STUDIA MATHEMATICA 221 (3) (2014)

Rough oscillatory singular integrals on $\mathbb{R}^n$

by

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Abstract. We establish sharp bounds for oscillatory singular integrals with an arbitrary real polynomial phase P. The kernels are allowed to be rough both on the unit sphere and in the radial direction. We show that the bounds grow no faster than $\log \deg(P)$, which is optimal and was first obtained by Papadimitrakis and Parissis (2010) for kernels without any radial roughness. Among key ingredients of our methods are an $L^1 \to L^2$ estimate and extrapolation.

1. Introduction and main results. Throughout this paper, $\mathbb{R}^n$, $n \geq 2$, is the n-dimensional Euclidean space and $\mathbb{S}^{n-1}$ the unit sphere in $\mathbb{R}^n$ equipped with the normalized Lebesgue surface measure $d\sigma$. Also, we let ξ' denote $\xi/|\xi|$ for $\xi \in \mathbb{R}^n \setminus \{0\}$, and p' denotes the exponent conjugate to p, that is, $1/p + 1/p' = 1$.

Let $K_{\Omega,h}(y)$ be a Calderón–Zygmund type kernel of the form $K_{\Omega,h}(y) = h(|y|)\Omega(y')|y|^{-n}$ where $h : [0,\infty) \to \mathbb{C}$ is a measurable function and Ω is an integrable function over $\mathbb{S}^{n-1}$ satisfying

$$\int_{\mathbb{S}^{n-1}} \Omega(u)\, d\sigma(u) = 0. \tag{1.1}$$

Let $\mathcal{P}(n;d)$ denote the set of all polynomials on $\mathbb{R}^n$ which have real coefficients and degrees not exceeding d. For $\gamma > 0$, let $\Delta_\gamma(\mathbb{R}_+)$ denote the collection of all measurable functions $h : [0,\infty) \to \mathbb{C}$ satisfying

$$\|h\|_{\Delta_\gamma} = \sup_{k\in\mathbb{Z}} \Big(\int_{2^k}^{2^{k+1}} |h(t)|^\gamma \, dt/t \Big)^{1/\gamma} < \infty,$$

and $\mathcal{L}_\gamma(\mathbb{R}_+)$ denote the collection of all measurable functions $h : [0,\infty) \to \mathbb{C}$ satisfying

2010 *Mathematics Subject Classification*: Primary 42B20; Secondary 26D05.
Key words and phrases: oscillatory singular integral, rough kernel, singular integral, Orlicz spaces, Block spaces, extrapolation, L^p boundedness.

DOI: 10.4064/sm221-3-4

$$L_\gamma(h) = \sup_{k\in\mathbb{Z}} \Big(\int_{2^k}^{2^{k+1}} |h(t)| \big(\log(2+|h(t)|)\big)^\gamma \, dt/t \Big) < \infty.$$

Also, we let $\mathcal{N}_\gamma(\mathbb{R}_+)$ denote the class of all measurable functions h on $\mathbb{R}_+$ such that

$$N_\gamma(h) = \sum_{m=1}^{\infty} m^\gamma 2^m d_m(h) < \infty$$

where $d_m(h) = \sup_{k\in\mathbb{Z}} 2^{-k}|E(k,m)|$ with

$$E(k,m) = \{t \in (2^k, 2^{k+1}] : 2^{m-1} < |h(t)| \le 2^m\} \quad \text{for } m \ge 2,$$
$$E(k,1) = \{t \in (2^k, 2^{k+1}] : |h(t)| \le 2\}.$$

For $\gamma \ge 1$ define $\mathcal{H}_\gamma(\mathbb{R}_+)$ to be the set of all measurable functions h on $\mathbb{R}_+$ satisfying the condition

$$\|h\|_{L^\gamma(\mathbb{R}_+, dr/r)} = \Big(\int_0^\infty |h(r)|^\gamma \, dr/r \Big)^{1/\gamma} < \infty.$$

Remark. It is easy to verify that the following inclusion relations hold and are proper:

(1) $\Delta_{\gamma_2}(\mathbb{R}_+) \subset \Delta_{\gamma_1}(\mathbb{R}_+)$ for $1 \le \gamma_1 < \gamma_2$;
(2) $\mathcal{H}_\infty(\mathbb{R}_+) = \Delta_\infty(\mathbb{R}_+)$ and $\mathcal{H}_\gamma(\mathbb{R}_+) \subset \Delta_\gamma(\mathbb{R}_+)$ for $1 < \gamma < \infty$;
(3) $\mathcal{N}_{\gamma_2}(\mathbb{R}_+) \subset \mathcal{N}_{\gamma_1}(\mathbb{R}_+)$ for $\gamma_1 < \gamma_2$;
(4) $\mathcal{L}_{\gamma_2}(\mathbb{R}_+) \subset \mathcal{L}_{\gamma_1}(\mathbb{R}_+)$ for $\gamma_1 < \gamma_2$;
(5) $\Delta_\gamma(\mathbb{R}_+) \subset \mathcal{N}_\alpha(\mathbb{R}_+) \subset \mathcal{L}_\alpha(\mathbb{R}_+)$ for any $\gamma \ge 1$ and $\alpha > 0$;
(6) for a given $\alpha > 1$, $\mathcal{L}_{\gamma+\alpha}(\mathbb{R}_+) \subset \mathcal{L}_\gamma(\mathbb{R}_+)$ for any $\gamma > 0$;
(7) $L(\log L)^\gamma(\mathbb{R}_+,\, dt/t) \subset \mathcal{N}_\gamma(\mathbb{R}_+)$ for all $\gamma > 0$ where $L(\log L)^\gamma(\mathbb{R}_+,\, dt/t)$ is the class of all measurable functions h on $\mathbb{R}_+$ which satisfy

$$\int_{\mathbb{R}_+} |h(t)| \big(\log(2+|h(t)|)\big)^\gamma \, dt/t < \infty.$$

Let $L(\log L)^\alpha(\mathbb{S}^{n-1})$ ($\alpha > 0$) denote the class of all functions Ω which satisfy

$$\|\Omega\|_{L(\log L)^\alpha(\mathbb{S}^{n-1})} = \int_{\mathbb{S}^{n-1}} |\Omega(x)| \big(\log(2+|\Omega(x)|)\big)^\alpha \, d\sigma(x) < \infty.$$

Now, let us recall the definition of the block space $B_q^{(0,v)}(\mathbb{S}^{n-1})$.

Definition. A *q-block* ($1 < q \le \infty$) on $\mathbb{S}^{n-1}$ is an L^q function b on $\mathbb{S}^{n-1}$ that satisfies the following conditions:

(i) $\operatorname{supp}(b) \subset I$;
(ii) $\|b\|_{L^q} \le |I|^{-1/q'}$ where $|I| = \sigma(I)$ and $I = B(x_0', \theta_0) = \{x' \in \mathbb{S}^{n-1} : |x' - x_0'| < \theta_0\}$ is a cap on $\mathbb{S}^{n-1}$ for some $x_0' \in \mathbb{S}^{n-1}$ and $\theta_0 \in (0,1]$.

