

Polynomial approximations and universality

by

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Abstract. We give another version of the recently developed abstract theory of universal series to exhibit a necessary and sufficient condition of polynomial approximation type for the existence of universal elements. This certainly covers the case of simultaneous approximation with a sequence of continuous linear mappings. In the case of a sequence of unbounded operators the same condition ensures existence and density of universal elements. Several known results, stronger statements or new results can be deduced in a unified way.

1. Introduction. The theory of universal series dates back almost a century and during that period several results concerning various types of these series have been obtained (see for instance [11], [8], [9] and the references therein). We refer to [7] for an excellent survey on universality. Recently Bayart, Grosse-Erdmann, Nestoridis and Papadimitropoulos have developed an abstract framework for the theory of universal series, from which they deduce in a unified way most of the known results [10], [2]. Let A be a complete metrizable vector subspace of $\mathbb{K}^{\mathbb{N}}$. It turns out that the existence of universal elements in A is equivalent to the existence of a function which approximates both a given function in the space where the universal property holds and zero in the space A where the universal function should live. The existence of such universal series also implies topological genericity of such elements. The present paper deals with a special case of this abstract theorem, where we impose a simultaneous approximation in A using a sequence of continuous linear mapping $T_n : A \rightarrow A$. From this, we exhibit a necessary and sufficient condition of polynomial approximation type for the existence of universal elements $a \in A$ so that $T_n a$ is dense in A and all approximations are realized with the same sequence of indices. Moreover, if A is a separable infinite-dimensional Banach space, we prove that the same condition implies the existence and the density of universal series under a sequence of

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unbounded operators on A . As a special case, we can consider sequences (T_n) of operators that are generated by a single linear operator T via iteration, that is, $T_n = T^n$. Hence the existence of such elements implies the hypercyclicity of T (a vector a is said to be a hypercyclic vector for T provided its orbit $\{T^n a : n \in \mathbb{N}\}$ is dense). Thus we deduce a hypercyclicity criterion for unbounded operators. Finally, all these results allow us to deduce and easily unify several known results, stronger statements or new results.

2. From abstract theory to polynomial approximation

2.1. Introduction. The theory of universality can be embedded in a very general framework. Let Y be a complete metrizable topological vector space and Z a metrizable topological vector space, endowed with translation invariant metrics d and ρ , respectively.

DEFINITION 2.1. Let M be a separable closed subspace of Z and $L_n : Y \rightarrow Z$, $n \in \mathbb{N}$, continuous linear mappings. An element $y \in Y$ belongs to the class $\mathcal{U}(\mathcal{L}; M)$ if $M \subset \overline{\{L_n y : n \in \mathbb{N}\}}$.

The most general theorem of the abstract theory is the following (Theorem 27 of [2]).

THEOREM 2.2 ([2]). *Under the above assumptions, if there exists a dense subset Y_0 of Y such that, for every $y \in Y_0$, $(L_n y)$ converges to an element in M , then the following are equivalent:*

- (α) $\mathcal{U}(\mathcal{L}; M) \neq \emptyset$.
- (β) For every $z \in M$ and $\varepsilon > 0$, there exist $n \geq 0$ and $y \in Y$ such that

$$\rho(L_n y, z) < \varepsilon \quad \text{and} \quad d(y, 0) < \varepsilon.$$
- (γ) $\mathcal{U}(\mathcal{L}; M)$ is a dense G_δ subset of Y .

A more restricted form of the abstract theory with straightforward applications is as follows.

Let X be a metrizable topological vector space over the field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . Let ρ be a translation invariant metric compatible with the operations $+$ and \cdot of the vector space X . Let $(x_k)_{k=0}^{+\infty}$ be a fixed sequence in X . We are interested in restricted universal series. We fix a subspace A of $\mathbb{K}^{\mathbb{N}}$ and assume that it carries a complete metrizable vector space topology induced by a translation invariant metric d , such that the following three properties hold:

- (P_1) the coordinate projections $A \rightarrow \mathbb{K}$, $a \mapsto a_m$, are continuous for all $m \in \mathbb{N}$,
- (P_2) the set of polynomials $G = \{a = (a_n)_{n \geq 0} \in \mathbb{K}^{\mathbb{N}} : \{n : a_n \neq 0\} \text{ is finite}\}$ is contained in A ,
- (P_3) G is dense in A .

As usual we denote by $(e_n)_{n \geq 0}$ the canonical basis of $\mathbb{K}^{\mathbb{N}}$.

DEFINITION 2.3. A sequence $a \in A$ belongs to the class \mathcal{U}_A if, for every $x \in X$, there exists a sequence (λ_n) in \mathbb{N} such that

$$(1) \quad \sum_{j=0}^{\lambda_n} a_j x_j \rightarrow x \quad \text{and} \quad \sum_{j=0}^{\lambda_n} a_j e_j \rightarrow a \quad \text{as } n \rightarrow +\infty.$$

Observe that $\mathcal{U}_A = \mathcal{U}(\mathcal{L}; X \times \{0\})$ with the map $L_n : A \rightarrow X \times A$, $a \mapsto (\sum_{j=0}^n a_j x_j, \sum_{j=0}^n a_j e_j - a)$. We set $Y_0 = G$, which is a dense subset of $A = Y$.

In this case, a condition of polynomial approximation type is equivalent to the existence of universal elements. Indeed, Theorem 2.2 takes the following form.

COROLLARY 2.4 ([2]). *Under the above assumptions the following are equivalent:*

- (1) $\mathcal{U}_A \neq \emptyset$.
- (2) For every $x \in X$ and $\varepsilon > 0$, there exist $n \in \mathbb{N}$ and $a_0, a_1, \dots, a_n \in \mathbb{K}$ such that

$$\rho\left(\sum_{j=0}^n a_j x_j, x\right) < \varepsilon \quad \text{and} \quad d\left(\sum_{j=0}^n a_j e_j, 0\right) < \varepsilon.$$

- (3) \mathcal{U}_A is a dense G_δ subset of A .

2.2. Polynomial approximation and continuous mappings I. Let us now consider a sequence $T = (T_n)_{n \in \mathbb{N}}$ of continuous linear mappings $T_n : A \rightarrow A$, such that for every $a \in G$, $(T_n a)$ converges to an element in A .

DEFINITION 2.5. An element $a \in A$ belongs to the class $\mathcal{U}_{A,T}$ if, for every $x \in X$ and $b \in A$, there exists an increasing sequence (λ_n) of positive integers such that

$$(2) \quad \rho\left(\sum_{j=0}^{\lambda_n} a_j x_j, x\right) \rightarrow 0, \quad d\left(\sum_{j=0}^{\lambda_n} a_j e_j, a\right) \rightarrow 0, \quad d(T_{\lambda_n}(a), b) \rightarrow 0$$

as $n \rightarrow +\infty$. In this situation, Theorem 2.2 becomes

COROLLARY 2.6. *Under the above assumptions the following are equivalent:*

- (i) $\mathcal{U}_{A,T} \neq \emptyset$.
- (ii) For every $x \in X$, $b \in A$ and $\varepsilon > 0$, there exist $n, m \in \mathbb{N}$ with $m \geq n$ and $a_0, a_1, \dots, a_n, a_{n+1}, \dots, a_m \in \mathbb{K}$ such that

$$\rho\left(\sum_{j=0}^n a_j x_j, x\right) < \varepsilon, \quad d\left(\sum_{j=0}^n a_j e_j, 0\right) < \varepsilon,$$

$$d\left(\sum_{j=n+1}^m a_j e_j, 0\right) < \varepsilon, \quad d\left(T_n\left(\sum_{j=0}^m a_j e_j\right), b\right) < \varepsilon.$$

(iii) $\mathcal{U}_{A,T}$ is a dense G_δ subset of A .

