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Limiting real interpolation methods for arbitrary Banach couples

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Abstract. We study limiting K- and J-methods for arbitrary Banach couples. They are related by duality and they extend the methods already known in the ordered case. We investigate the behaviour of compact operators and we also discuss the representation of the methods by means of the corresponding dual functional. Finally, some examples of limiting function spaces are given.

1. Introduction. The real interpolation method $(A_0, A_1)_{\theta,q}$, where $0 < \theta < 1$ and $1 \le q \le \infty$, plays an important role in the study of function spaces, operator theory and approximation theory, as one can see, for example, in the monographs by Butzer and Berens [6], Bergh and Löfström [4], Triebel [31, 32], Bennett and Sharpley [3] or Brudnyĭ and Krugljak [5]. The parameter θ takes values in the open interval $(0, 1)$. The space $(A_0, A_1)_{\theta,q}$ can be described by means of Peetre's K-functional or by means of its dual functional, the J-functional.

The extension of the real method which is obtained by replacing in the definition t^θ by a more general function $f(t)$ (see the paper by Gustavsson [26]) is also important. The case $f(t) = t^\theta g(t)$ is of special interest. Here, g is a power of $1 + |\log t|$ or, more generally, a slowly varying function (see the papers by Evans and Opic [20], Evans, Opic and Pick [21], Gogatishvili, Opic and Trebels [24] and Ahmed, Edmunds, Evans and Karadzhov [1]). Then θ can take the values 1 and 0, but in these limit cases the extra function $g(t)$ is essential to get a meaningful definition. However, if the Banach spaces are related by a continuous embedding, say $A_0 \hookrightarrow A_1$, then limiting spaces $(A_0, A_1)_{0,q;J}$ and $(A_0, A_1)_{1,q;K}$ can be defined without the help of any auxiliary function, just making a natural modification in the definition of the real interpolation method. These limiting methods have been intensively studied, as can be seen in the papers by Gomez and Milman [25],

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Cobos, Fernández-Cabrera, Kühn and Ullrich [9], Cobos, Fernández-Cabrera and Mastyło [11], Cobos and Kühn [15] and Cobos, Fernández-Cabrera and Martínez [10]. The space $(A_0, A_1)_{0,q;J}$ is very close to A_0, and $(A_0, A_1)_{1,q;K}$ to A_1. This fact is important in applications.

To be in the ordered case $A_0 \hookrightarrow A_1$ is basic for the arguments of these papers, but it is only a restriction from the point of view of interpolation theory. For this reason, it is natural to study the extension of limiting methods to arbitrary, not necessarily ordered, couples of Banach spaces $\bar{A} = (A_0, A_1)$. This question has been considered by Cobos, Fernández-Cabrera and Silvestre in [12, 13], but the main target in these papers was to describe the spaces that arise when interpolating $\{A_0, A_1, A_1, A_0\}$ by the methods associated to the unit square. Several limiting K- and J-methods were introduced with the property that along the diagonals of the square, the interpolated spaces are sums of limiting methods and real interpolation spaces in the K-case, while they are intersections of limiting methods and real interpolation spaces in the J-case.

In the present paper our aim is to develop a comprehensive theory of limiting methods for arbitrary couples. Following the pattern of the real method, this calls for selecting from the different methods introduced in [12, 13] just a family of K-methods and a family of J-methods which are related by duality and that allow one to produce a sufficiently rich theory. In terms of the interpolation of $\{A_0, A_1, A_1, A_0\}$, the choice we make corresponds to the methods that arise using the centre of the square.

We start by reviewing some general facts on interpolation theory in Section 2. Then, in Sections 3 and 4, we introduce the limiting K- and J-methods, respectively. We also establish their basic properties and we study their connection with the methods developed for the ordered case and with those considered in [12] and [13]. There is a price to be paid for having methods for general couples: They satisfy worse norm estimates for interpolated operators than in the ordered case and, as a consequence, interpolation properties of compact operators are also worse than in the ordered case. We deal with compact operators in Section 5. As we show there, given $T \in \mathcal{L}(\bar{A}, \bar{B})$, a sufficient condition for the interpolated operator by limiting methods to be compact is that both restrictions $T : A_0 \to B_0$ and $T : A_1 \to B_1$ are compact.

Section 6 is devoted to the description of the limiting K-spaces using the J-functional. This can be done provided that $1 \le q < \infty$. Some consequences of that description are also given. Duality between limiting K- and J-spaces is discussed in Section 7, while Section 8 contains some examples of limiting spaces. First, working with any σ-finite measure space, we characterise the limiting spaces generated by the couple (L_∞, L_1). Then we consider a couple formed by two weighted L_q-spaces and, as an application, we determine the spaces generated by the Sobolev couple (H^{s_0}, H^{s_1}). We also consider the case of the couple $(B^{s_0}_{p,q}, B^{s_1}_{p,q})$ of Besov spaces. Finally, we apply the limiting methods to obtain a Hausdorff–Young type result for the Zygmund space $L_2(\log L)_{-1/2}([0, 2\pi])$.

2. Preliminaries. Let $\bar{A} = (A_0, A_1)$ be a *Banach couple*, that is, two Banach spaces A_0, A_1 which are continuously embedded in some Hausdorff topological vector space. Form the sum A_0+A_1 and the intersection $A_0 \cap A_1$, which are Banach spaces endowed with the norms

$$\|a\|_{A_0+A_1} = \inf\{\|a_0\|_{A_0} + \|a_1\|_{A_1} : a = a_0 + a_1,\ a_j \in A_j\}$$

and

$$\|a\|_{A_0\cap A_1} = \max\{\|a\|_{A_0}, \|a\|_{A_1}\},$$

respectively. Clearly $A_0 \cap A_1 \hookrightarrow A_0 + A_1$. Here $\hookrightarrow$ means continuous embedding.

For $t > 0$, *Peetre's* K- *and* J-*functionals* are defined by

$$\begin{aligned} K(t,a) &= K(t,a;\bar{A}) \\ &= \inf\{\|a_0\|_{A_0} + t\|a_1\|_{A_1} : a = a_0 + a_1, a_j \in A_j\}, \qquad a \in A_0 + A_1, \end{aligned}$$

and

$$J(t,a) = J(t,a;\bar{A}) = \max\{\|a\|_{A_0}, t\|a\|_{A_1}\}, \qquad a \in A_0 \cap A_1.$$

Notice that $K(1,\cdot) = \|\cdot\|_{A_0+A_1}$ and $J(1,\cdot) = \|\cdot\|_{A_0\cap A_1}$.

Let $0 < \theta < 1$ and $1 \le q \le \infty$. The *real interpolation space* $\bar{A}_{\theta,q} = (A_0, A_1)_{\theta,q}$, viewed as a K-space, consists of all $a \in A_0 + A_1$ for which the norm

$$\|a\|_{\bar{A}_{\theta,q}} = \bigg(\int_0^\infty (t^{-\theta}K(t,a))^q \,\frac{dt}{t}\bigg)^{1/q}$$

is finite (when $q = \infty$ the integral should be replaced by a supremum). See [4, 3, 5, 31]. It follows from the equivalence theorem that $\bar{A}_{\theta,q}$ coincides with the collection of all those $a \in A_0 + A_1$ for which there is a strongly measurable function $u(t)$ with values in $A_0 \cap A_1$ such that

$$a = \int_0^\infty u(t)\,\frac{dt}{t} \qquad \text{(convergence in } A_0 + A_1)$$

and

$$\bigg(\int_0^\infty (t^{-\theta}J(t,u(t)))^q \,\frac{dt}{t}\bigg)^{1/q} < \infty.$$

Moreover,

$$\|a\|_{\bar{A}_{\theta,q;J}} = \inf\bigg\{\bigg(\int_0^\infty (t^{-\theta}J(t,u(t)))^q \,\frac{dt}{t}\bigg)^{1/q} : a = \int_0^\infty u(t)\,\frac{dt}{t}\bigg\}$$

is an equivalent norm to $\|\cdot\|_{\bar{A}_{\theta,q}}$. We refer to [35] for details on the Bochner integral.