Definition. The *block space* $B_q^{(0,\upsilon)}(\mathbb{S}^{n-1})$ is defined by

$$B_q^{(0,\upsilon)}(\mathbb{S}^{n-1}) = \Big\{\Omega \in L^1(\mathbb{S}^{n-1}) : \Omega = \sum_{\mu=1}^{\infty} \lambda_\mu b_\mu,\ M_q^{(0,\upsilon)}(\{\lambda_\mu\}) < \infty\Big\}$$

where each λ_μ is a complex number, each b_μ is a q-block supported on a cap I_μ on $\mathbb{S}^{n-1}$, $\upsilon > -1$, and

$$M_q^{(0,\upsilon)}(\{\lambda_\mu\}) = \sum_{\mu=1}^{\infty} |\lambda_\mu|\{1 + \log^{\upsilon+1}(|I_\mu|^{-1})\}. \tag{1.2}$$

We remark that for any $q > 1$, $0 < \alpha < \beta$, and $\upsilon > -1$, the following inclusions hold and are proper:

$$L^q(\mathbb{S}^{n-1}) \subset L(\log L)^\beta(\mathbb{S}^{n-1}) \subset L(\log L)^\alpha(\mathbb{S}^{n-1}) \subset L^1(\mathbb{S}^{n-1}), \tag{1.3}$$

$$\bigcup_{r>1} L^r(\mathbb{S}^{n-1}) \subset B_q^{(0,\upsilon)}(\mathbb{S}^{n-1}) \subset L^1(\mathbb{S}^{n-1}), \tag{1.4}$$

$$B_q^{(0,\upsilon_2)}(\mathbb{S}^{n-1}) \subset B_q^{(0,\upsilon_1)}(\mathbb{S}^{n-1}) \quad \text{for any } -1 < \upsilon_1 < \upsilon_2. \tag{1.5}$$

The question of relation between $B_q^{(0,\upsilon-1)}(\mathbb{S}^{n-1})$ and $L(\log^+ L)^\upsilon(\mathbb{S}^{n-1})$ (for $\upsilon > 0$) remains open.

By Plancherel's Theorem, the L^2 boundedness problem for oscillatory singular integral operators of convolution type directly leads to the consideration of the oscillatory singular integral $J(P)$ and its n-dimensional analogue $I_{\Omega,h}(P)$, which are defined by

$$J(P) = \text{p.v.} \int_{\mathbb{R}} e^{iP(t)} \frac{dt}{t} \quad \text{for } P \in \mathcal{P}(1;d), \tag{1.6}$$

$$I_{\Omega,h}(P) = \text{p.v.} \int_{\mathbb{R}^n} e^{iP(x)} K_{\Omega,h}(x)\, dx \quad \text{for } P \in \mathcal{P}(n;d). \tag{1.7}$$

One of the main problems regarding these oscillatory singular integrals is to obtain sharp estimates with constants depending only on the degree of the polynomial P. Also of importance is estimating $I_{\Omega,h}(P)$ under minimal conditions both on Ω and h. The study of these problems was initiated by Stein–Wainger [17] and Stein [16], and continued by Parissis [12], [13] and by Papadimitrakis–Parissis [11]. Stein and Wainger [17] proved that if $P \in \mathcal{P}(1;d)$, then $|J(P)| \le C_d$ for some constant C_d depending only on the degree d of the polynomial P and independent of its coefficients. Parissis [13] showed that the true order of magnitude of $J(P)$ is $\log d$, as stated in the following theorem:

Theorem A. *Let $J(P)$ be as above. Then there is an absolute constant C such that*

$$\Big|\sup_{P\in\mathcal{P}(1;d)} J(P)\Big| \le C(\log d+1). \tag{1.8}$$

On the other hand, Stein [16] studied the higher-dimensional singular integral $I_{\Omega,1}(P)$ and proved that if $h\equiv 1$ and $\Omega\in L^\infty(S^{n-1})$ satisfies (1.1), then for any $P\in\mathcal{P}(n;d)$, there exists a positive constant C_d depending only on the degree d of the polynomial P and independent of its coefficients such that

$$|I_{\Omega,1}(P)| \le C_d\|\Omega\|_{L^\infty(\mathbb{S}^{n-1})}. \tag{1.9}$$

Recently, Papadimitrakis and Parissis [11] improved Stein's result by showing that the constant C_d can be replaced by $C(\log d+1)$ for some absolute constant C and that the condition on Ω can be weakened to $\Omega\in L\log L(\mathbb{S}^{n-1})$. Their result can be stated as follows.

Theorem B. *Assume that $h\equiv 1$ and $\Omega\in L\log L(\mathbb{S}^{n-1})$ satisfies* (1.1). *Then there exists an absolute positive constant C such that*

$$\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,1}(P)| \le C(\log d+1)(1+\|\Omega\|_{L\log L(\mathbb{S}^{n-1})}). \tag{1.10}$$

It is worth mentioning that, by Theorem A, one can easily show that if Ω is an odd function on $\mathbb{S}^{n-1}$ and Ω is merely in $L^1(\mathbb{S}^{n-1})$, then

$$\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,1}(P)| \le C(\log d+1)\|\Omega\|_{L^1(\mathbb{S}^{n-1})}. \tag{1.11}$$

Very recently [2], Theorem B was improved by replacing the condition $\Omega\in L\log L(\mathbb{S}^{n-1})$ by the weaker condition $\Omega\in H^1(\mathbb{S}^{n-1})$ (the Hardy space on the unit sphere). At this point we remark that the investigation of kernels $K_{\Omega,h}$ which have the additional roughness in the radial direction due to the presence of h was started by R. Fefferman [8] and taken up by several other well-known authors. We remark that the method employed in [2] is no longer applicable if the kernel has roughness in the radial direction. So in light of the estimates in (1.9)–(1.10) and the inclusion relations in (1.3)–(1.5), the following questions arise naturally:

Question 1. *Does an estimate of $\sup_{P\in\mathcal{P}(n;d)}|I_{\Omega,h}(P)|$ as in* (1.10) *hold under conditions of the form $\Omega\in L(\log L)^\alpha(\mathbb{S}^{n-1})$ (for $0<\alpha\le 1$) and $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $\gamma>1$, and if so, what is the best possible value of the exponent α?*

Question 2. *Does an estimate of $\sup_{P\in\mathcal{P}(n;d)}|I_{\Omega,h}(P)|$ as in* (1.10) *hold under conditions of the form $\Omega\in B_q^{(0,\alpha)}(\mathbb{S}^{n-1})$ (for $\alpha>-1$) and $h\in\Delta_\gamma(\mathbb{R}_+)$ for some $\gamma>1$, and if so, what is the best possible value of the exponent α?*

The main purpose of this paper is to answer the above questions. Our approach will rely on some delicate estimates of $\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,h}(P)|$ and an extrapolation argument. The exact statements of our results are the following:

THEOREM 1.1. *Assume that* $\Omega \in L^q(\mathbb{S}^{n-1})$ *for some* $1 < q \le 2$ *and* $h \in \Delta_\gamma(\mathbb{R}_+)$ *for some* $1 < \gamma \le 2$. *Then*

$$\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,h}(P)| \le C(\log d+1)(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})} \tag{1.12}$$

where C *is an absolute positive constant. Moreover, the exponent* -1 *is the best possible.*

THEOREM 1.2. *Assume that* $\Omega \in L^q(\mathbb{S}^{n-1})$ *for some* $1 < q \le 2$ *and* $h \in \mathcal{H}_\gamma(\mathbb{R}_+)$ *for some* $\gamma > 1$. *Then*

$$\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,h}(P)| \le C(\log d+1)(q-1)^{-1/\gamma'}\|h\|_{L^\gamma(\mathbb{R}_+,dr/r)}\|\Omega\|_{L^q(\mathbb{S}^{n-1})} \tag{1.13}$$

where C *is an absolute positive constant.*

THEOREM 1.3.

(a) *Assume that* $\Omega \in L\log L(\mathbb{S}^{n-1})$ *and* $h \in \mathcal{N}_1(\mathbb{R}_+)$. *Then*

$$\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,h}(P)| \le C(\log d+1)(1+\|\Omega\|_{L\log L(\mathbb{S}^{n-1})}) \tag{1.14}$$

where C *is an absolute positive constant.*

(b) *The condition* $\Omega \in L\log L(\mathbb{S}^{n-1})$ *is the best possible in the sense that there exists an* Ω *with* $\Omega \in L(\log L)^{1-\varepsilon}(\mathbb{S}^{n-1})$ *for all* $\varepsilon > 0$ *and satisfying* (1.1) *such that* $\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,1}(P)| = \infty$.

