_0} : \mathcal{A} \to G$, defined by

$$\Phi_{i_0}(a) = S(w_k a, w_k),$$

does not depend on $k \in J$. It is clear that Φ_{i_0} is linear. Notice that

$$\|\Phi_{i_0}(a)\| = \|S(w_k a, w_k)\| \leq \|S\|\,\|w_k\|_W^2 \|a\| = \|S\|\,\|e_{i_0}\|\,\|a\|. \tag{5.1}$$

Let us remark that if $\mathcal{A} = \mathcal{K}(\mathcal{H})$ then each (minimal) projection has norm one, which is not true in general if $\mathcal{A} = \mathcal{HS}(\mathcal{H})$, but in both cases Φ_{i_0} is bounded and $\|\Phi_{i_0}\| \leq \|S\|\,\|e_{i_0}\|$. Furthermore,

$$\begin{aligned}
\Phi_{i_0}(\langle y e_{i_0}, x\rangle) &= S(w_k\langle y e_{i_0}, x\rangle, w_k) = \sum_{j\in J} S(w_k\langle y e_{i_0}, w_j\rangle\langle w_j, x\rangle, w_k)\\
&= \sum_{j\in J} S(w_k e_{i_0}\langle y, w_j\rangle e_{i_0}\langle w_j, x\rangle, w_k) = \sum_{j\in J} S(w_k(\lambda_j e_{i_0})\langle w_j, x\rangle, w_k)\\
&= \sum_{j\in J} \lambda_j S(w_k\langle w_j, x\rangle, w_k) = \sum_{j\in J} S(w_k\langle w_j, x\rangle, w_k(\overline{\lambda}_j e_{i_0}))\\
&= \sum_{j\in J} S(w_k\langle w_j, x\rangle, w_k(e_{i_0}\langle w_j, y\rangle e_{i_0})) = \sum_{j\in J} S(w_k\langle w_j, x\rangle, w_k\langle w_j, y e_{i_0}\rangle)\\
&= \sum_{j\in J} S(w_j\langle w_j, x\rangle, w_j\langle w_j, y e_{i_0}\rangle) = S\Big(\sum_{j\in J} w_j\langle w_j, x\rangle, \sum_{j\in J} w_j\langle w_j, y e_{i_0}\rangle\Big)\\
&= S(x, y e_{i_0}).
\end{aligned}$$

Since $i_0 \in I$ is arbitrary, it follows that

$$S(x, y) = S\Big(x, \sum_{i\in I} y e_i\Big) = \sum_{i\in I} S(x, y e_i) = \sum_{i\in I} \Phi_i(\langle y e_i, x\rangle). \tag{5.2}$$

Define $\Phi : \langle W, W\rangle \to G$ by

$$\Phi(a) = \sum_{i\in I} \Phi_i(e_i a).$$

By (5.2), the mapping Φ is well-defined and

$$S(x, y) = \Phi(\langle y, x\rangle)$$

for all $x, y \in W$. Since all Φ_i are linear, Φ is linear as well. Uniqueness of such Φ is obvious.

If $\mathcal{H}$ is finite-dimensional, then $\sum_{i\in I}\Phi_i(e_i a)$ converges for all $a\in\mathcal{A}$. If $\dim\mathcal{H}=\dim_{\mathcal{A}}W=\aleph_0$, then by Lemma 5.1 there exists an orthonormal basis $\{v_i : i\in I\}$ for W such that $\langle v_i,v_i\rangle=\xi_i\otimes\xi_i$ for all $i\in I$. By Remark 5.2, $\sum_{j\in I}v_j a\in W$ for all $a\in\mathcal{A}$, so for all $a,b\in\mathcal{A}$ we have

$$\begin{aligned}S\Big(\sum_{j\in I}v_j b,\sum_{j\in I}v_j a^*\Big)&=\sum_{i\in I}\Phi_i\Big(\Big\langle\sum_{j\in I}v_j a^* e_i,\sum_{j\in I}v_j b\Big\rangle\Big)\\&=\sum_{i\in I}\Phi_i\Big(\sum_{j\in I}e_i a\langle v_j,v_j\rangle b\Big)=\sum_{i\in I}\Phi_i\Big(\sum_{j\in I}e_i a(\xi_j\otimes\xi_j)b\Big)=\sum_{i\in I}\Phi_i(e_i ab).\end{aligned}$$

