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The Daugavet property and translation-invariant subspaces

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Abstract. Let G be an infinite, compact abelian group and let Λ be a subset of its dual group Γ. We study the question which spaces of the form $C_\Lambda(G)$ or $L^1_\Lambda(G)$ and which quotients of the form $C(G)/C_\Lambda(G)$ or $L^1(G)/L^1_\Lambda(G)$ have the Daugavet property.

We show that $C_\Lambda(G)$ is a rich subspace of $C(G)$ if and only if $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set. If $L^1_\Lambda(G)$ is a rich subspace of $L^1(G)$, then $C_\Lambda(G)$ is a rich subspace of $C(G)$ as well. Concerning quotients, we prove that $C(G)/C_\Lambda(G)$ has the Daugavet property if Λ is a Rosenthal set, and that $L^1_\Lambda(G)$ is a poor subspace of $L^1(G)$ if Λ is a nicely placed Riesz set.

1. Introduction. I. K. Daugavet [3] proved in 1963 that all compact operators T on $C[0,1]$ satisfy the norm identity

$$\|\mathrm{Id} + T\| = 1 + \|T\|,$$

which has become known as the *Daugavet equation*. C. Foiaş and I. Singer [5] extended this result to all weakly compact operators on $C[0,1]$ and A. Pełczyński [5, p. 446] observed that their argument can also be used for weakly compact operators on $C(K)$ provided that K is a compact space without isolated points. Shortly afterwards, G. Ya. Lozanovskiĭ [20] showed that the Daugavet equation holds for all compact operators on $L^1[0,1]$, and J. R. Holub [12] extended this result to all weakly compact operators on $L^1(\Omega, \Sigma, \mu)$ where μ is a σ-finite non-atomic measure. V. M. Kadets, R. V. Shvidkoy, G. G. Sirotkin, and D. Werner [17] proved that the validity of the Daugavet equation for weakly compact operators already follows from the corresponding statement for operators of rank one. This result led to the following definition: A Banach space X is said to have the *Daugavet property* if every operator $T : X \to X$ of rank one satisfies the Daugavet equation.

Examples include the aforementioned spaces $C(K)$ and $L^1(\Omega, \Sigma, \mu)$, certain function algebras such as the disk algebra $A(\mathbb{D})$ or the algebra of bounded analytic functions H^∞ [28, 29], and non-atomic C^*-algebras [23].

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If X has the Daugavet property, not only all weakly compact operators on X satisfy the Daugavet equation but also all strong Radon–Nikodým operators [17], meaning operators T for which $\overline{T[B_X]}$ is a Radon–Nikodým set, and operators not fixing a copy of ℓ^1 [27]. Furthermore, X fails the Radon–Nikodým property [29], contains a copy of ℓ^1 [17], does not have an unconditional basis [13], and does not even embed into a space with an unconditional basis [17].

The listed properties give the impression that spaces with the Daugavet property are "big". It is therefore an interesting question which subspaces of a space X with the Daugavet property inherit this property. One approach is to look at closed subspaces Y such that the quotient space X/Y is "small". For this purpose, V. M. Kadets and M. M. Popov [15] introduced on $C[0,1]$ and $L^1[0,1]$ the class of *narrow* operators, a generalization of the class of compact operators, and called a subspace *rich* if the corresponding quotient map is narrow. This concept was transferred to spaces with the Daugavet property by V. M. Kadets, R. V. Shvidkoy, and D. Werner [18]. Rich subspaces inherit the Daugavet property and the class of narrow operators includes all weakly compact operators, all strong Radon–Nikodým operators, and all operators which do not fix copies of ℓ^1 [18].

If Y is a rich subspace of a Banach space X with the Daugavet property, then not only Y inherits the Daugavet property but also every closed subspace of X which contains Y. In view of this property, V. M. Kadets, V. Shepelska, and D. Werner introduced a similar notion for quotients of X and called a closed subspace Y *poor* if X/Z has the Daugavet property for every closed subspace $Z \subset Y$. They also showed that poverty is a dual property to richness [16].

Let us consider an infinite, compact abelian group G with its Haar measure m. Since G has no isolated points and m has no atoms, the spaces $C(G)$ and $L^1(G)$ have the Daugavet property. Using the group structure of G, we can translate functions that are defined on G and look at closed, translation-invariant subspaces of $C(G)$ or $L^1(G)$. These subspaces can be described via subsets Λ of the dual group Γ and are of the form $C_\Lambda(G) = \{f \in C(G) : \operatorname{spec} f \subset \Lambda\}$ and $L^1_\Lambda(G) = \{f \in L^1(G) : \operatorname{spec} f \subset \Lambda\}$, where

$$\operatorname{spec} f = \{\gamma \in \Gamma : \hat{f}(\gamma) \neq 0\}.$$

We are going to study the question which closed, translation-invariant subspaces of $C(G)$ and $L^1(G)$ and which quotients of the form $C(G)/C_\Lambda(G)$ or $L^1(G)/L^1_\Lambda(G)$ have the Daugavet property. We will show that $C_\Lambda(G)$ is a rich subspace of $C(G)$ if and only if $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set, and that $C_\Lambda(G)$ is rich in $C(G)$ if $L^1_\Lambda(G)$ is rich in $L^1(G)$. We will prove that $C(G)/C_\Lambda(G)$ has the Daugavet property if $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$, and that $L^1(G)/L^1_\Lambda(G)$ has the Daugavet property if $C_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $C(G)$. We will furthermore identify a big class of poor, translation-invariant subspaces of $L^1(G)$.

2. Preliminaries. Let $\mathbb{T}$ be the *circle group*, i.e., the multiplicative group of all complex numbers with absolute value one. In what follows, G will be an infinite, compact abelian group with addition as group operation and e_G as identity element. $\mathcal{B}(G)$ will denote its Borel σ-algebra, m its normalized Haar measure, Γ its (discrete) *dual group*, i.e., the group of all continuous homomorphisms from G into $\mathbb{T}$, and Λ a subset of Γ. Linear combinations of elements of Γ are called *trigonometric polynomials* and we set $T(G) = \operatorname{lin}\Gamma$. We will write $\mathbf{1}_G$ for the identity element of Γ, which coincides with the function identically equal to one.

LEMMA 2.1. *If O is an open neighborhood of e_G, then there exists a covering of G by disjoint Borel sets $B_1, \dots, B_n$ with $B_k - B_k \subset O$ for $k = 1, \dots, n$.*

Proof. Let V be an open neighborhood of e_G with $V - V \subset O$. Since G is compact, we can choose $x_1, \dots, x_n \in G$ with $G = \bigcup_{k=1}^n (x_k + V)$. Set $B_1 = x_1 + V$ and $B_k = (x_k + V) \setminus \bigcup_{l=1}^{k-1} B_l$ for $k = 2, \dots, n$. Then $B_1, \dots, B_n$ is a covering of G by disjoint Borel sets and for every $k \in \{1, \dots, n\}$,

$$B_k - B_k \subset (x_k + V) - (x_k + V) \subset V - V \subset O. \ \blacksquare$$

$L^1(G)$ and $M(G)$, the space of all regular Borel measures on G, are commutative Banach algebras with respect to convolution, and $L^1(G)$ is a closed ideal of $M(G)$ [25, Theorems 1.1.7, 1.3.2, and 1.3.5]. If $\mu \in M(G)$, its *Fourier–Stieltjes transform* is defined by

$$\hat{\mu}(\gamma) = \int_G \overline{\gamma}\, d\mu \qquad (\gamma \in \Gamma),$$

and the map $\mu \mapsto \hat{\mu}$ is injective, multiplicative, and continuous [25, Theorems 1.3.3 and 1.7.3]. $L^1(G)$ does not have a unit, unless G is discrete. But we always have an approximate unit [11, Remark VIII.32.33(c) and Theorem VIII.33.12].

PROPOSITION 2.2. *There is a net $(v_j)_{j \in J}$ in $L^1(G)$ with the following properties:*

(i) $\|f - f * v_j\|_1 \to 0$ *for every* $f \in L^1(G)$;
(ii) $\|f - f * v_j\|_\infty \to 0$ *for every* $f \in C(G)$;
(iii) $v_j \geq 0$, $v_j \in T(G)$ *and* $\hat{v}_j \geq 0$ *for every* $j \in J$;
(iv) $\|v_j\|_1 = 1$ *for every* $j \in J$;
(v) $\hat{v}_j(\gamma) \to 1$ *for every* $\gamma \in \Gamma$.

If $f : G \to \mathbb{C}$ is a function and x an element of G, the *translate* f_x of f is defined by

$$f_x(y) = f(y - x) \quad (y \in G).$$

A subspace X of $L^1(G)$ or $C(G)$ is called *translation-invariant* if X contains with a function f all possible translates f_x. As already mentioned in the introduction, all closed, translation-invariant subspaces of $C(G)$ or $L^1(G)$ are of the form $C_\Lambda(G)$ or $L^1_\Lambda(G)$ [11, Theorem IX.38.7], where Λ is a subset of Γ. We define $T_\Lambda(G)$, $L^\infty_\Lambda(G)$, and $M_\Lambda(G)$ analogously. Note that by Proposition 2.2 the space $T_\Lambda(G)$ is $\|\cdot\|_\infty$-dense in $C_\Lambda(G)$ and $\|\cdot\|_1$-dense in $L^1_\Lambda(G)$.

We will need the following characterization of the Daugavet property [17, Lemma 2.2].

LEMMA 2.3. *Let X be a Banach space. The following assertions are equivalent:*

(i) *X has the Daugavet property.*
(ii) *For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$ there is some $y \in S_X$ such that* $\operatorname{Re} x^*(y) \geq 1 - \varepsilon$ *and* $\|x + y\| \geq 2 - \varepsilon$.
(iii) *For every $x \in S_X$, $x^* \in S_{X^*}$, and $\varepsilon > 0$ there is some $y^* \in S_{X^*}$ such that* $\operatorname{Re} y^*(x) \geq 1 - \varepsilon$ *and* $\|x^* + y^*\| \geq 2 - \varepsilon$.

