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Stable rank and real rank of compact transformation group C^*-algebras

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Abstract. Let (G, X) be a transformation group, where X is a locally compact Hausdorff space and G is a compact group. We investigate the stable rank and the real rank of the transformation group C^*-algebra $C_0(X) \rtimes G$. Explicit formulae are given in the case where X and G are second countable and X is locally of finite G-orbit type. As a consequence, we calculate the ranks of the group C^*-algebra $C^*(\mathbb{R}^n \rtimes G)$, where G is a connected closed subgroup of $\mathrm{SO}(n)$ acting on $\mathbb{R}^n$ by rotation.

Introduction. For a C^*-algebra A, the stable rank $\mathrm{sr}(A)$ was introduced by Rieffel [24] with a view to applications in K-theory. In particular, the condition that $\mathrm{sr}(A) \leq 2$ is relevant for cancellation in $K_0(A)$ [2, 25]. The real rank $\mathrm{RR}(A)$, defined by Brown and Pedersen [4], is somewhat related to the stable rank but also has some different properties. There are some large classes of C^*-algebras with real rank zero (see, e.g., [5, Section V.7]). Both the stable rank and the real rank (in particular, the condition $\mathrm{RR}(A) = 0$) have played a significant role in the classification theory of C^*-algebras [14, 26].

For unital A, the stable rank $\mathrm{sr}(A)$ is either ∞ or the smallest possible integer n such that each n-tuple in A^n can be approximated in norm by n-tuples $(b_1, \ldots, b_n)$ such that $\sum_{i=1}^n b_i^* b_i$ is invertible. Similarly, the real rank $\mathrm{RR}(A)$ is either ∞ or the smallest non-negative integer n such that each $(n+1)$-tuple of self-adjoint elements in A^{n+1} can be approximated in norm by $(n+1)$-tuples $(b_0, b_1, \ldots, b_n)$ of self-adjoint elements such that $\sum_{i=0}^n b_i^2$ is invertible. For non-unital A, these ranks are defined to be those of the unitization of A. For the algebra of continuous functions on a compact Hausdorff

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space X one has $\mathrm{RR}(C(X)) = \dim X$ and $\mathrm{sr}(C(X)) = \lfloor (\dim X)/2 \rfloor + 1$, where $\dim X$ is the covering dimension of X (see [23]). Thus the real rank and the stable rank can be considered as non-commutative analogues of the real and complex dimension of topological spaces.

There is now an extensive literature on stable rank and real rank, especially for group C^*-algebras (see [1, 6, 18, 29, 30] and the references therein). It seems that less is known in general about transformation group C^*-algebras $C_0(X) \rtimes G$, although some important and difficult examples have been calculated. For example, it is known that irrational rotation algebras and Bunce–Deddens algebras have stable rank 1 and real rank 0 (see [5] and the references therein). On the other hand, it has been claimed that if G is a second countable, connected compact group, and X is a second countable locally compact Hausdorff space, then $\mathrm{RR}(C_0(X) \rtimes G) \leq \dim(X/G)$ [21, Theorem 2.1] and $\mathrm{sr}(C_0(X) \rtimes G) \leq 1 + \lfloor \dim(X/G)/2 \rfloor$ [21, Proposition 2.3]. However, as we shall point out at the end of Section 1, the proofs in [21] contain serious errors. Nevertheless, the above inequality for the stable rank follows from [27, Theorem 3.4 and Proposition 3.1] provided that G is a compact Lie group and X is locally of finite G-orbit type.

In this paper, we establish explicit formulae for the stable rank and the real rank of $C_0(X) \rtimes G$ when G is a second countable compact group, X is a second countable locally compact Hausdorff space and X is locally of finite G-orbit type (Theorems 2.3 and 2.4). Let $\mathcal{H}$ be a representative system of the conjugacy classes of stability groups for the action of G on X and, for $H \in \mathcal{H}$, let X_H denote the set of points in X with stability group equal to H and let $N_G(H) = \{g \in G : g^{-1}Hg = H\}$, the normaliser of H in G. We then show, for example, that $\mathrm{RR}(C_0(X) \rtimes G) = 0$ if and only if $\dim(X/G) = 0$, and if $1 \leq \dim(X/G) < \infty$, then

$$\mathrm{RR}(C_0(X) \rtimes G) = \max\left\{1, \max_{H \in \mathcal{H}}\left\{\left\lceil \frac{\dim(X_H/N_G(H))}{2[G:H]-1} \right\rceil\right\}\right\},$$

where $\frac{d}{\infty}$ is understood as 0. In Theorem 3.1 we apply the results of Section 2 to determine the ranks of $C_0(\mathbb{R}^n) \rtimes G$, where G is a connected closed subgroup of $\mathrm{SO}(n)$ acting on $\mathbb{R}^n$ by rotation.

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1. Preliminaries: transformation groups and covering dimension. Let (G, X) be a topological transformation group. That is, G is a topological group and X is a Hausdorff space together with a continuous map $G \times X \to X$, $(g, x) \mapsto g \cdot x$, such that $e \cdot x = x$ and $h \cdot (g \cdot x) = (hg) \cdot x$ for all $g, h \in G$ and $x \in X$. For $x \in X$, let S_x denote the *stabiliser* of x, that is, $S_x = \{g \in G : g \cdot x = x\}$, and let $G \cdot x = \{g \cdot x : g \in G\}$ denote the orbit of x. The quotient space X/G is always endowed with the quotient topology. Note that the quotient map $X \to X/G$ is continuous and open. If G is compact then the quotient map is also closed and proper, and X/G is Hausdorff [3]. The transformation group (G, X) is said to be *locally compact* if both G and X are locally compact.

A subgroup S of G is called a *principal stability group* if there exists a dense subset D of X such that S_x is conjugate to S for every $x \in D$. The space X is *of finite G-orbit type* if there exist subgroups $H_1, \dots, H_n$ of G such that for each $x \in X$, S_x is conjugate to one of the H_j. Moreover, X is said to be *locally of finite G-orbit type* if every point in X has a (G-invariant) neighbourhood which is of finite G-orbit type. If G is a compact Lie group and X is a topological manifold, then X is locally of finite G-orbit type [3, Remark after IV.1.2], and if in addition X/G is connected, then there exists a principal stability group [15].

For every closed subgroup H of G, let $X_H = \{x \in X : S_x = H\}$. We shall several times use the fact that if G is compact and $g^{-1}Hg \subseteq H$ for some $g \in G$, then actually $g^{-1}Hg = H$ so that $g \in N_G(H)$. This can be seen as follows. The closed subsemigroup of G generated by g is already a group (see [11, (9.28)(a)]). Hence there exists a sequence $(n_j)_j$ of natural numbers such that $g^{n_j} \to e$. Since

$$g^{-n}Hg^n \subseteq g^{-n+1}Hg^{n-1} \subseteq \cdots \subseteq H$$

for all $n \in \mathbb{N}$, it follows that $g^{-1}Hg = H$.