Hence $\sum_{i\in I}\Phi_i(e_i a)$ converges for all $a\in\mathcal{A}^2=\mathcal{A}$. This means that in the cases when $\dim\mathcal{H}$ is finite or $\dim\mathcal{H}=\dim_{\mathcal{A}}W=\aleph_0$, we can extend Φ from $\langle W,W\rangle$ to $\mathcal{A}$ if we define

$$\Phi(a)=\sum_{i\in I}\Phi_i(e_i a).$$

Finally, let us prove that $\Phi:\mathcal{A}\to G$ is bounded. If $\mathcal{H}$ is finite-dimensional, this immediately follows from (5.1). Now assume that $\dim\mathcal{H}=\dim_{\mathcal{A}}W=\aleph_0$. Then for every $a\in\mathcal{A}$, by Remark 5.2, we have

$$\begin{aligned}\|\Phi(a^*a)\|&=\Big\|\Phi\Big(\Big\langle\sum_{i\in I}v_i a,\sum_{i\in I}v_i a\Big\rangle\Big)\Big\|=\Big\|S\Big(\sum_{i\in I}v_i a,\sum_{i\in I}v_i a\Big)\Big\|\\&\le\|S\|\Big\|\sum_{i\in I}v_i a\Big\|_W^2=\|S\|\Big\|\Big\langle\sum_{i\in I}v_i a,\sum_{i\in I}v_i a\Big\rangle\Big\|=\|S\|\,\|a^*a\|.\end{aligned}$$

Thus Φ is bounded on positive elements on $\mathcal{A}$. Therefore it is bounded on $\mathcal{A}$. ∎

We are now ready to prove our main result of this section.

Theorem 5.4. *Let W be a Hilbert $\mathcal{A}$-module such that* $\dim_{\mathcal{A}}W\ge 2$. *Let G be a uniquely 2-divisible abelian group, and $f:W\to G$ be an o.a.m. Then:*

(i) *There exist a unique additive mapping $T:W\to G$ and a unique symmetric biadditive orthogonality preserving mapping $B:W\times W\to G$ such that*

$$f(x)=T(x)+B(x,x)\quad\textit{for all } x\in W.$$

(ii) *If G is a normed space and f is continuous, then there are a unique continuous additive mapping $T:W\to G$ and a unique linear mapping $\Phi:\langle W,W\rangle\to G$ such that*

$$f(x)=T(x)+\Phi(\langle x,x\rangle)\quad\textit{for all } x\in W.$$

Furthermore, if $\mathcal{H}$ is finite-dimensional or $\dim\mathcal{H}=\dim_{\mathcal{A}}W=\aleph_0$, *then Φ can be extended to a continuous linear mapping on $\mathcal{A}$.*

Proof. (i) First assume that W is either finite-dimensional with $\dim_{\mathcal{A}} W = 2n$, or $\dim_{\mathcal{A}} W \geq \aleph_0$. If $\dim_{\mathcal{A}} W = 2n$ then let V be a closed submodule of W such that $\dim_{\mathcal{A}} W = n$; if $\dim_{\mathcal{A}} W \geq \aleph_0$ then let V be a closed submodule of W such that $\dim_{\mathcal{A}} V = \dim_{\mathcal{A}} V^{\perp} = \dim_{\mathcal{A}} W$. Let $\{w_i : i \in I\}$ be an orthonormal basis for W such that $\langle w_i, w_i \rangle = e$ for all $i \in I$, where e is a fixed rank one projection in $\mathcal{A}$. Let $\{w_i : i \in I_1 \subseteq I\}$ be an orthonormal basis for V and $\{w_i : i \in I_2 \subseteq I\}$ be an orthonormal basis for $V^{\perp}$. Let $\varphi : V \to V^{\perp}$ be an isomorphism between the bases of V and $V^{\perp}$. It remains to apply Theorem 4.1(i). Notice that $V \oplus \varphi(V) = V \oplus V^{\perp} = W$.

