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Medians, continuity, and vanishing oscillation

by

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Abstract. We consider properties of medians as they pertain to the continuity and vanishing oscillation of a function. Our approach is based on the observation that medians are related to local sharp maximal functions restricted to a cube of $\mathbb{R}^n$.

In considering the problem of the resistance of materials to certain types of deformations, F. John was led to the study of quasi-isometric mappings. The setting is essentially as follows. Let $Q_0 \subset \mathbb{R}^n$ be a cube and f a continuous function on Q_0. Assume that to each subcube Q of Q_0 with sides parallel to those of Q_0 there is assigned a constant c_Q and let μ_Q be the function of the real variable M given by

$$\mu_Q(M) = \frac{|\{y \in Q : |f(y) - c_Q| > M\}|}{|Q|}.$$

Let $\phi(M) = \sup_{Q \subset Q_0} \mu_Q(M)$, $0 < s < 1/2$, and λ a number such that $\phi(\lambda) \le s$. Then under these assumptions

$$\phi(M) \le Ae^{-BM/\lambda}$$

holds for all nonnegative M where A, B are universal functions of s and the dimension n. Thus the space of functions of bounded mean oscillation (BMO) was introduced and the John–Nirenberg inequality established [5].

Strömberg [11] adopted this setting when studying spaces close to BMO and discussed different ways of describing the oscillation of a function f on cubes. He also incorporated the value $s = 1/2$ above, which corresponds to the notion of median value $m_f(Q) = m_f(1/2, Q)$ of f over Q. The local maximal functions of Strömberg are of particular interest because they allow for pointwise estimates for Calderón–Zygmund singular integral operators [4, 6].

In this paper we consider properties of medians as they pertain to the continuity and vanishing oscillation of a function. Our approach is based on

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the observation that medians are related to local sharp maximal functions restricted to a cube $Q_0 \subset \mathbb{R}^n$ with parameter $0 < s \le 1/2$ by means of the expression

$$M^{\sharp,\phi}_{0,s,Q_0} f(x) = \sup_{x\in Q,\, Q\subset Q_0} \inf_c \frac{m_{|f-c|}(1-s,Q)}{\phi(|Q|)} \sim \sup_{x\in Q,\, Q\subset Q_0} \frac{m_{|f-m_f(1-s,Q)|}(1-s,Q)}{\phi(|Q|)},$$

where ϕ equals 1 in the case of functions with bounded median oscillation with parameter s (bmo_s), and satisfies appropriate conditions in the case of functions with vanishing median oscillation with parameter s (vmo_s). Strömberg showed that $\mathrm{bmo}_s = \mathrm{BMO}$, and, similarly, we show here that for sufficiently small s, $\mathrm{vmo}_s = \mathrm{VMO}$, the space of functions of vanishing mean oscillation. Moreover, since VMO is known to contain bounded discontinuous functions [8, 10], we complete the picture by giving criteria for continuity of functions equivalent to a bounded function on a cube in terms of medians.

The paper is organized as follows. In Section 1 we introduce the notions of median and maximal median with respect to a parameter $0 < s < 1$; when $s \neq 1/2$ we refer to these medians as *biased with parameter* s. In Section 2 we consider the a.e. convergence of maximal biased medians in the spirit of Fujii's results [3] for $s = 1/2$. In Section 3, motivated by similar results involving averages [9], we characterize continuity in terms of maximal biased medians with parameter $> 1/2$. In Section 4 we extend the Strömberg decomposition of cubes to parameters $> 1/2$. Finally, in Section 5 we consider the spaces of functions with vanishing median oscillation, establish a John–Nirenberg type inequality they satisfy, and show that, as anticipated, they coincide with VMO for sufficiently small s.

1. Medians and maximal medians. In what follows we restrict our attention to cubes with sides parallel to the coordinate axes.

DEFINITION 1.1. For a cube $Q \subset \mathbb{R}^n$, $0 < s < 1$, and a real-valued measurable function f on Q we say that $m_f(s,Q)$ is a *median value of f over Q with parameter s* if

$$|\{y \in Q : f(y) < m_f(s,Q)\}| \le s|Q| \tag{1.1}$$

and

$$|\{y \in Q : f(y) > m_f(s,Q)\}| \le (1-s)|Q|. \tag{1.2}$$

When $s = 1/2$, $m_f(1/2,Q) = m_f(Q)$ corresponds to a median value of f over Q. The set of median values of f is one point or a closed interval as the example $f = \chi_{[1/2,1)}$ on $[0,1]$ shows. It is therefore convenient to work with maximal medians, which are uniquely defined [1, 3]. More precisely, we have

DEFINITION 1.2. For a cube $Q \subset \mathbb{R}^n$, $0 < s < 1$, and a real-valued measurable function f on Q, we say that $M_f(s,Q)$ is the *maximal median of f over Q with parameter s* if

$$M_f(s,Q) = \sup\{M : |\{y \in Q : f(y) < M\}|\} \leq s|Q|.$$

The reader will have no difficulty in proving the sup above is assumed, that is to say,

$$|\{y \in Q : f(y) < M_f(s,Q)\}| \leq s|Q|.$$

To justify the nomenclature of maximal median we verify that $M_f(s,Q)$ satisfies the conditions that characterize medians. (1.1) is guaranteed since $M_f(s,Q)$ is the maximum value for which it holds. As for (1.2), let $B_n = \{y \in Q : f(y) \geq M_f(s,Q) + 1/n\}$ and note that $|B_n| \leq (1-s)|Q|$, all n, and $\{y \in Q : f(y) > M_f(s,Q)\} \subset \liminf_n B_n$. Then $|\{y \in Q : f(y) > M_f(s,Q)\}| \leq \liminf_n |B_n| \leq (1-s)|Q|$.

Hereafter when considering a median we mean the maximal median and denote it simply by $m_f(s,Q)$. Clearly maximal medians satisfy

$$|\{y \in Q : f(y) \leq m_f(s,Q)\}| \geq s|Q| \tag{1.3}$$

and

$$|\{y \in Q : f(y) \geq m_f(s,Q)\}| \geq (1-s)|Q|. \tag{1.4}$$

We summarize the basic properties of maximal medians that are of interest to us in the following proposition.