LEMMA 1.1. *Let (G, X) be a topological transformation group where G is compact and X is of finite G-orbit type. Then, for some stability group S, the set $\{x \in X : S_x \text{ is conjugate to } S\}$ is open in X.*

Proof. Let $H_1, \dots, H_n$ be representatives of the different conjugacy classes of stability groups. Towards a contradiction, assume that for each $1 \le k \le n$, there exist $j_k \in \{1, \dots, n\}$ and $h_k \in G$ such that $j_k \neq k$ and $h_k H_{j_k} h_k^{-1} \subseteq H_k$. Then, with $j_0 = 1$, we find sequences $(j_k)_k \subseteq \{1, \dots, n\}$ and $(g_k)_k \subseteq G$ such that $j_k \neq j_{k-1}$ and

$$H_{j_0} \supseteq g_1 H_{j_1} g_1^{-1} \supseteq \cdots \supseteq g_k H_{j_k} g_k^{-1} \supseteq \cdots$$

for all $k \in \mathbb{N}$. There must exist $l, k \in \{1, \dots, n\}$ with $l < k$ and $j_l = j_k$. Then $H_{j_l} \supseteq g_l^{-1} g_k H_{j_l} g_k^{-1} g_l^{-1}$, which in turn implies that $g_l H_{j_l} g_l^{-1} = g_k H_{j_k} g_k^{-1}$. Since $j_k \neq j_{k-1}$, we have $l < k-1$ and hence

$$g_l H_{j_l} g_l^{-1} \supseteq g_{k-1} H_{j_{k-1}} g_{k-1}^{-1} \supseteq g_k H_{j_k} g_k^{-1}.$$

This shows that H_{j_k} and $H_{j_{k-1}}$ are conjugate, a contradiction.

Thus H_1, say, has the property that $gH_jg^{-1} \not\subseteq H_1$ for any $2 \le j \le n$ and $g \in G$. Let

$$C = \{x \in X : S_x \text{ contains a conjugate of } H_j \text{ for some } 2 \le j \le n\}.$$

Then C is closed in X. Indeed, let $(x_\lambda)_\lambda$ be a net in C converging to some $x \in X$ and let $g_\lambda \in G$ and $j_\lambda \in \{2, \dots, n\}$ be such that $S_{x_\lambda} \supseteq g_\lambda H_{j_\lambda} g_\lambda^{-1}$. After passing to a subnet if necessary, we can assume that $j_\lambda = j$ for all λ and $g_\lambda \to g$ for some $g \in G$. It follows that $gH_j g^{-1} \subseteq S_x$, whence $x \in C$.

Finally, observe that

$$X \setminus C = \{x \in X : S_x \text{ is conjugate to } H_1\}.$$

Clearly, if $x \in X \setminus C$, then S_x is not conjugate to any of $H_2, \dots, H_n$ and hence must be conjugate to H_1. On the other hand, if S_x is conjugate to H_1 then, for each $j \geq 2$, S_x cannot contain a conjugate of H_j because H_1 does not. ∎

Of course, in the setting of Lemma 1.1, if there exists a principal stability group, then the subgroup S must be one. Note, however, that a principal stability group need not exist when X is of finite G-orbit type. A simple example is provided by $G = \mathrm{SO}(n) \times \mathrm{SO}(m)$ acting by rotation on $X = (\mathbb{R}^n \times \{0\}) \cup (\{0\} \times \mathbb{R}^m) \subseteq \mathbb{R}^n \times \mathbb{R}^m$ for $m, n \geq 3$.

Let $\mathcal{K}(G)$ denote the set of all closed subgroups of G endowed with Fell's topology [9]. Recall that a base for this topology is formed by the sets

$$U(C, \mathcal{F}) = \{H \in \mathcal{K}(G) : H \cap C = \emptyset,\ H \cap V \neq \emptyset \text{ for all } V \in \mathcal{F}\},$$

where C is a compact subset of G and $\mathcal{F}$ is a finite family of non-empty open subsets of G. The following lemma might well be known. We include the short proof for the reader's convenience.

Lemma 1.2. *Let (G, X) be a topological transformation group with G compact. Suppose there exists a closed subgroup H of G such that the stabiliser of every point in X is conjugate to H. Then the stabiliser map $x \mapsto S_x$ from X into $\mathcal{K}(G)$ is continuous.*

Proof. It suffices to show that if $(x_\lambda)_\lambda$ is any net in X converging to some $x \in X$, then for some subnet of $(x_\lambda)_\lambda$, the stability groups converge to S_x in $\mathcal{K}(G)$. Choose $g_\lambda \in G$ and $g \in G$ such that $S_{x_\lambda} = g_\lambda^{-1} H g_\lambda$ for all λ and $S_x = gHg^{-1}$. Since G is compact, after passing to a subnet if necessary, we can assume that $g_\lambda \to g_0$ for some $g_0 \in G$. Then, since H is compact, $g_\lambda^{-1} H g_\lambda \to g_0^{-1} H g_0$ by definition of the topology on $\mathcal{K}(G)$. On the other hand, for $h \in H$, $(g_\lambda^{-1} h g_\lambda) \cdot x_\lambda = x_\lambda \to x$. It follows that $(g_0 g)^{-1} H g_0 g \subseteq H$, and this implies that $g_0^{-1} H g_0 = gHg^{-1} = S_x$, as was to be shown. ∎

Similar arguments show that if G has a principal stability subgroup S (for example if G is a closed subgroup of $\mathrm{SO}(n)$ acting on $\mathbb{R}^n \setminus \{0\}$ by rotation) then continuity of the stabiliser map implies that every stability subgroup is conjugate to S.

Corollary 1.3. *Suppose that (G, X) is a locally compact transformation group where G is compact and X is locally of finite G-orbit type. Then* *for any closed subgroup H of G, the space X_H (with the relative topology) is locally compact.*

Proof. Suppose first of all that X is of finite G-orbit type. By Lemma 1.1 there is a representative system $H_1, \ldots, H_r$ of the conjugacy classes of stability groups such that the G-invariant sets

$$V_j = \{x \in X : S_x \text{ is conjugate to } H_j\}$$

have the property that V_1 is open in X (and hence is locally compact) and for $1 \le j \le r-1$, V_{j+1} is open in $X \setminus \bigcup_{k=1}^{j} V_k$. Since $X \setminus \bigcup_{k=1}^{j} V_k$ is closed in X, it follows that V_{j+1} is locally compact. For $1 \le j \le r$, the stabiliser map $x \mapsto S_x$ is continuous on V_j by Lemma 1.2. Therefore X_{H_j} is closed in V_j and hence locally compact.

For the general case, suppose that H is a closed subgroup of G and that $x \in X_H$. Let U be a relatively compact, G-invariant open neighbourhood of x in X. Then U is of finite orbit type, and hence $X_H \cap U = U_H$ is a locally compact space. So there is a compact set N which is a neighbourhood of x in the space $X_H \cap U$. Since U is open in X, N is a neighbourhood of x in the space X_H. ∎

For any topological space X, let $\mathcal{C}(X)$ denote the set of all compact subsets of X. Our general reference for the covering dimension $\dim X$ of X is [23].

LEMMA 1.4. *Let X be a locally compact Hausdorff space and Y a discrete space. Let G be a compact group acting on X and Y and hence on the product space $X \times Y$. Then*

$$\sup\{\dim C : C \in \mathcal{C}((X \times Y)/G)\} = \sup\{\dim K : K \in \mathcal{C}(X/G)\}.$$

Proof. Since Y is discrete, $(X \times Y)/G$ is a disjoint union of open subsets $(X \times G \cdot y)/G$, $y \in Y$. Hence, for any $C \in \mathcal{C}((X \times Y)/G)$, since C meets only finitely many of the disjoint sets $(X \times G \cdot y)/G$,

$$\dim C = \max\{\dim(C \cap (X \times G \cdot y)/G) : y \in Y\}.$$

To prove the lemma, we can therefore assume that $Y = G \cdot y$ for some $y \in Y$. Since G is compact, Y is finite and the normal subgroup H of G defined by

$$H = \{g \in G : g \cdot z = z \text{ for all } z \in Y\}$$

has finite index in G. Now, since the action of H on Y is trivial, the spaces $(X \times Y)/H$ and $(X/H) \times Y$ are canonically homeomorphic. We conclude that, replacing X with X/H and G with G/H, it suffices to prove the lemma in the case of a finite group G.