Now assume that W is finite-dimensional with $\dim_{\mathcal{A}} W = 2n+1$. From the above we deduce that the desired conclusion is true for f restricted to any $2n$-dimensional closed submodule of W. Let X be a closed submodule of W such that $\dim_{\mathcal{A}} X = 1$. Then $\dim_{\mathcal{A}} X^{\perp} = 2n$. Let Z be a closed submodule of W such that $\dim_{\mathcal{A}} Z = 2$, and $X \subset Z$. The statement is true on both Z and $X^{\perp}$, hence on W.

(ii) Combining the proofs of (i) above and Theorem 4.1(ii), we can find a unique continuous additive mapping $T : W \to G$ and a unique continuous sesquilinear orthogonality preserving mapping $S : W \times W \to G$ such that

$$f(x) = T(x) + S(x,x) \quad \text{for all } x \in W.$$

Then we apply Proposition 5.3 to end the proof. ∎

Let us emphasize that in Theorem 5.4 the additional assumption that f is odd implies that the corresponding mapping B (or Φ) must be zero, so f is additive. Analogously, if f is even then T is zero, thus $f(x) = B(x,x)$ (or $f(x) = \Phi(\langle x, x \rangle)$) for all $x \in W$. Recall that for each o.a.m. f the mapping $x \mapsto f(x) - f(-x)$ is an odd o.a.m. and the mapping $x \mapsto f(x) + f(-x)$ is an even o.a.m.

Furthermore, if we assume that in Theorem 5.4 the mapping f has the form $f(x) = F(x, \ldots, x)$, where F is n-additive (i.e., additive in all n variables), then from the above we conclude that $n \leq 2$.

We should mention that the condition $\dim_{\mathcal{A}} W \geq 2$ is essential in Theorem 5.4, as shown in the following example.

Example 5.5. Let $(\mathcal{H}, (\cdot,\cdot))$ be an infinite-dimensional Hilbert space. Then $\mathcal{H}$ is a Hilbert $\mathcal{A}$-module with respect to the $\mathcal{A}$-valued inner product given by $\langle \xi, \eta \rangle = \eta \otimes \xi$. It is known that $\dim_{\mathcal{A}} \mathcal{H} = 1$ ([6, Example 1] and [5, Example 2.3]). Notice that $\langle \xi, \eta \rangle = 0$ if and only if $\xi = 0$ or $\eta = 0$. Then every odd mapping on $\mathcal{H}$ (taking values in a uniquely 2-divisible abelian group) is orthogonally additive, but not additive in general. For example, fix $0 \neq \eta_0 \in \mathcal{H}$ and define $f(\xi) = (\xi, \eta_0)\xi \otimes \xi$ for all $\xi \in \mathcal{H}$.

The next example shows that in the case when $\mathcal{H}$ is infinite-dimensional and $\dim_{\mathcal{A}} W$ is finite, the mapping Φ from Theorem 5.4 cannot be extended to a continuous linear mapping on $\mathcal{A}$.

Example 5.6. Let $\mathcal{H}$ be an infinite-dimensional separable Hilbert space. Let $W = \mathcal{H} \oplus \mathcal{H}$ be a Hilbert $\mathcal{A}$-module with the coordinate operations and the $\mathcal{A}$-valued inner product given by

$$\langle (\xi_1, \xi_2), (\eta_1, \eta_2) \rangle = \eta_1 \otimes \xi_1 + \eta_2 \otimes \xi_2.$$

Then $\dim_{\mathcal{A}} W = 2$ (see [6, Theorem 3] and [8, Section 2]). To distinguish the above notation, we use $(\cdot, \cdot)_{\mathcal{H}}$ for the inner product on $\mathcal{H}$. Define $f : W \to \mathbb{C}$ by

$$f((\xi_1, \xi_2)) = (\xi_1, \xi_1)_{\mathcal{H}} + (\xi_2, \xi_2)_{\mathcal{H}} \quad \text{for all } \xi_1, \xi_2 \in \mathcal{H}.$$