If $A_0 \hookrightarrow A_1$, the space $\bar{A}_{1,q;K}$ is defined as the set of all those $a \in A_1$ which have a finite norm

$$\|a\|_{\bar{A}_{1,q;K}} = \bigg(\int_1^\infty (t^{-1}K(t,a))^q \,\frac{dt}{t}\bigg)^{1/q}.$$

Note that the integral is not on $(0,\infty)$ but only on $(1,\infty)$. This change is essential for a meaningful definition. In this ordered case, the space $\bar{A}_{0,q;J}$ consists of all those elements $a \in A_1$ which can be represented as $a = \int_1^\infty v(t)\,dt/t$ (convergence in A_1), where $v(t)$ is a strongly measurable function with values in A_0 and such that

$$\bigg(\int_1^\infty J(t,v(t))^q \,\frac{dt}{t}\bigg)^{1/q} < \infty.$$

We set

$$\|a\|_{\bar{A}_{0,q;J}} = \inf\bigg\{\bigg(\int_1^\infty J(t,v(t))^q \,\frac{dt}{t}\bigg)^{1/q} : a = \int_1^\infty v(t)\,\frac{dt}{t}\bigg\}.$$

The spaces $\bar{A}_{1,q;K}$ and $\bar{A}_{0,q;J}$ correspond to the limit values $\theta = 1, 0$. They are studied in [9, 15] and the papers cited there. Throughout the following sections, we shall extend these constructions to general Banach couples.

Subsequently, for $1 \le q \le \infty$, we let ℓ_q be the usual space of q-summable scalar sequences, and c_0 is the space of null sequences. Given any sequence (λ_m) of positive numbers and any sequence (W_m) of Banach spaces, we write $\ell_q(\lambda_m W_m)$ for the space of all vector-valued sequences $w = (w_m)$ with $w_m \in W_m$ and such that

$$\|w\|_{\ell_q(\lambda_m W_m)} = \Big(\sum_m (\lambda_m \|w_m\|_{W_m})^q\Big)^{1/q} < \infty.$$

If for each m the space W_m is equal to the scalar field $\mathbb{K}$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), we simply write $\ell_q(\lambda_m)$. The space $c_0(\lambda_m W_m)$ is defined similarly.

As usual, if X, Y are non-negative quantities depending on certain parameters, we write $X \lesssim Y$ if there is a constant $c > 0$ independent of the parameters involved in X and Y such that $X \le cY$. If $X \lesssim Y$ and $Y \lesssim X$, we write $X \sim Y$.

3. Limiting K-spaces. We start by introducing the limiting K-spaces that we will consider in the following.

Definition 3.1. Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. The space $\bar{A}_{q;K} = (A_0, A_1)_{q;K}$ is formed by all those $a \in A_0 + A_1$

which have a finite norm

$$\|a\|_{\bar{A}_{q;K}} = \Big(\int_0^1 K(t,a)^q \,\frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty (t^{-1}K(t,a))^q \,\frac{dt}{t}\Big)^{1/q}.$$

Since

$$K(t,a;A_0,A_1) = tK(t^{-1},a;A_1,A_0),$$

it is clear that

$$(A_0,A_1)_{q;K} = (A_1,A_0)_{q;K}. \tag{3.1}$$

Moreover, one can show that $\bar{A}_{q;K}$ is complete. Next we show that $\bar{A}_{q;K}$ is an intermediate space between A_0 and A_1, and that it is larger than any real interpolation space.

LEMMA 3.2. *Let $\bar{A} = (A_0,A_1)$ be a Banach couple, let $0<\theta<1$ and $1\le q,r\le\infty$. Then*

$$A_0\cap A_1 \hookrightarrow (A_0,A_1)_{\theta,r} \hookrightarrow (A_0,A_1)_{q;K} \hookrightarrow A_0+A_1.$$

Moreover, $(A_0,A_1)_{\infty;K} = A_0+A_1$ with equivalent norms.

Proof. It is well-known that

$$A_0\cap A_1 \hookrightarrow (A_0,A_1)_{\theta,r} \hookrightarrow (A_0,A_1)_{\theta,\infty} \quad \text{(see [4] or [31])}.$$

Take any $a\in(A_0,A_1)_{\theta,\infty}$. We have

$$\Big(\int_0^1 K(t,a)^q \,\frac{dt}{t}\Big)^{1/q} \le \Big(\int_0^1 t^{\theta q} \,\frac{dt}{t}\Big)^{1/q} \|a\|_{\bar{A}_{\theta,\infty}} = c_1\|a\|_{\bar{A}_{\theta,\infty}}$$

and

$$\Big(\int_1^\infty (t^{-1}K(t,a))^q \,\frac{dt}{t}\Big)^{1/q} \le \Big(\int_1^\infty t^{(\theta-1)q} \,\frac{dt}{t}\Big)^{1/q} \|a\|_{\bar{A}_{\theta,\infty}} = c_2\|a\|_{\bar{A}_{\theta,\infty}}.$$

Hence, $(A_0,A_1)_{\theta,\infty}\hookrightarrow(A_0,A_1)_{q;K}$.

Assume now that $a\in(A_0,A_1)_{q;K}$. Using that $K(t,a)$ is a non-decreasing function of t, we derive with $c_3 = (\int_1^\infty t^{-q}\,dt/t)^{-1/q}$ that

$$\begin{aligned}\|a\|_{A_0+A_1} &= c_3\Big(\int_1^\infty t^{-q} \,\frac{dt}{t}\Big)^{1/q} K(1,a) \le c_3\Big(\int_1^\infty (t^{-1}K(t,a))^q \,\frac{dt}{t}\Big)^{1/q}\\ &\le c_3\|a\|_{\bar{A}_{q;K}}.\end{aligned}$$

Finally, if $q=\infty$ we have

$$\|a\|_{\bar{A}_{\infty;K}} = \sup_{0<t\le 1} K(t,a) + \sup_{1<t<\infty} t^{-1}K(t,a) = \|a\|_{A_0+A_1},$$

as desired. ∎

Remark 3.3. In the ordered case where $A_0 \hookrightarrow A_1$, if we disregard the term with the integral over $(0,1)$ in Definition 3.1, then we recover the spaces $\bar{A}_{1,q;K}$ introduced in the previous section. Notice that $\bar{A}_{q;K}$ extends $\bar{A}_{1,q;K}$ to arbitrary couples because if $A_0 \hookrightarrow A_1$ we have

$$\left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} \le \left(\int_0^1 t^q \frac{dt}{t}\right)^{1/q} \|a\|_{A_1} \le c\|a\|_{\bar{A}_{1,q;K}}.$$

So, $\bar{A}_{q;K} = \bar{A}_{1,q;K}$ with equivalence of norms.

Next we show the connection between $\bar{A}_{q;K}$ and the limiting spaces introduced in [12]. Let $\tilde{A}_{1,q;K}$ and $\tilde{A}_{0,q;K}$ be the collections of all those $a \in A_0 + A_1$ which have a finite norm

$$\|a\|_{\tilde{A}_{1,q;K}} = \sup_{0<t\le 1} t^{-1}K(t,a) + \left(\int_1^\infty (t^{-1}K(t,a))^q \frac{dt}{t}\right)^{1/q}$$

and

$$\|a\|_{\tilde{A}_{0,q;K}} = \left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} + \sup_{1\le t<\infty} K(t,a),$$

respectively. We refer to [12] for details on $\tilde{A}_{1,q;K}$ and $\tilde{A}_{0,q;K}$.