THEOREM 1.4.

(a) *Assume that* $\Omega \in B_q^{(0,0)}(\mathbb{S}^{n-1})$ *for some* $q > 1$ *and* $h \in \mathcal{N}_1(\mathbb{R}_+)$. *Then*

$$\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,h}(P)| \le C(\log d+1)(1+\|\Omega\|_{B_q^{(0,0)}(\mathbb{S}^{n-1})}) \tag{1.15}$$

where C *is an absolute positive constant.*

(b) *The condition* $\Omega \in B_q^{(0,0)}(\mathbb{S}^{n-1})$ *is the best possible in the sense that there exists an* $\Omega \in B_q^{(0,\upsilon)}(\mathbb{S}^{n-1})$ *for any* υ, $-1 < \upsilon < 0$, *which satisfies* (1.1) *and is such that* $\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,1}(P)| = \infty$.

THEOREM 1.5.

(a) *Assume that* $h \in \mathcal{H}_\gamma(\mathbb{R}_+)$ *for some* $\gamma > 1$ *and* $\Omega \in L(\log L)^{1/\gamma'}(\mathbb{S}^{n-1})$. *Then*

$$\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,h}(P)| \le C(\log d+1)(1+\|\Omega\|_{L(\log L)^{1/\gamma'}(\mathbb{S}^{n-1})}). \tag{1.16}$$

(b) *Assume that* $h \in \mathcal{H}_\gamma(\mathbb{R}_+)$ *for some* $\gamma > 1$ *and* $\Omega \in B_q^{(0,-1/\gamma)}(\mathbb{S}^{n-1})$ *for some* $q > 1$. *Then*

$$\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,h}(P)| \le C(\log d + 1)(1 + \|\Omega\|_{B_q^{(0,-1/\gamma)}(\mathbb{S}^{n-1})}) \tag{1.17}$$

where C *is an absolute positive constant.*

Remarks. (1) In Theorems 1.3–1.4, the condition $h \in \mathcal{N}_1(\mathbb{R}_+)$ is the least stringent condition known to date.

(2) In Theorem 1.5(a), the condition $\Omega \in L(\log L)^{1/\gamma'}(\mathbb{S}^{n-1})$ is weaker than the condition $\Omega \in L(\log L)(\mathbb{S}^{n-1})$ in Theorem 1.3(a), which is due to fact that the condition imposed on h is more restrictive than in Theorem 1.3(a). A similar situation occurs in both Theorems 1.4(a) and 1.5(b).

The paper is organized as follows. A few lemmas will be recalled or proved in Section 2. In Section 3 we prove the optimality of the conditions imposed on Ω. Section 4 contains the proofs of the main results. In Section 5, we present an alternative proof of Theorem A.

Throughout this paper, we let C denote a constant which is independent of the essential variables. Its value may change from line to line.

2. Some lemmas. We start this section by recalling the following result from [11], which was established on the basis of a result of Carbery and Wright [5].

Lemma 2.1. *Let* P *be a real homogeneous polynomial of degree* d *on* $\mathbb{R}^n$. *Then*

$$\int_{\mathbb{S}^{n-1}} \frac{\|P\|_{L^\infty(\mathbb{S}^{n-1})}^{1/(2d)}}{\|P(x)\|^{1/(2d)}}\, d\sigma(x) \le C$$

for some absolute constant C, *independent of* P *and* d.

We shall need the following lemma from [3].

Lemma 2.2. *Let* $h(t) = b_0 + b_1 t + \cdots + b_d t^d$ *be a real polynomial of degree at most* d, *and let* $\psi \in C^1[a,b]$. *Then for any* j_0 *with* $1 \le j_0 \le d$, *there exists a positive constant* C *independent of* a, b *and of the coefficients of* $b_0, \ldots, b_d$, *and also independent of* d, *such that*

$$\Big|\int_a^b e^{ih(t)}\psi(t)\, dt\Big| \le C|b_{j_0}|^{-1/d}\Big\{\sup_{a\le t\le b}|\psi(t)| + \int_a^b |\psi'(t)|\, dt\Big\}$$

for $0 < a < b \le 1$.

One of the main ingredients in the proof of the main results in this paper is the following lemma, which is similar in spirit to Lemmas 3.3 and 3.4 in [7].

LEMMA 2.3. *Let P be a homogeneous polynomial of degree d on $\mathbb{R}^n$ satisfying*

$$\int_{\mathbb{S}^{n-1}} P(x)\,d\sigma(x) = 0. \tag{2.1}$$

Then there exists a positive constant C, independent of P and d, such that

$$\sup_{\omega\in\mathbb{R}} \int_{\mathbb{S}^{n-1}} \frac{\|P\|_{L^\infty(\mathbb{S}^{n-1})}^{\delta_d}}{|P(x)-\omega|^{\delta_d}}\,d\sigma(x) \le C \tag{2.2}$$

where $\delta_d = 1/(2d)$ if d is even and $\delta_d = 1/(4d)$ if d is odd.

Proof. We need to consider two cases.

CASE 1: d is even. Let $\|\cdot\|_p$ denote the L^p norm on $\mathbb{S}^{n-1}$ and

$$\langle F, G\rangle = \int_{\mathbb{S}^{n-1}} F(x)G(x)\,d\sigma(x).$$

Then by (2.1) we have $\langle P, \omega\rangle = 0$ for any constant ω. Thus

$$\begin{aligned}\|P-\omega\|_2^2 &= \langle P-\omega, P-\omega\rangle = \langle P,P\rangle - 2\langle P,\omega\rangle + \omega^2\\ &= \langle P,P\rangle + \omega^2 \ge \|P\|_2^2,\end{aligned}$$

which implies that

$$\|P\|_2 \le \inf\{\|P-\omega\|_2 : \omega\in\mathbb{R}\}.$$

In the rest of the proof we shall use very frequently the following inequalities: For any homogeneous polynomial P of degree d, we have

$$\|P\|_{L^p(\mathbb{S}^{n-1})}^{1/d} \le \|P\|_{L^q(\mathbb{S}^{n-1})}^{1/d} \le C\|P\|_{L^p(\mathbb{S}^{n-1})}^{1/d} \tag{2.3}$$

for some absolute constant C independent of d and for $1\le p\le q\le\infty$. The first inequality in (2.3) is clear, while the second follows from [5, Corollary, p. 234] and from the fact that P is a homogeneous polynomial.

Since $P(x)-\omega|x|^d$ is a homogeneous polynomial of degree d (and $\omega|x|^d = \omega$ for $x\in\mathbb{S}^{n-1}$), by (2.3) we have

$$\|P\|_\infty^{1/(2d)} \le C\inf\{\|P-\omega\|_\infty^{1/(2d)} : \omega\in\mathbb{R}\} \tag{2.4}$$

with C independent of d. Thus,

$$\int_{\mathbb{S}^{n-1}} \frac{\|P-\omega\|_{L^\infty(\mathbb{S}^{n-1})}^{1/(2d)}}{|P(x)-\omega|^{1/(2d)}}\,d\sigma(x) \le C. \tag{2.5}$$

By (2.4)–(2.5) we get

$$\int_{\mathbb{S}^{n-1}} \frac{\|P\|_{L^\infty(\mathbb{S}^{n-1})}^{1/(2d)}}{|P(x)-\omega|^{1/(2d)}}\,d\sigma(x) \le C \tag{2.6}$$

for every $\omega\in\mathbb{R}$, which ends the proof of (2.2) when d is even.