PROPOSITION 1.1. *Let $Q \subset \mathbb{R}^n$ be a cube, $0 < s, t < 1$, and f, g real-valued measurable functions on Q. Then the following properties hold:*

(i) *For $s < t$,*

$$m_f(s,Q) \leq m_f(t,Q). \tag{1.5}$$

(ii) *If $f \leq g$ a.e., then*

$$m_f(s,Q) \leq m_g(s,Q). \tag{1.6}$$

(iii) *For a constant c,*

$$m_f(s,Q) - c = m_{f-c}(s,Q). \tag{1.7}$$

(iv) *If $f, g \geq 0$ a.e., $0 < s, s_1 < 1$, and $0 < t < s + s_1 - 1$, then*

$$m_{f+g}(t,Q) \leq m_f(s,Q) + m_g(s_1,Q). \tag{1.8}$$

(v) *In general, $m_{-|f|}(s,Q) \leq m_f(s,Q) \leq m_{|f|}(s,Q)$. And if $m_f(s,Q) \leq 0$, then*

$$|m_f(s,Q)| \leq m_{|f|}(1-s,Q). \tag{1.9}$$

Thus for general f,

$$|m_f(s,Q)| \le m_{|f|}(s,Q), \qquad 1/2 \le s < 1. \tag{1.10}$$

(vi) *If $f \ge 0$ is locally integrable and f_Q denotes the average of f over Q, then*

$$m_f(s,Q) \le \frac{1}{1-s}\, f_Q. \tag{1.11}$$

Proof. (i) Since $|\{y \in Q : f(y) < m_f(s,Q)\}| \le s|Q| < t|Q|$, (1.5) holds.

(ii) Up to a set of measure zero $\{y \in Q : g(y) < m_f(s,Q)\} \subset \{y \in Q : f(y) < m_f(s,Q)\}$. Therefore $|\{y \in Q : g(y) < m_f(s,Q)\}| \le s|Q|$, and so $m_f(s,Q) \le m_g(s,Q)$.

(iii) Since $\{y \in Q : f(y) < m_f(s,Q)\} = \{y \in Q : f(y)-c < m_f(s,Q)-c\}$ it readily follows that $m_f(s,Q) - c \le m_{f-c}(s,Q)$. And since $\{y \in Q : f(y)-c < m_{f-c}(s,Q)\} = \{y \in Q : f(y) < m_{f-c}(s,Q)+c\}$, $m_{f-c}(s,Q)+c \le m_f(s,Q)$. Note that, in particular, $m_c(s,Q) = c$.

(iv) For the sake of argument suppose that f, g are measurable functions on Q such that $m_{f+g}(t,Q) - (m_f(s,Q) + m_g(s_1,Q)) > 2\eta > 0$. Then $\{y \in Q : f(y) < m_f(s,Q)+\eta\} \cap \{y \in Q : g(y) < m_g(s_1,Q)+\eta\} \subset \{y \in Q : f(y)+g(y) < m_{f+g}(t,Q)\}$. Now, since $|\{y \in Q : f(y) < m_f(s,Q) + \eta\}| \ge s|Q|$, $|\{y \in Q : g(y) < m_g(s_1,Q) + \eta\}| \ge s_1|Q|$, and $|\{y \in Q : f(y) + g(y) < m_f(t,Q)\}| \le t|Q|$, it readily follows that $s + s_1 \le 1 + t$, which is not the case.

(v) Since $-|f| \le f \le |f|$, by (1.6), $m_{-|f|}(s,Q) \le m_f(s,Q) \le m_{|f|}(s,Q)$. Now, if $m_f(s,Q) \le 0$ note that $\{y \in Q : |f(y)| < -m_f(s,Q)\} = \{y \in Q : m_f(s,Q) < -|f(y)|\} \subset \{y \in Q : m_f(s,Q) < f(y)\}$, and therefore $|\{y \in Q : |f(y)| < -m_f(s,Q)\}| \le (1-s)|Q|$. Consequently, $|m_f(s,Q)| \le m_{|f|}(1-s,Q)$. And since for $s \ge 1/2$, $1-s \le s$, by (1.5) we have $|m_f(s,Q)| \le m_{|f|}(s,Q)$ for that range of s.

(vi) We may assume that $m_f(s,Q) \ne 0$. Then by (1.4) and Chebyshev's inequality,

$$(1-s)|Q| \le |\{y \in Q : f(y) \ge m_f(s,Q)\}| \le \frac{1}{m_f(s,Q)} \int_Q f(y)\,dy,$$

and the conclusion follows. ■

The restriction $1/2 \le s < 1$ is necessary for (1.10) to hold. Let $Q = [0,1]$ and $f(x) = -2\chi_{[0,1/2)}(x) + \chi_{[1/2,1)}(x)$; then for $0 < s < 1/2$, $m_f(s,Q) = -2$ but $m_{|f|}(s,Q) = 1 < 2$. And, in contrast to averages, the restriction $0 < t < s + s_1 - 1$ is necessary for (1.8) to hold. To see this let $Q = [0,1]$, and pick $1/2 < s_1 \le s < 1$, and $t = s + s_1 - 1 > 0$. If $f = \chi_{[0,1-s]}$ and $g = \chi_{[1-s_1,2(1-s_1)]}$, $\{y \in Q : f(y) + g(y) < 1\} = (1-s, 1-s_1) \cup (2(1-s_1), 1]$ has measure $(s-s_1) + 1 - 2(1-s_1) = t$, and therefore, although $m_f(s,Q) = m_g(s_1,Q) = 0$, $m_{f+g}(t,Q) = 1$.

Finally, maximal medians can be expressed in terms of distribution functions or nonincreasing rearrangements. Recall that the *nonincreasing rearrangement* f^* of f at level $\lambda > 0$ is given by $f^*(\lambda) = \inf\{\alpha > 0 : |\{x \in \mathbb{R}^n : |f(x)| > \alpha\}| \le \lambda\}$ and satisfies

$$|\{y \in \mathbb{R}^n : |f(y)| > f^*(u)\}| \le u, \qquad u > 0. \tag{1.12}$$

We then have

PROPOSITION 1.2. *Let $Q \subset \mathbb{R}^n$ be a cube, $0 < s < 1$, and f a measurable function on Q. Then*

$$m_{|f|}(1-s, Q) = \inf\{\alpha > 0 : |\{y \in Q : |f(y)| > \alpha\}| < s|Q|\} = (f\chi_Q)^*(s|Q|).$$

Proof. Let $\overline{\alpha} = \inf\{\alpha > 0 : |\{y \in Q : |f(y)| > \alpha\}| < s|Q|\}$. Then for all $\varepsilon > 0$ it readily follows that $|\{y \in Q : |f(y)| \le \overline{\alpha} + \varepsilon\}| > (1-s)|Q|$, which together with (1.3) implies $m_{|f|}(1-s, Q) \le \overline{\alpha} + \varepsilon$. Thus $m_{|f|}(1-s, Q) \le \overline{\alpha}$.

Next, by (1.12), $|\{y \in Q : |f(y)| > (f\chi_Q)^*(s|Q|)\}| \le s|Q|$, which gives $\overline{\alpha} \le (f\chi_Q)^*(s|Q|)$.

Finally, since for $\varepsilon > 0$, $|\{y \in Q : |f(y)| > (f\chi_Q)^*(s|Q|) - \varepsilon\}| > s|Q|$, by (1.2) it readily follows that $(f\chi_Q)^*(s|Q|) - \varepsilon < m_{|f|}(1-s, Q)$, and consequently $(f\chi_Q)^*(s|Q|) \le m_{|f|}(1-s, Q)$. ■

Other equivalent expressions appearing in the literature include those in [2].