3. Structure-preserving isometries. The Daugavet property depends crucially on the norm of a space and is preserved under isometries but in general not under isomorphisms. Considering translation-invariant subspaces of $C(G)$ and $L^1(G)$, it would be useful to know isometries that map translation-invariant subspaces onto translation-invariant subspaces.

DEFINITION 3.1. Let G_1 and G_2 be locally compact abelian groups with dual groups Γ_1 and Γ_2. Let $H : G_1 \to G_2$ be a continuous homomorphism. The *adjoint homomorphism* $H^* : \Gamma_2 \to \Gamma_1$ is defined by

$$H^*(\gamma) = \gamma \circ H \qquad (\gamma \in \Gamma_2).$$

The adjoint homomorphism H^* is continuous [10, Theorem VI.24.38], $H^{**} = H$ [10, VI.24.41(a)], and $H^*[\Gamma_2]$ is dense in Γ_1 if and only if H is one-to-one [10, VI.24.41(b)].

LEMMA 3.2. *Let $H : G \to G$ be a continuous and surjective homomorphism. Then H is measure-preserving, i.e., each Borel set B of G satisfies $m(H^{-1}[B]) = m(B)$.*

Proof. Denote by μ the push-forward of m under H. It is easy to see that μ is regular and $\mu(G) = 1$. Since the Haar measure is uniquely determined, it suffices to show that μ is translation-invariant.

Fix $B \in \mathcal{B}(G)$ and $x \in G$. H is surjective and thus there is $y \in G$ with $H(y) = x$. It is not difficult to check that $H^{-1}[B + H(y)] = H^{-1}[B] + y$. Using this equality, we get

$$\begin{aligned}\mu(B+x) &= m(H^{-1}[B+H(y)]) = m(H^{-1}[B]+y)\\ &= m(H^{-1}[B]) = \mu(B). \ \blacksquare\end{aligned}$$

PROPOSITION 3.3. *Let $H : \Gamma \to \Gamma$ be a one-to-one homomorphism and let Λ be a subset of Γ. Then $C_\Lambda(G) \cong C_{H[\Lambda]}(G)$ and $L^1_\Lambda(G) \cong L^1_{H[\Lambda]}(G)$.*

Proof. If we define $T : C(G) \to C(G)$ by

$$T(f) = f \circ H^* \quad (f \in C(G)),$$

then T is well-defined and an isometry because H^* is continuous and surjective. (Note that $H^*[G]$ is compact and therefore closed.) For every trigonometric polynomial $f = \sum_{k=1}^n a_k \gamma_k$ and every $x \in G$ we get

$$T(f)(x) = \sum_{k=1}^n a_k \gamma_k(H^*(x)) = \sum_{k=1}^n a_k H(\gamma_k)(x).$$

Hence for every $\Lambda \subset \Gamma$, T maps the space $T_\Lambda(G)$ onto $T_{H[\Lambda]}(G)$ and by density the space $C_\Lambda(G)$ onto $C_{H[\Lambda]}(G)$.

Let us look at the same T but now as an operator from $L^1(G)$ into itself. It is again an isometry because H^* is measure-preserving by Lemma 3.2. For every $\Lambda \subset \Gamma$, it still maps the space $T_\Lambda(G)$ onto $T_{H[\Lambda]}(G)$ and so by density $L^1_\Lambda(G)$ onto $L^1_{H[\Lambda]}(G)$. ■

COROLLARY 3.4. *Let $H : \Gamma \to \Gamma$ be a one-to-one homomorphism. If $C_\Lambda(G)$ has the Daugavet property, then $C_{H[\Lambda]}(G)$ has the Daugavet property as well. Analogously, if $L^1_\Lambda(G)$ has the Daugavet property, then $L^1_{H[\Lambda]}(G)$ has the Daugavet property as well.*

Let us give an example. Every one-to-one homomorphism on $\mathbb{Z}$ is of the form $k \mapsto nk$ where n is a fixed non-zero integer. So $C_\Lambda(\mathbb{T}) \cong C_{n\Lambda}(\mathbb{T})$ and $L^1_\Lambda(\mathbb{T}) \cong L^1_{n\Lambda}(\mathbb{T})$ for every non-zero integer n.

4. Rich subspaces

DEFINITION 4.1. Let X be a Banach space with the Daugavet property and let E be an arbitrary Banach space. An operator $T \in L(X, E)$ is called *narrow* if for any $x, y \in S_X$, $x^* \in X^*$, and $\varepsilon > 0$ there is an element $z \in S_X$ such that $\|T(y-z)\| + |x^*(y-z)| \le \varepsilon$ and $\|x+z\| \ge 2-\varepsilon$. A closed subspace Y of X is said to be *rich* if the quotient map $\pi : X \to X/Y$ is narrow.

A rich subspace inherits the Daugavet property. But even a little bit more is true [18, Theorem 5.2].

Proposition 4.2. *Let X be a Banach space with the Daugavet property and let Y be a rich subspace. Then (Y,X) is a Daugavet pair, i.e., for every $x\in S_X$, $y^*\in S_{Y^*}$, and $\varepsilon>0$ there is some $y\in S_Y$ with $\operatorname{Re} y^*(y)\ge 1-\varepsilon$ and $\|x+y\|\ge 2-\varepsilon$.*

Proof. Fix $x\in S_X$, $y^*\in S_{Y^*}$, and $\varepsilon>0$. Choose $\delta>0$ with $\frac{1-3\delta}{1+\delta}\ge 1-\varepsilon$ and $z\in S_Y$ with $\operatorname{Re} y^*(z)\ge 1-\delta$. Since Y is a rich subspace of X, there exists $x_0\in S_X$ with $d(x_0,Y)=d(z-x_0,Y)<\delta$, $|y^*(z-x_0)|\le\delta$, and $\|x+x_0\|\ge 2-\delta$. Fix $y_0\in Y$ with $\|x_0-y_0\|\le\delta$ and set $y=y_0/\|y_0\|$. Then

$$\operatorname{Re} y^*(y_0)\ge \operatorname{Re} y^*(z)-|y^*(z-x_0)|-\|x_0-y_0\|\ge 1-3\delta$$

and

$$\|x_0-y\|\le\|x_0-y_0\|+\|y_0-y\|\le 2\delta.$$

So by our choice of δ we get $\operatorname{Re} y^*(y)\ge 1-\varepsilon$ and $\|x+y\|\ge 2-\varepsilon$. ∎

Let us recall the following characterizations of narrow operators on $C(K)$ spaces [18, Theorem 3.7] and on $L^1(\Omega,\Sigma,\mu)$ spaces [14, Theorem 2.1], [18, Theorem 6.1].

Proposition 4.3. *Let K be a compact space without isolated points and let E be a Banach space. An operator $T\in L(C(K),E)$ is narrow if and only if for every non-empty open set O and every $\varepsilon>0$ there is a function $f\in S_{C(K)}$ with $f|_{K\setminus O}=0$ and $\|T(f)\|\le\varepsilon$.*

Remark. In Proposition 4.3, the function f can be chosen to be real-valued and non-negative. This was proven for $C(K,\mathbb{R})$ in [15, Lemma 1.4]. The same proof works with minor modifications for $C(K,\mathbb{C})$ as well.

Proposition 4.4. *Let (Ω,Σ,μ) be a non-atomic probability space and let E be a Banach space. A function $f\in L^1(\Omega)$ is said to be a* balanced ε-peak *on $A\in\Sigma$ if f is real-valued, $f\ge -1$, $\chi_A f=f$, $\int_\Omega f\,d\mu=0$, and $\mu(\{f=-1\})\ge\mu(A)-\varepsilon$. An operator $T\in L(L^1(\Omega),E)$ is narrow if and only if for every $A\in\Sigma$ and every $\delta,\varepsilon>0$ there is a balanced ε-peak f on A with $\|T(f)\|\le\delta$.*

Corollary 4.5. *If $C_\Lambda(G)$ is a rich subspace of $C(G)$, then for every $x\in G$, every open neighborhood O of e_G, and every $\varepsilon>0$ there exists a real-valued and non-negative $f\in S_{C(G)}$ with $f(x)=1$, $f|_{G\setminus(x+O)}=0$, and $d(f,C_\Lambda(G))\le\varepsilon$.*

Proof. Let V be a symmetric open neighborhood of e_G with $V+V\subset O$. Since $C_\Lambda(G)$ is a rich subspace of $C(G)$, we can pick a real-valued, non-negative $g\in S_{C(G)}$ with $g|_{G\setminus V}=0$ and $d(g,C_\Lambda(G))\le\varepsilon$. Fix $x_0\in V$ with $g(x_0)=1$ and set $f=g_{x-x_0}$. This function is still at a distance of at most ε from $C_\Lambda(G)$ because $C_\Lambda(G)$ is translation-invariant. Furthermore, $f(x)=1$

and $f|_{G\setminus(x+O)} = 0$ by our choice of V. In fact, if we pick $y \in G$ with $f(y) \neq 0$, we get $g(y-x+x_0) = f(y) \neq 0$. Consequently, $y - x + x_0 \in V$ and $y \in x - x_0 + V \subset x + V + V \subset x + O$. ■

We have seen in Proposition 4.2 that a rich subspace inherits the Daugavet property. But even more is true. A closed subspace Y of X is rich if and only if every closed subspace Z of X with $Y \subset Z \subset X$ has the Daugavet property [18, Theorem 5.12]. In order to prove that a translation-invariant subspace Y of $C(G)$ or $L^1(G)$ is rich, we do not have to consider all subspaces of $C(G)$ or $L^1(G)$ containing Y but only the translation-invariant ones.