CASE 2: d is odd. Now we need to prove

$$\sup_{\omega\in\mathbb{R}} \int_{\mathbb{S}^{n-1}} \frac{\|P\|_{L^\infty(\mathbb{S}^{n-1})}^{1/(4d)}}{|P(x)-\omega|^{1/(4d)}}\, d\sigma(x) \le C. \tag{2.7}$$

By (2.3) there exists a constant $c_0 \ge 1$ independent of d such that

$$\|P\|_2^{1/(2d)} \le \|P\|_\infty^{1/(2d)} \le c_0\|P\|_2^{1/(2d)} \tag{2.8}$$

for every homogeneous polynomial P of degree d, and hence

$$\|\tilde{P}\|_2^{1/(4d)} \le \|\tilde{P}\|_\infty^{1/(4d)} \le c_0\|\tilde{P}\|_2^{1/(4d)} \tag{2.9}$$

for every homogeneous polynomial $\tilde{P}$ of degree $2d$. Thus (2.7) is equivalent to

$$\sup_{\omega\in\mathbb{R}} \int_{\mathbb{S}^{n-1}} \frac{\|P\|_{L^2(\mathbb{S}^{n-1})}^{1/(4d)}}{|P(x)-\omega|^{1/(4d)}}\, d\sigma(x) \le C. \tag{2.10}$$

By a scaling argument, (2.10) will follow from the next proposition. ∎

PROPOSITION 2.4. *Let $d \in \mathbb{N}$ be odd. If P is a homogeneous polynomial of degree d and $\|P\|_2 = 1$, then*

$$\sup_{\omega\in\mathbb{R}} \int_{\mathbb{S}^{n-1}} \frac{1}{|P(x)-\omega|^{1/(4d)}}\, d\sigma(x) \le C \tag{2.11}$$

where C is independent of d.

Proof. By (2.8) and since $\|P\|_2 = 1$, we have

$$1 \le \|P\|_\infty \le c_0^{(2d)}. \tag{2.12}$$

If $|\omega| > 2c_0^{(2d)}$, then for every $x \in \mathbb{S}^{n-1}$ we have $|P(x)-\omega|^{1/(4d)} \ge (c_0^{(2d)})^{1/(4d)} \ge 1$ and hence (2.11) holds. We now assume that $|\omega| \le 2c_0^{(2d)}$. Let $\phi_\omega(x) = (P(x))^2 - \omega^2|x|^{2d}$. Since ϕ_ω is a homogeneous polynomial of degree $2d$, by Lemma 2.1 we obtain

$$\sup_{\omega\in\mathbb{R}} \int_{\mathbb{S}^{n-1}} \frac{\|\phi_\omega\|_{L^\infty(\mathbb{S}^{n-1})}^{1/(4d)}}{|\phi_\omega(x)|^{1/(4d)}}\, d\sigma(x) \le C. \tag{2.13}$$

For any $x \in \mathbb{S}^{n-1}$, by (2.12) we have

$$|\phi_\omega(x)|^{1/(4d)} = |P(x)+\omega|^{1/(4d)}|P(x)-\omega|^{1/(4d)} \le 2c_0|P(x)-\omega|^{1/(4d)}. \tag{2.14}$$

Now we notice that

$$\langle \phi_1, 1\rangle = \int_{\mathbb{S}^{n-1}} \phi_1(x)\, d\sigma(x) = \|P\|_2^2 - 1 = 0.$$

Thus,

$$\begin{aligned}(2.15)\qquad \|\phi_\omega\|_2^2 &= \|\phi_1 + (1-\omega^2)\|_2^2 \\ &= \langle \phi_1, \phi_1\rangle + 2(1-\omega^2)\langle \phi_1, 1\rangle + (1-\omega^2)^2 \\ &= \|\phi_1\|_2^2 + (1-\omega^2)^2 \ge \|\phi_1\|_2^2.\end{aligned}$$

Since d is odd, there exists an $x_0 \in \mathbb{S}^{n-1}$ with $P(x_0) = 0$. By (2.9) and (2.15) we obtain

$$\begin{aligned}(2.16)\qquad 1 &= \big|(P(x_0))^2 - |x_0|^{2d}\big|^{1/(4d)} = |\phi_1(x_0)|^{1/(4d)} \\ &\le \|\phi_1\|_\infty^{1/(4d)} \le c_0\|\phi_1\|_2^{1/(4d)} \le c_0\|\phi_\omega\|_\infty^{1/(4d)}.\end{aligned}$$

By (2.14) and (2.16), we have

$$(2.17)\qquad \frac{1}{|P(x)-\omega|^{1/(4d)}} \le \frac{2c_0^2\|\phi_\omega\|_\infty^{1/(4d)}}{|\phi_\omega(x)|^{1/(4d)}}.$$

By (2.13) and (2.17) we see that (2.11) holds when $|\omega| \le 2c_0^{(2d)}$. The proof of the proposition is now complete. ∎

3. Proof of the optimality of the conditions imposed on Ω

Proof of Theorem 1.3(b). Let $\mathcal{P}^*(n)$ denote the set of all polynomials P_a on $\mathbb{R}^n$ given by $P_a(x) = a \cdot x$ where $a = (a_1, \ldots, a_n) \in \mathbb{S}^{n-1}$ and $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$. Let $P_a \in \mathcal{P}^*(n)$ for some $a \in \mathbb{S}^{n-1}$. Then

$$\begin{aligned}I_{\Omega,1}(P_a) &= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\varepsilon\le|x|\le R} e^{iP_a(x)}\frac{\Omega(x/|x|)}{|x|^n}\,dx \\ &= \lim_{\substack{\varepsilon\to 0\\ R\to\infty}} \int_{\mathbb{S}^{n-1}}\int_\varepsilon^R e^{-i2\pi t(a\cdot x)}\,\Omega(x)\,\frac{dt}{t}\,d\sigma(x).\end{aligned}$$

Since

$$\int_\varepsilon^R (e^{-2\pi it(a\cdot x)} - \cos(2\pi t))\,\frac{dt}{t} \to \log|a\cdot x|^{-1} - i\frac{\pi}{2}\operatorname{sgn}(a\cdot x)$$

as $R \to \infty$ and $\varepsilon \to 0$, the integral is bounded, uniformly in ε and R, by $C\big(1 + \big|\log|a\cdot x|\big|\big)$.

Thus, using (1.1) and Lebesgue's Dominated Convergence Theorem, we get

$$I_{\Omega,1}(P_a) = \int_{\mathbb{S}^{n-1}} \Omega(x)\Big(\log|a\cdot x|^{-1} - i\frac{\pi}{2}\operatorname{sgn}(a\cdot x)\Big)\,d\sigma(x) = m_\Omega(a),$$

and hence

$$\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,1}(P)| \ge m_\Omega(a) \qquad \text{for any } a = (a_1, \ldots, a_n) \in \mathbb{S}^{n-1}.$$

Now consider the homogeneous Calderón–Zygmund singular integral operator T_Ω given by

$$T_\Omega f(x) = \text{p.v.} \int_{\mathbb{R}^n} \Omega(y')|y|^{-n} f(x-y)\,dy.$$

It is well-known that $\widehat{T_\Omega f}(\xi) = m_\Omega(\xi)\hat{f}(\xi)$ and the convolution operator T_Ω is bounded from $L^2(\mathbb{R}^n)$ to itself if and only if $m_\Omega \in L^\infty(\mathbb{R}^n)$.