2. Convergence of medians. The examples following Proposition 1.1 suggest that medians rely more heavily on the distribution of the values of f than do averages. On the other hand, averages and medians are not always at odds. In particular, using (1.11) the reader should have no difficulty in verifying that the following version of the Lebesgue differentiation theorem holds: If f is a locally integrable function on $\mathbb{R}^n$ and $1/2 \le s < 1$, then

$$\lim_{x \in Q,\, Q \to x} m_f(s, Q) = f(x)$$

at every Lebesgue point x of f.

Thus, in some sense $m_f(s, Q)$ is a good substitute for f_Q for small Q. In fact, a more careful argument gives that the biased maximal medians $m_f(s, Q)$ of an arbitrary measurable function f converge to f a.e., a fact observed by Fujii [3] for the case $s = 1/2$.

THEOREM 2.1. *Let f be a real-valued, finite a.e. measurable function on $\mathbb{R}^n$, and $0 < s < 1$. Then*

$$\lim_{x \in Q,\, Q \to x} m_f(s, Q) = f(x) \quad \text{a.e.} \tag{2.1}$$

In particular, (2.1) *holds at every point of continuity x of f.*

Proof. For $k \geq 1$ and an integer j, let $E_{k,j} = \{x \in \mathbb{R}^n : (j-1)/2^k \leq f(x) < j/2^k\}$, $a_{k,j} = (j-1)/2^k$, and put $S_k(x) = \sum_{j=-\infty}^{\infty} a_{k,j}\chi_{E_{k,j}}(x)$. Note that since f is finite a.e., $\mathbb{R}^n = \bigcup_{k,j} E_{k,j}$ except possibly for a set of measure 0, and when $f(x)$ is finite we have $0 \leq f(x) - S_k(x) \leq 2^{-k}$, which gives $m_{S_k}(s,Q) \leq m_f(s,Q) \leq m_{S_k}(s,Q) + 2^{-k}$ for all cubes Q. Let $A_{k,j} = \{x \in E_{k,j} : x \text{ is a point of density for } E_{k,j}\}$, $A_k = \bigcup_{j=-\infty}^{\infty} A_{k,j}$. Since f is finite a.e., $|\mathbb{R}^n \setminus A_k| = 0$ for all k, and if $A = \bigcup_{k=1}^{\infty} A_k$, also $|\mathbb{R}^n \setminus A| = 0$.

We claim that the limit in question exists for $x \in A$. Given $\varepsilon > 0$, pick k such that $2^{-k+1} < \varepsilon$. Then $x \in A_{k,j}$ for some j, and

$$\lim_{x \in Q,\, Q \to x} \frac{|A_{k,j} \cap Q|}{|Q|} = 1.$$

Let $\delta = \max\{s, 1-s\}$ and note that for all cubes Q with small enough measure containing x,

$$\frac{|A_{k,j} \cap Q|}{|Q|} > \delta.$$

We restrict our attention to such small cubes Q containing x. Note that for these cubes $m_{S_k}(s,Q) = a_{k,j}$. Indeed, on the one hand, since $S_k(y) = a_{k,j}$ for $y \in A_{k,j}$, $|\{y \in Q : S_k(y) < a_{k,j}\}| \leq |A_{k,j}^c \cap Q| < s|Q|$, and therefore $a_{k,j} \leq m_{S_k}(s,Q)$. And, on the other, since for $\varepsilon > 0$, $\{y \in Q : S_k(y) < a_{k,j} + \varepsilon\} \supset A_{k,j} \cap Q$, it follows that $|\{y \in Q : S_k(y) < a_{k,j} + \varepsilon\}| \geq |A_{k,j} \cap Q| \geq s|Q|$. Hence, $m_{S_k}(s,Q) \leq a_{k,j} + \varepsilon$, and since ε is arbitrary, $m_{S_k}(s,Q) \leq a_{k,j}$.

Then, since $a_{k,j} = m_{S_k}(s,Q) = S_k(x)$ for $x \in A_{k,j}$,

$$\begin{aligned}|m_f(s,Q) - f(x)| &\leq |m_f(s,Q) - m_{S_k}(s,Q)| + |m_{S_k}(s,Q) - f(x)| \\ &\leq 2^{-k} + (f(x) - S_k(x)) \leq 2^{-k+1} < \varepsilon.\end{aligned}$$

In other words, $|m_f(s,Q) - f(x)| < \varepsilon$ for $x \in A$ and all Q with small enough measure containing x.

Now, at a point of continuity x of f, given $\varepsilon > 0$, let $\delta > 0$ be such that $|f(y) - f(x)| \leq \varepsilon$ for $y \in B(x,\delta)$. Then for y in a cube Q containing x and contained in $B(x,\delta)$ we have $-\varepsilon \leq f(y) - f(x) \leq \varepsilon$, and consequently $-\varepsilon = m_{-\varepsilon}(s,Q) \leq m_{f-f(x)}(s,Q) = m_f(s,Q) - f(x) \leq m_\varepsilon(s,Q) = \varepsilon$, hence $|m_f(s,Q) - f(x)| \leq \varepsilon$. ∎

3. A median characterization for continuity. We say that a measurable function f on a cube $Q_0 \subset \mathbb{R}^n$ is *equivalent* to a continuous function on Q_0 if the values of f can be modified on a set of Lebesgue measure 0 so as to coincide with a continuous function on Q_0; similarly for f equivalent to a bounded function on a cube. In this section we characterize those measurable functions equivalent to a bounded function on a cube that are equivalent to a continuous function on that cube in terms of medians, keeping in mind that in the case of locally integrable functions the condition involves the consideration of oscillations involving two nonoverlapping cubes [9].

DEFINITION 3.1. For $0 < s < 1$ and nonoverlapping cubes $Q_1, Q_2 \subset \mathbb{R}^n$, let

$$\Psi_s(f, Q_1, Q_2) = \frac{|Q_1|}{|Q_1 \cup Q_2|}\, m_f(s, Q_1) + \frac{|Q_2|}{|Q_1 \cup Q_2|}\, m_f(s, Q_2),$$

and

$$\Omega(f, s, \delta) = \sup_{\operatorname{diam}(Q_1 \cup Q_2) \le \delta} \inf_c \Psi_s(|f - c|, Q_1, Q_2).$$

Ψ_s is a weighted average of maximal medians of f in the spirit of averages and $\Omega(f, s, \delta)$ is related to the oscillation of a measurable function on a cube, as shown by the following result.

THEOREM 3.1. *Let $Q_0 \subset \mathbb{R}^n$ be a cube, $1/2 < s < 1$, f a measurable function on Q_0 that is equivalent to a bounded function there, and $\omega(f, \delta)$, $\delta > 0$, the essential modulus of continuity of f defined by*

$$\omega(f, \delta) = \sup_{|h| \le \delta} \Big(\operatorname*{ess\,sup}_{x, x+h \in Q_0} |f(x+h) - f(x)| \Big).$$

Then we have $\Omega(f, s, \delta) = \omega(f, \delta)/2$.