We claim that f is an orthogonally additive mapping. Indeed, let $\xi_1, \xi_2, \eta_1, \eta_2 \in \mathcal{H}$ be such that

$$0 = \langle (\xi_1, \xi_2), (\eta_1, \eta_2) \rangle = \eta_1 \otimes \xi_1 + \eta_2 \otimes \xi_2.$$

Then one can easily check that

$$(\xi_1, \eta_1)_{\mathcal{H}} + (\eta_1, \xi_1)_{\mathcal{H}} + (\xi_2, \eta_2)_{\mathcal{H}} + (\eta_2, \xi_2)_{\mathcal{H}} = 0.$$

Thus

$$\begin{aligned} f((\xi_1, \xi_2) + (\eta_1, \eta_2)) &= f((\xi_1 + \eta_1, \xi_2 + \eta_2)) \\ &= (\xi_1 + \eta_1, \xi_1 + \eta_1)_{\mathcal{H}} + (\xi_2 + \eta_2, \xi_2 + \eta_2)_{\mathcal{H}} \\ &= (\xi_1, \xi_1)_{\mathcal{H}} + (\xi_1, \eta_1)_{\mathcal{H}} + (\eta_1, \xi_1)_{\mathcal{H}} + (\eta_1, \eta_1)_{\mathcal{H}} \\ &\quad + (\xi_2, \xi_2)_{\mathcal{H}} + (\xi_2, \eta_2)_{\mathcal{H}} + (\eta_2, \xi_2)_{\mathcal{H}} + (\eta_2, \eta_2)_{\mathcal{H}} \\ &= (\xi_1, \xi_1)_{\mathcal{H}} + (\xi_2, \xi_2)_{\mathcal{H}} + (\eta_1, \eta_1)_{\mathcal{H}} + (\eta_2, \eta_2)_{\mathcal{H}} \\ &= f((\xi_1, \xi_2)) + f((\eta_1, \eta_2)). \end{aligned}$$

This shows that f is an orthogonally additive mapping. It is clear that f is even.

We now verify that f is continuous. Indeed, if (ξ_n, η_n) is a sequence in W converging to (ξ_0, η_0), then $\langle (\xi_n - \xi_0, \eta_n - \eta_0), (\xi_n - \xi_0, \eta_n - \eta_0) \rangle \to 0$. This implies $(\xi_n - \xi_0) \otimes (\xi_n - \xi_0) + (\eta_n - \eta_0) \otimes (\eta_n - \eta_0) \to 0$, and so $(\xi_n - \xi_0) \otimes (\xi_n - \xi_0) \to 0$ and $(\eta_n - \eta_0) \otimes (\eta_n - \eta_0) \to 0$. Hence $(\xi_n, \xi_n)_{\mathcal{H}} \to (\xi_0, \xi_0)_{\mathcal{H}}$ and $(\eta_n, \eta_n)_{\mathcal{H}} \to (\eta_0, \eta_0)_{\mathcal{H}}$. Therefore, $f((\xi_n, \eta_n)) \to f((\xi_0, \eta_0))$, proving the continuity of f.

In what follows, we prove that there is no continuous linear mapping $\Phi : \mathcal{A} \to \mathbb{C}$ such that $f(w) = \Phi(\langle w, w \rangle)$ for all $w \in W$. To the contrary, assume that there is such a mapping Φ. Let $\{\xi_n : n \in \mathbb{N}\}$ be an orthonormal basis for $\mathcal{H}$ and let $E_n = \xi_n \otimes \xi_n$ for all $n \in \mathbb{N}$. For each $n \in \mathbb{N}$, we set $T_n = \sum_{k=1}^{n} k^{-1} E_k \in \langle W, W\rangle$. Since the sequence (T_n) converges to $T = \sum_{k=1}^{\infty} k^{-1} E_k \in \mathcal{A}$ and Φ is continuous, we conclude that the sequence $(\Phi(T_n))$ converges as well. However,