From the result of M. Weiss and A. Zygmund [18] we deduce that there exists $\Omega \in L(\log L)^{1-\varepsilon}(\mathbb{S}^{n-1})$ for any $\varepsilon > 0$ such that m_Ω is unbounded on $\mathbb{R}^n$. Hence we have $\sup_{P\in\mathcal{P}(n;d)} |I_{\Omega,1}(P)| = \infty$, which ends the proof of Theorem 1.3(b). ∎

Proof of Theorem 1.4(b). We argue as in the proof of Theorem 1.3(b). By a counterexample in [1] we deduce that there exists $\Omega \in B_q^{(0,\upsilon)}(\mathbb{S}^{n-1})$ for any υ, $-1 < \upsilon < 0$, such that m_Ω is unbounded on $\mathbb{R}^n$, which in turn ends the proof of Theorem 1.4(b). ∎

4. Proofs of theorems

Proof of Theorem 1.1. Assume that $h \in \Delta_\gamma(\mathbb{R}_+)$ for some $\gamma \in (1,2]$, and assume that $\Omega \in L^q(\mathbb{S}^{n-1})$ for some $q \in (1,2]$ and satisfies (1.1). Let

$$A_d = A_d(\Omega, h, n) = \sup_{\substack{0<\varepsilon<R\\ P\in\mathcal{P}(n;d)}} |J_{\varepsilon,R}(P)|$$

where

$$J_{\varepsilon,R}(P) = \int_{\varepsilon\le|x|\le R} e^{iP(x)} K_{\Omega,h}(x)\,dx.$$

We need to show that

$$A_d \le C(\log d + 1)(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})} \tag{4.1}$$

for some absolute positive constant C. We shall first prove (4.1) for the case $d = 2^m$ for some integer $m \ge 0$; then the general case will be an immediate consequence.

Switching to polar coordinates, we get

$$J_{\varepsilon,R}(P) = \int_{\mathbb{S}^{n-1}} \int_\varepsilon^R e^{iP(tx)} h(t)\Omega(x)\,\frac{dt}{t}\,d\sigma(x).$$

Write

$$P(tx) = \sum_{j=1}^{d} P_j(x)t^j + R(t) \tag{4.2}$$

with

$$\int_{\mathbb{S}^{n-1}} P_j(x)\,d\sigma(x) = 0 \tag{4.3}$$

for every $j = 1, \ldots, d$ where $P_j(x)$ is homogeneous of degree j. We remark that when j is odd, then (4.3) holds automatically. When j is even, one may need to add/subtract a constant multiple of $|x|^j$, which is itself a homogeneous polynomial of degree j. Since Ω satisfies (1.1), we may assume without loss of generality that $R(t) \equiv 0$. Let

$$m_j = \|P_j\|_{L^\infty(\mathbb{S}^{n-1})} \quad \text{and} \quad Q(tx) = \sum_{j=1}^{d/2} P_j(x)t^j.$$

Since ε and R are arbitrary positive numbers and P is a polynomial of degree d, by a dilation in t we may assume, without loss of generality, that $\max_{d/2<j\le d} m_j = 1$. Also, there is $d/2 < j_0 \le d$ such that $m_{j_0} = 1$. Now, $J_{\varepsilon,R}(P)$ can be written as

$$\begin{aligned} |J_{\varepsilon,R}(P)| &\le \Big| \int_{\mathbb{S}^{n-1}} \int_{\varepsilon}^{R_0} e^{iP(tx)} h(t)\Omega(x)\,\frac{dt}{t}\,d\sigma(x)\Big| \\ &\quad + \Big| \int_{\mathbb{S}^{n-1}} \int_{R_0}^{R} e^{iP(tx)} h(t)\Omega(x)\,\frac{dt}{t}\,d\sigma(x)\Big| \\ &= I_1 + I_2 \end{aligned} \tag{4.4}$$

where R_0 will be defined later.

Let us first estimate I_1:

$$\begin{aligned} I_1 &\le \int_{\mathbb{S}^{n-1}} \int_0^{R_0} |e^{iP(tx)} - e^{iQ(tx)}|\,|h(t)|\,|\Omega(x)|\,\frac{dt}{t}\,d\sigma(x) \\ &\quad + \Big| \int_{\mathbb{S}^{n-1}} \int_{\varepsilon}^{R_0} e^{iQ(tx)} h(t)\Omega(x)\,\frac{dt}{t}\,d\sigma(x)\Big| \\ &\le \|\Omega\|_{L^1(\mathbb{S}^{n-1})} \sum_{d/2<j\le d} m_j \Big(\int_0^{R_0} t^j |h(t)|\,\frac{dt}{t}\Big) + A_{d/2}. \end{aligned}$$

By Hölder's inequality and letting $R_0 = d^{1/(d\gamma')}$ we get

$$I_1 \le C(\log\theta)\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma} + A_{d/2}. \tag{4.5}$$

Now we estimate I_2. It is clear that

$$I_2 \le \int_1^R |h(t)| \Big| \int_{\mathbb{S}^{n-1}} \Omega(x) e^{iP(tx)}\,d\sigma(x)\Big| \,\frac{dt}{t}.$$

Let $\theta = 2^{q'\gamma'}$. For each fixed $R > 1$ we have a unique $k_0 \in \mathbb{Z}_+$ such that

$\theta^{k_0-1} \le R < \theta^{k_0}$. Hence

$$I_2 \le \sup_{k_0\in\mathbb{Z}_+} \int_{\theta^{k_0-1}}^{\theta^{k_0}} |h(t)| \Big| \int_{\mathbb{S}^{n-1}} e^{iP(tx)} \Omega(x)\, d\sigma(x)\Big| \frac{dt}{t} + \sup_{k_0\in\mathbb{Z}_+} \Big(\sum_{k=k_0+1}^{\infty} I_{k,\theta}\Big) = J_1 + J_2 \tag{4.6}$$

where

$$I_{k,\theta} = \int_{\theta^{k-1}}^{\theta^k} |h(t)| \Big| \int_{\mathbb{S}^{n-1}} e^{iP(tx)} \Omega(x)\, d\sigma(x)\Big| \frac{dt}{t}.$$

It is easy to see that

$$J_1 \le C(\log\theta)\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma}. \tag{4.7}$$

Therefore, it remains to estimate J_2.

To this end, we first estimate $I_{k,\theta}$:

$$\begin{aligned} I_{k,\theta} &\le \Big(\int_{\theta^{k-1}}^{\theta^k} |h(t)| \frac{dt}{t}\Big)^{1/\gamma} \Big(\int_{\theta^{k-1}}^{\theta^k} \Big| \int_{\mathbb{S}^{n-1}} e^{iP(tx)} \Omega(x)\, d\sigma(x)\Big|^{\gamma'} \frac{dt}{t}\Big)^{1/\gamma'} \\ &\le C(\log\theta)^{1/\gamma}\|h\|_{\Delta_\gamma} |\Omega|^{1-2/\gamma'}_{L^q(\mathbb{S}^{n-1})} \Big(\int_{\theta^{k-1}}^{\theta^k} \Big| \int_{\mathbb{S}^{n-1}} e^{iP(tx)} \Omega(x)\, d\sigma(x)\Big|^{2} \frac{dt}{t}\Big)^{1/\gamma'}. \end{aligned} \tag{4.8}$$

Now

$$\begin{aligned} &\int_{\theta^{k-1}}^{\theta^k} \Big| \int_{\mathbb{S}^{n-1}} e^{iP(tx)} \Omega(x)\, d\sigma(x)\Big|^{2} \frac{dt}{t} \\ &\quad = \int_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \Omega(x)\overline{\Omega(y)} \Big(\int_{\theta^{k-1}}^{\theta^k} e^{i(P(tx)-P(ty))} \frac{dt}{t}\Big)\, d\sigma(x)\, d\sigma(y). \end{aligned} \tag{4.9}$$

By a change of variable and invoking Lemma 2.2 we get

$$\Big| \int_{\theta^{k-1}}^{\theta^k} e^{i(P(tx)-P(ty))} \frac{dt}{t}\Big| \le C\theta\cdot\theta^{-kj_0/d}|P_{j_0}(x)-P_{j_0}(y)|^{-1/d}.$$