Proof. For nonoverlapping cubes $Q_1, Q_2 \subset Q_0$, let

$$\Theta = \operatorname{ess\,osc}(f, Q_1 \cup Q_2) = \operatorname*{ess\,sup}_{Q_1 \cup Q_2} f - \operatorname{ess\,inf}_{Q_1 \cup Q_2} f.$$

Let $\delta > 0$. If $x \in Q_1 \cup Q_2 \subset Q_0$ is such that $x + h \in Q_0$ where $|h| \le \delta$, since

$$\operatorname*{ess\,sup}_{x, x+h \in Q_0} |f(x+h) - f(x)| \ge \operatorname*{ess\,sup}_{Q_1 \cup Q_2} f - \operatorname*{ess\,inf}_{Q_1 \cup Q_2} f,$$

taking the sup over $|h| \le \delta$ it readily follows that $\omega(f, \delta) \ge \Theta$. Moreover, since for $y \in Q_1$ and an arbitrary constant c,

$$|f(y) - c| \le \max\Big\{ \operatorname*{ess\,sup}_{Q_1 \cup Q_2} f - c, c - \operatorname*{ess\,inf}_{Q_1 \cup Q_2} f \Big\},$$

picking $c = (\sup \inf_{Q_1 \cup Q_2} f + \operatorname{ess\,inf}_{Q_1 \cup Q_2} f)/2$, it follows that $|f(y) - c| \le \Theta/2$, and, consequently, $m_{|f-c|}(s, Q_1) \le m_{\Theta/2}(s, Q_1) = \Theta/2$; similarly we have $m_{|f-c|}(s, Q_2) \le \Theta/2$. Therefore,

$$\inf_c \Psi_s(|f - c|, Q_1, Q_2) \le \frac{|Q_1|}{|Q_1 \cup Q_2|} \frac{\Theta}{2} + \frac{|Q_2|}{|Q_1 \cup Q_2|} \frac{\Theta}{2} = \frac{\Theta}{2},$$

and consequently $\Omega(f, s, \delta) \le \Theta/2 \le \omega(f, \delta)/2$.

Conversely, let $0 < t < 2s - 1$. Then for fixed $\delta > 0$, given $\varepsilon > 0$, pick h with $|h| < \delta$ such that $\operatorname{ess\,sup}_{x, x+h \in Q_0} |f(x+h) - f(x)| \ge \omega(f, \delta) - \varepsilon$. Then $E = \{x \in Q_0 : x + h \in Q_0 \text{ and } |f(x+h) - f(x)| \ge \omega(f, \delta) - \varepsilon\}$ has positive measure. Let $x \in E$ be a point of density of E and a small enough so that $Q(x,a), Q(x+h,a)$ are nonoverlapping and

$$\frac{|E \cap Q(x,a)|}{|Q(x,a)|} > 1 - t.$$

Now, since $\{y \in Q(x+h,a) : g(y) < M\} = \{y \in Q(x,a) : g(y+h) < M\}$ and $|Q(x,a)| = |Q(x+h,a)|$, it readily follows that $m_{|f-c|}(s, Q(x+h,a)) = m_{|f(\cdot+h)-c|}(s, Q(x,a))$, and consequently, since $|f(y+h)-f(y)| \le |f(y+h)-c| + |f(y)-c|$, by (1.8) and (1.6), $m_{|f-c|}(s, Q(x,a)) + m_{|f-c|}(s, Q(x+h,a)) \ge m_{|f-c|+|f(\cdot+h)-c|}(t, Q(x,a)) \ge m_{|f(\cdot+h)-f|}(t, Q(x,a))$. Therefore,

$$\begin{aligned}\Psi_s(|f-c|, Q(x,a), Q(x+h,a))& \\ \ge \frac{1}{2} m_{|f-c|}(s, Q(x,a)) &+ \frac{1}{2} m_{|f-c|}(s, Q(x+h,a)) \\ &\ge \frac{1}{2} m_{|f(\cdot+h)-f|}(t, Q(x,a)).\end{aligned}$$

Finally, since $|E \cap Q(x,a)| = |\{y \in Q(x,a) : |f(y+h) - f(y)| \ge \omega(f,\delta) - \varepsilon\}| > (1-t)|Q(x,a)|$, it readily follows that $m_{|f(\cdot+h)-f|}(t, Q(x,a)) \ge \omega(f,\delta) - \varepsilon$, which, since ε is arbitrary, implies $\Psi_s(|f-c|, Q(x,a), Q(x+h,a)) \ge \omega(f,\delta)/2$. Thus $\Omega(f,s,\delta + (\sqrt{2a})^n) \ge \omega(f,\delta)$, and letting $a \to 0$, $\Omega(f,s,\delta) \ge \omega(f,\delta)/2$. ∎

THEOREM 3.2. *Let $Q_0 \subset \mathbb{R}^n$ be a cube, $1/2 < s < 1$, and f a measurable function on Q_0 that is equivalent to a bounded function there. Then f is equivalent to a continuous function on Q_0 iff $\lim_{\eta\to 0^+} \Omega(f,s,\eta) = 0$.*

The proof follows at once from Theorem 3.1. Note that by Proposition 1.2 the conclusion can also be stated in terms of rearrangements.

4. A decomposition of cubes. Strömberg's essential tool in dealing with the oscillation of functions and local maximal functions is a decomposition of cubes [11]. In this section we extend the results to biased medians with parameters $> 1/2$.

We begin by introducing the local sharp maximal function restricted to a cube.

DEFINITION 4.1. Let $Q_0 \subset \mathbb{R}^n$ and $0 < s \le 1/2$. For a measurable function f on Q_0, the *local sharp maximal function restricted to Q_0 of f*, written $M^\sharp_{0,s,Q_0} f(x)$, is defined at $x \in Q_0$ as

(4.1)
$$M^\sharp_{0,s,Q_0} f(x) = \sup_{x\in Q,\, Q\subset Q_0} \inf_c \inf\{\alpha \ge 0 : |\{y \in Q : |f(y)-c| > \alpha\}| < s|Q|\}.$$

When $Q_0 = \mathbb{R}^n$, $M^\sharp_{0,s,\mathbb{R}^n} f(x) = M^\sharp_{0,s} f(x)$ denotes the *local sharp maximal function of f at $x \in \mathbb{R}^n$.*

The range $0 < s \leq 1/2$ is necessary since for $s > 1/2$, $M^\sharp_{0,s,Q} f(x) = 0$ for a function f that takes two different values.

Local maximal functions, as well as maximal functions defined in terms of rearrangements, can be expressed in terms of medians. Let $\omega_s(f, Q) = \inf_c((f - c)\chi_Q)^*(s|Q|)$. Then by Proposition 1.2,

$$M^\sharp_{0,s,Q_0} f(x) = \sup_{x\in Q,\, Q\subset Q_0} \omega_s(f, Q) = \sup_{x\in Q,\, Q\subset Q_0} \inf_c m_{|f-c|}(1 - s, Q).$$

The first expression above is used by Lerner [6, 7].