$$\begin{aligned}\Phi(T_n) &= \sum_{k=1}^{n} \frac{1}{k}\Phi(E_k) = \frac{1}{2}\sum_{k=1}^{n} \frac{1}{k}\Phi(2E_k) = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}\Phi(\xi_k \otimes \xi_k + \xi_k \otimes \xi_k) \\ &= \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k}\Phi(\langle(\xi_k,\xi_k),(\xi_k,\xi_k)\rangle) = \frac{1}{2}\sum_{k=1}^{n}\frac{1}{k} f((\xi_k,\xi_k)) = \sum_{k=1}^{n}\frac{1}{k}\end{aligned}$$

does not converge, a contradiction.

The following result is an immediate consequence of Theorem 5.4 (see [6, Example 2]).

Corollary 5.7. *Let $\mathcal{H}$ be a Hilbert space with $2 \le \dim \mathcal{H} \le \aleph_0$, and the orthogonality on $\mathcal{A}$ be defined by*

$$x \perp y \;\Leftrightarrow\; x^* y = 0.$$

Assume that G is a uniquely 2-divisible abelian group and that $f : \mathcal{A} \to G$ is an o.a.m. Then:

(i) *There exist a unique additive mapping $T : \mathcal{A} \to G$ and a unique symmetric biadditive orthogonality preserving mapping $B : \mathcal{A}\times\mathcal{A} \to G$ such that*

$$f(x) = T(x) + B(x, x) \quad \text{for all } x \in \mathcal{A}.$$

(ii) *If G is a normed space and f is continuous, then T is continuous and there exists a unique continuous linear mapping $\Phi : \mathcal{A} \to G$ such that*

$$f(x) = T(x) + \Phi(x^* x) \quad \text{for all } x \in \mathcal{A}.$$

The following example demonstrates that it is essential that the underlying algebra is $\mathcal{K}(\mathcal{H})$ (instead of just being a C^*-algebra of compact operators) in Corollary 5.7 (and Theorem 5.4). The same example shows that we cannot take an arbitrary H^*-algebra instead of $\mathcal{HS}(\mathcal{H})$.

Example 5.8. Let $\mathcal{H}$ be a separable Hilbert space. Fix an orthonormal basis $\{\xi_i\}$ for $\mathcal{H}$. As usual, we represent operators on $\mathcal{H}$ as matrices with respect to $\{\xi_i\}$. Let $\mathcal{D}$ be the norm closed subalgebra of $\mathcal{A}$ consisting of all diagonal operators. Then $\mathcal{D}$ is a Hilbert module over itself; notice that $\mathcal{D}$ is commutative. Now, $\langle x, y\rangle = 0$ if and only if $x^*y = y^*x = yx^* = xy^* = 0$. Define $f : \mathcal{D} \to \mathcal{D}$ by $f(x) = x(x^*)^2$. Then f is an odd orthogonally additive mapping, but it is clearly not additive.

6. O.a.m. on Hilbert C^*-modules over a C^*-algebra of compact operators and Hilbert H^*-modules. Let $\mathcal{A}$ be an arbitrary C^*-algebra of compact operators. By [4, Theorem 1.4.5],

$$\mathcal{A} = \bigoplus_{j\in J} \mathcal{K}(\mathcal{H}_j) = \Big\{(a_j) \in \prod_{j\in J} \mathcal{K}(\mathcal{H}_j) : \lim_{j\in J} \|a_j\| = 0\Big\}.$$

Let W be a Hilbert $\mathcal{A}$-module. We may assume that W is full. If W_j denotes the closed linear span of $W\mathcal{K}(\mathcal{H}_j)$, then each W_j is a (full) Hilbert $\mathcal{K}(\mathcal{H}_j)$-module and W is the outer direct sum of W_j's:

$$W = \bigoplus_{j\in J} W_j = \Big\{(w_j) \in \prod_{j\in J} W_j : \lim_{j\in J} \|w_j\| = 0\Big\}$$

(see [6, Introduction] or [19]).