Proposition 3.4. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. Then*

$$\bar{A}_{q;K} = \tilde{A}_{0,q;K} + \tilde{A}_{1,q;K} \quad \textit{with equivalent norms.}$$

Proof. Let $a = x_0 + x_1$ with $x_0 \in \tilde{A}_{0,q;K}$ and $x_1 \in \tilde{A}_{1,q;K}$. Then

$$\begin{aligned}\left(\int_1^\infty (t^{-1}K(t,a))^q \frac{dt}{t}\right)^{1/q} &\le \left(\int_1^\infty (t^{-1}K(t,x_0))^q \frac{dt}{t}\right)^{1/q} \\ &\quad + \left(\int_1^\infty (t^{-1}K(t,x_1))^q \frac{dt}{t}\right)^{1/q} \\ &\le \left(\int_1^\infty t^{-q} \frac{dt}{t}\right)^{1/q} \|x_0\|_{\tilde{A}_{0,q;K}} + \|x_1\|_{\tilde{A}_{1,q;K}} \\ &\le c_1(\|x_0\|_{\tilde{A}_{0,q;K}} + \|x_1\|_{\tilde{A}_{1,q;K}}).\end{aligned}$$

Similarly,

$$\left(\int_0^1 K(t,a)^q \frac{dt}{t}\right)^{1/q} \le c_2(\|x_0\|_{\tilde{A}_{0,q;K}} + \|x_1\|_{\tilde{A}_{1,q;K}}).$$

This yields the continuous embedding $\tilde{A}_{0,q;K} + \tilde{A}_{1,q;K} \hookrightarrow \bar{A}_{q;K}$.

Conversely, let $a \in \bar{A}_{q;K}$ and take any representation $a = x_0 + x_1$ with $x_j \in A_j$ $(j = 0,1)$ and $\|x_0\|_{A_0} + \|x_1\|_{A_1} \le 2K(1,a) = 2\|a\|_{A_0+A_1}$. We claim that $x_j \in \tilde{A}_{j,q;K}$ for $j = 0, 1$. Indeed,

$$\begin{aligned}
\|x_0\|_{\tilde{A}_{0,q;K}} &= \Big(\int_0^1 K(t,x_0)^q \,\frac{dt}{t}\Big)^{1/q} + \sup_{1\le t<\infty} K(t,x_0) \\
&\le \Big(\int_0^1 K(t,a)^q \,\frac{dt}{t}\Big)^{1/q} + \Big(\int_0^1 K(t,x_1)^q \,\frac{dt}{t}\Big)^{1/q} + \|x_0\|_{A_0} \\
&\le \|a\|_{\bar{A}_{q;K}} + \Big(\int_0^1 t^q \,\frac{dt}{t}\Big)^{1/q} \|x_1\|_{A_1} + \|x_0\|_{A_0} \\
&\le \|a\|_{\bar{A}_{q;K}} + c_1\|a\|_{A_0+A_1} \le c_2\|a\|_{\bar{A}_{q;K}}
\end{aligned}$$

where we have used Lemma 3.2 in the last inequality. For x_1 we obtain

$$\begin{aligned}
\|x_1\|_{\tilde{A}_{1,q;K}} &\le \|x_1\|_{A_1} + \Big(\int_1^\infty (t^{-1}K(t,a))^q \,\frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty (t^{-1}K(t,x_0))^q \,\frac{dt}{t}\Big)^{1/q} \\
&\le \|x_1\|_{A_1} + \|a\|_{\bar{A}_{q;K}} + \Big(\int_1^\infty t^{-q} \,\frac{dt}{t}\Big)^{1/q} \|x_0\|_{A_0} \\
&\le c_3\|a\|_{\bar{A}_{q;K}}.
\end{aligned}$$

Hence, $a \in \tilde{A}_{0,q;K} + \tilde{A}_{1,q;K}$ and $\|a\|_{\tilde{A}_{0,q;K}+\tilde{A}_{1,q;K}} \le (c_2 + c_3)\|a\|_{\bar{A}_{q;K}}$. This completes the proof. ∎

As a direct consequence of Proposition 3.4 and [12, Thm. 4.1] it follows that

$$\bar{A}_{q;K} = (A_0, A_1, A_1, A_0)_{(1/2,1/2),q;K} \tag{3.2}$$

where $(\cdot,\cdot,\cdot,\cdot)_{(\alpha,\beta),q;K}$ stands for the K-method associated to the unit square (see [17, 22]).

Besides the relation described in Remark 3.3, the following lemma shows another interesting connection between $\bar{A}_{q;K}$ and the space $\bar{A}_{1,q;K}$ defined for ordered couples.

LEMMA 3.5. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. Then*

$$(A_0, A_1)_{q;K} = (A_0 \cap A_1, A_0 + A_1)_{1,q;K} \quad \textit{with equivalence of norms.}$$

Proof. Let $\bar{K}(t,a) = K(t,a; A_0 \cap A_1, A_0 + A_1)$ and $K(t,a) = K(t,a; A_0, A_1)$. According to [29, Thm. 3], for $1 < t < \infty$ and $a \in A_0 + A_1$, we have

$$\bar{K}(t,a) \sim tK(t^{-1},a) + K(t,a).$$

Consequently,

$$\begin{aligned}\|a\|_{(A_0\cap A_1,A_0+A_1)_{1,q;K}} &= \Big(\int_1^\infty (t^{-1}\bar{K}(t,a))^q\,\frac{dt}{t}\Big)^{1/q}\\ &\sim \Big(\int_0^1 K(t,a)^q\,\frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty (t^{-1}K(t,a))^q\,\frac{dt}{t}\Big)^{1/q}\\ &= \|a\|_{(A_0,A_1)_{q;K}}. \ \blacksquare\end{aligned}$$

Let $\bar{B} = (B_0, B_1)$ be another Banach couple. By writing $T \in \mathcal{L}(\bar{A}, \bar{B})$ we mean that T is a linear operator from A_0+A_1 into B_0+B_1 whose restrictions $T : A_j \to B_j$ are bounded for $j = 0, 1$. It is not hard to check that for any $1 \le q \le \infty$, the restriction $T : \bar{A}_{q;K} \to \bar{B}_{q;K}$ is also bounded with

$$\|T\|_{\bar{A}_{q;K},\bar{B}_{q;K}} \le \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}.$$

In the ordered case where $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$, it is shown in [9, Thm. 7.9] that if $1 \le q < \infty$ there is a constant $c > 0$ such that for any $T \in \mathcal{L}(\bar{A}, \bar{B})$, we have

$$\|T\|_{\bar{A}_{1,q;K},\bar{B}_{1,q;K}} \le c\|T\|_{A_1,B_1}\Big(1 + \max\Big\{0, \log\frac{\|T\|_{A_0,B_0}}{\|T\|_{A_1,B_1}}\Big\}\Big). \tag{3.3}$$

However, estimate (3.3) does not hold in the general case as the following example shows.

COUNTEREXAMPLE 3.6. Let $1 \le q < \infty$. Consider the couples $(\ell_q(e^{-n}), \ell_q)$ and $(\mathbb{K}, \mathbb{K})$, where sequences are indexed by $\mathbb{N}$. For $k \in \mathbb{N}$, let T_k be the linear operator defined by $T_k\xi = e^{-k}\xi_k$. Clearly, $T_k \in \mathcal{L}((\ell_q(e^{-n}), \ell_q), (\mathbb{K}, \mathbb{K}))$ with $\|T\|_{\ell_q(e^{-n}),\mathbb{K}} = 1$ and $\|T\|_{\ell_q,\mathbb{K}} = e^{-k}$. According to Lemma 3.5 and [9, Lemma 7.2 and Remark 7.3], we have

$$(\ell_q(e^{-n}), \ell_q)_{q;K} = (\ell_q, \ell_q(e^{-n}))_{1,q;K} = \ell_q(n^{1/q}e^{-n}).$$

Moreover, $(\mathbb{K}, \mathbb{K})_{q;K} = \mathbb{K}$ with equivalence of norms. Hence,

$$\|T_k\|_{(\ell_q(e^{-n}),\ell_q)_{q;K},(\mathbb{K},\mathbb{K})_{q;K}} \sim k^{-1/q}.$$

Since there is no $c > 0$ such that $k^{-1/q} \le cke^{-k}$ for all $k \in \mathbb{N}$, it follows that (3.3) does not hold in general outside the ordered case.