Let $q_X : X \to X/G$ and $q_{X \times Y} : X \times Y \to (X \times Y)/G$ denote the quotient maps. Both q_X and $q_{X \times Y}$ are continuous, open, proper surjections. Now, if $K \in \mathcal{C}(X/G)$ then $q_X^{-1}(K)$ is a compact space and $q_X^{-1}(t)$ is finite for every

$t \in X/G$. Since compact spaces are paracompact, $\dim K = \dim q_X^{-1}(K)$ by [23, Chapter 9, Proposition 2.16]. Let $C = q_{X\times Y}(q_X^{-1}(K) \times Y)$. Then using [23, 9.2.16] again, we obtain

$$\dim C = \dim(q_X^{-1}(K) \times Y) = \dim(q_X^{-1}(K)).$$

Conversely, let $C \in \mathcal{C}((X \times Y)/G)$ be given. Then $q_{X\times Y}^{-1}(C)$ is compact and hence equals $\bigcup_{j=1}^n X_j \times \{y_j\}$, where $y_1, \dots, y_n$ are distinct elements of Y and $X_1, \dots, X_n$ are compact subsets of X. Since the sets $X \times \{y\}$, $y \in Y$, are disjoint and clopen,

$$\dim q_{X\times Y}^{-1}(C) = \max_{1\le j\le n} \dim(X_j \times \{y_j\}) = \dim X_{j_0},$$

say. Let $K = q_X(X_{j_0})$; then applying [23, 9.2.16] twice more, we get

$$\dim C = \dim q_{X\times Y}^{-1}(C) = \dim X_{j_0} \le \dim q_X^{-1}(K) = \dim K,$$

as required. ∎

Lemma 1.5. *Let (G, X) be a locally compact transformation group where G is compact, and let H be a closed subgroup of G. Then*

(i) *$X_H/N_G(H)$ is homeomorphic to $G \cdot X_H/G$;*
(ii) *if X/G is second countable and X is locally of finite G-orbit type, then*
$$\dim(X_H/N_G(H)) \le \dim(X/G).$$

Proof. (i) Define a map ϕ from $X_H/N_G(H)$ into X/G by $\phi(N_G(H)\cdot x) = G \cdot x$. Then ϕ is continuous since the quotient map $X \to X/G$ is continuous and the quotient map $X_H \to X_H/N_G(H)$ is open. If $x_1, x_2 \in X_H$ are such that $x_2 = g \cdot x_1$ for some $g \in G$, then

$$H = S_{x_2} = gS_{x_1}g^{-1} = gHg^{-1}$$

and hence $g \in N_G(H)$. Thus ϕ is injective and its range is the subspace $G \cdot X_H/G$ of X/G. To prove that $\phi : X_H/N_G(H) \to G \cdot X_H/G$ is open, let U be an open subset of $X_H/N_G(H)$, and let $q_1 : X_H \to X_H/N_G(H)$ and $q_2 : G \cdot X_H \to G \cdot X_H/G$ denote the quotient maps. Let $x \in q_2^{-1}(\phi(U))$ and let $(x_\alpha)_\alpha$ be any net in $G \cdot X_H$ converging to x. For each α, $x_\alpha = g_\alpha \cdot y_\alpha$ where $g_\alpha \in G$, $y_\alpha \in X_H$. Also, there exists $y \in q_1^{-1}(U) \subseteq X_H$ such that $q_2(x) = \phi(q_1(y)) = q_2(y)$, and so $x = g_0 \cdot y$ for some $g_0 \in G$. Passing to a subnet of $(g_\alpha)_\alpha$ and to the corresponding subnet of $(x_\alpha)_\alpha$, we may assume that $g_\alpha \to g_1 \in G$. Then $y_\alpha \to (g_1^{-1}g_0) \cdot y$ and therefore

$$H \subseteq S_{(g_1^{-1}g_0)\cdot y} = (g_1^{-1}g_0)S_y(g_1^{-1}g_0)^{-1} = (g_1^{-1}g_0)H(g_1^{-1}g_0)^{-1}.$$

This implies that $H = (g_1^{-1}g_0)H(g_1^{-1}g_0)^{-1}$, whence $g_1^{-1}g_0 \in N_G(H)$. Eventually $(g_0^{-1}g_1)\cdot y_\alpha \in q_1^{-1}(U)$ and so eventually $q_2(x_\alpha) = q_2(y_\alpha) \in \phi(U)$. Thus the original net $(x_\alpha)_\alpha$ is frequently in $q_2^{-1}(\phi(U))$. This shows that $q_2^{-1}(\phi(U))$ is open, and hence so is $\phi(U)$.

(ii) Recall that a second countable, locally compact Hausdorff space Y is normal [23, Chapter 1, Lemma 4.7] and every locally compact subspace Z of such a space Y is an F_σ-set in Y and hence $\dim Z \leq \dim Y$ [23, Chapter 3, Corollary 6.3]. By (i) and Corollary 1.3, we obtain

$$\dim(X_H/N_G(H)) = \dim(G \cdot X_H/G) \leq \dim(X/G),$$

as required. ∎

REMARK 1.6. (i) Let X be a second countable locally compact Hausdorff space. Then $\dim X = \sup\{\dim C : C \in \mathcal{C}(X)\}$. Indeed, each such C is closed and hence $\dim C \leq \dim X$. On the other hand, since X is normal and σ-compact, $\dim X \leq \sup\{\dim C : C \in \mathcal{C}(X)\}$ by the countable sum theorem [23, Chapter 3, Theorem 2.5].

(ii) Let X be a second countable locally compact Hausdorff space, and let X_j, $1 \leq j \leq n$, be pairwise disjoint subsets of X such that $X = \bigcup_{j=1}^n X_j$ and $\bigcup_{j=k}^n X_j$ is closed in X for every $2 \leq k \leq n$. Then

$$\dim X = \max\{\dim X_j : 1 \leq j \leq n\}.$$

This follows from $n-1$ applications of the fact that

$$\dim X = \max\{\dim Y, \dim(X \setminus Y)\}$$

for any closed subset Y of a second countable, locally compact Hausdorff space X [23, 3.5.8, 3.6.7, 2.2.9].

Finally, as promised in the introduction, we point out the errors in the proof of [21, Theorem 2.1]. The first one is the claim that, if F is the set of fixed points in X, then $C_0(X \setminus F) \rtimes G$ is isomorphic to $C_0((X \setminus F)/G) \otimes \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators on a separable Hilbert space. But this would imply that $C_0(X \setminus F) \rtimes G$ was a continuous trace C^*-algebra, which need not be the case (see the example below). The second problem concerns the application of [21, Proposition 1.10] to the short exact sequence which begins $0 \to C_0(X \setminus F) \rtimes G \to C_0(X) \rtimes G \to \cdots$. In fact, neither $C_0(X \setminus F) \rtimes G$ nor $C_0(X) \rtimes G$ need satisfy the hypotheses of [21, Proposition 1.10]. In particular, in the example below, neither of these algebras has Hausdorff spectrum. In addition, this example illustrates the sets X_H.