An efficient choice for c in the infimum above is $m_{|f-m_f(1-s,Q)|}(1-s, Q)$. Indeed, for $Q \subset Q_0$ and a constant c, since $1 - s \geq 1/2$, by (1.7) and (1.10),

$$|m_f(1 - s, Q) - c| \leq m_{|f-c|}(1 - s, Q). \tag{4.2}$$

Then, since $|f(y) - m_f(1 - s, Q)| \leq |f(y) - c| + |c - m_f(1 - s, Q)|$, by (1.5), (1.7), and (4.2),

$$\begin{aligned} m_{|f-m_f(1-s,Q)|}(1 - s, Q) &\leq m_{|f-c|}(1 - s, Q) + |c - m_f(1 - s, Q)| \\ &\leq m_{|f-c|}(1 - s, Q) + m_{|f-c|}(1 - s, Q) = 2m_{|f-c|}(1 - s, Q), \end{aligned}$$

and consequently

$$\begin{aligned} \inf_c m_{|f-c|}(1 - s, Q) &\leq m_{|f-m_f(1-s,Q)|}(1 - s, Q) \\ &\leq 2 \inf_c m_{|f-c|}(1 - s, Q). \end{aligned} \tag{4.3}$$

The decomposition of cubes relies on three lemmas which we prove next.

Lemma 4.1. *Let $Q \subset \mathbb{R}^n$ be a cube, $0 < s \leq 1/2$, $1/2 \leq t \leq 1 - s$, and f a measurable function on Q. Then for any $\eta > 0$,*

$$|\{y \in Q : |f(y) - m_f(t, Q)| \geq 2 \inf_{x\in Q} M^\sharp_{0,s,Q} f(x) + \eta\}| < s|Q|. \tag{4.4}$$

Proof. For fixed c, let $\alpha(c) = m_{|f-c|}(1 - s, Q)$. Then by (4.2) and (1.5),

$$|m_f(t, Q) - c| \leq m_{|f-c|}(t, Q) \leq m_{|f-c|}(1 - s, Q) = \alpha(c), \tag{4.5}$$

and by (1.4),

$$|\{y \in Q : |f(y) - c| \geq \alpha(c) + \varepsilon\}| < s|Q|, \quad \varepsilon > 0. \tag{4.6}$$

Let $m = \inf_c \alpha(c)$ and pick $\{c_k\}$ such that $m \leq \alpha(c_k) \leq m + 1/k$, all k. Then by (4.5),

$$\begin{aligned} |f(y) - c_k| &\geq |f(y) - m_f(t, Q)| - |m_f(t, Q) - c_k| \\ &\geq |f(y) - m_f(t, Q)| - \alpha(c_k), \end{aligned}$$

and consequently, since $2m + \eta \geq \alpha(c_k) + (\eta - 2/k)$, $\{y \in Q : |f(y) - m_f(t, Q)| \geq 2m + \eta\} \subset \{y \in Q : |f(y) - c_k| \geq \alpha(c_k) + \varepsilon_k\}$, where we have chosen k sufficiently large so that $\varepsilon_k = \eta - 2/k > 0$. Then by (4.6),

$|\{y \in Q : |f(y) - m_f(t,Q)| > 2m + \eta\}| < s|Q|$. Finally, since $M^{\#}_{0,s,Q} f(x) \geq m$ for all $x \in Q$, (4.4) holds. ■

LEMMA 4.2. *Let $Q \subset \mathbb{R}^n$ be a cube, $0 < s \leq 1/2$, $1/2 \leq t \leq 1 - s$, $\eta > 0$, and f a measurable function on an open cube containing Q. Then for any family of cubes $\{Q_\varepsilon\}$ with $(1-\varepsilon)Q \subset Q_\varepsilon \subset (1+\varepsilon)Q$,*

$$\limsup_{\varepsilon \to 0^+} |m_f(t,Q) - m_f(t,Q_\varepsilon)| \leq 2 \inf_{x \in Q} M^{\sharp}_{0,s,Q} f(x) + \eta.$$

Proof. Let $A = \inf_{x \in Q} M^{\sharp}_{0,s,Q} f(x)$. For the sake of argument assume there is a sequence $\varepsilon_k \to 0$ such that $|m_f(t,Q) - m_f(t,Q_{\varepsilon_k})| > 2A + \eta$ for all k. Then by (1.10),

$$2A + \eta < |m_f(t,Q) - m_f(t,Q_{\varepsilon_k})| \leq m_{|f - m_f(t,Q)|}(t, Q_{\varepsilon_k}),$$

and consequently, by (1.4),

$$|\{y \in Q_{\varepsilon_k} : |f(y) - m_f(t,Q)| > 2A + \eta\}| \geq (1-t)|Q_{\varepsilon_k}|. \tag{4.7}$$

Since $Q_{\varepsilon_k} \subset (1+\varepsilon_k)Q$, the left-hand side of (4.7) is bounded above by

$$\begin{aligned}|\{y \in (1+\varepsilon_k)Q &: |f(y) - m_f(t,Q)| > 2A + \eta\}| \\ &\leq ((1+\varepsilon_k)^n - 1)|Q| + |\{y \in Q : |f(y) - m_f(t,Q)| > 2A + \eta\}|,\end{aligned}$$

and since $(1-\varepsilon_k)Q \subset Q_{\varepsilon_k}$, the right-hand side of (4.7) is bounded below by

$$(1-t)(1-\varepsilon_k)^n |Q|.$$

Combining these estimates it follows that

$$|\{y \in Q : |f(y) - m_f(t,Q)| > 2A + \eta\}| \geq ((1-t)(1-\varepsilon_k)^n - ((1+\varepsilon_k)^n - 1))|Q|.$$

Now, by (4.4) there exists $\delta > 0$ such that $|\{y \in Q : |f(y) - m_f(t,Q)| \geq 2A + \eta\}| = (s - \delta)|Q|$, and so

$$s - \delta \geq ((1-t)(1-\varepsilon_k)^n - ((1+\varepsilon_k)^n - 1)).$$

Thus, letting $k \to \infty$ implies $s - \delta \geq 1 - t \geq s$, which is not the case. ■

LEMMA 4.3. *Let $Q_0, Q_1 \subset \mathbb{R}^n$ be cubes with $Q_0 \subset Q_1$ and $|Q_1| \leq 2^k |Q_0|$ for some integer k, $0 < s \leq 1/2$, $1/2 \leq t \leq 1 - s$, and f a measurable function on Q_1. Then*

$$|m_f(t,Q_0) - m_f(t,Q_1)| \leq 10k \inf_{x \in Q_0} M^{\sharp}_{0,s,Q_1} f(x). \tag{4.8}$$

Proof. By the triangle inequality it suffices to prove the case $k = 1$. Let $A = \inf_{x \in Q_0} M^{\sharp}_{0,s,Q_1} f(x)$. For the sake of argument suppose that (4.8) does not hold. Then if $A > 0$, by Lemma 4.2, for any fixed $0 < \eta < A/2$, there exists a cube Q_2 such that $Q_0 \subset Q_2 \subset Q_1$ and

$$|m_f(t,Q_2) - m_f(t,Q_0)| > 4A + 2\eta, \quad |m_f(t,Q_2) - m_f(t,Q_1)| > 4A + 2\eta.$$

And if $A = 0$, then $|m_f(t, Q_0) - m_f(t, Q_1)| > 0$ and there exists a cube Q_2 such that $Q_0 \subset Q_2 \subset Q_1$ and

$$|m_f(t, Q_2) - m_f(t, Q_0)| > 2\eta, \qquad |m_f(t, Q_2) - m_f(t, Q_1)| > 2\eta$$

for η sufficiently small.