By combining the last estimate with the trivial estimate

$$\Big| \int_{\theta^{k-1}}^{\theta^k} e^{i(P(tx)-P(ty))} \frac{dt}{t}\Big| \le \log\theta,$$

we obtain

$$\Big| \int_{\theta^{k-1}}^{\theta^k} e^{i(P(tx)-P(ty))} \frac{dt}{t}\Big| \le C(\log\theta)\theta^{-kj_0/\delta_d}|P_{j_0}(x)-P_{j_0}(y)|^{-\delta_d/q'} \tag{4.10}$$

where $\delta_j = 1/(2j)$ if j is even and $\delta_j = \delta_d = 1/(4j)$ if j is odd. Thus, by (4.8)–(4.10) and since $\|P_{j_0}\|_{L^\infty(\mathbb{S}^{n-1})} = 1$, we have

$$I_{k,\theta} \le C(\log\theta)\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})} \times \biggl(\int\limits_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}} \frac{\|P_{j_0}\|^{\delta_d}_{L^\infty(\mathbb{S}^{n-1})}}{|P_{j_0}(x)-P_{j_0}(y)|^{\delta_d}}\,d\sigma(x)\,d\sigma(y)\biggr)^{1/q'}. \tag{4.11}$$

Now since $\|P_{j_0}\|_{L^\infty(\mathbb{S}^{n-1})} = 1$ and $d/2 < j_0 \le d$ we get

$$\frac{\|P_{j_0}\|^{\delta_d}_{L^\infty(\mathbb{S}^{n-1})}}{|P_{j_0}(x)-P_{j_0}(y)|^{\delta_d}} \le C\frac{\|P_{j_0}\|^{\delta_{j_0}}_{L^\infty(\mathbb{S}^{n-1})}}{|P_{j_0}(x)-P_{j_0}(y)|^{\delta_{j_0}}}.$$

Thus by (4.11) and Lemma 2.3 we have

$$I_{k,\theta} \le C(\log\theta)\theta^{-kj_0\delta_{j_0}}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})},$$

which in turn implies

$$J_2 \le C(\log\theta)\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})}. \tag{4.12}$$

By (4.4)–(4.6) and (4.12) we obtain

$$A_d \le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})} + A_{d/2}.$$

Since $d = 2^m$, we get

$$A_{2^m} \le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})} + A_{2^{m-1}},$$

and hence by induction on m we have

$$A_{2^m} \le Cm(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})} + A_1. \tag{4.13}$$

Now, we need to estimate A_1. To this end, we notice that any $P \in \mathcal{P}(n;1)$ has a non-constant term of the form $a\cdot x$ for some $a \in \mathbb{R}^n$. It is easy to see that

$$\begin{aligned}|J_{\varepsilon,R}(P)| &\le \biggl|\int\limits_{\mathbb{S}^{n-1}}\int\limits_{\varepsilon}^{1} e^{it(a'\cdot x)}h(t)\Omega(x)\,\frac{dt}{t}\,d\sigma(x)\biggr| \\ &\quad+\biggl|\int\limits_{\mathbb{S}^{n-1}}\int\limits_{1}^{R} e^{it(a'\cdot x)}h(t)\Omega(x)\,\frac{dt}{t}\,d\sigma(x)\biggr| \\ &= L_1 + L_2\end{aligned} \tag{4.14}$$

where $a' = a/|a|$. By (1.1) we have

$$\begin{aligned}L_1 &\le \int\limits_{\mathbb{S}^{n-1}}\int\limits_{\varepsilon}^{1} |e^{it(a'\cdot x)}-1|\,|h(t)|\,|\Omega(x)|\,\frac{dt}{t}\,d\sigma(x) \\ &\le \|\Omega\|_{L^1(\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma}.\end{aligned} \tag{4.15}$$

To estimate L_2 we proceed as for J_2 above. For each fixed $R>1$ we have a unique $k_0\in\mathbb{Z}_+$ such that $\theta^{k_0-1}\le R<\theta^{k_0}$. Hence

$$\begin{aligned}(4.16)\qquad L_2 &\le \sup_{k_0\in\mathbb{Z}_+}\int\limits_{\theta^{k_0-1}}^{\theta^{k_0}} |h(t)|\Big|\int\limits_{\mathbb{S}^{n-1}} e^{it(a'\cdot x)}\Omega(x)\,d\sigma(x)\Big|\,\frac{dt}{t}+\sup_{k_0\in\mathbb{Z}_+}\sum_{k=k_0+1}^{\infty}H_k\\ &= E_1+E_2\end{aligned}$$

where

$$H_k=\int\limits_{\theta^{k-1}}^{\theta^k}|h(t)|\Big|\int\limits_{\mathbb{S}^{n-1}} e^{it(a'\cdot x)}\Omega(x)\,d\sigma(x)\Big|\,\frac{dt}{t}.$$

It is easy to see that

$$(4.17)\qquad E_1\le C(\log\theta)\|\Omega\|_{L^q(\mathbb{S}^{n-1})}\|h\|_{\Delta_\gamma}.$$

By Hölder's inequality we have

$$\begin{aligned}(4.18)\qquad H_k &\le C(\log\theta)^{1/\gamma}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})}^{1-2/\gamma'}\\ &\quad\times\Big(\int\limits_{\theta^{k-1}}^{\theta^k}\Big|\int\limits_{\mathbb{S}^{n-1}} e^{it(a'\cdot x)}\Omega(x)\,d\sigma(x)\Big|^2\frac{dt}{t}\Big)^{1/\gamma'}.\end{aligned}$$

Since

$$\begin{aligned}&\int\limits_{\theta^{k-1}}^{\theta^k}\Big|\int\limits_{\mathbb{S}^{n-1}} e^{iP(tx)}\Omega(x)\,d\sigma(x)\Big|^2\frac{dt}{t}\\ &\qquad=\int\limits_{\mathbb{S}^{n-1}\times\mathbb{S}^{n-1}}\Omega(x)\overline{\Omega(y)}\Big(\int\limits_{\theta^{k-1}}^{\theta^k} e^{ita'\cdot(x-y)}\,\frac{dt}{t}\Big)\,d\sigma(x)\,d\sigma(y),\end{aligned}$$

by Hölder's inequality, (4.17) and the estimate

$$(4.19)\qquad \Big|\int\limits_{\theta^{k-1}}^{\theta^k} e^{ita'\cdot(x-y)}\,\frac{dt}{t}\Big|\le C(\log\theta)\theta^{-k}|a'\cdot(x-y)|^{-1/(2q')},$$

we get

$$H_k\le C(\log\theta)\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})},$$

and hence

$$(4.20)\qquad E_2\le C(\log\theta)\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})}.$$

Now (4.14)–(4.16), (4.20), and the definition of A_1 lead to

$$(4.21)\qquad A_1\le C(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})}.$$

Hence, by (4.13) and (4.21) we obtain

$$(4.22)\qquad A_{2^m}\le C(m+1)(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})}.$$

The case of the general d is now easy. Choose a positive integer m so that $2^{m-1} < d \le 2^m$. By definition of A_d and since $\mathcal{P}(n;d) \subset \mathcal{P}(n;2^m)$, we have

$$\begin{aligned} A_d \le A_{2^m} &\le C(m+1)(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})} \\ &\le C(\log d+1)(q-1)^{-1}(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}\|\Omega\|_{L^q(\mathbb{S}^{n-1})}, \end{aligned}$$

which completes the proof of Theorem 1.1. ∎

Proof of Theorem 1.2. We use exactly the same method as in the proof of Theorem 1.1, except for two minor modifications which occur in several places: First, we need to replace $\theta = 2^{q'\gamma'}$ by $\theta = 2^{q'}$. Next, we notice that by Hölder's inequality we have

$$\int_{\theta^{k-1}}^{\theta^k} |h(t)|\,\frac{dt}{t} \le (\log\theta)^{1/\gamma'}\Big(\int_{\theta^{k-1}}^{\theta^k} |h(t)|^\gamma\,\frac{dt}{t}\Big)^{1/\gamma} \le (\log\theta)^{1/\gamma'}\|h\|_{L^\gamma(\mathbb{R}_+,dr/r)}.$$