LEMMA 4.6. *Let X be a Banach space. If for every $\varepsilon > 0$ there is a Banach space Y that has the Daugavet property and is $(1+\varepsilon)$-isomorphic to X, then X has the Daugavet property.*

Since the proof is straightforward, it will be omitted.

PROPOSITION 4.7. *Suppose Λ is a subset of Γ such that $C_\Theta(G)$ has the Daugavet property for all $\Lambda \subset \Theta \subset \Gamma$. Then $C_\Lambda(G)$ is a rich subspace of $C(G)$. The analogous statement is valid for subspaces of $L^1(G)$.*

Proof. We will only prove the result for subspaces of $C(G)$. The proof for subspaces of $L^1(G)$ works the same way.

It suffices to show that for arbitrary $f_1, f_2 \in S_{C(G)}$ the linear span of $C_\Lambda(G)$, f_1 and f_2 has the Daugavet property [18, Lemma 5.6]. In order to do this, we are going to prove that $X = \operatorname{lin}\{C_\Lambda(G) \cup \{f_1, f_2\}\}$ meets the assumptions of Lemma 4.6.

Fix $\varepsilon > 0$, and suppose that f_1 does not belong to $C_\Lambda(G)$ and f_2 does not belong to $\operatorname{lin}\{C_\Lambda(G) \cup \{f_1\}\}$; the other cases can be treated similarly. Then X is isomorphic to $C_\Lambda(G) \oplus_1 \operatorname{lin}\{f_1\} \oplus_1 \operatorname{lin}\{f_2\}$ and there exists $M > 0$ with

$$M(\|h\|_\infty + |\alpha| + |\beta|) \le \|h + \alpha f_1 + \beta f_2\|_\infty \quad (h \in C_\Lambda(G),\ \alpha, \beta \in \mathbb{C}).$$

Using the density of $T(G)$ in $C(G)$, we choose $g_1, g_2 \in S_{T(G)}$ with $\|f_k - g_k\|_\infty \le M\varepsilon$ for $k = 1, 2$. If we define $T : X \to \operatorname{lin}\{C_\Lambda(G) \cup \{g_1, g_2\}\}$ by

$$T(h + \alpha f_1 + \beta f_2) = h + \alpha g_1 + \beta g_2 \quad (h \in C_\Lambda(G),\ \alpha, \beta \in \mathbb{C}),$$

then T is surjective and $\|T^{-1}\|\|T\| \le (1+\varepsilon)/(1-\varepsilon)$ since

$$\begin{aligned}\|T(h + \alpha f_1 + \beta f_2) - (h + \alpha f_1 + \beta f_2)\|_\infty &\le M\varepsilon(|\alpha| + |\beta|)\\ &\le \varepsilon\|h + \alpha f_1 + \beta f_2\|_\infty\end{aligned}$$

for $h \in C_\Lambda(G)$ and $\alpha, \beta \in \mathbb{C}$.

To complete the proof, we have to show that $Y = \operatorname{lin}\{C_\Lambda(G) \cup \{g_1, g_2\}\}$ has the Daugavet property. Set $\Delta = \operatorname{spec} g_1 \cup \operatorname{spec} g_2$. Since g_1 and g_2 are trigonometric polynomials, the set Δ is finite. By assumption, $C_{\Lambda\cup\Delta}(G)$

has the Daugavet property. The space Y is a finite-codimensional subspace of $C_{\Lambda\cup\Delta}(G)$, and has therefore the Daugavet property as well [17, Theorem 2.14]. ∎

Not all translation-invariant subspaces of $C(G)$ or $L^1(G)$ which have the Daugavet property must be rich. The subspace $C_{2\mathbb{Z}}(\mathbb{T})$ has the Daugavet property because $C(\mathbb{T}) \cong C_{2\mathbb{Z}}(\mathbb{T})$ by Corollary 3.4. But every $f \in C_{2\mathbb{Z}}(\mathbb{T})$ satisfies

$$f(t) = f(-t) \quad (t \in \mathbb{T}),$$

and therefore $C_{2\mathbb{Z}}(\mathbb{T})$ cannot be a rich subspace of $C(\mathbb{T})$. Similarly, $L^1_{2\mathbb{Z}}(\mathbb{T})$ has the Daugavet property but is not a rich subspace of $L^1(\mathbb{T})$.

If X is a Banach space with the Daugavet property, then all operators on X which do not fix ℓ^1 are narrow [18, Theorem 4.13]. This implies that Y is a rich subspace of X if the quotient space X/Y contains no copy of ℓ^1 or if $(X/Y)^*$ has the Radon–Nikodým property [18, Proposition 5.3]. Let us apply these results to translation-invariant subspaces of $C(G)$ or $L^1(G)$.

Definition 4.8.

(a) Λ is called a *Rosenthal set* if every equivalence class of $L^\infty_\Lambda(G)$ contains a continuous member, i.e., $L^\infty_\Lambda(G) = C_\Lambda(G)$.
(b) Λ is called a *Riesz set* if every $\mu \in M_\Lambda(G)$ is absolutely continuous with respect to the Haar measure, i.e., $M_\Lambda(G) = L^1_\Lambda(G)$.

Every Sidon set is a Rosenthal set. (Recall that $\Lambda \subset \Gamma$ is said to be a *Sidon set* if there exists a constant $M > 0$ such that $\sum_{\gamma\in\Lambda} |\hat{f}(\gamma)| \le M\|f\|_\infty$ for all $f \in T_\Lambda(G)$.) But $\bigcup_{n=1}^\infty (2n)!\{1,\ldots,2n\}$ is an example of a Rosenthal set which is not a Sidon set [24, Corollary 4]. Every Rosenthal set is a Riesz set [21, Théorème 3] and it is a classical result due to F. and M. Riesz that $\mathbb{N}$ is a Riesz set [26, Theorem 17.13].

Proposition 4.9. *If Λ is a Riesz set, then $C_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $C(G)$, and if Λ is a Rosenthal set, then $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$.*

Proof. Suppose Λ is a Riesz set. Since $T_{\Gamma\setminus\Lambda^{-1}}(G)$ is dense in $C_{\Gamma\setminus\Lambda^{-1}}(G)$, we have $C_{\Gamma\setminus\Lambda^{-1}}(G)^\perp = M_\Lambda(G)$. Hence $M_\Lambda(G)$ can be identified with the dual space of $C(G)/C_{\Gamma\setminus\Lambda^{-1}}(G)$. Since Λ is a Riesz set, $M_\Lambda(G)$ has the Radon–Nikodým property [21, Théorème 2] and $C_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $C(G)$ [18, Proposition 5.3].

Suppose now that Λ is a Rosenthal set. We apply the same reasoning as before and use the fact that $L^\infty_\Lambda(G)$ has the Radon–Nikodým property if Λ is a Rosenthal set [21, Théorème 1]. ∎

In Section 5, we will give an example of a non-Rosenthal set Λ such that $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$. In the case of translation-invariant subspaces of $C(G)$, the previous result can be strengthened.

DEFINITION 4.10. A measure $\mu \in M(G)$ is said to be *diffuse* or *non-atomic* if $\mu(B) = 0$ for all countable sets $B \subset G$. We denote by $M_{\mathrm{diff}}(G)$ the set of all diffuse members of $M(G)$. A subset Λ of Γ is called a *semi-Riesz set* if every $\mu \in M_\Lambda(G)$ is diffuse.

If G is infinite, then the Haar measure on G is diffuse and every Riesz set of Γ is a semi-Riesz set. The set $\{\sum_{k=0}^{n} \varepsilon_k 4^k : n \in \mathbb{N},\ \varepsilon_k \in \{-1, 0, 1\}\}$ is an example of a proper semi-Riesz set [28, p. 126].

D. Werner showed that $C_{\Gamma\setminus\Lambda^{-1}}(G)$ has the Daugavet property if Λ is a semi-Riesz set [28, Theorem 3.7]. Combining this result with the fact that every subset of a semi-Riesz set is still a semi-Riesz set, by Proposition 4.7 we get the following corollary.

COROLLARY 4.11. *If Λ is a semi-Riesz set, then $C_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $C(G)$.*

The converse implication is also valid.

PROPOSITION 4.12. *If $C_\Lambda(G)$ is a rich subspace of $C(G)$, then $C_\Lambda(G)^\perp$ consists of diffuse measures.*

Proof. Let $[\mu]$ denote the equivalence class of μ in $M(G)/C_\Lambda(G)^\perp$. It suffices to show the following: For every $x \in G$, every $\alpha \in \mathbb{C}$, and every $\mu \in M(G)$ with $\mu(\{x\}) = 0$ we have $\|[\alpha\delta_x] + [\mu]\| = |\alpha| + \|[\mu]\|$. Indeed, if the preceding statement is true, for every $\mu \in C_\Lambda(G)^\perp$ and every $x \in G$ we get

$$0 = \|[\mu]\| = \|[\mu(\{x\})\delta_x] + [\mu - \mu(\{x\})\delta_x]\| = |\mu(\{x\})| + \|[\mu - \mu(\{x\})\delta_x]\|.$$

Hence $|\mu(\{x\})| = 0$ and μ is a diffuse measure.

Fix $x \in G$, $\alpha \in \mathbb{C} \setminus \{0\}$, $\mu \in M(G)$ with $\mu(\{x\}) = 0$, and $\varepsilon > 0$. Choose $f \in S_{C_\Lambda(G)}$ with $\operatorname{Re}\int_G f\,d\mu \ge \|[\mu]\| - \varepsilon$. Since $|\mu|$ is a regular Borel measure and f is a continuous function, there is an open neighborhood O of e_G with $|\mu|(x + O) < \varepsilon$ and $|f(x) - f(x + y)| < \varepsilon$ for all $y \in O$. As $C_\Lambda(G)$ is a rich subspace of $C(G)$, by Corollary 4.5 we can pick a real-valued, non-negative $g_0 \in S_{C(G)}$ with $g_0(x) = 1$, $g_0|_{G\setminus(x+O)} = 0$, and $d(g_0, C_\Lambda(G)) < \varepsilon$.