EXAMPLE 1.7. Let $X = \mathbb{R}^n \times \mathbb{R}^m$, $n, m \geq 3$, and let $G = \mathrm{SO}(n) \times \mathrm{SO}(m)$ act on X in the obvious manner. Then $F = X_G = \{(0,0)\}$. Let $v = (1, 0, \ldots, 0) \in \mathbb{R}^n$ and $w = (1, 0, \ldots, 0) \in \mathbb{R}^m$, and embed $\mathrm{SO}(n-1)$ into $\mathrm{SO}(n)$ as the subgroup fixing v and $\mathrm{SO}(m-1)$ into $\mathrm{SO}(m)$ as the subgroup fixing w. Then for every non-zero $x \in X$ there exist $s, t \neq 0$ such that x belongs to the G-orbit of one of the vectors (tv, sw), $(tv, 0)$ and $(0, sw)$. Thus, apart from G itself, there are three conjugacy classes of stability subgroups, with representatives

$$H_1 = S_{(tv,sw)} = \mathrm{SO}(n-1) \times \mathrm{SO}(m-1),$$
$$H_2 = S_{(tv,0)} = \mathrm{SO}(n-1) \times \mathrm{SO}(m),$$
$$H_3 = S_{(0,sw)} = \mathrm{SO}(n) \times \mathrm{SO}(m-1).$$

Then $N_G(H_j) = H_j$ for $1 \le j \le 3$ and $X_{H_1} = \{(tv, sw) : t, s \neq 0\}$, $X_{H_2} = \{(tv, 0) : t \neq 0\}$ and $X_{H_3} = \{(0, sw) : s \neq 0\}$.

The fact that $S_{(tv,w)} = \mathrm{SO}(n-1) \times \mathrm{SO}(m-1)$ for all $t \neq 0$ shows that the map $x \mapsto S_x$ is not continuous at $(0, w)$. It follows from [7, Corollary 2] that neither $C_0(X \setminus F) \rtimes G$ nor $C_0(X) \rtimes G$ has Hausdorff spectrum.

2. The ranks of compact transformation group C^*-algebras. We note here that if J is a closed ideal of a C^*-algebra A, then $\mathrm{sr}(J), \mathrm{sr}(A/J) \le \mathrm{sr}(A)$ [24, Section 4] and similarly for the real rank [8, Théorème 1.4]. Also, recall that, for a non-negative real number t, $\lfloor t \rfloor$ denotes the greatest integer n such that $n \le t$ and $\lceil t \rceil$ is the least integer m such that $m \ge t$.

Let (G, X) be a locally compact transformation group and let $C_0(X) \rtimes G$ denote the associated crossed product C^*-algebra. We shall frequently use the fact that if U is an open G-invariant subset of X then $C_0(U) \rtimes G$ embeds in $C_0(X) \rtimes G$ as a closed ideal with quotient canonically isomorphic to $C_0(X \setminus U) \rtimes G$. This is a well known consequence of the universal property of (full) crossed products (see, for example, [13]).

For $x \in X$ and a representation τ of S_x in the Hilbert space $\mathcal{H}(\tau)$, let $\pi_{x,\tau}$ be the $*$-representation of $C_0(X)$ in $\mathcal{H}(\tau)$ defined by $\pi_{x,\tau}(f) = f(x)\mathrm{id}_{\mathcal{H}(\tau)}$, $f \in C_0(X)$. Then $(\pi_{x,\tau}, \tau)$ is a covariant representation of $(C_0(X), S_x, \alpha)$, where α denotes the action of G on $C_0(X)$. If τ is irreducible, then the induced representation $\mathrm{ind}_{S_x}^G(\pi_{x,\tau}, \tau)$ of $C_0(X) \rtimes G$ is irreducible. If σ and τ are two irreducible representations of S_x, then $\mathrm{ind}_{S_x}^G(\pi_{x,\sigma}, \sigma)$ and $\mathrm{ind}_{S_x}^G(\pi_{x,\tau}, \tau)$ are equivalent if and only if σ and τ are equivalent. Moreover, if G is compact, then every irreducible representation of $C_0(X) \rtimes G$ is obtained in this way as an induced representation.

PROPOSITION 2.1. *Let (G, X) be a transformation group, where X is a locally compact Hausdorff space and G is a compact group. Suppose that there exists a subgroup H of G such that the stabiliser of every point in X is conjugate to H.*

(a) *Suppose that H has finite index in G. Then*

(i) $\mathrm{sr}(C_0(X) \rtimes G)$
$$= \sup\left\{1 + \left\lceil \frac{1}{[G:H]} \left\lfloor \tfrac{1}{2} \dim C \right\rfloor \right\rceil : C \in \mathcal{C}(X_H/N_G(H))\right\};$$

(ii) $$\mathrm{RR}(C_0(X) \rtimes G) = \sup\left\{\left\lceil \frac{\dim C}{2[G:H]-1} \right\rceil : C \in \mathcal{C}(X_H/N_G(H))\right\}.$$

Suppose, in addition, that X is second countable. Then

$$\begin{aligned}\text{(iii) } \operatorname{sr}(C_0(X) \rtimes G) &= 1 + \left\lceil \frac{1}{[G:H]} \left\lfloor \tfrac{1}{2} \dim(X_H/N_G(H)) \right\rfloor \right\rceil \\ &= 1 + \left\lceil \frac{1}{[G:H]} \left\lfloor \tfrac{1}{2} \dim(X/G) \right\rfloor \right\rceil;\end{aligned}$$

$$\text{(iv) } \operatorname{RR}(C_0(X) \rtimes G) = \left\lceil \frac{\dim(X_H/N_G(H))}{2[G:H]-1} \right\rceil = \left\lceil \frac{\dim(X/G)}{2[G:H]-1} \right\rceil.$$

(b) *Suppose that H has infinite index in G and that both X and G are second countable and* $\dim(X/G) < \infty$. *Then*

$$\begin{aligned}\text{(i) } \operatorname{sr}(C_0(X) \rtimes G) &= \min\bigl\{2, 1 + \bigl\lfloor \tfrac{1}{2} \dim(X_H/N_G(H)) \bigr\rfloor\bigr\} \\ &= \min\bigl\{2, 1 + \bigl\lfloor \tfrac{1}{2} \dim(X/G) \bigr\rfloor\bigr\};\end{aligned}$$

$$\begin{aligned}\text{(ii) } \operatorname{RR}(C_0(X) \rtimes G) &= \min\{1, \dim(X_H/N_G(H))\} \\ &= \min\{1, \dim(X/G)\}.\end{aligned}$$

Proof. Let $A = C_0(X) \rtimes G$ and note that, since the stabiliser map $x \mapsto S_x$ from X into $\mathcal{K}(G)$ is continuous (Lemma 1.2) and G is compact, A has continuous trace [7, Corollary 2]. By hypothesis, for each $x \in X$ there exists $g_x \in G$ such that $S_x = g_x H g_x^{-1}$. Thus there is a well defined G-equivariant map π from X to the left coset space $G/N_G(H)$ given by $\pi(x) = g_x N_G(H)$. Clearly, $\pi^{-1}(\{eH\}) = X_H$. An argument similar to (but simpler than) the proof of Lemma 1.2 shows that π is continuous. Hence, by [10, Theorem 17] the two C^*-algebras A and $C_0(X_H) \rtimes N_G(H)$ are Morita equivalent and hence their spectra are homeomorphic. On the other hand, since $(N_G(H), X_H)$ is a compact transformation group for which all of the stability subgroups are equal to H, it follows from [32, Corollary 5.12] that the continuous trace C^*-algebra $C_0(X_H) \rtimes N_G(H)$ has spectrum homeomorphic to the quotient space $(X_H \times \widehat{H})/N_G(H)$. Indeed, elements of $X_H \times \widehat{H}$ induce irreducible representations of $C_0(X_H) \rtimes N_G(H)$ as described above, and for $g \in N_G(H)$, $x \in X$ and $\tau \in \widehat{H}$, the representation induced from (x, τ) is equivalent to the one induced from $g \cdot (x, \tau) = (g \cdot x, g \cdot \tau)$.