4. Limiting J-spaces. Now we turn our attention to J-spaces.

DEFINITION 4.1. Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. The space $\bar{A}_{q;J} = (A_0, A_1)_{q;J}$ is formed by all those $a \in A_0 + A_1$ for which there exists a strongly measurable function $u(t)$ with values in A_0+A_1

such that

$$a = \int_0^\infty u(t)\,\frac{dt}{t} \qquad \text{(convergence in } A_0 + A_1) \tag{4.1}$$

and

$$\Big(\int_0^1 (t^{-1}J(t,u(t)))^q\,\frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty J(t,u(t))^q\,\frac{dt}{t}\Big)^{1/q} < \infty. \tag{4.2}$$

The norm $\|a\|_{\bar{A}_{q;J}}$ in $\bar{A}_{q;J}$ is the infimum of the expression in (4.2) over all representations (4.1) satisfying (4.2).

These spaces were introduced in [13] under the notation $\bar{A}_{\{1,0\},q;J}$. It can be checked that the spaces $\bar{A}_{q;J}$ are complete. Next we show that they are intermediate spaces with respect to the couple $\bar{A}$ and that they are smaller than any space $\bar{A}_{\theta,r}$.

Lemma 4.2. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let $0 < \theta < 1$ and $1 \le q, r \le \infty$. Then $A_0 \cap A_1 \hookrightarrow \bar{A}_{q;J} \hookrightarrow \bar{A}_{\theta,r} \hookrightarrow A_0 + A_1$. Moreover, $\bar{A}_{1;J} = A_0 \cap A_1$ with equivalent norms.*

Proof. Let $a \in A_0 \cap A_1$. Take $u(t) = a\chi_{(1,e)}$. Then $a = \int_0^\infty u(t)\,dt/t$ and we obtain

$$\|a\|_{\bar{A}_{q;J}} \le \Big(\int_1^e J(t,a)^q\,\frac{dt}{t}\Big)^{1/q} \le c\|a\|_{A_0\cap A_1}.$$

Suppose now that $a \in \bar{A}_{q;J}$ and let $a = \int_0^\infty u(t)\,dt/t$ be a representation satisfying (4.2). Then it is also a representation of a in $\bar{A}_{\theta,1}$ because, using Hölder's inequality, we have

$$\int_0^1 t^{-\theta} J(t,u(t))\,\frac{dt}{t} \le \Big(\int_0^1 (t^{-1}J(t,u(t)))^q\,\frac{dt}{t}\Big)^{1/q}\Big(\int_0^1 t^{(1-\theta)q'}\,\frac{dt}{t}\Big)^{1/q'}$$

and

$$\int_1^\infty t^{-\theta} J(t,u(t))\,\frac{dt}{t} \le \Big(\int_1^\infty J(t,u(t))^q\,\frac{dt}{t}\Big)^{1/q}\Big(\int_1^\infty t^{-\theta q'}\,\frac{dt}{t}\Big)^{1/q'}.$$

Therefore, $\bar{A}_{q;J} \hookrightarrow \bar{A}_{\theta,1}$. Since $\bar{A}_{\theta,1} \hookrightarrow \bar{A}_{\theta,r} \hookrightarrow A_0 + A_1$ (see [4] or [31]), it follows that $A_0 \cap A_1 \hookrightarrow \bar{A}_{q;J} \hookrightarrow \bar{A}_{\theta,r} \hookrightarrow A_0 + A_1$.

Finally, let $q = 1$ and $a \in \bar{A}_{1;J}$. Take any representation $a = \int_0^\infty u(t)\,dt/t$ in $\bar{A}_{1;J}$. Then the integral is absolutely convergent in $A_0 \cap A_1$ because, since $J(t,v)$ is a non-decreasing function of t and $t^{-1}J(t,v)$ is non-increasing, we get

$$\int_0^\infty \|u(t)\|_{A_0\cap A_1}\,\frac{dt}{t} = \int_0^1 J(1,u(t))\,\frac{dt}{t} + \int_1^\infty J(1,u(t))\,\frac{dt}{t}$$
$$\le \int_0^1 t^{-1}J(t,u(t))\,\frac{dt}{t} + \int_1^\infty J(t,u(t))\,\frac{dt}{t}.$$

Consequently, $a \in A_0 \cap A_1$ and $\|a\|_{A_0\cap A_1} \le \|a\|_{\bar{A}_{1,J}}$. ∎

It is shown in [13, Theorem 4.1] that

$$\bar{A}_{q;J} = (A_0, A_1, A_1, A_0)_{(1/2,1/2),q;J}, \tag{4.3}$$

where $(\cdot,\cdot,\cdot,\cdot)_{(\alpha,\beta),q;J}$ is the J-method defined by the unit square (see [17]). In particular if $A_0 \hookrightarrow A_1$ and $\bar{A}_{0,q;J}$ is the space introduced in Section 2, we have $\bar{A}_{q;J} = \bar{A}_{0,q;J}$.

For $1 < q \le \infty$, the spaces $\bar{A}_{q;J}$ can also be described using the K-functional. In fact, according to [13, Theorem 3.10], we have

$$\bar{A}_{q;J} = \bar{A}_{\{\mathfrak{f},\mathfrak{g}\},q;K} \quad \text{with equivalence of norms,} \tag{4.4}$$

where $\bar{A}_{\{\mathfrak{f},\mathfrak{g}\},q;K}$ is formed by all those $a \in A_0 + A_1$ such that

$$\|a\|_{\bar{A}_{\{\mathfrak{f},\mathfrak{g}\},q;K}} = \left(\int_0^1 \left(\frac{K(t,a)}{t(1-\log t)}\right)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left(\frac{K(t,a)}{1+\log t}\right)^q \frac{dt}{t}\right)^{1/q} < \infty.$$

Equality (4.4) is not true if $q = 1$. Indeed, let $\{0\} \ne A_0 \hookrightarrow A_1$, with the embedding of norm ≤ 1. Thus $K(t,a) = t\|a\|_{A_1}$ if $0 < t \le 1$. By Lemma 4.2, $\bar{A}_{1;J} = A_0$. However, $\bar{A}_{\{\mathfrak{f},\mathfrak{g}\},1;K} = \{0\}$ because for any $a \ne 0$ we obtain

$$\int_0^1 \frac{K(t,a)}{t(1-\log t)}\,\frac{dt}{t} = \|a\|_{A_1}\int_0^1 \frac{1}{1-\log t}\,\frac{dt}{t} = \infty.$$

In fact, the $(1;J)$-method cannot be described using the K-functional. Indeed, recall that for any Banach couple (A_0, A_1), one has $K(t,a;A_0,A_1) = K(t,a;A_0^\sim, A_1^\sim)$, where $A_j^\sim$ is the Gagliardo completion of A_j in $A_0 + A_1$ (see [3, Theorem 5.1.5]). Hence, if the $(1;J)$-method could be described using the K-functional, we would have, for any Banach couple,

$$A_0 \cap A_1 = (A_0, A_1)_{1;J} = (A_0^\sim, A_1^\sim)_{1;J} = A_0^\sim \cap A_1^\sim.$$

However, if we take $A_0 = c_0$ and $A_1 = \ell_\infty(2^{-n})$, then $A_0^\sim = \ell_\infty$ and clearly $A_0 \cap A_1 = c_0 \ne \ell_\infty = A_0^\sim \cap A_1^\sim$.

The following result is based on the K-description of $\bar{A}_{q;J}$.