Now let $\mathcal{A}$ be an arbitrary H^*-algebra. By [3, Theorems 4.2 and 4.3], $\mathcal{A}$ is the orthogonal sum $\bigoplus_{j\in J} \mathcal{A}_j$ where each $\mathcal{A}_j$ is a simple H^*-algebra which is a minimal closed ideal of $\mathcal{A}$ and $\mathcal{A}_j = \mathcal{HS}(\mathcal{H}_j)$ for some Hilbert space $\mathcal{H}_j$. Then every $a \in \mathcal{A}$ can be written as $a = \sum_{j\in J} a_j$ with $a_j \in \mathcal{HS}(\mathcal{H}_j)$ and $\|a\|^2 = \sum_{j\in J} \|a_j\|^2$. Let W be a Hilbert $\mathcal{A}$-module. We may assume that W is faithful (i.e. it has zero annihilator in $\mathcal{A}$). According to [8, Theorem 2.3], there exists a family $\{W_j : j \in J\}$ such that each W_j is a (faithful) Hilbert $\mathcal{HS}(\mathcal{H}_j)$-module and W is the mixed product of W_j's:

$$W = \mathop{\times}_{j\in J} W_j = \Big\{(w_j) \in \prod_{j\in J} W_j : \sum_{j\in J} \|w_j\|^2 < \infty\Big\}.$$

THEOREM 6.1. *Let $\mathcal{A} = \bigoplus_{j\in J} \mathcal{A}_j$ be a C^*-algebra of compact operators, resp. an H^*-algebra, with $\mathcal{A}_j = \mathcal{K}(\mathcal{H}_j)$, resp. $\mathcal{A}_j = \mathcal{HS}(\mathcal{H}_j)$. Let $W = \bigoplus_{j\in J} W_j$ be a Hilbert $\mathcal{A}$-module with W_j a Hilbert $\mathcal{A}_j$-module such that $\dim_{\mathcal{A}_j} W_j = \dim \mathcal{H}_j = \aleph_0$ for each $j \in J$. Let G be a normed space and let $f : W \to G$ be a continuous o.a.m. Then there exist a continuous additive mapping $T : W \to G$ and a continuous linear mapping $\Phi : \mathcal{A} \to G$ such that*

$$f(x) = T(x) + \Phi(\langle x, x\rangle) \quad \textit{for all } x \in W.$$

Proof. Define $f_j = f|_{W_j}$ for each $j \in J$. Then $f_j : W_j \to G$ is a continuous o.a.m. By Theorem 5.4, there exist a continuous additive mapping $T_j : W_j \to G$ and a continuous linear mapping $\Phi_j : \mathcal{A}_j \to G$ such that

$$f_j(x_j) = T_j(x_j) + \Phi_j(\langle x_j, x_j\rangle) \quad \text{for all } x_j \in W_j.$$

Define $T : W \to G$ by

$$T(x) = \tfrac{1}{2}(f(x) - f(-x)).$$

If we write $x = \sum_{j\in J} x_j$ with $x_j \in W_j$, then

$$T(x) = \frac{1}{2}\sum_{j\in J}(f_j(x_j) - f_j(-x_j)) = \sum_{j\in J} T_j(x_j).$$

This implies that T is an additive mapping; it is continuous since f is continuous.

Let $\{w_i : i \in I\}$ be an orthonormal basis for W and let $\{{w_i}^j : i \in I_j\} \subseteq \{w_i : i \in I\}$ be an orthonormal basis for W_j. By Lemma 5.1, without loss of generality we can assume $\langle {w_i}^j, {w_i}^j\rangle = {\xi_i}^j \otimes {\xi_i}^j$ where $\{{\xi_i}^j : i \in I_j\}$ is an orthonormal basis for $\mathcal{H}_j$. Let $a_j \in \mathcal{A}_j$. Then