Let g be an element of $C_\Lambda(G)$ with $\|g - g_0\|_\infty \le \varepsilon$. If we set $h_0 = f + (|\alpha|/\alpha - f(x))g_0$ and $h = f + (|\alpha|/\alpha - f(x))g$, then $h \in C_\Lambda(G)$ and $\|h - h_0\|_\infty \le 2\varepsilon$. Furthermore,

$$\alpha h_0(x) = |\alpha| \tag{4.1}$$

and

$$\operatorname{Re}\int_G h_0\,d\mu = \operatorname{Re}\int_G (f + (|\alpha|/\alpha - f(x))g_0)\,d\mu \tag{4.2}$$
$$\geq \|[\mu]\| - \varepsilon - 2\int_G g_0\,d|\mu| = \|[\mu]\| - \varepsilon - 2\int_{x+O} g_0\,d|\mu|$$
$$\geq \|[\mu]\| - \varepsilon - 2|\mu|(x+O) \geq \|[\mu]\| - 3\varepsilon.$$

Let us estimate the norm of h. For $y \in G \setminus (x+O)$ we get

$$\begin{aligned}|h(y)| &= |f(y) + (|\alpha|/\alpha - f(x))g(y)| \\ &\leq \|f\|_\infty + 2\|g|_{G\setminus(x+O)}\|_\infty \leq 1 + 2\varepsilon,\end{aligned}$$

and for $y \in x + O$,

$$\begin{aligned}|h(y)| &= |f(y) + (|\alpha|/\alpha - f(x))g(y)| \\ &\leq |f(y) - f(x)g_0(y)| + g_0(y) + 2\|g - g_0\|_\infty \\ &\leq |f(y) - f(x)| + |f(x)|(1 - g_0(y)) + g_0(y) + 2\varepsilon \\ &\leq \varepsilon + (1 - g_0(y)) + g_0(y) + 2\varepsilon = 1 + 3\varepsilon.\end{aligned}$$

Hence $\|h\|_\infty \leq 1 + 3\varepsilon$. Combining this estimate with (4.1) and (4.2), we get

$$\begin{aligned}(1+3\varepsilon)\|[\alpha\delta_x] + [\mu]\| &\geq \Big|\int_G h\,d(\alpha\delta_x + \mu)\Big| \\ &\geq \Big|\int_G h_0\,d(\alpha\delta_x + \mu)\Big| - 2\varepsilon\|\alpha\delta_x + \mu\| \\ &\geq |\alpha| + \|[\mu]\| - 3\varepsilon - 2\varepsilon\|\alpha\delta_x + \mu\|.\end{aligned}$$

We can choose $\varepsilon > 0$ arbitrarily small, and so $\|[\alpha\delta_x] + [\mu]\| = |\alpha| + \|[\mu]\|$. ∎

Combining the last two results, we get the following characterization of the closed, translation-invariant subspaces of $C(G)$ that are rich.

Theorem 4.13. *The space $C_\Lambda(G)$ is a rich subspace of $C(G)$ if and only if $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set.*

A linear projection P on a Banach space X is called an *L-projection* if

$$\|x\| = \|P(x)\| + \|x - P(x)\| \quad (x \in X).$$

A closed subspace of X is called an *L-summand* if it is the range of an L-projection. In the proof of Proposition 4.12 we showed that every Dirac measure δ_x still has norm one and still spans an L-summand if we consider it as an element of $C_\Lambda(G)^*$. Such subspaces are called *nicely embedded* and were studied by D. Werner [28]. His proof of the fact that $C_\Lambda(G)$ has the Daugavet property if $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set is as well based on the observation that then $C_\Lambda(G)$ is nicely embedded.

Let us present an alternative proof of Theorem 4.13 for the case that G is metrizable. It is based on results of V. M. Kadets and M. M. Popov [15]. We say that an operator $T \in L(C(G), E)$ vanishes at a point $x \in G$ and write $x \in \operatorname{van} T$ if there exist a sequence $(O_n)_{n\in\mathbb{N}}$ of open neighborhoods of x with $\operatorname{diam} O_n \to 0$ and a sequence $(f_n)_{n\in\mathbb{N}}$ of real-valued and non-negative functions satisfying $f_n \in S_{C(G)}$, $f_n|_{G\setminus O_n} = 0$, $(f_n)_{n\in\mathbb{N}}$ converges pointwise to $\chi_{\{x\}}$, and $\|T(f_n)\| \to 0$. An operator T is narrow if and only if $\operatorname{van} T$ is dense in G [15, Lemma 1.6]. Furthermore, $x \in \operatorname{van} T$ if and only if for any functional $e^* \in E^*$ the point x is not an atom of the measure corresponding to $T^*(e^*)$ [15, Lemma 1.7]. Let Λ be a subset of Γ, let $\pi : C(G) \to C(G)/C_\Lambda(G)$ be the canonical quotient map, and note that

$$\operatorname{ran}(\pi^*) = C_\Lambda(G)^\perp = M_{\Gamma\setminus\Lambda^{-1}}(G).$$

If $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set, then every element of $M_{\Gamma\setminus\Lambda^{-1}}(G)$ is a diffuse measure. Therefore $\operatorname{van}\pi = G$, and π is a narrow operator. Conversely, if π is narrow, it is an easy consequence of Corollary 4.5 that $\operatorname{van}\pi = G$. Therefore, $M_{\Gamma\setminus\Lambda^{-1}}(G)$ must consist of diffuse measures and $\Gamma \setminus \Lambda^{-1}$ is a semi-Riesz set.

Let Λ be a subset of $\mathbb{Z}$ and let $\lambda_1, \lambda_2, \ldots$ be an enumeration of Λ with $|\lambda_1| \le |\lambda_2| \le \cdots$. We say that Λ is *uniformly distributed* if

$$\frac{1}{n}\sum_{k=1}^{n} t^{\lambda_k} \to 0 \quad (t \in \mathbb{T},\ t \ne 1).$$

R. Demazeux proved that $C_\Lambda(\mathbb{T})$ is a rich subspace of $C(\mathbb{T})$ if Λ is uniformly distributed [4, Théorème I.1.7]. Theorem 4.13 shows that $\mathbb{Z}\setminus(-\Lambda)$ is a semi-Riesz set if Λ is uniformly distributed.

THEOREM 4.14. *If $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$, then Λ is a semi-Riesz set.*

Proof. The proof is based on arguments used by G. Godefroy, N. J. Kalton, and D. Li [8, Proposition III.10] and N. J. Kalton [19, Theorem 5.4].

Suppose that Λ is not a semi-Riesz set. We will show that $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is not a rich subspace of $L^1(G)$.

Let $\mu \in M_\Lambda(G)$ be a non-diffuse measure and assume that $\mu(\{e_G\}) = 1$, i.e., $\mu = \delta_{e_G} + \nu$ with $\nu(\{e_G\}) = 0$. (If μ is not of this form, fix $x \in G$ with $\mu(\{x\}) \ne 0$ and consider the measure $\mu(\{x\})^{-1}(\mu * \delta_{-x}) \in M_\Lambda(G)$.) Let $R, S, T : L^1(G) \to L^1(G)$ be the convolution operators defined by $R(f) = \mu * f$, $S(f) = \nu * f$, and $T(f) = |\nu| * f$. Note that $R = \mathrm{Id} + S$. Recall that for every $\lambda \in M(G)$ and $f \in L^1(G)$ we have

$$(\lambda * f)(x) = \int_G f(x-y)\, d\lambda(y)$$

for m-almost all $x \in G$ [10, Theorem V.20.12]. Therefore,

$$\|S(\chi_E)\|_1 \le \|T(\chi_E)\|_1 \quad (E \in \mathcal{B}(G)).$$

We will first show that there exists $A \in \mathcal{B}(G)$ with $m(A) > 0$ such that $R|_{L^1(A)}$ is an isomorphism onto its image. (We write $L^1(A)$ for the subspace $\{f \in L^1(G) : \chi_A f = f\}$.) Since $\nu(\{e_G\}) = 0$, we can choose a sequence $(O_n)_{n\in\mathbb{N}}$ of open neighborhoods of e_G with $|\nu|(O_n) \to 0$. For each $n \in \mathbb{N}$, use Lemma 2.1 to find a covering of G by disjoint Borel sets $B_{n,1}, \dots, B_{n,N_n}$ with $B_{n,k} - B_{n,k} \subset O_n$ for $k = 1, \dots, N_n$. For every $n \in \mathbb{N}$ set

$$R_n = \sum_{k=1}^{N_n} P_{B_{n,k}} R P_{B_{n,k}}, \quad S_n = \sum_{k=1}^{N_n} P_{B_{n,k}} S P_{B_{n,k}}, \quad T_n = \sum_{k=1}^{N_n} P_{B_{n,k}} T P_{B_{n,k}},$$

where for every $E \in \mathcal{B}(G)$, P_E denotes the projection from $L^1(G)$ onto $L^1(E)$ defined by $P_E(f) = \chi_E f$. For every $n \in \mathbb{N}$ let ρ_n be the map defined by

$$\rho_n(E) = \|T_n(\chi_E)\|_1 \quad (E \in \mathcal{B}(G)).$$

Since T_n is continuous and maps positive functions to positive functions, it is a consequence of the monotone convergence theorem that ρ_n is a positive Borel measure on G. Every ρ_n is absolutely continuous with respect to the Haar measure m and has Radon–Nikodým derivative ω_n. For each $n \in \mathbb{N}$, we get

$$\begin{aligned}
\rho_n(G) = \|T_n(\chi_G)\|_1 &= \sum_{k=1}^{N_n} \int_{B_{n,k}} T(\chi_{B_{n,k}})(x)\, dm(x) \\
&= \sum_{k=1}^{N_n} \int_{B_{n,k}} \int_G \chi_{B_{n,k}}(x-y)\, d|\nu|(y)\, dm(x) \\
&= \sum_{k=1}^{N_n} \int_{B_{n,k}} |\nu|(x - B_{n,k})\, dm(x) \\
&\le \sum_{k=1}^{N_n} \int_{B_{n,k}} |\nu|(B_{n,k} - B_{n,k})\, dm(x) \le |\nu|(O_n).
\end{aligned}$$