Proof. Observe that $\mathcal{U}_{A,T} = \mathcal{U}(\mathcal{L}; X \times A \times \{0\})$ with $L_n : \mathbb{K}^{\mathbb{N}} \rightarrow X \times A \times A$, $a \mapsto (\sum_{j=0}^n a_j x^j, T_n(a), \sum_{j=0}^n a_j e_j - a)$. Then it suffices to prove that condition (β) of Theorem 2.2 is equivalent to condition (ii).

$(\beta) \Rightarrow$ (ii): Assume that, for every $z = (x, b, 0) \in X \times A \times \{0\}$ and $\varepsilon > 0$, there exist $n \geq 0$ and $y \in A$ such that

$$\begin{aligned} \rho\left(\sum_{j=0}^n y_j x_j, x\right) &< \varepsilon/2, & d(T_n(y), b) &< \varepsilon/2, \\ d\left(\sum_{j=0}^n y_j e_j, y\right) &< \varepsilon/2, & d(y, 0) &< \varepsilon/4. \end{aligned}$$

Using the fact that G is dense in A and L_n is a continuous map, it follows immediately that there exists a polynomial $a = \sum_{j=0}^m a_j e_j$, with $m \geq n$, such that $d(a, y) < \varepsilon/4$ and

$$\begin{aligned} \rho\left(\sum_{j=0}^n y_j x_j, \sum_{j=0}^n a_j x_j\right) &< \varepsilon/2, & d(T_n(y), T_n(a)) &< \varepsilon/2, \\ d\left(\sum_{j=0}^n y_j e_j, \sum_{j=0}^n a_j e_j\right) &< \varepsilon/4. \end{aligned}$$

By using the triangle inequality, we get

$$\begin{aligned} \rho\left(\sum_{j=0}^n a_j x_j, x\right) &< \varepsilon/2 + \varepsilon/2 = \varepsilon, & d(T_n(a), b) &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \\ d\left(\sum_{j=n+1}^m a_j e_j, 0\right) &\leq d(a, y) + d\left(y, \sum_{j=0}^n y_j e_j\right) + d\left(\sum_{j=0}^n y_j e_j, \sum_{j=0}^n a_j e_j\right) \\ &< \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon, \\ d\left(\sum_{j=0}^n a_j e_j, 0\right) &\leq d\left(\sum_{j=0}^n y_j e_j, \sum_{j=0}^n a_j e_j\right) + d\left(\sum_{j=0}^n y_j e_j, y\right) + d(y, 0) \\ &< \varepsilon/4 + \varepsilon/2 + \varepsilon/4 = \varepsilon. \end{aligned}$$

(ii) \Rightarrow (β) : This is trivial. ■

2.3. Polynomial approximation and continuous mappings II.

As in the abstract theory developed in [2], we need a small step forward in the abstract theory to cover the case of universal Taylor series. Let us first introduce the following formalism. We keep the notations X and G .

Denote by E a complete metrizable topological vector space, with topology induced by a translation invariant metric d . Let $(\tilde{e}_k)_{k=0}^{+\infty}$ be a fixed sequence in E and suppose that, for any $n \in \mathbb{N}$, we have a continuous linear map $\phi_n : E \rightarrow \mathbb{K}$. For every $a \in G$, we consider the polynomial $g_a = \sum_{j=0}^{+\infty} a_j \tilde{e}_j$ (which belongs to E) and we set $\deg(g_a) = \max\{j \in \mathbb{N} : a_j \neq 0\}$. We shall make the following assumptions.

- (E₁) The set $\{g_a : a \in G\}$ is dense in E .
- (E₂) For every $a \in G$ and every $j \in \mathbb{N}$, $\phi_j(g_a) = a_j$.

Let $T = (T_n)_{n \in \mathbb{N}}$ be a sequence of continuous linear mappings $T_n : E \rightarrow E$ such that, for every $a \in G$, $(T_n(g_a))$ converges to an element in E .

DEFINITION 2.7. An element $f \in E$ belongs to the class $\mathcal{U}_{E,T}$ if, for every $x \in X$ and $y \in E$, there exists an increasing sequence (λ_n) of positive integers such that $\rho(\sum_{j=0}^{\lambda_n} \phi_j(f)x_j, x) \rightarrow 0$, $d(\sum_{j=0}^{\lambda_n} \phi_j(f)\tilde{e}_j, f) \rightarrow 0$ and $d(T_{\lambda_n}(f), y) \rightarrow 0$ as $n \rightarrow +\infty$.

Hence we have the following extension of Corollary 2.6, whose proof is similar.

COROLLARY 2.8. *Under the above assumptions the following are equivalent:*

- (i) $\mathcal{U}_{E,T} \neq \emptyset$.
- (ii) For every $x \in X$, $y \in E$ and $\varepsilon > 0$, there exist $n, m \in \mathbb{N}$ with $m \geq n$ and $a_0, a_1, \dots, a_n, a_{n+1}, \dots, a_m \in \mathbb{K}$ such that

$$\begin{aligned} \rho\left(\sum_{j=0}^n a_j x_j, x\right) < \varepsilon, & \quad d\left(\sum_{j=0}^n a_j \tilde{e}_j, 0\right) < \varepsilon, \\ d\left(\sum_{j=n+1}^m a_j \tilde{e}_j, 0\right) < \varepsilon, & \quad d\left(T_n\left(\sum_{j=0}^m a_j \tilde{e}_j\right), y\right) < \varepsilon. \end{aligned}$$

- (iii) $\mathcal{U}_{E,T}$ is a dense G_δ subset of E .

As in Section 1.3 of [2], it is possible to have universality not only in a single set X , but also in a countable union of such sets. Hence we assume that $(X_k)_{k \geq 1}$ is a sequence of metrizable topological vector spaces over \mathbb{K} , equipped with translation invariant metrics ρ_k . For any $k \geq 1$, let $(x_{k,j})_{j=0}^{+\infty}$ be a fixed sequence in X_k . We keep the same assumptions (E_i) , $i = 1, 2$, with trivial modifications.

DEFINITION 2.9. An element $f \in E$ belongs to the class $\mathcal{U}_{E,T}$ if, for every $k \geq 1$, every $x \in X_k$ and $y \in E$, there exists an increasing sequence (λ_n) of positive integers such that $\rho_k(\sum_{j=0}^{\lambda_n} \phi_j(f)x_{k,j}, x) \rightarrow 0$, $d(\sum_{j=0}^{\lambda_n} \phi_j(f)\tilde{e}_j, f) \rightarrow 0$ and $d(T_{\lambda_n}(f), y) \rightarrow 0$ as $n \rightarrow +\infty$.

We obtain the following result.