Lemma 4.3. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. Then $(A_0, A_1)_{q;J} = (A_0 \cap A_1, A_0 + A_1)_{0,q;J}$ with equivalence of norms. In particular, $(A_0, A_1)_{q;J} = (A_1, A_0)_{q;J}$.*

Proof. If $q=1$, we have $(A_0,A_1)_{1;J} = A_0\cap A_1 = (A_0\cap A_1, A_0+A_1)_{0,1;J}$. If $q\neq 1$, set $\bar{K}(t,a) = K(t,a;A_0\cap A_1, A_0+A_1)$. By [9, Theorem 4.2], we obtain

$$\|a\|_{(A_0\cap A_1,A_0+A_1)_{0,q;J}} \sim \bigg(\int_1^\infty \bigg(\frac{\bar{K}(t,a)}{1+\log t}\bigg)^q \frac{dt}{t}\bigg)^{1/q}.$$

Using [29, Theorem 3], a change of variable and (4.4), we derive that

$$\bigg(\int_1^\infty \bigg(\frac{\bar{K}(t,a)}{1+\log t}\bigg)^q \frac{dt}{t}\bigg)^{1/q} \sim \bigg(\int_0^1 \bigg(\frac{K(s,a)}{s(1-\log s)}\bigg)^q \frac{ds}{s}\bigg)^{1/q} + \bigg(\int_1^\infty \bigg(\frac{K(t,a)}{1+\log t}\bigg)^q \frac{dt}{t}\bigg)^{1/q} \sim \|a\|_{(A_0,A_1)_{q;J}}.$$

Consequently, $(A_0,A_1)_{q;J} = (A_0\cap A_1, A_0+A_1)_{0,q;J}$. Finally, the last equality implies that $(A_0,A_1)_{q;J} = (A_1,A_0)_{q;J}$. ∎

It is easy to check that if $T\in\mathcal{L}(\bar{A},\bar{B})$, then $T:\bar{A}_{q;J}\to\bar{B}_{q;J}$ is bounded with $\|T\|_{\bar{A}_{q;J},\bar{B}_{q;J}} \le \max\{\|T\|_{A_0,B_0}, \|T\|_{A_1,B_1}\}$.

5. Compact operators. Interpolation of compact operators is a classical question that has attracted the attention of many authors (see [7] and the references given there). As concerns the real method, the final result was obtained in 1992 by Cwikel [18] and Cobos, Kühn and Schonbek [16], who proved that if $T\in\mathcal{L}(\bar{A},\bar{B})$ and any of the restrictions $T:A_j\to B_j$ $(j=0,1)$ is compact, then the interpolated operator $T:(A_0,A_1)_{\theta,q}\to(B_0,B_1)_{\theta,q}$ is also compact.

For limiting methods in the ordered case where $A_0\hookrightarrow A_1$ and $B_0\hookrightarrow B_1$, it was proved by Cobos, Fernández-Cabrera, Kühn and Ullrich [9] that compactness of $T:A_1\to B_1$ implies that $T:\bar{A}_{1,q;K}\to\bar{B}_{1,q;K}$ is also compact, whereas compactness of $T:A_0\to B_0$ is not enough (see [9, Counterexample 7.11 and Theorem 7.14].

In the general case, the bad behaviour of the $(q;K)$-method suggests poorer properties with respect to interpolation of compact operators. Next, we show with an example based on [9, Counterexample 7.11] that in contrast to the ordered case, if $T\in\mathcal{L}(\bar{A},\bar{B})$ and $T:A_1\to B_1$ is compact, then $T:\bar{A}_{q;K}\to\bar{B}_{q;K}$ may fail to be compact.

Counterexample 5.1. Let $1\le q<\infty$ and consider the couples $\bar{A} = (\ell_q(3^{-n}),\ell_q)$, $\bar{B} = (\ell_q(2^{-n}),\ell_q)$. Let D be the diagonal operator defined by $D(\xi_n) = ((2/3)^n\xi_n)$. Then $D:\ell_q(3^{-n})\to\ell_q(2^{-n})$ is bounded and $D:\ell_q\to\ell_q$ is compact. However, according to (3.1) and [9, Lemma 7.2 and Remark 7.3], we have

$$(\ell_q(3^{-n}),\ell_q)_{q;K} = \ell_q(n^{1/q}3^{-n}) \quad\text{and}\quad (\ell_q(2^{-n}),\ell_q)_{q;K} = \ell_q(n^{1/q}2^{-n}),$$

and it is not hard to check that $D : \ell_q(n^{1/q}3^{-n}) \to \ell_q(n^{1/q}2^{-n})$ fails to be compact.

Nevertheless, if the first couple reduces to a single Banach space, then the behaviour of the $(q;K)$-method improves.

PROPOSITION 5.2. *Let A be a Banach space, let $\bar{B} = (B_0, B_1)$ be a Banach couple and let $1 \le q \le \infty$. If T is a linear operator such that $T : A \to B_j$ is bounded for $j = 0, 1$ and one of these restrictions is compact, then $T : A \to \bar{B}_{q;K}$ is also compact.*

Proof. Clearly, $T : A \to B_0 + B_1$ compactly and $T : A \to B_0 \cap B_1$ boundedly. If $1 \le q < \infty$, using Lemma 3.5 and [9, Theorem 7.14], we derive that $T : A \to (B_0 \cap B_1, B_0 + B_1)_{1,q;K} = (B_0, B_1)_{q;K}$ is compact. If $q = \infty$, the result follows from the last part of Lemma 3.2. ∎

In order to establish the compactness result in the general case, given any Banach couple (A_0, A_1), we write (A_0^o, A_1^o) for the Banach couple formed by the closures of $A_0 \cap A_1$ in A_j for $j = 0, 1$.

THEOREM 5.3. *Let $\bar{A} = (A_0, A_1)$, $\bar{B} = (B_0, B_1)$ be Banach couples, let $1 \le q \le \infty$ and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. If $T : A_j \to B_j$ is compact for $j = 0, 1$, then $T : (A_0^o, A_1^o)_{q;K} \to (B_0, B_1)_{q;K}$ is compact as well.*

Proof. The result follows from (3.2) and [23, Corollary 4.4]. ∎

REMARK 5.4. We will show at the end of Section 6 that if $q < \infty$ then $(A_0^o, A_1^o)_{q;K} = (A_0, A_1)_{q;K}$.

Next we turn our attention to the $(q;J)$-method. In the ordered case where $A_0 \hookrightarrow A_1$ and $B_0 \hookrightarrow B_1$, if $T \in \mathcal{L}(\bar{A}, \bar{B})$ and $T : A_0 \to B_0$ compactly, then $T : \bar{A}_{0,q;J} \to \bar{B}_{0,q;J}$ is compact (see [9, Theorem 6.4]). However, in the general case, compactness of $T : A_0 \to B_0$ does not imply that $T : \bar{A}_{q;J} \to \bar{B}_{q;J}$ is compact. An example can be given by reversing the order of the couples in [9, Counterexample 6.2] and using Lemma 4.3.

The following results give sufficient conditions for interpolation of compact operators for the $(q;J)$-method.

PROPOSITION 5.5. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple, let B be a Banach space and let $1 \le q \le \infty$. If T is a linear operator such that $T : A_j \to B$ is bounded for $j = 0, 1$ and any of these two restrictions is compact, then $T : \bar{A}_{q;J} \to B$ is also compact.*

Proof. It is clear that $T : A_0 \cap A_1 \to B$ is compact. If $q = 1$, the result follows using that $\bar{A}_{1;J} = A_0 \cap A_1$. Assume now that $1 < q \le \infty$. We infer that $T : A_0 + A_1 \to B$ is bounded because

$$\begin{aligned}\|T(a_0 + a_1)\|_B &\le \|Ta_0\|_B + \|Ta_1\|_B \\ &\le \max\{\|T\|_{A_0}, \|T\|_{A_1}\}(\|a_0\|_{A_0} + \|a_1\|_{A_1}).\end{aligned}$$

Hence, applying [9, Theorem 6.4] to the couples $(A_0 \cap A_1, A_0 + A_1), (B, B)$ and using Lemma 4.3, we conclude that

$$T : \bar{A}_{q;J} = (A_0 \cap A_1, A_0 + A_1)_{0,q;J} \to B$$

is also compact. ■

We finish this section with a consequence of (4.3) and [17, Theorem 6.1].