Thus in both cases the sets $\{y \in Q_k : |f(y) - m_f(t, Q_k)| \le 2A + \eta\}$, $k = 0, 1, 2$, are pairwise disjoint subsets of Q_1, and consequently

$$\begin{aligned}\{y \in Q_0 : |f(y) - m_f(t, Q_0)| \le 2A + \eta\}& \\ \cup \{y \in Q_2 : |f(y) - m_f(t, Q_2)| \le 2A + \eta\}& \\ \subset \{y \in Q_1 : |f(y) - m_f(t, Q_1)| > 2A + \eta\}&.\end{aligned}$$

Therefore, since $\inf_{x \in Q_k} M^\sharp_{0,s,Q_k} f(x) \le A$ for $k = 0, 1, 2$, by Lemma 4.1,

$$(1 - s)|Q_0| + (1 - s)|Q_2| < s|Q_1|.$$

Thus $2(1 - s)|Q_0| < s|Q_1| < 2s|Q_0|$, and consequently $1 < 2s$, which is not the case. ∎

We are now ready to consider the decomposition of cubes relative to medians.

PROPOSITION 4.1. *Let $Q \subset \mathbb{R}^n$ be a cube, $0 < s \le 1/2$, $1/2 \le t \le 1 - s$, $\delta, \beta > 0$, and f a measurable function on Q. Then if $|m_f(t, Q)| \le \delta$, there exists a (possibly empty) family $\{Q_k\}$ of nonoverlapping dyadic subcubes of Q so that*

(1) $Q_k \not\subset \{y \in Q : M^\sharp_{0,s,Q} f(y) > \beta\}$,
(2) $\delta < |m_f(t, Q_k)| \le \delta + 10n\beta$,
(3) $|f(x)| \le \delta$ *for a.e.* $x \in Q \setminus (\{y \in Q : M^{\#}_{0,s,Q} f(y) > \beta\} \cup \bigcup_k Q_k)$.

Proof. If $M^\sharp_{0,s,Q} f(y) > \beta$ for all $y \in Q$ we pick $\{Q_k\}$ as the empty family. Otherwise subdivide Q dyadically into 2^n subcubes and note that by Lemma 4.3, for each dyadic subcube Q', $|m_f(t, Q')| \le \delta + 10n \inf_{x \in Q'} M^\sharp_{0,s,Q} f(x)$. Thus for each of these subcubes Q' one of the following holds:

(a) $Q' \subset \{y \in Q : M^\sharp_{0,s,Q} f(y) > \beta\}$: we discard Q'.
(b) Q' satisfies conditions (1) and (2) above: we collect this Q'.
(c) $Q' \not\subset \{y \in Q : M^\sharp_{0,s,Q} f(y) > \beta\}$ but $|m_f(t, Q')| \le \delta$: we subdivide Q' and continue in this fashion.

Finally, a.e. $x \in Q \setminus (\{y \in Q : M^\sharp_{0,s,Q} f(y) > \beta\} \cup \bigcup_k Q_k)$ is contained in arbitrarily small cubes $\{Q_k(x)\}$ containing x so that $|m_f(t, Q_k(x))| \le \delta$. By Theorem 2.1 it readily follows that $|f(x)| \le \delta$ for a.e. such x. ∎

We can be more precise in the description of the cubes above when $M^\sharp_{0,s,Q} f \in L^\infty(Q)$. Observe that $f - m_f(t, Q)$ satisfies $m_{f - m_f(t,Q)}(t, Q) = 0$ and $M^\sharp_{0,s,Q} f(x) = M^\sharp_{0,s,Q}(f - m_t(t, Q))(x)$ for all $x \in Q$, which means that the decomposition for $f - m_f(t,Q)$ holds for any $\delta > 0$. Let $\{Q_j\}$ and $\{Q_k\}$ denote the families of cubes obtained from the decomposition with parameters $\beta \geq \|M^\sharp_{0,s,Q}f\|_{L^\infty(Q)}$ and $\delta_1 = 4\beta + 2\eta$ and $\delta_2 = 2\delta_1 + 10n\beta$, respectively.

Observe that by construction we have $\bigcup_k Q_k \subset \bigcup_j Q_j$. To see this consider a dyadic subcube Q' of Q that has not been discarded; this depends on β and not on δ_1 or δ_2. If Q' is a Q_j, then $\delta_1 < |m_{f-m_f(t,Q)}(t,Q')| \leq \delta_1 + 10n\beta < \delta_2$ and Q' is not a Q_k. So any Q_k contained in Q' arises from subsequent subdivisions of Q'. On the other hand, if Q' is not a Q_j, then $|m_{f-m_f(t,Q)}(t,Q')| \leq \delta_1 < \delta_2$ and Q' is not a Q_k either. Since this relation is maintained at every level of the successive dyadic subdivisions, the Q_k's arise from subdivisions of Q_j's.

From here on the argument proceeds as in Lemma 4.3. Note that

$$\delta_1 < |m_{f-m_f(t,Q)}(t,Q_j)| = |m_f(t,Q_j) - m_f(t,Q)| \leq \delta_1 + 10n\beta = \delta_2 - \delta_1,$$

and

$$\delta_2 < |m_{f-m_f(t,Q)}(t,Q_k)| = |m_f(t,Q) - m_f(t,Q_k)|.$$

Therefore,

$$\begin{aligned}|m_f(t,Q_j) - m_f(t,Q_k)| &\geq |m_f(t,Q_k) - m_f(t,Q)| - |m_f(t,Q_j) - m_f(t,Q)|\\ &> \delta_2 - (\delta_2 - \delta_1) = \delta_1 = 4\beta + 2\eta.\end{aligned}$$

Thus it readily follows that the sets $\{y \in Q : |f(y) - m_f(t,Q)| \leq 2\beta + \eta\}$, $\{y \in Q_j : |f(y) - m_f(t,Q_j)| \leq 2\beta + \eta\}$, and $\{y \in Q_k : |f(y) - m_f(t,Q_k)| \leq 2\beta + \eta\}$ are nonoverlapping, and so

$$\begin{aligned}&\{y \in Q_j : |f(y) - m_f(t,Q_j)| \leq 2\beta + \eta\}\\ &\quad\cup \{y \in Q_k : |f(y) - m_f(t,Q_k)| \leq 2\beta + \eta\}\\ &\qquad\subset \{y \in Q : |f(y) - m_f(t,Q)| > 2\beta + \eta\}.\end{aligned} \tag{4.9}$$