COROLLARY 2.10. *Under the above assumptions the following are equivalent:*

- (i) $\mathcal{U}_{E,T} \neq \emptyset$.
- (ii) *For every $k \geq 1$, $x \in X_k$, $y \in E$ and $\varepsilon > 0$, there exist $n, m \in \mathbb{N}$ with $m \geq n$ and $a_0, a_1, \dots, a_n, a_{n+1}, \dots, a_m \in \mathbb{K}$ such that*

$$\begin{aligned} \rho_k\left(\sum_{j=0}^n a_j x_{k,j}, x\right) < \varepsilon, & \quad d\left(\sum_{j=0}^n a_j \tilde{e}_j, 0\right) < \varepsilon, \\ d\left(\sum_{j=n+1}^m a_j \tilde{e}_j, 0\right) < \varepsilon, & \quad d\left(T_n\left(\sum_{j=0}^m a_j \tilde{e}_j\right), y\right) < \varepsilon. \end{aligned}$$

- (iii) $\mathcal{U}_{E,T}$ is a dense G_δ subset of E .

For the proof we notice that a combination of Corollary 2.8 and Baire's Theorem yields the implication (i) \Rightarrow (ii). The rest is straightforward.

2.4. Polynomial approximation and unbounded operators. Observe first that condition (ii) of Corollary 2.6 can be replaced by another condition of polynomial approximation type.

LEMMA 2.11. *Under the assumptions of Corollary 2.6, assertion (ii) is equivalent to the following condition:*

- (ii') *For every $q \in \mathbb{N}$, $x \in X$, $b \in A$ and $\varepsilon > 0$, there exist $p, n, m \in \mathbb{N}$ with $m \geq n \geq p \geq q$ and $a_p, a_{p+1}, \dots, a_n, a_{n+1}, \dots, a_m \in \mathbb{K}$ such that*

$$\begin{aligned} \rho\left(\sum_{j=p}^n a_j x_j, x\right) < \varepsilon, & \quad d\left(\sum_{j=p}^n a_j e_j, 0\right) < \varepsilon, & \quad d\left(\sum_{j=n+1}^m a_j e_j, 0\right) < \varepsilon, \\ d\left(T_n\left(\sum_{j=p}^m a_j e_j\right), b\right) < \varepsilon, & \quad d\left(T_i\left(\sum_{j=p}^m a_j e_j\right), 0\right) < \varepsilon, & \quad i = 1, \dots, q. \end{aligned}$$

Proof. Indeed, (ii') \Rightarrow (ii) is trivial. The converse implication is not much more difficult: using the continuity of the mappings T_i , there exists $0 < \varepsilon' < \varepsilon/2$ such that for any $h \in A$ satisfying $d(h, 0) < \varepsilon'$, we have $d(T_i(h), 0) < \varepsilon$, for every $i = 1, \dots, q$. Since condition (ii) implies the existence of universal elements, let $a \in \mathcal{U}_{A,T}$. Observe that, for any $a' \in G$, we have $a + a' \in \mathcal{U}_{A,T}$, thus there exists some $p \geq q$ so that $c = (0, \dots, 0, a_p, a_{p+1}, \dots)$ belongs to $\mathcal{U}_{A,T}$ and $d(c, 0) < \varepsilon'/2$. Since $c \in \mathcal{U}_{A,T}$, we may find $n > p$ such that

$$(3) \quad \rho\left(\sum_{j=q}^n a_j x_j, x\right) < \varepsilon, \quad d\left(\sum_{j=p}^n a_j e_j, c\right) < \varepsilon/2, \quad d(T_n(c), b) < \varepsilon/2.$$

Since the operator T_n is continuous, there exists $0 < \varepsilon'' < \varepsilon'/2$ such that for any $h \in A$ the inequality $d(c, h) < \varepsilon''$ implies $d(T_n(c), T_n(h)) < \varepsilon'$. Since

$c \in \mathcal{U}_{A,T}$, we may find $m > n$ such that $d(\sum_{j=p}^m a_j e_j, c) < \varepsilon''$. It follows $d(T_n(\sum_{j=p}^m a_j e_j), T_n(c)) < \varepsilon'$. By the triangle inequality, we obtain

$$\begin{aligned} d\left(T_n\left(\sum_{j=p}^m a_j e_j\right), b\right) &\leq d\left(T_n\left(\sum_{j=p}^m a_j e_j\right), T_n(c)\right) + d(T_n(c), b) < \varepsilon' + \varepsilon/2 < \varepsilon, \\ d\left(\sum_{j=p}^n a_j e_j, 0\right) &\leq d\left(\sum_{j=p}^n a_j e_j, c\right) + d(c, 0) < \varepsilon/2 + \varepsilon'/2 < \varepsilon, \\ d\left(\sum_{j=n+1}^m a_j e_j, 0\right) &\leq d\left(\sum_{j=p}^m a_j e_j, c\right) + d\left(\sum_{j=p}^n a_j e_j, c\right) < \varepsilon'' + \varepsilon/2 < \varepsilon. \end{aligned}$$

Moreover,

$$d\left(\sum_{j=p}^m a_j e_j, 0\right) \leq d\left(\sum_{j=p}^m a_j e_j, c\right) + d(c, 0) < \varepsilon'' + \varepsilon'/2 < \varepsilon',$$

which implies $d(T_i(h), 0) < \varepsilon$ for every $i = 1, 2, \dots, q$. Therefore we have (ii'). ■

Clearly we can replace conditions (ii) of Corollaries 2.8 and 2.10 by (ii') with obvious modifications.

Assume now that A is a separable infinite-dimensional Banach space and let (T_n) be a sequence of densely defined and closed linear operators. We keep the notations X, G of the previous sections and we suppose that G is a dense subset of the domain of T_n for all positive integers n , such that for every $a \in G$, $(T_n a)$ converges to an element in A . Then condition (ii') implies the density of the set $\mathcal{U}_{A,T}$ (see Definition 2.5) of universal elements.

THEOREM 2.12. *Let (T_n) be a sequence of densely defined and closed linear operators and suppose that G is a dense subset of the domain of T_n for all positive integers n , and that for every $a \in G$, $(T_n a)$ converges to an element in A . Assume that*

(ii') *for every $q \in \mathbb{N}$, $x \in X$, $b \in A$ and $\varepsilon > 0$, there exist $p, n, m \in \mathbb{N}$ with $m \geq n \geq p \geq q$ and $a_p, a_{p+1}, \dots, a_n, a_{n+1}, \dots, a_m \in \mathbb{K}$ such that*

$$\begin{aligned} \rho\left(\sum_{j=p}^n a_j x_j, x\right) &< \varepsilon, \quad \left\|\sum_{j=p}^n a_j e_j\right\| < \varepsilon, \quad \left\|\sum_{j=n+1}^m a_j e_j\right\| < \varepsilon, \\ \left\|T_n\left(\sum_{j=p}^m a_j e_j\right) - b\right\| &< \varepsilon, \quad \left\|T_i\left(\sum_{j=p}^m a_j e_j\right)\right\| < \varepsilon, \quad i = 1, \dots, q. \end{aligned}$$

Then $\mathcal{U}_{A,T}$ is a dense subset of A .

Proof. If we take a_j in \mathbb{Q} or in $\mathbb{Q} + i\mathbb{Q}$, the assumption implies that X is separable. Let (y_s) (resp. (\tilde{y}_t)) be a dense sequence in X (resp. in A). Let us consider an enumeration $(y_{s_i}, \tilde{y}_{t_i})$ of all pairs (y_s, \tilde{y}_t) . For notational convenience, we will just write (y_l, \tilde{y}_l) instead of $(y_{s_i}, \tilde{y}_{t_i})$. Let $h = \sum_{j \geq 0} h_j e_j$ be in G and $\varepsilon > 0$.