Assume first that G, H and X satisfy the hypotheses of (b). Then, since X and G are second countable, A is separable and $\dim \pi = \aleph_0$ for every $\pi \in \widehat{A}$. Moreover, A has continuous trace. By Lemmas 1.4 and 1.5, for every compact subset C of $(X_H \times \widehat{H})/N_G(H)$,

$$\begin{aligned}\dim C &\le \sup\{\dim K : K \in \mathcal{C}(X_H/N_G(H))\} \\ &\le \dim(X_H/N_G(H)) \le \dim(X/G).\end{aligned}$$

Since $\widehat{A}$ is a second countable, locally compact Hausdorff space and is homeomorphic to $(X_H \times \widehat{H})/N_G(H)$, Lemmas 1.4 and 1.5 and Remark 1.6(i) show that $\dim \widehat{A} \le \dim(X/G) < \infty$. The following formulae are given in [1, Proposition 1.2(b)]:

$$\mathrm{RR}(A) = \min\{1, \sup\{\dim C : C \in \mathcal{C}(\widehat{A})\}\},$$
$$\mathrm{sr}(A) = \min\{2, 1 + \lfloor \tfrac{1}{2} \sup\{\dim C : C \in \mathcal{C}(\widehat{A})\} \rfloor\}.$$

The statements in (b) now follow from Lemmas 1.4 and 1.5 together with Remark 1.6(i) again.

Now suppose that H has finite index in G. For $d \in \mathbb{N}$, define $\widehat{H}_d = \{\tau \in \widehat{H} : \dim \tau = d\}$. Then $X_H \times \widehat{H}$ is the disjoint union of the open and closed sets $X_H \times \widehat{H}_d$, $d \in \mathbb{N}$, and each $X_H \times \widehat{H}_d$ is $N_G(H)$-invariant. It therefore follows that $(X_H \times \widehat{H})/N_G(H)$ is the disjoint union of open and closed sets $(X_H \times \widehat{H}_d)/N_G(H)$, $d \in \mathbb{N}$.

Let A_d denote the closed ideal of A with $\widehat{A}_d = (X_H \times \widehat{H}_d)/N_G(H)$. Note that points of $(X_H \times \widehat{H}_d)/N_G(H)$ give rise to irreducible representations of A, with dimension $d[G : H]$, by the induction process described at the start of this section. Hence each A_d is $d[G : H]$-homogeneous. Since the sets $\widehat{A}_d$ are open and closed, A is isomorphic to the c_0-direct sum of the A_d. Thus

$$\mathrm{RR}(A) = \sup_{d \in \mathbb{N}} \{\mathrm{RR}(A_d)\}, \qquad \mathrm{sr}(A) = \sup_{d \in \mathbb{N}} \{\mathrm{sr}(A_d)\}.$$

Since A_d is $d[G : H]$-homogeneous, the formulae in [1, Proposition 1.2(a)] for the real and stable ranks of an n-homogeneous C^*-algebra give

$$\mathrm{RR}(A_d) = \sup\left\{ \left\lceil \frac{\dim C}{2d[G:H] - 1} \right\rceil : C \in \mathcal{C}(\widehat{A}_d) \right\},$$
$$\mathrm{sr}(A_d) = \sup\left\{ 1 + \left\lceil \frac{1}{d[G:H]} \lfloor \tfrac{1}{2} \dim C \rfloor \right\rceil : C \in \mathcal{C}(\widehat{A}_d) \right\}.$$

Since $\widehat{H}$ is discrete and $N_G(H)$ is compact, for every $d \in \mathbb{N}$ with $\widehat{H}_d \neq \emptyset$,

$$\sup\{\dim C : C \in \mathcal{C}(\widehat{A}_d)\} = \sup\{\dim K : K \in \mathcal{C}(X_H/N_G(H))\}$$

by Lemma 1.4. The formulae in (a)(i),(ii) follow by taking $d = 1$, and the formulae in (iii) and (iv) follow from Remark 1.6(i) and Lemma 1.5(i). ∎

The following simple lemma, together with the countable sum theorem for the covering dimension, will allow us to obtain results in Theorems 2.3 and 2.4 when X is only locally of finite G-orbit type.

LEMMA 2.2. *Let (G, X) be a transformation group, where G is a compact group and X is a locally compact Hausdorff space. Then*

(i) $\mathrm{sr}(C_0(X) \rtimes G) = \sup\{\mathrm{sr}(C(G \cdot K) \rtimes G) : K \in \mathcal{C}(X)\}$;
(ii) $\mathrm{RR}(C_0(X) \rtimes G) = \sup\{\mathrm{RR}(C(G \cdot K) \rtimes G) : K \in \mathcal{C}(X)\}$.

Proof. Let $A = C_0(X) \rtimes G$ and let $\mathcal{V}$ denote the set of all relatively compact open subsets of X, directed by inclusion. Then A is the inductive limit of the family of ideals $C_0(G\cdot V)\rtimes G$, $V\in\mathcal{V}$, and hence

$$\mathrm{sr}(A)\le\sup_{V\in\mathcal{V}}\{\mathrm{sr}(C_0(G\cdot V)\rtimes G)\},\qquad \mathrm{RR}(A)\le\sup_{V\in\mathcal{V}}\{\mathrm{RR}(C_0(G\cdot V)\rtimes G)\}$$

(compare [24, Theorem 5.1] and [12, Lemma 4.1(i)]). On the other hand, $C_0(G\cdot V)\rtimes G$ is an ideal of $C(G\cdot\overline{V})\rtimes G$, and for every $K\in\mathcal{C}(X)$, $C(G\cdot K)\rtimes G$ is a quotient of A. It follows that

$$\begin{aligned}\mathrm{sr}(A)&\le\sup_{V\in\mathcal{V}}\{\mathrm{sr}(C_0(G\cdot V)\rtimes G)\}\le\sup_{V\in\mathcal{V}}\{\mathrm{sr}(C(G\cdot\overline{V})\rtimes G)\}\\&\le\sup_{K\in\mathcal{C}(X)}\{\mathrm{sr}(C(G\cdot K)\rtimes G)\}\le\mathrm{sr}(A),\end{aligned}$$

and similarly for the real rank. ■

In the following two theorems, $\mathcal{H}$ denotes a representative system of the conjugacy classes of stability groups, and $\mathcal{H}_{\mathrm{fin}}=\{H\in\mathcal{H}:[G:H]<\infty\}$.