$$\begin{aligned}\Phi_j(a_j^* a_j) &= \Phi_j\Big(\sum_{i\in I_j} a_j^*({\xi_i}^j \otimes {\xi_i}^j)a_j\Big) = \sum_{i\in I_j}\Phi_j(a_j^*\langle {w_i}^j, {w_i}^j\rangle a_j)\\ &= \sum_{i\in I_j}\Phi_j(\langle {w_i}^j a_j, {w_i}^j a_j\rangle) = \Phi_j\Big(\Big\langle \sum_{i\in I_j} {w_i}^j a_j, \sum_{i\in I_j} {w_i}^j a_j\Big\rangle\Big).\end{aligned}$$

Notice that $\sum_{i\in I} w_i a$ converges for all $a \in \mathcal{A}$. In fact, if $a = \sum_{j\in J} a_j$ with $a_j \in \mathcal{A}_j$, then Remark 5.2 implies that $\sum_{i\in I_j} w_i^j a_j \in W_j$ and

$$\Big\langle \sum_{i\in I_j} w_i^j a_j, \sum_{i\in I_j} w_i^j a_j\Big\rangle = a_j^* a_j.$$

Then

$$\Big\| \sum_{i\in I_j} w_i^j a_j\Big\|_{W_j} = \Big\|\Big\langle \sum_{i\in I_j} w_i^j a_j, \sum_{i\in I_j} w_i^j a_j\Big\rangle\Big\|^{1/2} = \|a_j^* a_j\|^{1/2} = \|a_j\|.$$

Hence $\sum_{j\in J}\sum_{i\in I_j} {w_i}^j a_j \in W$, that is, $\sum_{i\in I} w_i a \in W$. For $a \in \mathcal{A}$ define $\Phi(a^*a) = \frac{1}{2}(f(\sum_{i\in I} w_i a) + f(-\sum_{i\in I} w_i a))$. If we write $a = \sum_{j\in J} a_j$ with $a_j \in \mathcal{A}_j$, then

$$\begin{aligned}\Phi(a^*a) &= \frac{1}{2}\sum_{j\in J}\Big(f_j\Big(\sum_{i\in I_j} {w_i}^j a_j\Big) + f_j\Big(-\sum_{i\in I_j} {w_i}^j a_j\Big)\Big)\\ &= \sum_{j\in J}\Phi_j\Big(\Big\langle \sum_{i\in I_j} {w_i}^j a_j, \sum_{i\in I_j} {w_i}^j a_j\Big\rangle\Big) = \sum_{j\in J}\Phi_j(a_j^* a_j).\end{aligned}$$

Using the fact that every $a \in A$ can be written as a linear combination of four positive elements (i.e., those of the form x^*x for some $x \in \mathcal{A}$), we define $\Phi(a) = \sum_{j\in J}\Phi_j(a_j)$ for all $a = \sum_{j\in J} a_j \in A$. Since each Φ_j is linear, Φ is linear as well. Since

$$\|\Phi(a^*a)\| \le \|f\|\,\Big\|\sum_{i\in I} w_i a\Big\| = \|f\|\,\|a\|$$

for every $a \in \mathcal{A}$, we see that Φ is continuous. Finally,

$$f(x) = T(x) + \Phi(\langle x, x\rangle) \quad \text{for all } x \in W. \blacksquare$$

Corollary 6.2. *Let $\mathcal{A} = \bigoplus_{j\in J} \mathcal{A}_j$ be a C^*-algebra of compact operators, resp. an H^*-algebra, with $\mathcal{A}_j = \mathcal{K}(\mathcal{H}_j)$, resp. $\mathcal{A}_j = \mathcal{HS}(\mathcal{H}_j)$, such that $2 \le \dim \mathcal{H}_j \le \aleph_0$ for each $j \in J$. Let G be a normed space and let $f : \mathcal{A} \to G$ be a continuous o.a.m., with respect to the orthogonality defined by*

$$x \perp y \ \Leftrightarrow \ x^*y = 0.$$

Then there exist a unique continuous additive mapping $T : \mathcal{A} \to G$ and a unique continuous linear mapping $\Phi : \mathcal{A} \to G$ such that