Step 1: There exists $u_1 \in A$ such that $T_n(h) \rightarrow u_1$ as $n \rightarrow +\infty$. There exists an integer $q_1 > \max\{j : h_j \neq 0\}$ such that $\|T_n(h) - u_1\| < 1/2^2$ for every $n \geq q_1$. By assumption, there exist $p_1, n_1, m_1 \in \mathbb{N}$ with $m_1 \geq n_1 \geq p_1 \geq q_1$ and $a_{p_1}, a_{p_1+1}, \dots, a_{n_1}, a_{n_1+1}, \dots, a_{m_1} \in \mathbb{K}$ such that

$$\rho\left(\sum_{j=p_1}^{n_1} a_j x_j, y_1 - h\right) < 1/2^2, \quad \left\| \sum_{j=p_1}^{n_1} a_j e_j \right\| < \varepsilon/2^2,$$

$$\left\| \sum_{j=n_1+1}^{m_1} a_j e_j \right\| < \varepsilon/2^2, \quad \left\| T_{n_1}\left(\sum_{j=p_1}^{m_1} a_j e_j\right) - \tilde{y}_1 + u_1 \right\| < 1/2^2.$$

Step 2: There exists $u_2 \in A$ such that $T_n(h + \sum_{j=p_1}^{m_1} a_j e_j) \rightarrow u_2$ as $n \rightarrow +\infty$. We may find $q_2 \geq m_1 + 1$ such that $\|T_n(h + \sum_{j=p_1}^{m_1} a_j e_j) - u_2\| < 1/2^3$ for every $n \geq q_2$. By assumption, there exist $p_2, n_2, m_2 \in \mathbb{N}$ with $m_2 \geq n_2 \geq p_2 \geq q_2$ and $a_{p_2}, a_{p_2+1}, \dots, a_{n_2}, a_{n_2+1}, \dots, a_{m_2} \in \mathbb{K}$ such that

$$\rho\left(\sum_{j=p_2}^{n_2} a_j x_j, y_2 - h - \sum_{j=p_1}^{m_1} a_j x_j\right) < 1/2^3, \quad \left\| \sum_{j=p_2}^{n_2} a_j e_j \right\| < \varepsilon/2^3,$$

$$\left\| \sum_{j=n_2+1}^{m_2} a_j e_j \right\| < \varepsilon/2^3, \quad \left\| T_{n_2}\left(\sum_{j=p_2}^{m_2} a_j e_j\right) - \tilde{y}_2 + u_2 \right\| < 1/2^3,$$

$$\left\| T_{n_1}\left(\sum_{j=p_2}^{m_2} a_j e_j\right) \right\| < 1/2^3.$$

Assume that, for every $i = 1, \dots, l-1$, the integers $m_i \geq n_i \geq p_i \geq q_i$ and the coefficients a_{p_i}, \dots, a_{m_i} are constructed.

Step l: There exists $u_l \in A$ such that $T_n(h + \sum_{i=1}^{l-1} \sum_{j=p_i}^{m_i} a_j e_j) \rightarrow u_l$ as $n \rightarrow +\infty$. We find $q_l \geq m_{l-1} + 1$ such that $\|T_n(h + \sum_{i=1}^{l-1} \sum_{j=p_i}^{m_i} a_j e_j) - u_l\| < 1/2^{l+1}$ for every $n \geq q_l$. By assumption, there exist $p_l, n_l, m_l \in \mathbb{N}$ with $m_l \geq n_l \geq p_l \geq q_l$ and $a_{p_l}, a_{p_l+1}, \dots, a_{n_l}, a_{n_l+1}, \dots, a_{m_l} \in \mathbb{K}$ such that

$$\rho\left(\sum_{j=p_l}^{n_l} a_j x_j, y_l - h - \sum_{i=1}^{l-1} \sum_{j=p_i}^{m_i} a_j x_j\right) < 1/2^{l+1}, \quad \left\| \sum_{j=p_l}^{n_l} a_j e_j \right\| < \varepsilon/2^{l+1},$$

$$\left\| \sum_{j=n_l+1}^{m_l} a_j e_j \right\| < \varepsilon/2^{l+1}, \quad \left\| T_{n_l}\left(\sum_{j=p_l}^{m_l} a_j e_j\right) - \tilde{y}_l + u_l \right\| < 1/2^{l+1},$$

$$\left\| T_{n_i} \left(\sum_{j=p_i}^{m_i} a_j e_j \right) \right\| < 1/2^{l+1}, \quad i = 1, \dots, l-1.$$

We set $a = h + \sum_{i=1}^{+\infty} \sum_{j=p_i}^{m_i} a_j e_j = \sum_{j=0}^{+\infty} a_j e_j$. Clearly we get

$$\|a - h\| \leq \sum_{i=1}^{+\infty} \left\| \sum_{j=p_i}^{m_i} a_j e_j \right\| < \sum_{i=1}^{+\infty} \varepsilon/2^i = \varepsilon, \quad \text{so} \quad \|a\| \leq \|h\| + \varepsilon.$$

Let $x \in X, b \in A$ and $\eta > 0$. We will find $N \in \mathbb{N}$ so that $\rho(\sum_{j=0}^N a_j e_j, x) < \eta$, $\|\sum_{j=0}^N a_j e_j - a\| < \eta$ and $\|T_N(a) - b\| < \eta$. First we let $l \geq 1$ be a fixed integer and observe that

$$\begin{aligned} \left(\sum_{i=l+1}^M \sum_{j=p_i}^{m_i} a_j e_j, T_{n_l} \left(\sum_{i=l+1}^M \sum_{j=p_i}^{m_i} a_j e_j \right) \right) &= \left(\sum_{i=l+1}^M \sum_{j=p_i}^{m_i} a_j e_j, \sum_{i=l+1}^M T_{n_l} \left(\sum_{j=p_i}^{m_i} a_j e_j \right) \right) \\ &\rightarrow \left(\sum_{i=l+1}^{+\infty} \sum_{j=p_i}^{m_i} a_j e_j, \sum_{i=l+1}^{+\infty} T_{n_l} \left(\sum_{j=p_i}^{m_i} a_j e_j \right) \right) \end{aligned}$$

as $M \rightarrow +\infty$. Since T_{n_l} is closed, the vector $\sum_{i=l+1}^{+\infty} \sum_{j=p_i}^{m_i} a_j e_j$ is in the domain of T_{n_l} and

$$T_{n_l} \left(\sum_{i=l+1}^{+\infty} \sum_{j=p_i}^{m_i} a_j e_j \right) = \sum_{i=l+1}^{+\infty} T_{n_l} \left(\sum_{j=p_i}^{m_i} a_j e_j \right).$$

It follows that a belongs to the domain of T_{n_l} . Clearly we may find $l \in \mathbb{N}$ such that $\rho(x, y_l) < \eta/2$, $\|b - \tilde{y}_l\| < \eta/2$ and $2 < 2^l \eta$. By construction, we have $\rho(h + \sum_{i=1}^{l-1} \sum_{j=p_i}^{m_i} a_j e_j + \sum_{j=p_l}^{n_l} a_j e_j, y_l) < 1/2^{l+1}$, which implies

$$\rho \left(\sum_{j=0}^{n_l} a_j e_j, x \right) \leq \rho \left(\sum_{j=0}^{n_l} a_j e_j, y_l \right) + \rho(y_l, x) < 1/2^{l+1} + \eta/2 \leq \eta.$$