Now, since $\inf_{x\in Q_j} M^\sharp_{0,s,Q_j}f(x) \leq \beta$ for each Q_j, by (4.4) it follows that

$$(1-s)\sum_j |Q_j| \leq \sum_j |\{y \in Q_j : |f(y) - m_f(t,Q_j)| \leq 2\beta + \eta\}|,$$

and a similar estimate holds with the Q_k's in place of the Q_j's. Finally, since the sets on the left-hand side of (4.9) are pairwise disjoint for all j and k, and since $\bigcup_k Q_k \subset \bigcup_j Q_j$ and $\inf_{x\in Q} M^\sharp_{0,s,Q}f(x) \leq \beta$, by Lemma 4.1,

$$2\sum_k |Q_k| \leq \sum_k |Q_k| + \sum_j |Q_j| \leq \frac{s}{1-s}|Q|,$$

and consequently, since $2(1-s) \geq 1$,

$$\sum_k |Q_k| \leq \frac{s}{2(1-s)}|Q| \leq s|Q|. \tag{4.10}$$

5. Vanishing median oscillation. We say that a measurable function f defined on a cube $Q_0 \subset \mathbb{R}^n$ is of *vanishing median oscillation with parameter s in Q_0* ($\mathrm{vmo}_s(Q_0)$) if

$$\phi_s(u) = \sup_{Q \subset Q_0,\, |Q| \le u} \inf_c m_{|f-c|}(1-s, Q)$$

satisfies $\lim_{u \to 0^+} \phi_s(u) = 0$.

Note that by (1.11),

$$\inf_c m_{|f-c|}(1-s, Q) \le \frac{1}{s} \frac{1}{|Q|} \int_Q |f(y) - f_Q|\, dy,$$

and therefore $\lim_{u\to 0^+} \phi_s(u) = 0$ for all s whenever $f \in \mathrm{VMO}(Q_0)$. Here we show that the spaces actually coincide for $s \le 2^{-n}$.

Now, ϕ_s is a nonnegative, nondecreasing continuous function that vanishes at the origin, and vmo_s may be described in terms of such functions ϕ as follows. Let

$$\|f\|_{s,\phi,Q_0} = \sup_{Q \subset Q_0} \inf_c \frac{m_{|f-c|}(1-s,Q)}{\phi(|Q|)} \sim \sup_{Q\subset Q_0} \frac{m_{|f-m_f(1-s,Q)|}(1-s,Q)}{\phi(|Q|)},$$

and $\mathrm{bmo}_{s,\phi}(Q_0) = \{f : f$ is defined and measurable on Q_0, and $\|f\|_{s,\phi,Q_0} < \infty\}$. Then $\mathrm{vmo}_s(Q_0) = \bigcup_\phi \mathrm{bmo}_{s,\phi}(Q_0)$.

Now fix Q_0 and $0 < s \le 1/2^n$. Let $\phi : \mathbb{R}^+ \to \mathbb{R}^+$ be continuous, nondecreasing, and $\phi(0) = 0$, and define $\Psi_{|Q_0|} : [0, 2^n|Q_0|] \to \mathbb{R}^+$ by

$$\Psi_{|Q_0|}(u) = \int_u^{2^n|Q_0|} \frac{\phi(v)}{v}\, dv. \tag{5.1}$$

We can then prove a strengthened version of the John–Nirenberg inequality.

THEOREM 5.1. *Let $f \in \mathrm{bmo}_{s,\phi}(Q_0)$ for some ϕ as above and $\Psi_{|Q_0|}(u)$ be given by* (5.1). *Then there exist constants c_1, c_2 independent of f and of all subcubes $Q \subset Q_0$ so that*

$$|\{y \in Q : |f(y) - m_f(1-s,Q)| > \lambda\}| \le c_1 \Psi_{|Q|}^{-1}(c_2\lambda/\|f\|_{s,\phi,Q}), \qquad \lambda > 0. \tag{5.2}$$

Proof. If $\|f\|_{s,\phi,Q_0} = 0$, clearly $M^\sharp_{0,s,Q_0} f(x) = 0$ for all $x \in Q_0$ and by Lemma 4.3, the medians of f over all subcubes of Q_0 are constant. Then by Theorem 2.1, f is a.e. constant, and the conclusion holds in this case. Otherwise, since $\|f - c\|_{s,\phi,Q_0} = \|f\|_{s,\phi,Q_0}$ and $\|cf\|_{s.\phi,Q_0} = |c|\, \|f\|_{s,\phi,Q_0}$ for all constants c, we may assume that $\|f\|_{s,\phi,Q_0} = 1$ and $m_f(1-s,Q_0) = 0$. Then by (4.3),

$$m_{|f-m_f(1-s,Q)|}(t,Q) \le 2\phi(|Q|) \le 2\phi(|Q_0|) \quad \text{ for all } Q \subset Q_0,$$

and consequently $\|M^\sharp_{0,s,Q_0} f\|_{L^\infty(Q_0)} \le 2\phi(|Q_0|)$. Pick now $\beta_0 = 2\phi(|Q_0|)$, and note that since $\phi(u) > 0$ for $u > 0$, $\delta_0 = (10n+9)\beta_0$ works in Proposition 4.1 and in the comments that follow it. Since $|\{y \in Q_0 : M^\sharp_{0,s,Q_0} f(y) > \beta_0\}| = 0$ we get a (first-generation) family $\{Q^1_j\}$ of nonoverlapping subcubes of Q_0 so that

(1) $\delta_0 < |m_f(t, Q^1_j)| \le \delta_0 + 10n\beta_0$ for all j,
(2) $|f(x)| \le \delta_0$ for a.e. $x \in Q_0 \setminus \bigcup_j Q^1_j$, and
(3) $\sum_j |Q^1_j| \le s|Q_0|$.

Now we fix one cube Q^1_j of this family, which for simplicity we denote Q^1, and define $g = f - m_f(t, Q^1)$. Note that $m_g(t, Q^1) = 0$ and $g - m_g(t, Q) = f - m_f(t, Q)$ for all $Q \subset Q^1$. Then as above $m_{|g-m_g(t,Q)|}(t, Q) \le 2\phi(|Q|)$ for all $Q \subset Q^1$, and thus $\|M^\sharp_{0,s,Q^1} g\|_{L^\infty(Q^1)} \le 2\phi(|Q|) = 2\phi(|Q_0|/2^n)$.

We then pick (first-generation) parameters $\beta_1 = 2\phi(|Q_0|/2^n)$ and $\delta_1 = (10n+9)\beta_1$, which gives $|\{y \in Q^1 : M^\sharp_{0,s,Q^1} g(y) > \beta_1\}| = 0$. As before, we get a (second-generation) nonoverlapping family $\{Q^2_j\} \subset Q^1$ so that

(1) $\delta_1 < |m_g(t, Q^2_j)| \le \delta_1 + 10n\beta_1$ for all j,
(2) $|g(x)| \le \delta_1$ for a.e. $x \in Q^1 \setminus \bigcup_j Q^2_j$, and
(3) $\sum_j |Q^2_j| \le s|Q^1|$.