Therefore, $\rho_n(G) \to 0$, and in particular $\omega_n \to 0$ in m-measure. So there exists a Borel set B_0 of G with $m(B_0) > 0$ and $n_0 \in \mathbb{N}$ satisfying

$$\omega_{n_0}(x) \le 1/2 \quad (x \in B_0).$$

Consequently, $\|S_{n_0}(\chi_E)\|_1 \le \|T_{n_0}(\chi_E)\|_1 \le \frac{1}{2} m(E)$ for all Borel sets $E \subset B_0$, and $\| S_{n_0}|_{L^1(B_0)}\| \le 1/2$. Therefore $(\mathrm{Id} + S_{n_0})|_{L^1(B_0)} = R_{n_0}|_{L^1(B_0)}$ is an isomorphism onto its image. Fix $k_0 \in \{1, \dots, N_{n_0}\}$ with $m(B_0 \cap B_{n_0,k_0}) > 0$

and set $A = B_0 \cap B_{n_0,k_0}$. Then $R|_{L^1(A)}$ is an isomorphism onto its image because $\|R_{n_0}(f)\|_1 \le \|R(f)\|_1$ for all $f \in L^1(G)$.

We will now finish the proof by showing that $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is not a rich subspace of $L^1(G)$. Let $\pi : L^1(G) \to L^1(G)/\ker(R)$ be the canonical quotient map and let $\tilde{R} : L^1(G)/\ker(R) \to L^1(G)$ be a bounded operator with $R = \tilde{R} \circ \pi$. Since $R|_{L^1(A)}$ is an isomorphism, $\pi|_{L^1(A)}$ is bounded from below. By Proposition 4.4, π cannot be a narrow operator. Then $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is contained in $\ker(R)$, and is therefore not a rich subspace of $L^1(G)$. ∎

Corollary 4.15. *If $L^1_\Lambda(G)$ is a rich subspace of $L^1(G)$, then $C_\Lambda(G)$ is a rich subspace of $C(G)$.*

The space $C_{\mathbb{N}}(\mathbb{T})$ is a rich subspace of $C(\mathbb{T})$, but $L^1_{\mathbb{N}}(\mathbb{T})$ has the Radon–Nikodým property and therefore not the Daugavet property. So the converse of Corollary 4.15 is not true.

5. Products of compact abelian groups. Let G_1 and G_2 be compact abelian groups with normalized Haar measures m_1 and m_2. The direct product $G = G_1 \times G_2$ is again a compact abelian group if we endow it with the product topology. If $f : G_1 \to \mathbb{C}$ and $g : G_2 \to \mathbb{C}$, we denote by $f \otimes g$ the function $(x, y) \mapsto f(x)g(y)$. The dual group of G can now be identified with $\Gamma_1 \times \Gamma_2$ because every $\gamma \in \Gamma$ is of the form $\gamma_1 \otimes \gamma_2$ with $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$ [25, Theorem 2.2.3]. Furthermore, the Haar measure on G coincides with the product measure $m_1 \times m_2$ [10, Example IV.15.17(i)].

Proposition 5.1. *Let G_1 be an infinite, compact abelian group, let G_2 be an arbitrary compact abelian group, let Λ_1 be a subset of Γ_1, and let Λ_2 be a subset of Γ_2.*

(a) *Suppose that $C_{\Lambda_1}(G_1)$ is a rich subspace of $C(G_1)$ and $C_{\Lambda_2}(G_2)$ is a rich subspace of $C(G_2)$ (or, if G_2 is finite, that $\Lambda_2 = \Gamma_2$). Then $C_{\Lambda_1\times\Lambda_2}(G_1 \times G_2)$ is a rich subspace of $C(G_1 \times G_2)$.*
(b) *Suppose that $C_{\Lambda_1}(G_1)$ is a rich subspace of $C(G_1)$ and Λ_2 is non-empty. Then $C_{\Lambda_1\times\Lambda_2}(G_1 \times G_2)$ has the Daugavet property.*

Proof. Set $G = G_1 \times G_2$ and $\Lambda = \Lambda_1 \times \Lambda_2$.

We start with part (a). Let O be a non-empty open set of G, and $\varepsilon > 0$. By Proposition 4.3, we have to find $f \in S_{C(G)}$ with $f|_{G\setminus O} = 0$ and $d(f, C_\Lambda(G)) \le \varepsilon$.

Pick non-empty open sets $O_1 \subset G_1$ and $O_2 \subset G_2$ with $O_1 \times O_2 \subset O$, and $\delta > 0$ with $2\delta + \delta^2 \le \varepsilon$. By assumption, there exist $f_k \in S_{C(G_k)}$ and $g_k \in T_{\Lambda_k}(G_k)$ with $f_k|_{G_k\setminus O_k} = 0$ and $\|f_k - g_k\|_\infty \le \delta$ for $k = 1, 2$. If we set $f = f_1 \otimes f_2$ and $g = g_1 \otimes g_2$, then $f \in S_{C(G)}$, $g \in T_\Lambda(G)$, and $f|_{G\setminus O} = 0$.

Furthermore,

$$d(f, C_\Lambda(G)) \le \|f-g\|_\infty \le \|f_1\|_\infty \|f_2-g_2\|_\infty + \|g_2\|_\infty \|f_1-g_1\|_\infty \\ \le \delta + (1+\delta)\delta \le \varepsilon.$$

Let us now consider part (b). The space $C_{\Gamma_1\times\Lambda_2}(G)$ can canonically be identified with $C(G_1, C_{\Lambda_2}(G_2))$, the space of all continuous functions from G_1 into $C_{\Lambda_2}(G_2)$, and has therefore the Daugavet property [13, Theorem 4.4]. We will prove that $C_\Lambda(G)$ is a rich subspace of $C_{\Gamma_1\times\Lambda_2}(G)$. For this, it is sufficient to show that for every non-empty open set O of G_1, every $g \in T_{\Lambda_2}(G_2)$ with $\|g\|_\infty = 1$, and every $\varepsilon > 0$, there exists $f \in S_{C(G_1)}$ with $f|_{G_1\setminus O} = 0$ and $d(f\otimes g, C_\Lambda(G)) \le \varepsilon$ [1, Proposition 4.3(a)]. Since $C_{\Lambda_1}(G_1)$ is a rich subspace of $C(G_1)$, there exist $f \in S_{C(G_1)}$ and $h \in T_{\Lambda_1}(G_1)$ with $f|_{G_1\setminus O} = 0$ and $\|f-h\|_\infty \le \varepsilon$. Then $h\otimes g \in T_\Lambda(G)$ and

$$d(f\otimes g, C_\Lambda(G)) \le \|f\otimes g - h\otimes g\|_\infty \le \|f-h\|_\infty \|g\|_\infty \le \varepsilon. \blacksquare$$

PROPOSITION 5.2. *Let G be the product of two compact abelian groups G_1 and G_2 and denote by p the projection from $\Gamma = \Gamma_1 \times \Gamma_2$ onto Γ_1. If $C_\Lambda(G)$ is a rich subspace of $C(G)$, then $C_{p[\Lambda]}(G_1)$ is a rich subspace of $C(G_1)$ (or $p[\Lambda] = \Gamma_1$ if G_1 is finite).*

Proof. Let O be a non-empty open set of G_1, and $\varepsilon > 0$. By Proposition 4.3, we have to find $f \in S_{C(G_1)}$ with $f|_{G_1\setminus O} = 0$ and $d(f, C_{p[\Lambda]}(G_1)) \le \varepsilon$. (Note that this is sufficient in the case of finite G_1 as well.)

Since $C_\Lambda(G)$ is a rich subspace of $C(G)$, there exist $f_0 \in S_{C(G)}$ and $g_0 \in T_\Lambda(G)$ with $f_0|_{G\setminus(O\times G_2)} = 0$ and $\|f_0 - g_0\|_\infty \le \varepsilon$. Fix $(x_0, y_0) \in G$ with $|f_0(x_0, y_0)| = 1$. Setting $f = f_0(\,\cdot\,, y_0)$ and $g = g_0(\,\cdot\,, y_0)$, we get $f \in S_{C(G_1)}$, $g \in T_{p[\Lambda]}(G_1)$, and $f|_{G_1\setminus O} = 0$. Finally,

$$d(f, C_{p[\Lambda]}(G_1)) \le \|f-g\|_\infty \le \|f_0 - g_0\|_\infty \le \varepsilon. \blacksquare$$

PROPOSITION 5.3. *Let G_1 and G_2 be infinite compact abelian groups, let Λ_1 be a subset of Γ_1, and let Λ_2 be a subset of Γ_2.*

(a) *If $L^1_{\Lambda_2}(G_2)$ is a rich subspace of $L^1(G_2)$, then $L^1_{\Gamma_1\times\Lambda_2}(G_1\times G_2)$ is a rich subspace of $L^1(G_1\times G_2)$.*
(b) *Suppose that $L^1_{\Lambda_1}(G_1)$ is a rich subspace of $L^1(G_1)$ and Λ_2 is non-empty. Then $L^1_{\Lambda_1\times\Lambda_2}(G_1\times G_2)$ has the Daugavet property.*

Proof. Set $G = G_1 \times G_2$ and $\Lambda = \Lambda_1 \times \Lambda_2$.