Moreover,

$$\begin{aligned} \left\| a - \sum_{j=0}^{n_l} a_j e_j \right\| &= \left\| \sum_{j=n_{l+1}}^{m_l} a_j e_j + \sum_{i=l+1}^{+\infty} \sum_{j=p_i}^{m_i} a_j e_j \right\| \\ &\leq \left\| \sum_{j=n_{l+1}}^{m_l} a_j e_j \right\| + \sum_{i=l+1}^{+\infty} \left\| \sum_{j=p_i}^{m_i} a_j e_j \right\| \leq \varepsilon/2^{l+1} + \varepsilon/2^l < \eta. \end{aligned}$$

Finally, we get

$$(4) \quad \|T_{n_l}(a) - b\| \leq \left\| T_{n_l} \left(h + \sum_{i=1}^l \sum_{j=p_i}^{m_i} a_j e_j \right) - b \right\| + \left\| T_{n_l} \left(\sum_{i=l+1}^{+\infty} \sum_{j=p_i}^{m_i} a_j e_j \right) \right\|.$$

Combining (4) with the estimates

$$\begin{aligned} \left\| T_{n_l} \left(h + \sum_{i=1}^l \sum_{j=p_i}^{m_i} a_j e_j \right) - b \right\| &\leq \left\| T_{n_l} \left(h + \sum_{i=1}^{l-1} \sum_{j=p_i}^{m_i} a_j e_j \right) - u_l \right\| \\ &\quad + \left\| T_{n_l} \left(\sum_{j=p_l}^{m_l} a_j e_j \right) - b + u_l \right\| \\ &\leq 1/2^{l+1} + 1/2^{l+1} < \eta/2 \end{aligned}$$

and

$$\left\| T_{n_l} \left(\sum_{i=l+1}^{+\infty} \sum_{j=p_i}^{m_i} a_j e_j \right) \right\| \leq \sum_{i=l+1}^{+\infty} \left\| T_{n_l} \left(\sum_{j=p_i}^{m_i} a_j e_j \right) \right\| \leq \sum_{i=l+1}^{+\infty} 1/2^i = 1/2^l < \eta/2,$$

we deduce $\|T_{n_l}(a) - b\| < \eta$. This completes the proof. ■

As in Section 1.3 of [2], it is possible to have universality not only in a single set X , but also in a countable union of such sets. Hence we assume that $(X_k)_{k \geq 1}$ is a sequence of metrizable topological vector spaces over \mathbb{K} , equipped with translation invariant metrics ρ_k . For any $k \geq 1$, let $(x_{k,j})_{j=0}^{+\infty}$ be a fixed sequence in X_k . Assume that there exists a dense sequence $(y_{s,k})_{s=1}^{+\infty}$ in X_k , for every $k \geq 1$.

DEFINITION 2.13. An element $a = (a_0, a_1, \dots) \in A$ belongs to the class $\tilde{\mathcal{U}}_{A,T}$ if, for every $k \geq 1$, every $x \in X_k$ and $b \in A$, there exists an increasing sequence (λ_n) of positive integers such that

$$\rho_k \left(\sum_{j=0}^{\lambda_n} a_j x_{k,j}, x \right) \rightarrow 0, \quad d \left(\sum_{j=0}^{\lambda_n} a_j \tilde{e}_j, a \right) \rightarrow 0, \quad d(T_{\lambda_n}(a), b) \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

THEOREM 2.14. *Under the above assumptions, assume that*

(ii') *for every $q \in \mathbb{N}$, $k \geq 1$, $x \in X_k$, $b \in A$ and $\varepsilon > 0$, there exist $p, n, m \in \mathbb{N}$ with $m \geq n \geq p \geq q$ and $a_p, a_{p+1}, \dots, a_n, a_{n+1}, \dots, a_m \in \mathbb{K}$ such that*

$$\begin{aligned} \rho_k \left(\sum_{j=p}^n a_j x_{k,j}, x \right) < \varepsilon, \quad \left\| \sum_{j=p}^n a_j e_j \right\| < \varepsilon, \quad \left\| \sum_{j=n+1}^m a_j e_j \right\| < \varepsilon, \\ \left\| T_n \left(\sum_{j=p}^m a_j e_j \right) - b \right\| < \varepsilon, \quad \left\| T_i \left(\sum_{j=p}^m a_j e_j \right) \right\| < \varepsilon, \quad i = 1, \dots, q. \end{aligned}$$

Then $\tilde{\mathcal{U}}_{A,T}$ is a dense subset of A .

Proof. Let (\tilde{y}_t) be a dense sequence in A . Let $(y_{s_i, k_i}, \tilde{y}_{t_i}, \rho_{k_i})$ be an enumeration of all triplets $(y_{s,k}, \tilde{y}_t, \rho_k)$. We proceed as in the proof of Theorem 2.12. ■

REMARK 2.15. Theorem 2.12 seems to give a new sufficient condition for the existence of hypercyclic vectors of an unbounded operator. Let us recall that a vector a is *hypercyclic* for an unbounded operator T on a separable infinite-dimensional Banach space A if for every integer $n \geq 0$, $T^n a$ is in the domain of T and the orbit $\{a, Ta, T^2 a, \dots\}$ is dense in A . We deduce from Theorem 2.12 the following.

THEOREM 2.16. *Let $A \subset \mathbb{K}^{\mathbb{N}}$ be a separable infinite-dimensional Banach space and let T be a densely defined and closed linear operator on A for which T^n is a closed operator for all positive integers n . Suppose that the set of polynomials $G = \{a = (a_n)_{n \geq 0} \in \mathbb{K}^{\mathbb{N}} : \{n : a_n \neq 0\} \text{ is finite}\}$ is a dense subset of the domain of T^n for all positive integers n , and for every $a \in G$, $(T^n a)$ converges to an element in A . Then T is hypercyclic if for every $q \in \mathbb{N}$, $b \in A$ and $\varepsilon > 0$, there exist $p, n, m \in \mathbb{N}$ with $m \geq n \geq p \geq q$ and $a_p, a_{p+1}, \dots, a_n, a_{n+1}, \dots, a_m \in \mathbb{K}$ such that*

$$\left\| \sum_{j=p}^m a_j e_j \right\| < \varepsilon, \quad \left\| T^n \left(\sum_{j=p}^m a_j e_j \right) - b \right\| < \varepsilon$$

and

$$\left\| T^i \left(\sum_{j=p}^m a_j e_j \right) \right\| < \varepsilon, \quad i = 1, \dots, q.$$

We can compare this result with Theorem 2.1 of [3].