THEOREM 5.6. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples, let $T \in \mathcal{L}(\bar{A}, \bar{B})$ and let $1 \le q \le \infty$. If $T : A_j \to B_j$ is compact for $j = 0, 1$, then $T : \bar{A}_{q;J} \to \bar{B}_{q;J}$ is also compact.*

6. Description of K-spaces using the J-functional. In (4.4) we have pointed out that limiting J-spaces can be described by using the K-functional provided that $1 < q \le \infty$. In this section we study the description of limiting K-spaces using the J-functional.

DEFINITION 6.1. Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $1 \le q \le \infty$. Write $\rho(t) = 1 + |\log t|$ and $\mu(t) = t^{-1}(1 + |\log t|)$. The space $\bar{A}_{\{\rho,\mu\},q;J}$ is formed by all those elements $a \in A_0 + A_1$ for which there is a representation

$$a = \int_0^\infty u(t) \frac{dt}{t} \qquad \text{(convergence in } A_0 + A_1) \tag{6.1}$$

with $u(t)$ being a strongly measurable function with values in $A_0 \cap A_1$ and such that

$$\Big(\int_0^1 (\rho(t) J(t, u(t)))^q \frac{dt}{t} \Big)^{1/q} + \Big(\int_1^\infty (\mu(t) J(t, u(t)))^q \frac{dt}{t} \Big)^{1/q} < \infty. \tag{6.2}$$

The norm in $\bar{A}_{\{\rho,\mu\},q;J}$ is given by taking the infimum of the values in (6.2) over all possible representations (6.1) of a satisfying (6.2).

The following result shows the relationship between these spaces and limiting K-spaces.

THEOREM 6.2. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and $1 \le q < \infty$. Then $(A_0, A_1)_{q;K} = (A_0, A_1)_{\{\rho,\mu\},q;J}$ with equivalence of norms.*

Proof. Choose $a \in (A_0, A_1)_{\{\rho,\mu\},q;J}$ and a representation $a = \int_0^\infty u(s)\, ds/s$ such that

$$\Big(\int_0^1 (\rho(t) J(t, u(t)))^q \frac{dt}{t} \Big)^{1/q} + \Big(\int_1^\infty (\mu(t) J(t, u(t)))^q \frac{dt}{t} \Big)^{1/q} \le 2\|a\|_{\bar{A}_{\{\rho,\mu\},q;J}}.$$

For any $0 < t < \infty$, we have

$$K(t,a) \le \int_0^\infty K(t,u(s))\,\frac{ds}{s} \le \int_0^\infty \min(1,t/s)J(s,u(s))\,\frac{ds}{s}$$
$$= \int_0^t J(s,u(s))\,\frac{ds}{s} + \int_t^\infty \frac{t}{s}J(s,u(s))\,\frac{ds}{s}.$$

Hence,

$$\|a\|_{\bar{A}_{q,K}} = \left(\int_0^1 K(t,a)^q\,\frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty (t^{-1}K(t,a))^q\,\frac{dt}{t}\right)^{1/q}$$
$$\le \left(\int_0^1\left(\int_0^t J(s,u(s))\,\frac{ds}{s}\right)^q\frac{dt}{t}\right)^{1/q} + \left(\int_0^1\left(\int_t^\infty \frac{t}{s}J(s,u(s))\,\frac{ds}{s}\right)^q\frac{dt}{t}\right)^{1/q}$$
$$+ \left(\int_1^\infty\left(\frac{1}{t}\int_0^t J(s,u(s))\,\frac{ds}{s}\right)^q\frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty\left(\int_t^\infty \frac{1}{s}J(s,u(s))\,\frac{ds}{s}\right)^q\frac{dt}{t}\right)^{1/q}$$
$$= I_1 + I_2 + I_3 + I_4.$$

Let us estimate each of these terms separately. Let $h \in L_{q'}((0,1),dt/t)$ with $\|h\|_{L_{q'}} = 1$ and such that

$$I_1 = \left(\int_0^1\left(\int_0^t J(s,u(s))\,\frac{ds}{s}\right)^q\frac{dt}{t}\right)^{1/q} = \int_0^1 h(t)\int_0^t J(s,u(s))\,\frac{ds}{s}\,\frac{dt}{t}.$$

Using Fubini's theorem, Hölder's inequality, changing variables and applying Hardy's inequality (see [28]), we obtain

$$I_1 = \int_0^1\int_s^1 h(t)J(s,u(s))\,\frac{dt}{t}\,\frac{ds}{s} = \int_0^1 J(s,u(s))\rho(s)\,\frac{1}{\rho(s)}\int_s^1 h(t)\,\frac{dt}{t}\,\frac{ds}{s}$$
$$\le \left(\int_0^1(\rho(s)J(s,u(s)))^q\,\frac{ds}{s}\right)^{1/q}\left(\int_0^1\left(\frac{1}{1-\log s}\int_s^1 h(t)\,\frac{dt}{t}\right)^{q'}\frac{ds}{s}\right)^{1/q'}$$
$$\lesssim \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}}\left(\int_0^\infty\left(\frac{1}{1+x}\int_{e^{-x}}^1 h(t)\,\frac{dt}{t}\right)^{q'}dx\right)^{1/q'}$$
$$\le \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}}\left(\int_0^\infty\left(\frac{1}{x}\int_0^x h(e^{-s})\,ds\right)^{q'}dx\right)^{1/q'}$$
$$\lesssim \left(\int_0^\infty h(e^{-s})^{q'}\,ds\right)^{1/q'}\|a\|_{\{\rho,\mu\},q;J} = \|a\|_{\{\rho,\mu\},q;J}.$$

As for I_2, by Hölder's inequality and the fact that $s(1-\log s)$ is increasing in $(0,1)$, we get

$$
\begin{aligned}
I_2 \le{}& \Big(\int_0^1 \Big(t\int_t^1 \frac{1-\log s}{s(1-\log s)} J(s,u(s))\,\frac{ds}{s}\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&+ \Big(\int_0^1 \Big(t\int_1^\infty \frac{1}{s} J(s,u(s))\,\frac{ds}{s}\Big)^q \frac{dt}{t}\Big)^{1/q}\\
\le{}& \Big(\int_0^1 \Big(\frac{1}{1-\log t}\int_t^1 (1-\log s) J(s,u(s))\,\frac{ds}{s}\Big)^q \frac{dt}{t}\Big)^{1/q}\\
&+ \Big(\int_0^1 t^q \Big(\int_1^\infty \Big(\frac{1+\log s}{s} J(s,u(s))\Big)^q \frac{ds}{s}\Big)\\
&\qquad\qquad \cdot \Big(\int_1^\infty (1+\log s)^{-q'}\,\frac{ds}{s}\Big)^{q/q'} \frac{dt}{t}\Big)^{1/q}.
\end{aligned}
$$

The last integral is finite because $q' = (1-1/q)^{-1}$ is greater than 1. Changing variables and using Hardy's inequality, we derive

$$
\begin{aligned}
I_2 \lesssim{}& \Big(\int_0^\infty \Big(\frac{1}{1+v}\int_0^v (1+x) J(e^{-x},u(e^{-x}))\,dx\Big)^q dv\Big)^{1/q}\\
&+ \Big(\int_0^1 t^q \int_1^\infty (\mu(s) J(s,u(s)))^q \,\frac{ds}{s}\,\frac{dt}{t}\Big)^{1/q}\\
\lesssim{}& \Big(\int_0^\infty ((1+x) J(e^{-x},u(e^{-x})))^q\,dx\Big)^{1/q} + \|a\|_{\{\rho,\mu\},q;J} \lesssim \|a\|_{\{\rho,\mu\},q;J}.
\end{aligned}
$$