THEOREM 2.3. *Let (G,X) be a transformation group, where X is a locally compact Hausdorff space, G is a compact group and every stability group has finite index in G.*

(a) *Suppose that X is of finite G-orbit type. Then*

(i) $\mathrm{sr}(C_0(X)\rtimes G)$
$$=\max_{H\in\mathcal{H}}\sup\left\{1+\left\lceil\frac{1}{[G:H]}\left\lfloor\tfrac12\dim C\right\rfloor\right\rceil : C\in\mathcal{C}(X_H/N_G(H))\right\};$$

(ii) $\mathrm{RR}(C_0(X)\rtimes G)$
$$=\max_{H\in\mathcal{H}}\sup\left\{\left\lceil\frac{\dim C}{2[G:H]-1}\right\rceil : C\in\mathcal{C}(X_H/N_G(H))\right\}.$$

(b) *Suppose that X is second countable and locally of finite G-orbit type. Then*

(i) $$\mathrm{sr}(C_0(X)\rtimes G)=\sup_{H\in\mathcal{H}}\left\{1+\left\lceil\frac{1}{[G:H]}\left\lfloor\tfrac12\dim(X_H/N_G(H))\right\rfloor\right\rceil\right\};$$

(ii) $$\mathrm{RR}(C_0(X)\rtimes G)=\sup_{H\in\mathcal{H}}\left\{\left\lceil\frac{\dim(X_H/N_G(H))}{2[G:H]-1}\right\rceil\right\}.$$

Proof. (a) Let $A=C_0(X)\rtimes G$. To prove (i), we proceed by induction on the number of elements in $\mathcal{H}$. Notice first that by Lemma 1.1 there exists $H_0\in\mathcal{H}_{\mathrm{fin}}$ such that the set $U=\{x\in X:S_x \text{ is conjugate to } H_0\}$ is open in X. Then, as in the proof of Proposition 2.1(a), the ideal $I=C_0(U)\rtimes G$ has continuous trace and is isomorphic to a c_0-direct sum of homogeneous C^*-algebras. It follows by [20, Lemma 5(a)] that $\mathrm{sr}(A)=\max\{\mathrm{sr}(I),\mathrm{sr}(A/I)\}$.

Now $\mathrm{sr}(I)$ is given by formula (a)(ii) of Proposition 2.1. Since $\mathcal{H}\setminus\{H_0\}$ represents the conjugacy classes of stability groups for the action of G on $X\setminus U$, the inductive hypothesis together with $\mathrm{sr}(A)=\max\{\mathrm{sr}(I),\mathrm{sr}(A/I)\}$ yields the formula in (i).

The argument for part (ii) is very similar, but requires the formula $\mathrm{RR}(A) = \max\{\mathrm{RR}(I), \mathrm{RR}(A/I)\}$ where the ideal I is as above. The proof of this formula proceeds as in the case of stable rank [20, Lemma 5(a)], but uses the fact that the real rank does not increase on passing to a closed ideal and replaces the application of [28, Proposition 3.15] by [21, Lemma 1.9]. (Note that in the proof of [21, Lemma 1.9] there is sufficient surjectivity for the argument of [21, Proposition 1.3] to be satisfactorily applied. A full proof of [21, Proposition 1.3] has been given in [19, Proposition 1.6].)

(b) Since X is a σ-compact, locally compact Hausdorff space, there exists a sequence $(Y_n)_n$ of G-invariant compact subsets Y_n of X such that every G-invariant compact subset K of X is contained in some Y_n. Then $C(K) \rtimes G$ is a quotient of $C(Y_n) \rtimes G$, and hence Lemma 2.2 shows that

$$\mathrm{sr}(C_0(X) \rtimes G) = \sup_{n\in\mathbb{N}}\{\mathrm{sr}(C(Y_n) \rtimes G)\}, \tag{1}$$

$$\mathrm{RR}(C_0(X) \rtimes G) = \sup_{n\in\mathbb{N}}\{\mathrm{RR}(C(Y_n) \rtimes G)\}. \tag{2}$$

On the other hand, for any $H \in \mathcal{H}$,

$$X_H/N_G(H) = \bigcup_{n\in\mathbb{N}} (Y_n)_H/N_G(H).$$

Furthermore, each $(Y_n)_H$ is relatively closed in X_H and the quotient map is closed. By Corollary 1.3, $X_H/N_G(H)$ is a second countable, locally compact Hausdorff space and hence is normal. By the countable sum theorem

$$\dim(X_H/N_G(H)) = \sup_{n\in\mathbb{N}}\{\dim((Y_n)_H/N_G(H))\}. \tag{3}$$

The results now follow by applying part (a) to each Y_n and noting that $\dim((Y_n)_H/N_G(H))$ is the supremum of the dimensions of its compact subsets (Remark 1.6(i)). ■

THEOREM 2.4. *Let (G, X) be a transformation group, where X is a second countable, locally compact Hausdorff space and G is a second countable compact group. In addition, suppose that $\dim(X/G) < \infty$, that X is locally of finite G-orbit type and that $\mathcal{H}_{\mathrm{fin}} \neq \mathcal{H}$. Then*

(i) $\mathrm{sr}(C_0(X) \rtimes G) = 1$ *if and only if* $\dim(X/G) \le 1$ *and otherwise*
$$\mathrm{sr}(C_0(X) \rtimes G) = \max\left\{2, \max_{H\in\mathcal{H}_{\mathrm{fin}}}\left\{1 + \left\lceil \frac{1}{[G:H]} \left\lfloor \tfrac{1}{2}\dim(X_H/N_G(H))\right\rfloor \right\rceil\right\}\right\},$$
where the right hand side is interpreted as 2 *if* $\mathcal{H}_{\mathrm{fin}} = \emptyset$;

(ii) $\mathrm{RR}(C_0(X) \rtimes G) = 0$ *if and only if* $\dim(X/G) = 0$ *and otherwise*
$$\mathrm{RR}(C_0(X) \rtimes G) = \max\left\{1, \max_{H\in\mathcal{H}_{\mathrm{fin}}}\left\{\left\lceil \frac{\dim(X_H/N_G(H))}{2[G:H]-1}\right\rceil\right\}\right\},$$
where the right hand side is interpreted as 1 *if* $\mathcal{H}_{\mathrm{fin}} = \emptyset$.

Proof. Since X is a σ-compact locally compact Hausdorff space, there exists a sequence $(Y_n)_n$ of G-invariant compact subsets Y_n of X such that every G-invariant compact subset K of X is contained in some Y_n. As in the proof of Theorem 2.3(b), we obtain the equations (1), (2) and (3) above. Furthermore, by the countable sum theorem, $\dim(X/G) = \sup_{n\in\mathbb{N}}\{\dim(Y_n/G)\}$. Therefore, to prove the formulae of the theorem, we can assume that X is compact and hence of finite G-orbit type.

Let N be a closed normal subgroup of G of finite index in G such that $N \subseteq H$ for all $H \in \mathcal{H}_{\mathrm{fin}}$. Then the set

$$X_{\mathrm{fin}} = \{x \in X : [G : S_x] < \infty\} = \{x \in X : S_x \supseteq N\}$$

is closed in X, and $X \setminus X_{\mathrm{fin}}$ is a non-empty, open, G-invariant set. Applying Lemma 1.1 to $X \setminus X_{\mathrm{fin}}$, we conclude that there exists $S \in \mathcal{H} \setminus \mathcal{H}_{\mathrm{fin}}$ such that the set $U = \{x \in X : S_x \text{ is conjugate to } S\}$ is open in X. Let $A = C_0(X) \rtimes G$ and $I = C_0(U) \rtimes G$, and let $\mathcal{V}$ be the collection of all G-invariant, relatively compact, open subsets V of U such that $\overline{V} \subseteq U$, directed by inclusion. To each $V \in \mathcal{V}$ associate the ideals $I_V = C_0(V) \rtimes G$ and $J_V = C_0(X \setminus \overline{V}) \rtimes G$. Then $I_V \cdot J_V = \{0\}$ for each V, and since $\bigcup\{C_0(V) : V \in \mathcal{V}\}$ is dense in $C_0(U)$, it follows by standard arguments that $I = \overline{\bigcup\{I_V : V \in \mathcal{V}\}}$. Then

$$\mathrm{sr}(A) = \max\{\mathrm{sr}(A/I), \sup_{V\in\mathcal{V}}\{\mathrm{sr}(A/J_V)\}\}$$

by applying [28, Proposition 3.15] to the unitization of A, and

$$\mathrm{RR}(A) = \max\{\mathrm{RR}(A/I), \sup_{V\in\mathcal{V}}\{\mathrm{RR}(A/J_V)\}\}$$

by [21, Lemma 1.9] (see the remarks in the proof of Theorem 2.3(a)).