3. Applications

3.1. Universality and weighted backward shift. We set $E = A = \mathbb{C}^{\mathbb{N}}$ which is endowed with the classical metric $d(a, b) = \sum_{n=0}^{\infty} (1/2^n)(|a_n - b_n| / (1 + |a_n - b_n|))$, $a, b \in A$. We fix a compact set $K \subset \mathbb{C}$ such that $0 \notin K$ and K^c is connected. Let $X = H(\mathbb{C})$ be the space of entire functions endowed with the metric $\rho(f, g) = \sum_{n=0}^{\infty} (1/2^n)(\|f - g\|_n / (1 + \|f - g\|_n))$, where $\|f - g\|_n = \sup_K |f^{(n)} - g^{(n)}|$, $f, g \in H(\mathbb{C})$. The metric spaces A and X satisfy our assumptions. We set $x_j = (z \mapsto z^j)$. Let

$$T : A \rightarrow A, \quad a = (a_0, a_1, a_2, \dots) \mapsto (a_1, 2a_2, 3a_3, \dots).$$

For any $n \in \mathbb{N}^*$, we denote $T_n = T^n$.

PROPOSITION 3.1. *Let $K \subset \mathbb{C}$ be a compact set with $0 \notin K$ and K^c connected. Let T be the weighted backward shift defined by $T(a_0, a_1, a_2, a_3, \dots) = (a_1, 2a_2, 3a_3, \dots)$ for $a \in \mathbb{C}^{\mathbb{N}}$. Then there exists a sequence $a = (a_n) \in \mathbb{C}^{\mathbb{N}}$ such that for every $b \in \mathbb{C}^{\mathbb{N}}$ and every entire function h there exists $(\lambda_n) \in \mathbb{N}^{\mathbb{N}}$ such that for every $l \in \mathbb{N}$,*

$$T^{\lambda_n}(a) \rightarrow b \quad \text{and} \quad \sup_{z \in K} \left| \left(\sum_{j=0}^{\lambda_n} a_j z^j \right)^{(l)} - h^{(l)}(z) \right| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

The set of such sequences is G_δ and dense in $\mathbb{C}^{\mathbb{N}}$ endowed with the Cartesian topology.

Proof. We prove that condition (ii) of Corollary 2.6 holds.

Let $p \in \mathbb{N}$, $\varepsilon > 0$, $h \in X$ and $b \in A$. We fix $N \in \mathbb{N}$ such that $N > \max(p, \sup_{z \in K} |z|)$ and $\sum_{n=N+1}^{+\infty} 1/2^n < \varepsilon/4$. Since $0 \notin K$ and K^c is connected, we find a simply connected open set Ω such that $0 \notin \Omega$ and $K \subset \Omega$. By Runge's Theorem, there exists a sequence (q_k) of polynomials converging to $h(z)/z^{N+1}$ uniformly on compact subsets of Ω . We set $p_k(z) = z^{N+1}q_k(z)$. Clearly using Weierstrass' Theorem we obtain, for a large enough k_0 , $\sup_K |p_{k_0}^{(l)} - h^{(l)}| < \varepsilon/8$ for $l = 0, 1, \dots, N$. We have $p_{k_0} = \sum_{j=N+1}^{n_0} a_j z^j$. Therefore we get

$$(5) \quad d\left(\sum_{j=N+1}^{n_0} a_j e_j, 0\right) < \varepsilon/4, \quad \rho\left(\sum_{j=N+1}^{n_0} a_j z^j, h\right) \leq \sum_{l=0}^N 2^{-l} \|p_{k_0} - h\|_l + \varepsilon/4 < \varepsilon/2.$$

Consider $b' = (b_0, b_1, \dots, b_N, 0, \dots)$ and $k_1 > n_0$ such that

$$(6) \quad |b_0| N^{k_1-l} / (k_1 - l)! < \varepsilon/8 \quad \text{for } l = 0, 1, \dots, N.$$

We set

$$b'' = \left(0, \dots, 0, \frac{b_0}{k_1!}, \frac{1!b_1}{(k_1+1)!}, \dots, \frac{N!b_N}{(k_1+N)!}, 0, \dots\right),$$

where $b''_{k_1} = b_0/k_1!$. An easy calculation gives $T_{k_1}(b'') = b'$ and $d(b', b) < \varepsilon/4$. Denote

$$\sum_{j=N+1}^{n_0} a_j e_j + \sum_{j=k_1}^{k_1+N} \frac{b_{j-k_1}}{j!} e_j = \sum_{j=p}^m a_j e_j,$$

upon setting $m = k_1 + N$, $a_j = 0$ for $p \leq j \leq N$ and for $n_0 < j < k_1$. Let $n = k_1$; thus from (5) and (6), we get

$$(7) \quad \rho\left(\sum_{j=p}^n a_j z^j, h\right) \leq \rho\left(\sum_{j=N+1}^{n_0} a_j z^j, h\right) + \rho(a_n z^n, 0) < \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

We also have $d(\sum_{j=p}^n a_j e_j, 0) < \varepsilon/4$ and $d(\sum_{j=n+1}^m a_j e_j, 0) < \varepsilon/4$. Moreover, taking into account the property $T_n(\sum_{j=p}^{n-1} a_j e_j) = 0$, we obtain

$$d\left(T_n\left(\sum_{j=p}^m a_j e_j\right), b\right) = d\left(T_n\left(\sum_{j=n}^m a_j e_j\right), b\right) = d(b', b) < \varepsilon/4.$$

Hence condition (ii) of Corollary 2.6 is satisfied. ■

Let us recall that there exists a sequence (K_k) of compact set of \mathbb{C} with $0 \notin K_k$ and K_k^c connected such that every compact set $K \subset \mathbb{C}$ with $0 \notin K$

and K^c connected is contained in some K_k (see [9]). We combine the previous result with Corollary 2.10 to obtain the following.

THEOREM 3.2. *Let T be the weighted backward shift defined by the equality $T(a_0, a_1, a_2, a_3, \dots) = (a_1, 2a_2, 3a_3, \dots)$ for $a \in \mathbb{C}^{\mathbb{N}}$. Then there exists a sequence $a = (a_n) \in \mathbb{C}^{\mathbb{N}}$ such that, for every $b \in \mathbb{C}^{\mathbb{N}}$, every entire function h and every compact set $K \subset \mathbb{C}$ with $0 \notin K$ and K^c connected, there exists $(\lambda_n) \in \mathbb{N}^{\mathbb{N}}$ such that, for every $l \in \mathbb{N}$,*

$$T^{\lambda_n}(a) \rightarrow b \quad \text{and} \quad \sup_{z \in K} \left| \left(\sum_{j=0}^{\lambda_n} a_j z^j \right)^{(l)} - h^{(l)}(z) \right| \rightarrow 0 \quad \text{as } n \rightarrow +\infty.$$

The set of such sequences is G_δ and dense in $\mathbb{C}^{\mathbb{N}}$ endowed with the Cartesian topology.

REMARKS 3.3.

1. In the last theorem, we can require the approximation of partial sums for $l = 0$ alone, and by Mergelyan's Theorem we can assume that $h \in A(K)$. This gives a special case of Theorem 2.8 of [6].
2. Proposition 3.1 and Theorem 3.2 are also valid if we replace T by another weighted backward shift B_ω with $B_\omega(e_j) = \omega_{j-1}e_{j-1}$ for $j \geq 1$, whenever (ω_j) is a sequence of strictly positive real numbers satisfying $(\prod_{k=0}^{n-1} \omega_k)^{-1} C^n \rightarrow 0$ as $n \rightarrow +\infty$, for every $C > 0$. Under these conditions, B_ω is obviously a continuous map and the same proof works.
3. In Theorem 3.2 and Proposition 3.1 the case $l = 0$ implies the result for all $l = 0, 1, 2, \dots$ because $\mathbb{C} \setminus \{0\}$ is open.