We start with part (a). The space $L^1(G)$ can canonically be identified with the Bochner space $L^1(G_1, L^1(G_2))$, and $L^1_{\Gamma_1\times\Lambda_2}(G)$ with the subspace $L^1(G_1, L^1_{\Lambda_2}(G_2))$. Since $L^1_{\Lambda_2}(G_2)$ is a rich subspace of $L^1(G_2)$, the space $L^1(G_1, L^1_{\Lambda_2}(G_2))$ is rich in $L^1(G_1, L^1(G_2))$ [14, Lemma 2.8].

Let us now consider part (b). Identifying again $L^1_{\Gamma_1\times\Lambda_2}(G)$ with the Bochner space $L^1(G_1, L^1_{\Lambda_2}(G_2))$, we see that $L^1_{\Gamma_1\times\Lambda_2}(G)$ has the Daugavet property [17, Example after Theorem 2.3]. We will show that $L^1_\Lambda(G)$ is a rich subspace of $L^1_{\Gamma_1\times\Lambda_2}(G)$. For this, for every Borel set A of G_1, every $g \in T_{\Lambda_2}(G_2)$ with $\|g\|_1 = 1$, and every $\delta, \varepsilon > 0$ it is sufficient to find a balanced ε-peak f on A with $d(f \otimes g, L^1_\Lambda(G)) \le \delta$ [2, Theorem 2.4]. Since $L^1_{\Lambda_1}(G_1)$ is a rich subspace of $L^1(G_1)$, there exist a balanced ε-peak f on A and $h \in T_{\Lambda_1}(G_1)$ with $\|f - h\|_1 \le \delta$. Then $h \otimes g \in T_\Lambda(G)$ and

$$d(f \otimes g, L^1_\Lambda(G)) \le \|f \otimes g - h \otimes g\|_1 = \|f - h\|_1 \|g\|_1 \le \delta. \ \blacksquare$$

PROPOSITION 5.4. *Let G be the product of two compact abelian groups G_1 and G_2, and denote by p the projection from $\Gamma = \Gamma_1 \times \Gamma_2$ onto Γ_1. If $L^1_\Lambda(G)$ is a rich subspace of $L(G)$, then $L^1_{p[\Lambda]}(G_1)$ is a rich subspace of $L^1(G_1)$ (or $p[\Lambda] = \Gamma_1$ if G_1 is finite).*

Proof. If $p[\Lambda] = \Gamma_1$, we have nothing to show. So let us assume that there exists $\gamma \in \Gamma_1 \setminus p[\Lambda]$. Set $\vartheta = \overline{\gamma} \otimes \mathbf{1}_{G_2}$ and $\Theta = \vartheta\Lambda$. The map $f \mapsto \vartheta f$ is an isometry from $L^1(G)$ onto $L^1(G)$ and maps $L^1_\Lambda(G)$ onto $L^1_\Theta(G)$. Analogously, the map $f \mapsto \overline{\gamma} f$ is an isometry from $L^1(G_1)$ onto $L^1(G_1)$ and maps $L^1_{p[\Lambda]}(G_1)$ onto $L^1_{\overline{\gamma}p[\Lambda]}(G_1)$. Note that

$$\overline{\gamma}p[\Lambda] = p[(\overline{\gamma}, \mathbf{1}_{G_2})\Lambda] = p[\Theta]$$

and $\mathbf{1}_{G_1} \notin \overline{\gamma}p[\Lambda]$. Taking into account that $L^1_\Lambda(G)$ is a rich subspace of $L^1(G)$ if and only if $L^1_\Theta(G)$ is a rich subspace of $L^1(G)$ and that $L^1_{p[\Lambda]}(G_1)$ is a rich subspace of $L^1(G_1)$ if and only if $L^1_{p[\Theta]}(G_1)$ is a rich subspace of $L^1(G_1)$, we may assume that $\mathbf{1}_{G_1} \notin p[\Lambda]$.

Fix a Borel subset A of G_1 and $\delta, \varepsilon > 0$. By Proposition 4.4, we have to find a balanced ε-peak f on A with $d(f, L^1_{p[\Lambda]}(G_1)) \le \delta$. By assumption, $L^1_\Lambda(G)$ is a rich subspace of $L^1(G)$, and therefore there exist a balanced $(\varepsilon/3)$-peak f_0 on $A \times G_2$ and $g \in T_\Lambda(G)$ with $\|f_0 - g\|_1 \le \delta/6$. Set

$$B = \{y \in G_2 : m_1(\{f_0(\,\cdot\,, y) = -1\}) > m_1(A) - \varepsilon\},$$
$$C = \{y \in G_2 : \|f_0(\,\cdot\,, y) - g(\,\cdot\,, y)\|_1 \le \delta/2\}.$$

Note we may assume that B and C are measurable [10, Theorem III.13.8]. We then get

$$\begin{aligned} m_1(A) - \varepsilon/3 \le m(\{f_0 = -1\}) &= \int_{G_2}\int_{G_1} \chi_{\{f_0=-1\}}(x, y)\, dm_1(x)\, dm_2(y) \\ &= \int_{G_2} m_1(\{f_0(\,\cdot\,, y) = -1\})\, dm_2(y) \\ &\le m_2(B)m_1(A) + (1 - m_2(B))(m_1(A) - \varepsilon) \\ &= m_1(A) + m_2(B)\varepsilon - \varepsilon \end{aligned}$$

and

$$\frac{\delta}{6} \geq \|f_0 - g\|_1 = \int_{G_2} \|f_0(\,\cdot\,,y) - g(\,\cdot\,,y)\|_1 \, dm_2(y)$$
$$\geq \frac{\delta}{2}(1 - m_2(C)).$$

Hence $m_2(B) \geq 2/3$ and $m_2(C) \geq 2/3$. Therefore $B \cap C \neq \emptyset$ and we can choose $y_0 \in B \cap C$.

Let us gather the properties of $f_0(\,\cdot\,,y_0) \in L^1(G_1)$. It is clear that $f_0(\,\cdot\,,y_0)$ is real-valued, $f_0(\,\cdot\,,y_0) \geq -1$, and $\chi_A f_0(\,\cdot\,,y_0) = f_0(\,\cdot\,,y_0)$. As y_0 belongs to B and C, we have $m_1(\{f_0(\,\cdot\,,y_0) = -1\}) > m_1(A) - \varepsilon$ and $\|f_0(\,\cdot\,,y_0) - g(\,\cdot\,,y_0)\|_1 \leq \delta/2$. The function $g(\,\cdot\,,y_0)$ belongs to $T_{p[\Lambda]}(G)$ and $\mathbf{1}_{G_1} \notin p[\Lambda]$. So $\int_{G_1} g(x,y_0)\,dm_1(x) = 0$ and $|\int_{G_1} f_0(x,y_0)\,dm_1(x)| \leq \delta/2$. Modifying $f_0(\,\cdot\,,y_0)$ a little bit, we get a balanced ε-peak f on A with $\|f - g(\,\cdot\,,y_0)\|_1 \leq \delta$. ■

Set $\Lambda = \mathbb{Z} \times \{0\}$. Then Λ is not a Rosenthal set because $C_\Lambda(\mathbb{T}^2) \cong C(\mathbb{T})$ contains a copy of c_0 [22, Proof of Theorem 3]. But $\mathbb{Z}^2 \setminus (-\Lambda) = \mathbb{Z} \times (\mathbb{Z} \setminus \{0\})$ and $L^1_{\mathbb{Z}\times(\mathbb{Z}\setminus\{0\})}(\mathbb{T}^2)$ is a rich subspace of $L^1(\mathbb{T}^2)$ by Proposition 5.3(a). So the converse of Proposition 4.9 is not true.

Let us come back to examples of translation-invariant subspaces that have the Daugavet property but are not rich. The examples mentioned in Section 4 are of the following type: We take a one-to-one homomorphism $H : \Gamma \to \Gamma$ that is not onto. Then $C_{H[\Gamma]}(G)$ and $L^1_{H[\Gamma]}(G)$ have the Daugavet property but are not rich subspaces of $C(G)$ or $L^1(G)$. In this case $\bigcap_{\gamma \in H[\Gamma]} \ker(\gamma)$ contains $\ker(H^*) \neq \{e_G\}$. Set $\Lambda = \mathbb{Z} \times \{1\}$. Using Propositions 5.1(b) and 5.3(b), we see that $C_\Lambda(\mathbb{T}^2)$ and $L^1_\Lambda(\mathbb{T}^2)$ have the Daugavet property. But they are not rich subspaces of $C(\mathbb{T}^2)$ or $L^1(\mathbb{T}^2)$ by Propositions 5.2 and 5.4. Furthermore, $\bigcap_{\gamma\in\Lambda} \ker(\gamma) = \{(1,1)\}$.

6. Quotients with respect to translation-invariant subspaces. We will now study quotients of the form $C(G)/C_\Lambda(G)$ and $L^1(G)/L^1_\Lambda(G)$. The following lemma is the key ingredient for all results of this section.

LEMMA 6.1. *If we interpret $f \in C(G)$ as a functional on $M(G)$, then*

$$\|\, f|_{L^1_\Lambda(G)}\| = \|\, f|_{M_\Lambda(G)}\|.$$

Analogously, if we interpret $g \in L^1(G)$ as a functional on $L^\infty(G)$, then

$$\|\, g|_{C_\Lambda(G)}\| = \|\, g|_{L^\infty_\Lambda(G)}\|.$$

Proof. We will just show the first statement. The proof of the second statement works the same way.