The details are omitted. ∎

Proof of Theorem 1.3(a). We follow the extrapolation method of Yano [19], and we present a similar argument to [4] and [15]. Assume that $\Omega \in L(\log L)(\mathbb{S}^{n-1})$ and satisfies (1.1). Let $S_{\Omega,h} = \sup_{P\in\mathcal{P}(n;d)} I_{\Omega,h}(P)$. Fix $\gamma \in (1,2]$, $h \in \Delta_\gamma(\mathbb{R}_+)$, and put $T(\Omega) = S_{\Omega,h}$. Then we have $T(\Omega_1+\Omega_2) \le T(\Omega_1) + T(\Omega_2)$. Now, we decompose Ω as follows: For $m \in \mathbb{N}$, let $\mathbb{E}_m = \{x \in \mathbb{S}^{n-1} : 2^m \le |\Omega(x)| < 2^{m+1}\}$. For $m \in \mathbb{N}$, set $b_m = \Omega\chi_{\mathbb{E}_m}$ where χ_A is the characteristic function of a set A. Let

$$E(\Omega) = \{m \in \mathbb{N} : \|b_m\|_1 \ge 2^{-4m}\},$$

and define the sequence $\{\Omega_m\}_{m\in E(\Omega)\cup\{0\}}$ of functions by

$$\begin{aligned} \Omega_m(x) &= \|b_m\|_1^{-1}\Big(b_m(x) - \int_{\mathbb{S}^{n-1}} b_m(x)\,d\sigma(x)\Big) \quad \text{for } m \in E(\Omega), \\ \Omega_0(x) &= \Omega(x) - \sum_{m\in E(\Omega)} \|b_m\|_1\Omega_m(x). \end{aligned}$$

It is easy to verify that the following hold:

$$\sum_{m\in E(\Omega)} m\|b_m\|_1 \le \frac{1}{\sqrt{\log 2}}\|\Omega\|_{L(\log L)(\mathbb{S}^{n-1})}, \tag{4.23}$$

$$\int_{\mathbb{S}^{n-1}} \Omega_m(u)\,d\sigma(u) = 0 \quad \text{for all } m \in E(\Omega)\cup\{0\}, \tag{4.24}$$

$$\|\Omega_m\|_{1+1/m} \le 2^6 \quad \text{for } m \in E(\Omega) \quad \text{and} \quad \|\Omega_0\|_2 \le 2^2. \tag{4.25}$$

By (4.23)–(4.25) and invoking Theorem 1.1, we obtain

$$\begin{aligned}T(\Omega) &\le T(\Omega_0) + \sum_{m\in E(\Omega)} \|b_m\|_1 T(\Omega_m)\\ &\le C(\log d+1)(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}\\ &\quad\times\Big(\|\Omega_0\|_{L^2(\mathbb{S}^{n-1})} + \sum_{m\in E(\Omega)} m\|b_m\|_1\|\Omega_m\|_{1+1/m}\Big)\\ &\le C(\log d+1)(1+\|\Omega\|_{L\log L(\mathbb{S}^{n-1})})(\gamma-1)^{-1}\|h\|_{\Delta_\gamma}.\end{aligned}$$

Now, fix $\Omega \in L\log L(\mathbb{S}^{n-1})$ and let $L(h) = S_{\Omega,h}$. Decompose h as follows: For $m \in \mathbb{N}$, let $\mathbb{E}_m = \{x \in \mathbb{R}_+ : 2^m \le |h(x)| < 2^{m+1}\}$. For $m \in \mathbb{N}$, set $h_m = h\chi_{\mathbb{E}_m}$and set $D(h) = \{m \in \mathbb{N} : d_m(h) \ge 2^{-4m}\}$. Also, let $h_0 = h - \sum_{m\in D(h)} h_m$. Then it is easy to verify that

$$\|h_m\|_{\Delta_{1+1/m}} \le 2^m (d_m(h))^{m/(m+1)} \le 2^m d_m(h), \tag{4.26}$$

$$\|h_0\|_{\Delta_2} \le 32. \tag{4.27}$$

Now by (4.26)–(4.27) and applying Theorem 1.1, we get

$$\begin{aligned}L(h) &\le L(h_0) + \sum_{m\in D(h)} d_m(h) L(h_m)\\ &\le C(\log d+1)(1+\|\Omega\|_{L\log L(\mathbb{S}^{n-1})})\Big(32 + \sum_{m\in D(h)} m2^m d_m(h)\Big)\\ &\le C(\log d+1)(1+\|\Omega\|_{L\log L(\mathbb{S}^{n-1})})(1+N_1(h)). \ \blacksquare\end{aligned}$$

Proof of Theorem 1.4(a). Assume that $\Omega \in B_q^{(0,0)}(\mathbb{S}^{n-1})$ for some $q > 1$ satisfies (1.1). Without loss of generality, we may assume $1 < q \le 2$. Fix $\gamma \in (1,2]$ and $h \in \Delta_\gamma(\mathbb{R}_+)$. Let $S_{\Omega,h} = \sup_{P\in\mathcal{P}(n;d)} I_{\Omega,h}(P)$. Put $T(\Omega) = S_{\Omega,h}$. Since $\Omega \in B_q^{(0,0)}(\mathbb{S}^{n-1})$, we can write Ω as $\Omega = \sum_{\mu=1}^\infty \lambda_\mu b_\mu$ where $\lambda_\mu \in \mathbb{C}$, b_μ is a q-block supported on a cap I_μ on $\mathbb{S}^{n-1}$, and $M_q^{(0,0)}(\{\lambda_\mu\}) < \infty$. For each block function $b_\mu(\cdot)$, define

$$\tilde{\Omega}_\mu(x) = b_\mu(x) - \int_{\mathbb{S}^{n-1}} b_\mu(u)\,d\sigma(u).$$

Let $\mathbb{K} = \{\mu \in \mathbb{N} : |I_\mu| < e^{-(q-1)^{-1}}\}$ and $\tilde{\Omega}_0 = \Omega - \sum_{\mu\in\mathbb{K}} \lambda_\mu \tilde{\Omega}_\mu$. Also, for $\mu \in \mathbb{K}$ let $\alpha_\mu = \log(|I_\mu|^{-1})$ and $\beta = \sum_{\mu=1}^\infty |\lambda_\mu|$. Then it is easy to see that

$$\int_{\mathbb{S}^{n-1}} \tilde{\Omega}_\mu(u)\,d\sigma(u) = 0 \quad \text{for all } \mu \in \mathbb{K}\cup\{0\}, \tag{4.28}$$

$$\|\tilde{\Omega}_0\|_q \le \beta e^{1/q}, \tag{4.29}$$

$$\|\tilde{\Omega}_\mu\|_{1+1/\alpha_\mu} \le 4 \quad \text{for all } \mu \in \mathbb{K}. \tag{4.30}$$

By (4.28)–(4.30) and invoking Theorem 1.1, we get

$$\begin{aligned}T(\Omega) &\le T(\tilde{\Omega}_0) + \sum_{\mu\in\mathbb{K}} |\lambda_\mu| T(\tilde{\Omega}_\mu)\\ &\le C(\log d+1)\Big((q-1)^{-1}\|\tilde{\Omega}_0\|_q + \sum_{\mu\in\mathbb{K}} |\lambda_\mu| \log|I_\mu|^{-1} \|\tilde{\Omega}_\mu\|_{1+1/\alpha_\mu}\Big)\\ &\le C(\log d+1)\Big(\beta e^{1/q}(q-1)^{-1} + 4\sum_{\mu\in\mathbb{K}} |\lambda_\mu| \log|I_\mu|^{-1}\Big)\\ &\le C(\log d+1)(1+\|\Omega\|_{B_q^{(0,0)}(\mathbb{S}^{n-1})}). \ \blacksquare\end{aligned}$$