3.2. Universal Taylor series under derivatives. As usual, we denote by $\hat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ the Riemann sphere, and we now suppose that Ω is a simply connected domain in \mathbb{C} such that $\hat{\mathbb{C}} \setminus \overline{\Omega}$ is connected. Now $A^\infty(\Omega)$ denotes the set of functions $f \in H(\Omega)$ such that each derivative $f^{(l)}$ extends continuously on $\overline{\Omega}$. We endow this space with the family of seminorms $\sup_{|z| \leq n, z \in \overline{\Omega}} |f^{(l)}(z)|$, $l, n \in \mathbb{N}$. Thus $A^\infty(\Omega)$ becomes a Fréchet space equipped with the standard invariant metric d . We denote by $E = X^\infty(\Omega)$ the closure of the set of polynomials in $A^\infty(\Omega)$. The space $(X^\infty(\Omega), d)$ is complete. Notice that there exists a sequence of compact sets $K_k \subset \mathbb{C}$ with $K_k \cap \overline{\Omega} = \emptyset$, K_k^c connected, and such that every compact set $K \subset \mathbb{C}$ with $K \cap \overline{\Omega} = \emptyset$ and K^c connected is contained in some K_k [2]. We let X_k , $k \geq 1$, be the space of entire functions endowed with the standard metric ρ_k induced by the seminorms $\sup_{z \in K_k} |f^{(l)}(z)|$, $l \geq 0$. Set $L_n = \overline{\Omega} \cap \{z : |z| \leq n\}$ and let $\zeta \in \Omega$ be a fixed point in L_1 . Clearly $\bigcup_n L_n = \overline{\Omega}$. For $f \in E$ and $n \geq 0$, we set $\tilde{e}_n = (z \mapsto (z - \zeta)^n) = x_{k,n}$ and $\phi_n(f) = f^{(n)}(\zeta)/n!$. Clearly

the assumptions of Section 2.3 are satisfied. Let $T : E \rightarrow E$, $f \mapsto f'$. For any $n \in \mathbb{N}^*$, we denote $T_n = T^n$.

THEOREM 3.4. *There exists a holomorphic function $f \in A^\infty(\Omega)$ such that, if we denote $S_N(f, \zeta)(z) = \sum_{n=0}^N (f^{(n)}(\zeta)/n!)(z - \zeta)^n$, the following property holds: for every function $g \in X^\infty(\Omega)$, every entire function h and every compact set $K \subset \mathbb{C}$ with $K \cap \overline{\Omega} = \emptyset$ and K^c connected, there exists an increasing sequence $(\lambda_n) \in \mathbb{N}^{\mathbb{N}}$ such that for every compact set $L \subset \Omega$ and every $l \in \mathbb{N}$, we have*

$$\begin{aligned} \sup_{z \in L} |f^{(\lambda_n+l)}(z) - g^{(l)}(z)| &\rightarrow 0 \quad \text{as } n \rightarrow +\infty, \\ \sup_{z \in K} |S_{\lambda_n}^{(l)}(f, \zeta)(z) - h^{(l)}(z)| &\rightarrow 0 \quad \text{as } n \rightarrow +\infty. \end{aligned}$$

The set of such f is G_δ and dense in $X^\infty(\Omega)$.

Proof. It suffices to prove that condition (ii) of Corollary 2.10 holds. Let $k \geq 0$, $p \in \mathbb{N}$, $\varepsilon > 0$, $x \in X_k$ and $y \in E$. We fix $N \in \mathbb{N}$ such that $N > p$ and $\sum_{n=N+1}^{+\infty} 1/2^n < \varepsilon/4$. As in the proof of Theorem 6 of [2], we construct a polynomial $P(z) = \sum_{j=N+1}^{n_0} a_j(z - \zeta)^j$ satisfying

$$(8) \quad \begin{aligned} \sup_{z \in K_k} |P^{(l)}(z) - x^{(l)}(z)| &< \varepsilon/8, \\ \sup_{z \in L_N} |P^{(l)}(z)| &< \varepsilon/8, \end{aligned} \quad l = 0, \dots, N.$$

Since $y \in X^\infty(\Omega)$, the closure of the set of polynomials in $A^\infty(\Omega)$, there exists a polynomial $q(z) = \sum_{j=0}^{n_1} q_j(z - \zeta)^j$ such that

$$\sup_{L_N} |q^{(l)} - y^{(l)}| < \varepsilon/4, \quad l = 0, 1, \dots, N.$$

Take $R > \max(1, \sup_{z \in K_k \cup L_N} |z - \zeta|)$. We set

$$q_{k_1}(z) = \sum_{j=0}^{n_1} (j!q_j/(j+k_1!))(z - \zeta)^{j+k_1}$$

and choose $k_1 > n_0$ such that for $l = 0, 1, \dots, N$,

$$\sum_{i=0}^{n_1} (i!|q_i|/(i+k_1-l)!)R^{i+k_1-l} < \varepsilon/8.$$

Denote $\sum_{j=N+1}^{n_0} a_j(z - \zeta)^j + \sum_{j=k_1}^{n_1+k_1} (q_{j-k_1}/j!)(z - \zeta)^j = \sum_{j=p}^m a_j(z - \zeta)^j$, upon setting $m = k_1 + n_1$, $a_j = 0$ for $p \leq j \leq N$ and for $n_0 < j < k_1$. We set $n = k_1$ and we end the proof as in Proposition 3.1. Indeed, we have

$$\begin{aligned} \rho_k \left(\sum_{j=p}^n a_j(z - \zeta)^j, x \right) &\leq \rho_k \left(\sum_{j=p}^{n_0} a_j(z - \zeta)^j, x \right) + \rho_k(q_0(z - \zeta)^{k_1}/k_1!, 0) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \\ d \left(\sum_{j=p}^n a_j(z - \zeta)^j, 0 \right) &\leq d \left(\sum_{j=p}^{n_0} a_j(z - \zeta)^j, 0 \right) + d(q_0(z - \zeta)^{k_1}/k_1!, 0) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon, \\ d \left(\sum_{j=n+1}^m a_j(z - \zeta)^j, 0 \right) &= d \left(\sum_{j=1}^{n_1} a_j(z - \zeta)^j, 0 \right) < \varepsilon/2, \\ d \left(T_n \left(\sum_{j=p}^m a_j(z - \zeta)^j \right), y \right) &= d(q, y) < \varepsilon. \blacksquare \end{aligned}$$

REMARKS 3.5.

1. We refer the reader to [4] or [2] for results of the same type as Theorem 3.4.
2. In Theorem 3.4, we can require the approximation for $l = 0$ alone and we can choose a compact set $L \subset \Omega$ with L^c connected. Thus, by Mergelyan's Theorem, we can assume that $g \in A(L)$ and $h \in A(K)$. This yields a special case of a result of [5].

3.3. Universal Dirichlet series and unbounded operators. In this section, we consider ordinary Dirichlet series $\sum_{n=1}^{+\infty} a_n n^{-z}$ absolutely convergent for $\Re(z) > 0$.

DEFINITION 3.6. A compact set $K \subset \mathbb{C}$ is called *admissible for Dirichlet series* if K^c is connected and $K = K_1 \cup \dots \cup K_d$, where each K_i is contained in a strip $\{z \in \mathbb{C} : c_i \leq \Re(z) \leq b_i\}$ with $b_i - c_i < 1/2$, the strips being disjoint.