It is clear that $\| f|_{L^1_\Lambda(G)}\| \le \| f|_{M_\Lambda(G)}\|$ because $L^1_\Lambda(G) \subset M_\Lambda(G)$. In order to prove the reverse inequality, we may assume without loss of generality that $\| f|_{M_\Lambda(G)}\| = 1$. Fix $\varepsilon > 0$ and an approximate unit $(v_j)_{j\in J}$ of $L^1(G)$ that has the properties listed in Proposition 2.2. Pick $\mu \in M_\Lambda(G)$ with $\|\mu\| = 1$ and $|\int_G f\,d\mu| \ge 1 - \varepsilon/2$. Using $\hat{v}_j(\gamma) \to 1$ for every $\gamma \in \Gamma$, we can deduce that

$$\int_G g\,d(\mu * v_j) \to \int_G g\,d\mu$$

for every $g \in T(G)$. So μ is the weak*-limit of $(\mu * v_j)_{j\in J}$ because $T(G)$ is dense in $C(G)$. Fix $j_0 \in J$ with $|\int_G f\,d(\mu * v_{j_0})| \ge 1-\varepsilon$. Since $\mu * v_{j_0} \in L^1_\Lambda(G)$ and $\|\mu * v_{j_0}\|_1 \le 1$, we have $\| f|_{L^1_\Lambda(G)}\| \ge 1 - \varepsilon$. As $\varepsilon > 0$ was arbitrarily chosen, this finishes the proof. ∎

THEOREM 6.2. *If $L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$, then the quotient $C(G)/C_\Lambda(G)$ has the Daugavet property.*

Proof. Note that $C_\Lambda(G)^\perp = M_{\Gamma\setminus\Lambda^{-1}}(G)$ because $T_\Lambda(G)$ is dense in $C_\Lambda(G)$. We can therefore identify the dual space of $C(G)/C_\Lambda(G)$ with $M_{\Gamma\setminus\Lambda^{-1}}(G)$.

Fix $[f] \in C(G)/C_\Lambda(G)$ with $\|[f]\| = 1$, $\mu \in M_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|\mu\| = 1$, and $\varepsilon > 0$. By Lemma 2.3, we have to find $\nu \in M_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|\nu\| = 1$, $\operatorname{Re}\int_G f\,d\nu \ge 1-\varepsilon$, and $\|\mu+\nu\| \ge 2-\varepsilon$. Let $\mu = \mu_s + g\,dm$ be the Lebesgue decomposition of μ where μ_s and m are singular and $g \in L^1(G)$.

If we interpret f as a functional on $M(G)$, then by Lemma 6.1 we have

$$\| f|_{L^1_{\Gamma\setminus\Lambda^{-1}}(G)}\| = \| f|_{M_{\Gamma\setminus\Lambda^{-1}}(G)}\| = 1.$$

$L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $L^1(G)$, and so by Proposition 4.2 there exists a function $h \in L^1_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|h\|_1 = 1$, $\operatorname{Re}\int_G fh\,dm \ge 1-\varepsilon$, and $\|g/\|g\|_1 + h\|_1 \ge 2 - \varepsilon$. Setting $\nu = h\,dm$, we therefore get

$$\begin{aligned}
\|\mu+\nu\| &= \|\mu_s\| + \|g+h\|_1 \\
&= \|\mu_s\| + \big\|g/\|g\|_1 + h - (1-\|g\|_1)g/\|g\|_1\big\|_1 \\
&\ge \|\mu_s\| + \big\|g/\|g\|_1 + h\big\|_1 - (1-\|g\|_1) \\
&\ge \|\mu_s\| + (2-\varepsilon) - (1-\|g\|_1) \\
&= \|\mu\| + 1 - \varepsilon = 2-\varepsilon. \ \blacksquare
\end{aligned}$$

COROLLARY 6.3. *If Λ is a Rosenthal set, then $C(G)/C_\Lambda(G)$ has the Daugavet property.*

THEOREM 6.4. *If $C_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $C(G)$, then the quotient $L^1(G)/L^1_\Lambda(G)$ has the Daugavet property.*

Proof. Let us begin as in the proof of Theorem 6.2. We can identify the dual space of $L^1(G)/L^1_\Lambda(G)$ with $L^\infty_{\Gamma\setminus\Lambda^{-1}}(G)$, because $T_\Lambda(G)$ is dense in $L^1_\Lambda(G)$, and therefore $L^1_\Lambda(G)^\perp = L^\infty_{\Gamma\setminus\Lambda^{-1}}(G)$.

Fix $[f] \in L^1(G)/L^1_\Lambda(G)$ with $\|[f]\| = 1$, $g \in L^\infty_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|g\|_\infty = 1$, and $\varepsilon > 0$. By Lemma 2.3, we have to find $h \in L^\infty_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|h\|_\infty = 1$, $\operatorname{Re}\int_G fh\,dm \geq 1-\varepsilon$, and $\|g+h\|_\infty \geq 2-\varepsilon$.

Choose $\delta \in (0,1)$ with $(1-5\|f\|_1\delta)/(1+3\delta) \geq 1-\varepsilon/2$, $\eta > 0$ such that $\int_A |f|\,dm \leq \delta$ for all $A \in \mathcal{B}(G)$ with $m(A) \leq \eta$, and $t \in \mathbb{T}$ with

$$m(\{\operatorname{Re}(t^{-1}g) \geq 1-\varepsilon/2\}) > 0.$$

If we interpret f as a functional on $L^\infty(G)$, then by Lemma 6.1 we have

$$\|f|_{C_{\Gamma\setminus\Lambda^{-1}}(G)}\| = \|f|_{L^\infty_{\Gamma\setminus\Lambda^{-1}}(G)}\| = 1.$$

Pick $h_0 \in C_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|h_0\|_\infty = 1$ and $\operatorname{Re}\int_G fh_0\,dm \geq 1-\delta$. Since h_0 is uniformly continuous, there exists an open neighborhood O of e_G with

$$|h_0(x) - h_0(y)| \leq \delta \quad (x-y \in O)$$

and $m(O) \leq \eta$. By assumption, $C_{\Gamma\setminus\Lambda^{-1}}(G)$ is a rich subspace of $C(G)$, and so by Corollary 4.5 there exist a real-valued non-negative $p_0 \in S_{C(G)}$ with $p_0|_{G\setminus O} = 0$ and $p_0(e_G) = 1$ and $p \in C_{\Gamma\setminus\Lambda^{-1}}(G)$ with $\|p_0 - p\|_\infty \leq \delta$. Then $V = \{p_0 > 1-\delta\}$ is an open neighborhood of e_G and $V \subset O$. An easy compactness argument shows that there exists $x_0 \in G$ with

$$m(\{x \in x_0 + V : \operatorname{Re}(t^{-1}g(x)) \geq 1-\varepsilon/2\}) > 0.$$

If we set

$$h_1 = h_0 + (t - h_0(x_0))p_{x_0} \quad \text{and} \quad h = h_1/\|h_1\|_\infty,$$

then h is normalized and belongs by construction to $C_{\Gamma\setminus\Lambda^{-1}}(G)$. Let us estimate the norm of h_1. For $x \in G \setminus (x_0 + O)$ we get

$$|h_1(x)| = |h_0(x) + (t - h_0(x_0))p(x-x_0)| \leq \|h_0\|_\infty + 2\|p|_{G\setminus O}\|_\infty \leq 1 + 2\delta,$$

and for $x \in x_0 + O$,

$$\begin{aligned}
|h_1(x)| &= |h_0(x) + (t - h_0(x_0))p(x-x_0)| \\
&\leq |h_0(x) - h_0(x_0)p_0(x-x_0)| + p_0(x-x_0) + 2\|p-p_0\|_\infty \\
&\leq |h_0(x) - h_0(x_0)| + |h_0(x_0)|(1 - p_0(x-x_0)) + p_0(x-x_0) + 2\delta \\
&\leq \delta + (1 - p_0(x-x_0)) + p_0(x-x_0) + 2\delta = 1 + 3\delta.
\end{aligned}$$

Consequently, $\|h_1\|_\infty \leq 1 + 3\delta$. Let us check that h is as desired. We first observe that

$$\begin{aligned}
\operatorname{Re}\int_G fh_1\,dm &\geq \operatorname{Re}\int_G fh_0\,dm - 2\int_G |fp_{x_0}|\,dm \\
&\geq (1-\delta) - 2\int_{x_0+O} |f|\,dm - 2\|f\|_1\|p_0 - p\|_\infty \\
&\geq (1-\delta) - 2\delta - 2\|f\|_1\delta = 1 - (3 + 2\|f\|_1)\delta.
\end{aligned}$$

Therefore, $\operatorname{Re}\int_G fh\,dm \geq 1-\varepsilon$ by our choice of δ. If $x \in x_0+V$, we get

$$\begin{aligned}\operatorname{Re}(t^{-1}h_1(x)) &\geq \operatorname{Re}(t^{-1}h_0(x)) + \operatorname{Re}(1-t^{-1}h_0(x_0))p_0(x-x_0) - 2\|p_0-p\|_\infty \\ &\geq \operatorname{Re}(t^{-1}h_0(x)) + \operatorname{Re}(1-t^{-1}h_0(x_0))(1-\delta) - 2\delta \\ &\geq 1-3\delta-|h_0(x)-h_0(x_0)| \geq 1-4\delta,\end{aligned}$$

and hence $\operatorname{Re}(t^{-1}h(x)) \geq 1-\varepsilon/2$ by our choice of δ. Thus

$$\begin{aligned}m(\{|g+h| \geq 2-\varepsilon\}) &\geq m(\{\operatorname{Re}(t^{-1}(g+h)) \geq 2-\varepsilon\}) \\ &\geq m(\{\operatorname{Re}(t^{-1}g) \geq 1-\varepsilon/2\} \cap \{\operatorname{Re}(t^{-1}h) \geq 1-\varepsilon/2\}) \\ &\geq m(\{\operatorname{Re}(t^{-1}g) \geq 1-\varepsilon/2\} \cap (x_0+V)) > 0\end{aligned}$$

and $\|g+h\|_\infty \geq 2-\varepsilon$. ∎

COROLLARY 6.5. *If Λ is a semi-Riesz set, then $L^1(G)/L^1_\Lambda(G)$ has the Daugavet property.*

7. Poor subspaces of $L^1(G)$. In Section 6, we have seen some cases in which the quotient space $L^1(G)/L^1_\Lambda(G)$ has the Daugavet property. Recall that a closed subspace Y of a Banach space X with the Daugavet property is rich if and only if every closed subspace Z of X with $Y \subset Z \subset X$ has the Daugavet property. A similar notion for quotients of X was introduced by V. M. Kadets, V. Shepelska, and D. Werner [16].