As for I_3, using Hölder's inequality and Hardy's inequality for the function $s^{-1}J(s,u(s))\chi_{(1,\infty)}(s)$, we have

$$
\begin{aligned}
I_3 \le{}& \Big(\int_1^\infty \Big(t^{-1}\int_0^1 J(s,u(s))\,\frac{ds}{s}\Big)^q \frac{dt}{t}\Big)^{1/q} + \Big(\int_1^\infty \Big(t^{-1}\int_1^t J(s,u(s))\,\frac{ds}{s}\Big)^q \frac{dt}{t}\Big)^{1/q}\\
\le{}& \Big(\int_1^\infty t^{-q}\Big(\int_0^1 (\rho(s) J(s,u(s)))^q\,\frac{ds}{s}\Big)\Big(\int_0^1 \Big(\frac{1}{1-\log s}\Big)^{q'} \frac{ds}{s}\Big)^{q/q'} \frac{dt}{t}\Big)^{1/q}\\
&+ \Big(\int_1^\infty (s^{-1}J(s,u(s)))^q\,\frac{ds}{s}\Big)^{1/q} \lesssim \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}}\\
&+ \Big(\int_1^\infty \Big(\frac{1+\log s}{s} J(s,u(s))\Big)^q (1+\log s)^{-q}\,\frac{ds}{s}\Big)^{1/q}\\
\le{}& \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} + \sup_{1\le s<\infty} (1+\log s)^{-q}\|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} \lesssim \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}}.
\end{aligned}
$$

In order to estimate the last term I_4, we proceed as in the case of I_1. Choose $h \in L_{q'}((1,\infty), dt/t)$ with $\|h\|_{L_{q'}} = 1$ and such that

$$I_4 = \int_1^\infty h(t) \int_t^\infty s^{-1} J(s, u(s)) \,\frac{ds}{s}\,\frac{dt}{t}.$$

We obtain

$$\begin{aligned} I_4 &= \int_1^\infty s^{-1} J(s,u(s)) \int_1^s h(t)\,\frac{dt}{t}\,\frac{ds}{s} \\ &\le \Big(\int_1^\infty (\mu(s) J(s,u(s)))^q \,\frac{ds}{s}\Big)^{1/q} \Big(\int_1^\infty \Big(\frac{1}{1+\log s}\int_1^s h(t)\,\frac{dt}{t}\Big)^{q'} \frac{ds}{s}\Big)^{1/q'} \\ &\le \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} \Big(\int_0^\infty \Big(\frac{1}{1+x}\int_0^x h(e^y)\,dy\Big)^{q'} dx\Big)^{1/q'} \\ &\lesssim \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}} \Big(\int_0^\infty h(e^x)^{q'}\,dx\Big)^{1/q'} = \|a\|_{\bar{A}_{\{\rho,\mu\},q;J}}. \end{aligned}$$

Consequently, $(A_0, A_1)_{\{\rho,\mu\},q;J} \hookrightarrow (A_0, A_1)_{q,K}$.

Conversely, take any $a \in (A_0, A_1)_{q,K}$. Then

$$\int_0^1 K(t,a)^q \,\frac{dt}{t} + \int_1^\infty (t^{-1}K(t,a))^q \,\frac{dt}{t} < \infty. \tag{6.3}$$

Since $K(t,a)$ (respectively, $t^{-1}K(t,a)$) is non-decreasing (respectively, non-increasing) in t, it follows from (6.3) that

$$K(t,a) \to 0 \text{ as } t \to 0 \quad\text{and}\quad \frac{K(t,a)}{t} \to 0 \text{ as } t \to \infty. \tag{6.4}$$

For $\nu \in \mathbb{Z}$, put

$$\eta_\nu = \begin{cases} 2^{-2^{-\nu-1}} & \text{if } \nu < 0, \\ 1 & \text{if } \nu = 0, \\ 2^{2^{\nu-1}} & \text{if } \nu > 0. \end{cases}$$

We can find decompositions $a = a_{0,\nu} + a_{1,\nu}$ with $a_{j,\nu} \in A_j$, $j = 0,1$, such that

$$\begin{aligned} \|a_{0,\nu}\|_{A_0} + \eta_{\nu+1}\|a_{1,\nu}\|_{A_1} &\le 2K(\eta_{\nu+1}, a) \quad \text{if } \nu \le 1, \\ \eta_{\nu-1}^{-1}\|a_{0,\nu}\|_{A_0} + \|a_{1,\nu}\|_{A_1} &\le 2\widetilde{K}(\eta_{\nu-1}^{-1}, a) \quad \text{if } \nu > 1, \end{aligned}$$

where $\widetilde{K}(t,a) = K(t,a;A_1,A_0)$.

Let $u_\nu = a_{0,\nu} - a_{0,\nu-1} = a_{1,\nu-1} - a_{1,\nu} \in A_0 \cap A_1$, $\nu \in \mathbb{Z}$. Given any $N, M \in \mathbb{N}$, we have

$$\Big\| a - \sum_{\nu=-N}^{M} u_\nu \Big\|_{A_0+A_1} = \|a - a_{0,M} + a_{0,-N-1}\|_{A_0+A_1}$$
$$\leq \|a_{0,-N-1}\|_{A_0} + \|a_{1,M}\|_{A_1}.$$

By (6.4), the last two terms go to 0 as $N, M \to \infty$. Hence, $a = \sum_{\nu\in\mathbb{Z}} u_\nu$ in $A_0 + A_1$.

Let $L_\nu = (\eta_{\nu-1}, \eta_\nu]$, $\nu \in \mathbb{Z}$. We have

$$\int_{L_\nu} \frac{dt}{t} = \begin{cases} 2^{-\nu-1}\log 2 & \text{if } \nu < 0, \\ \log 2 & \text{if } \nu = 0, 1, \\ 2^{\nu-2}\log 2 & \text{if } \nu > 1. \end{cases}$$

Let

$$v(t) = \begin{cases} \dfrac{u_\nu}{2^{-\nu-1}\log 2} & \text{if } t \in L_\nu \text{ and } \nu < 0, \\ \dfrac{u_\nu}{\log 2} & \text{if } t \in L_\nu \text{ and } \nu = 0, 1, \\ \dfrac{u_\nu}{2^{\nu-2}\log 2} & \text{if } t \in L_\nu \text{ and } \nu > 1. \end{cases}$$

Then $a = \int_0^\infty v(t)\,dt/t$ (convergence in $A_0 + A_1$). Next we show that this is a suitable representation of a in the J-space.