Let $1 < p < +\infty$ and set $A = \ell^p$, the Banach space endowed with the norm $\|a\|_p = (\sum_{j \geq 1} |a_j|^p)^{1/p}$. We fix a compact set $K \subset \{z \in \mathbb{C} : \Re(z) \leq 0\}$ admissible for Dirichlet series. Consider the space $X = H(\mathbb{C})$ of entire functions equipped with the metric $\rho(f, g) = \sup_K |f - g|$, $f, g \in H(\mathbb{C})$. Define

$$T : A \rightarrow A, \quad a = (a_1, a_2, a_3, \dots) \mapsto (-\log(2)a_2, -\log(3)a_3, \dots).$$

For any $n \in \mathbb{N}^*$, we denote $T_n = T^n$. Observe that T can be viewed as the operator $S(\partial/\partial z(\sum_{n \geq 1} a_n n^{-z}))$ where S is the backward shift. Clearly the set of Dirichlet polynomials is a dense subset of the domain of T_n and for every $a \in G$, $T_n a$ converges to zero. Moreover, for any $n \in \mathbb{N}^*$, T_n is an unbounded and closed linear operator. We need a result on approximation by Dirichlet polynomials.

LEMMA 3.7. *Let $K \subset \{z \in \mathbb{C} : \Re(z) \leq 0\}$ be an admissible compact set. Let $g : K \rightarrow \mathbb{C}$ be a function continuous on K and holomorphic in $\overset{\circ}{K}$, and let $\varepsilon > 0$, $l \in \mathbb{N}$ and $q \in \mathbb{N}$. Then there exists a Dirichlet polynomial $h(z) = \sum_{j=q+1}^m a_j j^{-z}$ such that $\sup_{z \in K} |h(z) - g(z)| < \varepsilon$ and $(\sum_{j=q+1}^m |\log^l(j) a_j|^p)^{1/p} < \varepsilon$.*

Proof. This follows from Lemma 2 of [1] by taking $f = 0$ with slight modifications. Indeed, we may suppose that g is an entire function, by Mergelyan’s Theorem. Write $K = K_1 \cup \dots \cup K_d$ with $K_i \subset \{z \in \mathbb{C} : c_i \leq \Re(z) \leq b_i\}$, and $0 \geq b_1 \geq c_1 > b_2 \geq \dots > b_d \geq c_d$, $b_i - c_i < 1/2$. We fix $\sigma_1 \in \mathbb{R}$ such that $K_1 + \sigma_1 \subset \{z \in \mathbb{C} : 1/2 < \Re(z) < 1\}$ and $\sigma_1 p > 1$. Consider $n_1 \geq q$ such that $\sum_{j \geq n_1+1} (\log^l(j) j^{-\sigma_1})^p < (\varepsilon/2)^p$. We end as in the proof of Lemma 2 of [1, p. 174]. ■

THEOREM 3.8. *Let A be an ℓ^p , $1 < p < +\infty$. Under the above assumptions, there exists a sequence $a = (a_n) \in A$ such that for every $b \in A$ and for every entire function h there exists $(\lambda_n) \in \mathbb{N}^{\mathbb{N}}$ such that $T_{\lambda_n}(a) \rightarrow b$ and $\sup_{z \in K} |\sum_{j=1}^{\lambda_n} a_j j^{-z} - h(z)| \rightarrow 0$ as $n \rightarrow +\infty$. The set of such sequences is dense in A .*

Proof. It suffices to prove that condition (ii’) of Theorem 2.12 holds. Let $q \in \mathbb{N}$ and $\varepsilon > 0$. By Lemma 3.7, there exist integers n, p , with $n \geq p \geq q$, and complex numbers a_p, \dots, a_n such that $\sup_{z \in K} |\sum_{j=p}^n a_j j^{-z} - g(z)| < \varepsilon$ and $(\sum_{j=p}^n |\log^q(j) a_j|^p)^{1/p} < \varepsilon/2$. We fix $N \in \mathbb{N}$ such that $(\sum_{j \geq N+1} |b_j|^p)^{1/p} < \varepsilon$. Let $b' = (b_1, \dots, b_N, 0, \dots)$. Then $\|b - b'\|_p < \varepsilon$. Moreover, choose an integer $k > n$ so that

$$\left(\sum_{j=1}^N \frac{|b_j|^p}{\log^p(j+1) \dots \log^p(j+k-i)} \right)^{1/p} < \varepsilon/2, \quad i = 0, 1, \dots, q.$$

We set $m = N + k$ and $c = (c_j)_{j=1}^{\infty}$, with

$$c_j = \begin{cases} 0 & \text{for } j \in \{1, \dots, p-1\} \cup \{n+1, \dots, k\} \cup \{m+1, m+2, \dots\}, \\ a_j & \text{for } j = p, \dots, n, \\ \frac{(-1)^k b_{j-k}}{\log(j+1-k) \dots \log(j)} & \text{for } j = k+1, \dots, m. \end{cases}$$

Then we get

$$\begin{aligned} \sup_{z \in K} \left| \sum_{j=p}^k c_j j^{-z} - g(z) \right| &= \sup_{z \in K} \left| \sum_{j=p}^n a_j j^{-z} - g(z) \right| < \varepsilon, \\ \left(\sum_{j=p}^k |c_j|^p \right)^{1/p} &= \left(\sum_{j=p}^n |a_j|^p \right)^{1/p} < \varepsilon, \end{aligned}$$

$$\left(\sum_{j=k+1}^m |c_j|^p\right)^{1/p} = \left(\sum_{j=1}^N \frac{|b_j|^p}{\log^p(j+1) \dots \log^p(j+k)}\right)^{1/p} < \varepsilon,$$

$$\|T_k(c) - b\|_p = \|b' - b\|_p < \varepsilon.$$

Finally, the following estimate holds, for every $i = 1, \dots, q$:

$$\|T_i(c)\|_p \leq \left(\sum_{j=p}^n |\log^q(j) a_j|^p\right)^{1/p} + \left(\sum_{j=1}^N \frac{|b_j|^p}{\log^p(j+1) \dots \log^p(j+k-i)}\right)^{1/p}$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$

Therefore condition (ii') of Theorem 2.12 is satisfied. ■

Let us recall that there exists a sequence (K_k) of admissible compact sets contained in $\{z \in \mathbb{C} : \Re(z) \leq 0\}$ such that every admissible compact set $K \subset \{z \in \mathbb{C} : \Re(z) \leq 0\}$ is contained in some K_k (see [2, Lemma 5]). We combine the previous result with Theorem 2.14 and Mergelyan's Theorem to obtain the following.

THEOREM 3.9. *Let A be an ℓ^p , $1 < p < +\infty$. Under the above assumptions, there exists a sequence $a = (a_n) \in A$ such that, every $b \in A$, for every admissible compact set $K \subset \{z \in \mathbb{C} : \Re(z) \leq 0\}$ and every function h continuous on K and holomorphic in $\overset{\circ}{K}$ there exists $(\lambda_n) \in \mathbb{N}^{\mathbb{N}}$ such that $T_{\lambda_n}(a) \rightarrow b$ and $\sup_{z \in K} |\sum_{j=1}^{\lambda_n} a_j j^{-z} - h(z)| \rightarrow 0$ as $n \rightarrow +\infty$. The set of such sequences is dense in A .*

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