DEFINITION 7.1. Let X be a Banach space with the Daugavet property. A closed subspace Y of X is called *poor* if X/Z has the Daugavet property for every closed subspace $Z \subset Y$.

The poor subspaces of a Banach space with the Daugavet property can be described using a generalized concept of narrow operators [16]. In the case of $L^1(\Omega,\Sigma,\mu)$ this leads to the following characterization [16, Corollary 6.6]:

PROPOSITION 7.2. *Let (Ω,Σ,μ) be a non-atomic probability space. A subspace X of $L^1(\Omega)$ is poor if and only if for every $A \in \Sigma$ of positive measure and every $\varepsilon > 0$ there exists $f \in S_{L^\infty(\Omega)}$ with $\chi_A f = f$ and $\|f|_X\| \leq \varepsilon$, where we interpret f as a functional on $L^1(\Omega)$.*

Using this characterization, we can build a link to a property that was studied by G. Godefroy, N. J. Kalton, and D. Li [8]. In the following, (Ω,Σ,μ) denotes a non-atomic probability space and P the natural projection from $L^1(\Omega)^{**}$ onto $L^1(\Omega)$. For every $A \in \Sigma$, we write $L^1(A)$ for the subspace $\{f \in L^1(\Omega) : \chi_A f = f\}$ and P_A for the projection from $L^1(\Omega)$ onto $L^1(A)$ defined by $P_A(f) = \chi_A f$.

DEFINITION 7.3. A closed subspace X of $L^1(\Omega)$ is said to be *small* if there is no $A \in \Sigma$ of positive measure such that P_A maps X onto $L^1(A)$.

If X is a poor subspace of $L^1(\Omega)$, then X is small [16, Corollary 6.7]. The converse is valid too.

PROPOSITION 7.4. *If X is a small subspace of $L^1(\Omega)$, then X is a poor subspace of $L^1(\Omega)$.*

Proof. Fix $A \in \Sigma$ with $\mu(A) > 0$ and $\varepsilon > 0$. By Proposition 7.2, we have to find $f \in S_{L^\infty(\Omega)}$ with $\chi_A f = f$ and $\| f|_X \| \le \varepsilon$.

Since X is small, the projection $P_A : L^1(\Omega) \to L^1(A)$ does not map X onto $L^1(A)$. By the (proof of the) open mapping theorem, $\overline{P_A[\varepsilon^{-1}B_X]}$ is nowhere dense in $L^1(A)$. Pick $g \in B_{L^1(A)}$ with $g \notin \overline{P_A[\varepsilon^{-1}B_X]}$. The set $\overline{P_A[\varepsilon^{-1}B_X]}$ is absolutely convex, and so by the Hahn–Banach theorem there exists a function $f \in S_{L^\infty(A)}$ with

$$\sup\Big\{\Big|\int_A fh\,d\mu\Big| : h \in \frac{1}{\varepsilon}B_X\Big\} \le \operatorname{Re}\int_A fg\,d\mu.$$

Using this inequality, we get

$$\| f|_X \| = \sup\Big\{\Big|\int_A fh\,d\mu\Big| : h \in B_X\Big\} \le \varepsilon \operatorname{Re}\int_A fg\,d\mu \le \varepsilon. \ \blacksquare$$

An important tool in the study of small subspaces is the topology of convergence in measure.

DEFINITION 7.5.

(a) A subspace X of $L^1(\Omega)$ is called *nicely placed* if B_X is closed with respect to convergence in measure.
(b) Λ is said to be *nicely placed* if $L^1_\Lambda(G)$ is a nicely placed subspace of $L^1(G)$.
(c) Λ is said to be a *Shapiro set* if every subset of Λ is nicely placed.

These terms were coined by G. Godefroy [6, 7], who showed that every Shapiro set is a Riesz set [9, Proposition IV.4.5]. The natural numbers are a Shapiro set in $\mathbb{Z}$ [9, Example IV.4.11] and $\Lambda = \bigcup_{n=0}^{\infty}\{k2^n : |k| \le 2^n\}$ is a nicely placed Riesz set which is not a Shapiro set [9, Example IV.4.12].

LEMMA 7.6. *Let X be a nicely placed subspace of $L^1(G)$ and suppose that there exists $A \in \mathcal{B}(G)$ with $m(A) > 0$ such that P_A maps X onto $L^1(A)$, i.e., suppose that X is not small. Then there exists a continuous operator $T : L^1(A) \to X$ with $j_A = P_A T$ where $j_A : L^1(A) \to L^1(G)$ is the natural injection.*

Proof. This proof is a modification of a proof by G. Godefroy, N. J. Kalton, and D. Li [8, Lemma III.5]. We identify X^{**} with $X^{\perp\perp} \subset L^1(G)^{**}$ and recall that A. V. Bukhvalov and G. Ya. Lozanovskiĭ showed that $P[B_{X^{\perp\perp}}] = B_X$ if X is nicely placed in $L^1(G)$ [9, Theorem IV.3.4].

Denote by $\mathcal{N}$ the directed set of open neighborhoods of e_G. (We turn $\mathcal{N}$ into a directed set by setting $V \le W$ if and only if V contains W.) Let $\mathcal{U}$ be an ultrafilter on $\mathcal{N}$ which contains the filter base

$$\{\{W \in \mathcal{N} : V \le W\} : V \in \mathcal{N}\}.$$

P_A is an open map by the open mapping theorem. So we can fix $M > 0$ with $B_{L^1(A)} \subset MP_A[B_X]$. For every $V \in \mathcal{N}$, use Lemma 2.1 to choose disjoint Borel sets $B_{V,1}, \ldots, B_{V,N_V}$ with $A = \bigcup_{k=1}^{N_V} B_{V,k}$ and $B_{V,k} - B_{V,k} \subset V$ for $k = 1, \ldots, N_V$. Picking $f_{V,k} \in MB_X$ with $P_A(f_{V,k}) = m(B_{V,k})^{-1}\chi_{B_{V,k}}$ for $k = 1, \ldots, N_V$, we define $S_V : L^1(A) \to X$ by

$$S_V(f) = \sum_{k=1}^{N_V}\Big(\int_{B_{V,k}} f\,dm\Big) f_{V,k} \qquad (f \in L^1(A)).$$

As the norm of every S_V is bounded by M, we can define $S : L^1(A) \to X^{\perp\perp}$ by

$$S(f) = w^*\text{-}\lim_{V,\mathcal{U}} S_V(f) \qquad (f \in L^1(A))$$

and set $T = PS$.

Let us check that $j_A = P_A T$. Fix $f \in L^1(A)$. Since $C(G)$ is dense in $L^1(G)$, we may assume that f is the restriction to A of a continuous function. Let $(S_{\varphi(j)}(f))_{j\in J}$ be a subnet of $(S_V(f))_{V\in\mathcal{N}}$ with $S(f) = w^*\text{-}\lim_j S_{\varphi(j)}(f)$. Since f is uniformly continuous, it is easy to construct an increasing sequence $(j_n)_{n\in\mathbb{N}}$ in J with

$$\sup\big\{\|f - P_A S_{\varphi(j)}(f)\|_\infty : j \ge j_n\big\} \to 0. \tag{7.1}$$

Furthermore, there exists a sequence $(g_n)_{n\in\mathbb{N}}$ in $L^1(G)$ that converges m-almost everywhere to $PS(f)$ with $g_n \in \operatorname{co}\{S_{\varphi(j)}(f) : j \ge j_n\}$ for all $n \in \mathbb{N}$ [9, Lemma IV.3.1]. Hence by (7.1) for m-almost all $x \in A$ we have

$$T(f)(x) = PS(f)(x) = \lim_n g_n(x) = f(x),$$

and therefore $j_A = P_A T$. ∎

Theorem 7.7. *If Λ is a nicely placed Riesz set, then $L^1_\Lambda(G)$ is a small subspace of $L^1(G)$.*

Proof. Assume that Λ is a nicely placed Riesz set such that $L^1_\Lambda(G)$ is not a small subspace of $L^1(G)$.

Since $L^1_\Lambda(G)$ is not small, there exists a Borel set A of positive measure such that P_A maps $L^1_\Lambda(G)$ onto $L^1(A)$. Using Lemma 7.6, we find $T : L^1(A) \to L^1_\Lambda(G)$ with $j_A = P_A T$. This operator is an isomorphism onto its image and $L^1_\Lambda(G)$ contains a copy of $L^1(A)$. So $L^1_\Lambda(G)$ fails the Radon–Nikodým property. But this contradicts our assumption because Λ

is a Riesz set if and only if $L^1_\Lambda(G)$ has the Radon–Nikodým property [21, Théorème 2]. ∎

Corollary 7.8. *If Λ is a Shapiro set, then $L^1_\Lambda(G)$ is a poor subspace of $L^1(G)$.*

Theorem 7.7 can be strengthened if G is metrizable. Let Λ be nicely placed. Then $L^1_\Lambda(G)$ is a poor subspace of $L^1(G)$ if and only if Λ is a semi-Riesz set [8, Proposition III.10].

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