Since A/J_V is isomorphic to $C(\overline{V}) \rtimes G$, we obtain

$$\mathrm{sr}(A) = \max\{\mathrm{sr}(A/I), \sup_{V\in\mathcal{V}}\{\mathrm{sr}(C(\overline{V}) \rtimes G)\}\},$$
$$\mathrm{RR}(A) = \max\{\mathrm{RR}(A/I), \sup_{V\in\mathcal{V}}\{\mathrm{RR}(C(\overline{V}) \rtimes G)\}\}.$$

Since U is a countable union of sets $\overline{V}$, $X_S = U_S$ is a countable union of relatively closed sets $\overline{V} \cap X_S = \overline{V}_S$, and hence the normal space $X_S/N_G(S)$ (see Corollary 1.3) is a countable union of closed subsets $\overline{V}_S/N_G(S)$. Proposition 2.1(b)(i) and the countable sum theorem then imply that

$$\begin{aligned}\mathrm{sr}(A) &= \max\Big\{\mathrm{sr}(A/I), \sup_{V\in\mathcal{V}}\Big\{\min\Big\{2, 1 + \Big\lfloor \tfrac{1}{2}\dim(\overline{V}_S/N_G(S))\Big\rfloor\Big\}\Big\}\Big\}\\ &= \max\Big\{\mathrm{sr}(A/I), \min\Big\{2, 1 + \sup_{V\in\mathcal{V}}\Big\{\Big\lfloor \tfrac{1}{2}\dim(\overline{V}_S/N_G(S))\Big\rfloor\Big\}\Big\}\Big\}\\ &= \max\Big\{\mathrm{sr}(A/I), \min\Big\{2, 1 + \Big\lfloor \tfrac{1}{2}\dim(X_S/N_G(S))\Big\rfloor\Big\}\Big\}.\end{aligned}$$

Similarly, using Proposition 2.1(b)(ii), we obtain

$$\mathrm{RR}(A) = \max\{\mathrm{RR}(A/I), \min\{1, \dim(X_S/N_G(S))\}\}.$$

Proceeding inductively, we find that $\mathrm{sr}(A)$ is the maximum of $\mathrm{sr}(C(X_{\mathrm{fin}}) \rtimes G)$ and

$$\max_{H \in \mathcal{H} \setminus \mathcal{H}_{\mathrm{fin}}} \{\min\{2, 1 + \lfloor \tfrac{1}{2} \dim(X_H/N_G(H)) \rfloor\}\}.$$

Similarly, $\mathrm{RR}(A)$ is the maximum of $\mathrm{RR}(C(X_{\mathrm{fin}}) \rtimes G)$ and

$$\max_{H \in \mathcal{H} \setminus \mathcal{H}_{\mathrm{fin}}} \{\min\{1, \dim(X_H/N_G(H))\}\}.$$

Now, by Theorem 2.3,

$$\mathrm{sr}(C(X_{\mathrm{fin}}) \rtimes G) = \max_{H \in \mathcal{H}_{\mathrm{fin}}} \left\{ 1 + \left\lceil \frac{1}{[G:H]} \left\lfloor \tfrac{1}{2} \dim(X_H/N_G(H)) \right\rfloor \right\rceil \right\},$$

$$\mathrm{RR}(C(X_{\mathrm{fin}}) \rtimes G) = \max_{H \in \mathcal{H}_{\mathrm{fin}}} \left\{ \left\lceil \frac{\dim(X_H/N_G(H))}{2[G:H]-1} \right\rceil \right\}.$$

Recall that, by Lemma 1.5 and Remark 1.6(ii),

$$\dim(X/G) = \max_{H \in \mathcal{H}} \{\dim(X_H/N_G(H))\}.$$

The above formulae then immediately show that $\mathrm{sr}(A) = 1$ if and only if $\dim(X/G) \le 1$, and $\mathrm{RR}(A) = 0$ if and only if $\dim(X/G) = 0$.

Now, assume that $\dim(X/G) \ge 2$, so that $\dim(X_S/N_G(S)) \ge 2$ for at least one $S \in \mathcal{H}$. If this happens for some $S \in \mathcal{H} \setminus \mathcal{H}_{\mathrm{fin}}$, then

$$\mathrm{sr}(A) = \max\{2, \mathrm{sr}(C(X_{\mathrm{fin}}) \rtimes G)\}.$$

Otherwise, $\dim(X_H/N_G(H)) \le 1$ for all $H \in \mathcal{H} \setminus \mathcal{H}_{\mathrm{fin}}$, and hence

$$\mathrm{sr}(A) = \max\{1, \mathrm{sr}(C(X_{\mathrm{fin}}) \rtimes G)\} = \max\{2, \mathrm{sr}(C(X_{\mathrm{fin}}) \rtimes G)\}.$$

Finally, if $\dim(X/G) \ge 1$, the stated formula for $\mathrm{RR}(A)$ follows in the same manner. ∎

Recall that X_G is the set of fixed points in X under the action of G.

COROLLARY 2.5. *Let (G, X) be a transformation group, where X is a second countable locally compact Hausdorff space and G is a second countable compact group. Suppose that X is locally of finite G-orbit type and that* $\dim(X/G) < \infty$. *Then*

$$\mathrm{sr}(C_0(X) \rtimes G) \le 1 + \left\lfloor \tfrac{1}{2} \dim(X/G) \right\rfloor.$$

Suppose, in addition, that G is connected. If $\dim(X/G) \le 1$ *then we have* $\mathrm{sr}(C_0(X) \rtimes G) = 1$, *and if* $\dim(X/G) \ge 2$ *then*

$$\mathrm{sr}(C_0(X) \rtimes G) = \max\left\{2, 1 + \left\lfloor \tfrac{1}{2} \dim(X_G) \right\rfloor\right\},$$

unless $X_G = \emptyset$, in which case $\mathrm{sr}(C_0(X) \rtimes G) = 2$.

Proof. The first inequality follows from Theorem 2.3(b)(i), Theorem 2.4(i) and Lemma 1.5(ii). When G is connected, $\mathcal{H}_{\mathrm{fin}}$ is either empty or equal to $\{G\}$. If the action is trivial then Theorem 2.3(b)(i) yields $\mathrm{sr}(C_0(X) \rtimes G) =$

$1 + \left\lfloor \frac{1}{2} \dim X \right\rfloor$ (which also follows from the fact that $C_0(X) \rtimes G \cong C_0(X) \otimes C^*(G)$). If the action is non-trivial then Theorem 2.4(i) yields the required results. ∎

Corollary 2.6. *Let G and X be as in Corollary* 2.5. *Then*

$$\mathrm{RR}(C_0(X) \rtimes G) \leq \dim(X/G).$$

Suppose, *in addition*, *that G is connected. If* $\dim(X/G) = 0$ *then we have* $\mathrm{RR}(C_0(X) \rtimes G) = 0$, *and if* $\dim(X/G) \geq 1$ *then*

$$\mathrm{RR}(C_0(X) \rtimes G) = \max\{1, \dim(X_G)\},$$

unless $X_G = \emptyset$, *in which case* $\mathrm{RR}(C_0(X) \rtimes G) = 1$.