Proof of Theorem 1.5. The proof uses the same argument employed in the proofs of Theorems 1.3(a) and 1.4(a). The details are omitted. ■

5. Proof of the one-dimensional case of Theorem 1.1. In this section we shall present another proof of Theorem A. The idea of the proof is similar to that in the proof of Theorem 1.1. Let

$$K_d = \sup_{\substack{0<\varepsilon<R\\ P\in\mathcal{P}(1;d)}} |H_{\varepsilon,R}(P)| \quad \text{where} \quad H_{\varepsilon,R}(P) = \int_{\varepsilon\le|x|\le R} e^{iP(t)}\,\frac{dt}{t}.$$

We need to show that

$$K_d \le C(\log d+1) \tag{5.1}$$

for some absolute positive constant C. We shall first prove (5.1) for the case $d = 2^m$ for some integer $m \ge 0$, and then the general case will be an immediate consequence. Fix $P(t) = a_0 + a_1 t + \cdots + a_d t^d \in \mathcal{P}(1;d)$. We may assume, without loss of generality, that $a_0 = 0$. Let $Q(t) = a_1 t + \cdots + a_{d/2} t^{d/2}$. Let $|a_{j_0}| = \max_{d/2<j\le d} |a_j|$. Since ε and R are arbitrary positive numbers, by a dilation in t we may assume, without loss of generality, that $|a_{j_0}| = 1$. Now,

$$|H_{\varepsilon,R}(P)| \le \Big|\int_{\varepsilon\le|t|\le1} e^{iP(t)}\,\frac{dt}{t}\Big| + \Big|\int_{1\le|t|\le R} e^{iP(t)}\,\frac{dt}{t}\Big| = X_1 + X_2. \tag{5.2}$$

Let us first estimate X_1:

$$\begin{aligned}X_1 &\le \int_{0\le|t|\le1} |e^{iP(t)} - e^{iQ(t)}|\,\frac{dt}{t} + \Big|\int_{\varepsilon\le|t|\le1} e^{iQ(t)}\,\frac{dt}{t}\Big|\\ &\le \sum_{d/2<j\le d} \frac{|a_j|}{j} + C_{d/2}.\end{aligned}$$

Since $|a_j| \le 1$ for $d/2 < j \le d$, we get

$$X_1 \le C + K_{d/2}. \tag{5.3}$$

Now we estimate X_2. We notice that

$$X_2 \leq \left| \int\limits_{1\leq t\leq R} e^{iP(t)}\, \frac{dt}{t} \right| + \left| \int\limits_{-R\leq t\leq -1} e^{iP(t)}\, \frac{dt}{t} \right| = X_2^+ + X_2^-. \tag{5.4}$$

To estimate X_2^+, notice that for each fixed $R > 1$ we have a unique $j_0 \in \mathbb{Z}_+$ such that $2^{j_0-1} \leq R < 2^{j_0}$. Hence

$$X_2^+ \leq \sup_{j_0\in\mathbb{Z}_+} \left| \int\limits_{2^{j_0-1}}^{2^{j_0}} e^{iP(t)}\, \frac{dt}{t} \right| + \sup_{j_0\in\mathbb{Z}_+} \left| \sum_{k=j_0+1}^{\infty} \int\limits_{1/2}^{1} e^{iP(2^k t)}\, \frac{dt}{t} \right| = Y_1 + Y_2. \tag{5.5}$$

It is easy to see that

$$Y_1 \leq \log 2. \tag{5.6}$$

To estimate Y_2 observe that by Lemma 2.2 we have

$$\left| \int\limits_{1/2}^{1} e^{iP(2^k t)}\, \frac{dt}{t} \right| \leq C 2^{-k(j_0-1)/d},$$

which in turn implies

$$Y_2 \leq C \sup_{j_0\in\mathbb{Z}_+} \sum_{k=j_0+1}^{\infty} 2^{-k(j_0-1)/d} \leq C. \tag{5.7}$$

By (5.5)–(5.7) we obtain

$$X_2^+ \leq C. \tag{5.8}$$

Similarly, we get

$$X_2^- \leq C. \tag{5.9}$$

Therefore, by (5.2)–(5.4) and (5.8)–(5.9), we get

$$K_d \leq C + K_{d/2}.$$

Now, we argue as in the n-dimensional case to finish the proof of Theorem A. The details are omitted. ■

References

[1] H. M. Al-Qassem, A. Al-Salman and Y. Pan, *Singular integrals associated to homogeneous mappings with rough kernels*, Hokkaido Math. J. 33 (2004), 551–569.

[2] H. M. Al-Qassem, L. Cheng, A. Fukui and Y. Pan, *Weighted estimates for oscillatory singular integrals*, J. Funct. Spaces Appl. 2013, art. ID 543748, 3 pp.

[3] H. M. Al-Qassem, L. Cheng and Y. Pan, *Rough oscillatory singular integrals on* $\mathbb{R}^n$, Acta Math. Sinica (English Ser.) 27 (2011), 1881–1898.

[4] H. Al-Qassem and Y. Pan, *On certain estimates for Marcinkiewicz integrals and extrapolation*, Collect. Math. 60 (2009), 123–145.

[5] A. Carbery and J. Wright, *Distributional and L^q norm inequalities for polynomials over convex bodies in* $\mathbb{R}^n$, Math. Res. Lett. 8 (2001), 233–248.

[6] L. K. Chen and H. Lin, *A maximal operator related to a class of singular integrals*, Illinois J. Math. 34 (1990), 120–126.

[7] D. Fan and Y. Pan, *Singular integral operators with rough kernels supported by subvarieties*, Amer. J. Math. 119 (1997), 799–839.

[8] R. Fefferman, *A note on singular integrals*, Proc. Amer. Math. Soc. 74 (1979), 266–270.

[9] M. Keitoku and E. Sato, *Block spaces on the unit sphere in* $\mathbb{R}^n$, Proc. Amer. Math. Soc. 119 (1993), 453–455.

[10] S. Lu, M. Taibleson and G. Weiss, *Spaces Generated by Blocks*, Beijing Normal Univ. Press, Beijing, 1989.

[11] M. Papadimitrakis and I. R. Parissis, *Singular oscillatory integrals on* $\mathbb{R}^n$, Math. Z. 266 (2010), 169–179.

[12] I. R. Parissis, *A sharp bound for the Stein–Wainger oscillatory integral*, Proc. Amer. Math. Soc. 136 (2008), 963–972.

[13] I. R. Parissis, *Oscillatory integrals with polynomial phase*, Ph.D. thesis, Univ. of Crete, 2007.

[14] F. Ricci and E. M. Stein, *Harmonic analysis on nilpotent groups and singular integrals I: oscillatory integrals*, J. Funct. Anal. 73 (1987), 179–194.

[15] S. Sato, *Estimates for singular integrals and extrapolation*, Studia Math. 192 (2009), 219–233.

[16] E. M. Stein, *Oscillatory integrals in Fourier analysis*, in: Beijing Lectures in Harmonic Analysis (Beijing, 1984), Ann. of Math. Stud. 112, Princeton Univ. Press, Princeton, NJ, 1986, 307–355.

[17] E. M. Stein and S. Wainger, *The estimation of an integral arising in multiplier transformations*, Studia Math. 35 (1970), 101–104.

[18] M. Weiss and A. Zygmund, *An example in the theory of singular integrals*, Studia Math. 26 (1965), 101–111.

[19] S. Yano, *An extrapolation theorem*, J. Math. Soc. Japan 3 (1951), 296–305.

[20] A. Zygmund, *Trigonometric Series*, 2nd ed., Cambridge Univ. Press, Cambridge, 1977.

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Received August 6, 2013 (7828)