We can keep control of the cubes we are gathering and f. Indeed, clearly

$$\sum_j |Q^2_j| \le s \sum_k |Q^1_k| \le s^2 |Q_0|.$$

And as for f, we notice that for a.e. $x \in Q^1 \setminus \bigcup_j Q^2_j$,

$$\begin{aligned} |f(x)| &\le |f(x) - m_f(t, Q^1)| + |m_f(t, Q^1| = |g(x)| + |m_f(t, Q^1)| \\ &\le \delta_1 + \delta_0 + 10n\beta_0 \le (20n+9)(\beta_0 + \beta_1). \end{aligned}$$

Continuing in this fashion, the computation becomes clear: Having selected the $(k-1)$st generation of subcubes $\{Q^{k-1}\}$, we then select a kth generation of subcubes so that

(1) with $\beta_j = 2\phi(|Q_0|/2^{nj})$ and $\delta_j = (10n+9)\beta_j$, $0 \le j \le k-1$, we have $|f(x)| \le (10n+9)\sum_{j=0}^{k-1} \delta_j + 10n \sum_{j=0}^{k-2} \beta_j \le (20n+9)\sum_{j=0}^{k-1} \beta_j$ for a.e. $x \in Q^{k-1} \setminus \bigcup_j Q^k_j$, and
(2) $\sum_j |Q^k_j| \le s^k |Q_0|$.

This is all that is needed. Suppose first that $\lim_{u\to 0^+} \Psi_{|Q_0|}(u) = \infty$. Then, for $\lambda > 2(20n+9)\phi(|Q_0|)$, let k be the largest integer with the property that $2(20n+9)\sum_{j=0}^{k-1} \phi(|Q_0|/2^{nj}) < \lambda$ and observe that by (1) above $\{y \in Q_0 : |f(y)| > \lambda\} \subset \bigcup_j Q^k_j$, and so by (2) above $|\{y \in Q_0 : |f(y)| > \lambda\}| \le s^k |Q_0|$.

Furthermore, by this choice of k we have

$$\lambda < 2(20n+9)\sum_{j=0}^{k}\phi(|Q_0|/2^{nj}) \le \frac{2(20n+9)}{n\ln(2)}\int\limits_{|Q_0|/2^{kn}}^{2^n|Q_0|}\phi(u)\,\frac{du}{u} = c\Psi_{|Q_0|}\left(\frac{|Q_0|}{2^{kn}}\right)$$

where $c = 2(20n+9)/n\ln(2)$.

So,

$$\frac{|Q_0|}{2^{kn}} \le \Psi_{|Q_0|}^{-1}(c_2\lambda), \qquad c_2 = \frac{n\ln(2)}{2(4+10n)}.$$

Therefore,

$$|\{y \in Q_0 : |f(y)| > \lambda\}| \le s^k|Q_0| \le \frac{|Q_0|}{2^{kn}} \le \Psi_{|Q_0|}^{-1}(c_2\lambda),$$

as we wanted to show.

And, for $\lambda \le 2(20n+9)\phi(|Q_0|)$, pick c_1 so that $|Q_0| \le c_1\Psi_{|Q_0|}^{-1}(c_2\lambda)$; since $\{y \in Q_0 : |f(y)| > \lambda\} \subset Q_0$, the conclusion holds. Clearly the argument works for all $Q \subset Q_0$.

Finally, in case $\lim_{u\to 0^+}\Psi_{|Q_0|}(u) < \infty$, the above argument works for all integers k, and therefore f is an essentially bounded function on Q_0 that satisfies (5.2). A more thorough argument shows that f is equivalent to an essentially Lipschitz function on Q_0 [10]. ∎

It is now straightforward that if $f \in \mathrm{vmo}_s(Q_0)$, then $f \in \mathrm{VMO}(Q_0)$. Pick ϕ such that $f \in \mathrm{bmo}_{s,\phi,Q_0}$. Then by integrating (5.2) with respect to λ it follows that for all subcubes $Q \subset Q_0$,

$$\begin{aligned}\int\limits_Q |f(y) - m_f(1-s,Q)|\,dy &= \int\limits_0^\infty |\{y \in Q : |f(y) - m_f(1-s,Q)| > \lambda\}|\,d\lambda \\ &\le c_1\int\limits_0^\infty \Psi_{|Q|}^{-1}(c_2\lambda/\|f\|_{s,\phi,Q_0})\,d\lambda \\ &\le c\|f\|_{s,\phi,Q_0}|Q|\phi(2^n|Q|).\end{aligned}$$

Therefore,

$$\sup_{Q\subset Q_0,\,|Q|\le u}\frac{1}{|Q|}\int\limits_Q |f(y) - f_Q|\,dy \le c\phi(2^n u) \to 0$$

as $u \to 0^+$ and $f \in \mathrm{VMO}(Q_0)$.

References

[1] L. Carleson, *BMO—10 years' development*, in: 18th Scandinavian Congress of Mathematicians (Aarhus, 1980), Progr. Math. 11, Birkhäuser, Boston, 1981, 3–21.

[2] M. Cwikel, Y. Sagher, and P. Shvartsman, *A new look at the John–Nirenberg and John–Strömberg theorems for BMO*, J. Funct. Anal. 263 (2012), 129–166.

[3] N. Fujii, *A condition for a two-weight norm inequality for singular integral operators*, Studia Math. 98 (1991), 175–190.

[4] B. Jawerth and A. Torchinsky, *Local sharp maximal functions*, J. Approx. Theory 43 (1985), 231–270.

[5] F. John, *Quasi-isometric mappings*, in: Seminari 1962/63 Anal. Alg. Geom. e Topol., Vol. 2, Ist. Naz. Alta Mat., Ediz. Cremonese, Roma, 1965, 462–473.

[6] A. K. Lerner, *A pointwise estimate for the local sharp maximal function with applications to singular integrals*, Bull. London Math. Soc. 42 (2010), 843–856.

[7] A. K. Lerner, *A simple proof of the A_2 conjecture*, Int. Math. Res. Notices 2012, doi: 10.1093/imrn/rns145.

[8] D. Sarason, *Functions of vanishing mean oscillation*, Trans. Amer. Math. Soc. 207 (1975), 391–405.

[9] X. L. Shi and A. Torchinsky, *Functions of vanishing mean oscillation*, Math. Nachr. 133 (1987), 289–296.

[10] S. Spanne, *Some function spaces defined using the mean oscillation over cubes*, Ann. Scuola Norm. Sup. Pisa (3) 19 (1965), 593–608.

[11] J.-O. Strömberg, *Bounded mean oscillation with Orlicz norms and duality of Hardy spaces*, Indiana Univ. Math. J. 28 (1979), 511–544.

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