Proof. The proof is similar to that of Corollary 2.5, using Theorem 2.3(b)(ii) and Theorem 2.4(ii). ∎

3. Motion group C^*-algebras. In [1] we have investigated the real and stable ranks for several classes of group C^*-algebras. However, these did not include C^*-algebras of motion groups. We are now able to treat some examples of motion groups as an application of Theorem 2.4.

For $n \geq 2$, let $\mathrm{SO}(n)$ act on $\mathbb{R}^n$ by rotation. Then $C_0(\mathbb{R}^n) \rtimes \mathrm{SO}(n)$ is isomorphic to the group C^*-algebra of the classical motion group $\mathbb{R}^n \rtimes \mathrm{SO}(n)$. More generally, let K be any compact group and let $\alpha : k \mapsto \alpha_k$ be an injective homomorphism from K into the automorphism group $\mathrm{GL}(n, \mathbb{R})$ of $\mathbb{R}^n$. Then, by replacing a given scalar product $\langle \cdot, \cdot \rangle$ on $\mathbb{R}^n$ by the scalar product $\langle \cdot, \cdot \rangle_\alpha$ defined by

$$\langle v, w \rangle_\alpha = \int_K \langle \alpha_k(v), \alpha_k(w) \rangle \, dk, \qquad v, w \in \mathbb{R}^n,$$

we can always assume that K is a subgroup of $\mathrm{SO}(n)$.

In what follows, we consider $\mathrm{SO}(n-1)$ as embedded into $\mathrm{SO}(n)$ as the subgroup fixing the vector $v = (1, 0, \ldots, 0) \in \mathbb{R}^n$.

Theorem 3.1. *Let G be a connected closed subgroup of* $\mathrm{SO}(n)$, *where* $n \geq 2$, *and let* $(\mathbb{R}^n)_G$ *be the set of points in* $\mathbb{R}^n$ *which are fixed by G. Then*

(i) $\mathrm{sr}(C_0(\mathbb{R}^n) \rtimes G) = 1$ *if and only if* $G a \mathrm{SO}(n-1) a^{-1} = \mathrm{SO}(n)$ *for some* $a \in \mathrm{SO}(n)$, *and*

$$\mathrm{sr}(C_0(\mathbb{R}^n) \rtimes G) = \max\left\{2, 1 + \left\lfloor \tfrac{1}{2} \dim((\mathbb{R}^n)_G) \right\rfloor\right\}$$

otherwise;

(ii) $\mathrm{RR}(C_0(\mathbb{R}^n) \rtimes G) = \max\{1, \dim((\mathbb{R}^n)_G)\}$.

Proof. Note that $\dim(\mathbb{R}^n/G) \leq n$ by [22, Theorem 3.16]. Define $X = \mathbb{R}^n \setminus \{0\}$ and let $q : X \to X/G$ denote the quotient map. Since X/G is metrizable and q is a closed surjection,

$$\begin{aligned} n = \dim X &\le \dim(X/G) + \sup\{\dim(q^{-1}(q(x))) : x \in X\} \\ &= \dim(X/G) + \sup\{\dim(G \cdot x) : x \in X\} \\ &= \dim(X/G) + \sup\{\dim(G/S_x) : x \in X\} \end{aligned}$$

by [23, Chapter 9, Proposition 2.6]. For each $x \in X$, $S_x = G \cap a\mathrm{SO}(n-1)a^{-1}$ for some $a \in \mathrm{SO}(n)$, and conversely every such subgroup of G occurs as a stability group. Thus

$$n \le \dim(X/G) + \sup\{\dim(G/(G \cap a\mathrm{SO}(n-1)a^{-1})) : a \in \mathrm{SO}(n)\}.$$

Suppose first that $\dim(G/(G \cap a\mathrm{SO}(n-1)a^{-1})) \le n-2$ for all $a \in \mathrm{SO}(n)$. Then $\dim(\mathbb{R}^n/G) = \dim(X/G) \ge 2$ and Corollaries 2.5 and 2.6 show that

$$\mathrm{sr}(C_0(\mathbb{R}^n) \rtimes G) = \max\bigl\{2, 1 + \bigl\lfloor \tfrac{1}{2} \dim((\mathbb{R}^n)_G) \bigr\rfloor\bigr\}$$

and $\mathrm{RR}(C_0(\mathbb{R}^n) \rtimes G) = \max\{1, \dim((\mathbb{R}^n)_G)\}$.

Now suppose that $\dim(G/(G \cap a\mathrm{SO}(n-1)a^{-1})) \ge n-1$ for some $a \in \mathrm{SO}(n)$. Then, since $C_0(\mathbb{R}^n) \rtimes a^{-1}Ga$ and $C_0(\mathbb{R}^n) \rtimes G$ are isomorphic and

$$a^{-1}[G/(G \cap a\mathrm{SO}(n-1)a^{-1})]a = a^{-1}Ga/(a^{-1}Ga \cap \mathrm{SO}(n-1)),$$

we can assume that a is the identity of $\mathrm{SO}(n)$. Let $H = G \cap \mathrm{SO}(n-1)$ and $C = G\mathrm{SO}(n-1)/\mathrm{SO}(n-1)$. Since $gH \mapsto g\mathrm{SO}(n-1)$ provides a homeomorphism between G/H and the closed subspace C of $\mathrm{SO}(n)/\mathrm{SO}(n-1)$, it follows that

$$n-1 \le \dim(G/H) = \dim C \le \dim(\mathrm{SO}(n)/\mathrm{SO}(n-1)) = n-1.$$

Hence $\dim C = n-1$. Since $\mathrm{SO}(n)/\mathrm{SO}(n-1) = S^{n-1}$ is locally homeomorphic to $\mathbb{R}^{n-1}$, it follows from [17, Theorem IV.5] and a compactness argument that C has non-empty interior. Then $V = \{g \in G : g\mathrm{SO}(n-1) \in C^\circ\}$ is a non-empty open subset of G. This implies that

$$C = \bigcup_{g \in G} gV\mathrm{SO}(n-1) = \bigcup_{g \in G} gC^\circ$$

is open (and closed) in S^{n-1}, hence equal to S^{n-1}. It follows that $G \cdot \mathrm{SO}(n-1) = \mathrm{SO}(n)$ and hence $G \cdot v = S^{n-1}$. Therefore $X/G = (0, \infty)$ and Corollaries 2.5 and 2.6 imply that $\mathrm{sr}(C_0(\mathbb{R}^n) \rtimes G) = 1$ and $\mathrm{RR}(C_0(\mathbb{R}^n) \rtimes G) = \max\{1, \dim((\mathbb{R}^n)_G)\}$. ■

Regarding the proof above, note that $G \cdot \mathrm{SO}(n-1) = \mathrm{SO}(n)$ if and only if G acts transitively on the sphere S^{n-1}. Actions with this property are described in [16]. We are grateful to M. C. Crabb for drawing our attention to this reference.

Corollary 3.2. *For all $n \ge 2$, we have*

$$\mathrm{sr}(C^*(\mathbb{R}^n \rtimes \mathrm{SO}(n))) = \mathrm{RR}(C^*(\mathbb{R}^n \rtimes \mathrm{SO}(n))) = 1.$$

In [29], Sudo has obtained the comparable result $\mathrm{sr}(C^*(\mathbb{R}^n \rtimes \mathrm{Spin}(n))) = 1$, where $\mathrm{Spin}(n)$ is the simply connected covering group of $\mathrm{SO}(n)$.

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