$$f(x) = T(x) + \Phi(x^*x) \quad \text{for all } x \in \mathcal{A}.$$

Let us mention that Example 5.8 also provides a counterexample for Corollary 6.2 in the case when $\mathcal{A} = W = \bigoplus_{j\in J} \mathcal{A}_j$ with $\mathcal{A}_j = W_j = \mathcal{K}(\mathcal{H}_j)$ or $\mathcal{HS}(\mathcal{H}_j)$ and $\dim_{\mathcal{A}_j} W_j = \dim \mathcal{H}_j = 1$ for all $j \in J$.

7. O.a.m. on $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. The aim of this section is to prove an analogue of Corollary 5.7 for $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ instead of $\mathcal{K}(\mathcal{H})$ and $\mathcal{HS}(\mathcal{H})$.

Proposition 7.1. *Let $\mathcal{H}_1$, $\mathcal{H}_2$ be Hilbert spaces with $\dim \mathcal{H}_1, \dim \mathcal{H}_2 \ge 2$. Let G be a uniquely 2-divisible abelian group and let $f : \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \to G$ be an o.a.m., with respect to the orthogonality defined by*

$$x \perp y \ \Leftrightarrow \ x^*y = 0.$$

Then there exist a unique additive mapping $T : \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \to G$ and a symmetric biadditive orthogonality preserving mapping $B : \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \times \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2) \to G$ such that

$$f(x) = T(x) + B(x, x) \quad \text{for all } x \in \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2).$$

If $\mathcal{H}$ is a Hilbert space such that $\dim \mathcal{H} \ge 2$, G is a Banach space and f is continuous, then T is continuous and there exists a unique continuous linear mapping $\Phi : \mathcal{B}(\mathcal{H}) \to G$ such that

$$f(x) = T(x) + \Phi(x^*x) \quad \text{for all } x \in \mathcal{B}(\mathcal{H}).$$

Proof. Let us emphasize that $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$ is a Hilbert $\mathcal{B}(\mathcal{H}_1)$-module with respect to the inner product $\langle x, y\rangle = x^*y$. If $\mathcal{H}_2$ is infinite-dimensional, let $\mathcal{K}$ be a closed subspace of $\mathcal{H}_2$ such that both $\mathcal{K}$ and $\mathcal{K}^\perp$ are infinite-dimensional. If $\dim \mathcal{H}_2 = 2n$, let $\mathcal{K}$ be an n-dimensional subspace of $\mathcal{H}_2$. Let $U : \mathcal{K} \to \mathcal{K}^\perp$ be unitary. Then $\varphi : \mathcal{B}(\mathcal{H}_1, \mathcal{K}) \to \mathcal{B}(\mathcal{H}_1, \mathcal{K}^\perp)$, $\varphi(A) = UA$, is an isomorphism. Notice that $\mathcal{B}(\mathcal{H}_1, \mathcal{K}) \oplus \varphi(\mathcal{B}(\mathcal{H}_1, \mathcal{K})) = \mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$. It remains to apply Theorem 4.1(i).

If $\dim \mathcal{H}_2 = 2n + 1$, let $\mathcal{K}$ be a 1-dimensional subspace of $\mathcal{H}_2$. Then $\dim \mathcal{K}^\perp = 2n$ and, according to the above, the statement holds on $\mathcal{B}(\mathcal{H}_1, \mathcal{K}^\perp)$. Let $\mathcal{M}$ be a 2-dimensional subspace of $\mathcal{H}_2$ containing $\mathcal{K}$. Again, according to the above, the statement also holds true on $\mathcal{B}(\mathcal{H}_1, \mathcal{M})$, hence finally on $\mathcal{B}(\mathcal{H}_1, \mathcal{H}_2)$.

The second statement can be proved in a similar way, but using Theorem 4.1(ii) instead of Theorem 4.1(i), and then applying the results from [1] (see also [2, Theorem 1.1]) to represent S via Φ. ■

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