If $\nu < 0$ and $t \in L_\nu$, we have

$$\begin{aligned} J(t, v(t)) &= \frac{J(t, u_\nu)}{2^{-\nu-1}\log 2} \lesssim 2^{\nu+1} J(\eta_\nu, u_\nu) \\ &\leq 2^{\nu+1}(\|a_{0,\nu}\|_{A_0} + \|a_{0,\nu-1}\|_{A_0} + \eta_\nu\|a_{1,\nu-1}\|_{A_1} + \eta_\nu\|a_{1,\nu}\|_{A_1}) \\ &\lesssim 2^{\nu+1} K(\eta_{\nu+1}, a). \end{aligned}$$

Therefore,

$$\begin{aligned} \int_{L_\nu} [(1-\log t)J(t, v(t))]^q \frac{dt}{t} &\lesssim (2^{\nu+1}K(\eta_{\nu+1}, a))^q \int_{2^{-2^{-\nu}}}^{2^{-2^{-\nu-1}}} (1-\log t)^q \frac{dt}{t} \\ &\leq (2^{\nu+1}K(\eta_{\nu+1}, a))^q (1 + 2^{-\nu}\log 2)^q 2^{-\nu-1}\log 2 \\ &\lesssim 2^{-\nu-1} K(\eta_{\nu+1}, a)^q. \end{aligned}$$

Now we distinguish three subcases. If $\nu < -2$, we derive

$$\int_{L_\nu} [(1-\log t)J(t, v(t))]^q \frac{dt}{t} \lesssim K(\eta_{\nu+1}, a)^q \int_{L_{\nu+2}} \frac{dt}{t} \leq \int_{L_{\nu+2}} K(t, a)^q \frac{dt}{t}.$$

If $\nu = -2$, we get

$$\int_{L_{-2}} [(1 - \log t) J(t, v(t))]^q \frac{dt}{t} \lesssim K(\eta_{-1}, a)^q \int_{L_0} \frac{dt}{t} \leq \int_{L_0} K(t, a)^q \frac{dt}{t}.$$

In the remaining case $\nu = -1$, we obtain

$$\int_{L_{-1}} [(1 - \log t) J(t, v(t))]^q \frac{dt}{t} \lesssim K(\eta_0, a)^q \lesssim \int_{L_1} \left(\frac{K(t, a)}{t} \right)^q \frac{dt}{t}.$$

Suppose now $\nu > 1$. A change of variables yields

$$\begin{aligned} \int_{L_\nu} \left(\frac{1 + \log t}{t} J(t, v(t)) \right)^q \frac{dt}{t} &= \int_{\eta_\nu^{-1}}^{\eta_{\nu-1}^{-1}} ((1 - \log s) s J(1/s, v(1/s)))^q \frac{ds}{s} \\ &= \int_{\eta_\nu^{-1}}^{\eta_{\nu-1}^{-1}} ((1 - \log s) \widetilde{J}(s, v(1/s)))^q \frac{ds}{s}, \end{aligned}$$

where $\widetilde{J}(s, w) = J(s, w; A_1, A_0)$. If $s \in (\eta_\nu^{-1}, \eta_{\nu-1}^{-1}]$, then $1/s \in L_\nu$, and we get

$$\begin{aligned} \widetilde{J}(s, v(1/s)) = \frac{\widetilde{J}(s, u_\nu)}{2^{\nu-2} \log 2} &\lesssim \frac{\widetilde{J}(\eta_{\nu-1}^{-1}, u_\nu)}{2^{\nu-2}} \\ &\leq 2^{2-\nu} (\eta_{\nu-1}^{-1} (\|a_{0,\nu}\|_{A_0} + \|a_{0,\nu-1}\|_{A_0}) + \|a_{1,\nu-1}\|_{A_1} + \|a_{1,\nu}\|_{A_1}) \\ &\lesssim 2^{2-\nu} \widetilde{K}(\eta_{\nu-2}^{-1}, a). \end{aligned}$$

This implies that

$$\begin{aligned} \int_{L_\nu} \left(\frac{1 + \log t}{t} J(t, v(t)) \right)^q \frac{dt}{t} &\lesssim (2^{2-\nu} \widetilde{K}(\eta_{\nu-2}^{-1}, a))^q (1 + \log \eta_\nu)^q \int_{L_\nu} \frac{dt}{t} \\ &\lesssim \left(\frac{K(\eta_{\nu-2}, a)}{\eta_{\nu-2}} \right)^q \int_{L_\nu} \frac{dt}{t}. \end{aligned}$$

Now, if $\nu > 2$, we derive

$$\begin{aligned} \int_{L_\nu} \left(\frac{1 + \log t}{t} J(t, v(t)) \right)^q \frac{dt}{t} &\lesssim \left(\frac{K(\eta_{\nu-2}, a)}{\eta_{\nu-2}} \right)^q 2^{\nu-2} \\ &\lesssim \left(\frac{K(\eta_{\nu-2}, a)}{\eta_{\nu-2}} \right)^q \int_{L_{\nu-2}} \frac{dt}{t} \\ &\leq \int_{L_{\nu-2}} \left(\frac{K(t, a)}{t} \right)^q \frac{dt}{t}. \end{aligned}$$

If $\nu = 2$, we have

$$\int_{L_2} \left(\frac{1+\log t}{t} J(t, v(t))\right)^q \frac{dt}{t} \lesssim \left(\frac{K(\eta_0, a)}{\eta_0}\right)^q \int_{L_2} \frac{dt}{t}$$
$$\lesssim K(1/2, a)^q \int_{L_0} \frac{dt}{t} \lesssim \int_{L_0} K(t, a)^q \frac{dt}{t}.$$

Finally, we focus on the remaining two cases: $\nu = 0, 1$. If $\nu = 0$ and $t \in L_0$, then

$$J(t, v(t)) = \frac{J(t, u_0)}{\log 2} \lesssim \|a_{0,0}\|_{A_0} + \|a_{0,-1}\|_{A_0} + \|a_{1,-1}\|_{A_1} + \|a_{1,0}\|_{A_1} \lesssim K(2, a).$$

Hence,

$$\int_{L_0} ((1-\log t) J(t, v(t)))^q \frac{dt}{t} \lesssim K(2, a)^q \lesssim \int_{L_1} (t^{-1} K(t, a))^q \frac{dt}{t}.$$

If $\nu = 1$ and $t \in L_1$, then $J(t, v(t)) \lesssim K(4, a)$, and so

$$\int_{L_1} \left(\frac{1+\log t}{t} J(t, v(t))\right)^q \frac{dt}{t} \lesssim \left(\frac{K(4, a)}{4}\right)^q \int_{L_2} \frac{dt}{t} \lesssim \int_{L_2} \left(\frac{K(t, a)}{t}\right)^q \frac{dt}{t}.$$

With all these estimates, we have

$$\left(\int_0^1 ((1-\log t) J(t, v(t)))^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left(\frac{1+\log t}{t} J(t, v(t))\right)^q \frac{dt}{t}\right)^{1/q}$$
$$= \left(\sum_{\nu=-\infty}^{-1} \int_{L_\nu} ((1-\log t) J(t, v(t)))^q \frac{dt}{t} + \int_{L_0} ((1-\log t) J(t, v(t)))^q \frac{dt}{t}\right)^{1/q}$$
$$+ \left(\sum_{\nu=2}^{\infty} \int_{L_\nu} \left(\frac{1+\log t}{t} J(t, v(t))\right)^q \frac{dt}{t} + \int_{L_1} \left(\frac{1+\log t}{t} J(t, v(t))\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\lesssim \left(\sum_{\nu=-\infty}^{-2} \int_{L_{\nu+2}} K(t, a)^q \frac{dt}{t} + \int_{L_1} \left(\frac{K(t, a)}{t}\right)^q \frac{dt}{t}\right)^{1/q}$$
$$+ \left(\int_{L_0} K(t, a)^q \frac{dt}{t} + \sum_{\nu=3}^{\infty} \int_{L_{\nu-2}} \left(\frac{K(t, a)}{t}\right)^q \frac{dt}{t}\right)^{1/q}$$
$$\lesssim \left(\int_0^1 K(t, a)^q \frac{dt}{t}\right)^{1/q} + \left(\int_1^\infty \left(\frac{K(t, a)}{t}\right)^q \frac{dt}{t}\right)^{1/q}.$$

This shows that $(A_0, A_1)_{q,K} \hookrightarrow (A_0, A_1)_{\{\rho,\mu\},q;J}$ and completes the proof. ∎

REMARK 6.3. In the proof of Theorem 6.2, the assumption $q \neq \infty$ has allowed us to use Hardy's inequality, as well as guaranteed the convergence of certain integrals. So, it is essential for the arguments. In fact, the equality $\bar{A}_{\infty;K} = \bar{A}_{\{\rho,\mu\},\infty;J}$ does not hold in general: Assume that $A_0 \hookrightarrow A_1$ with the closure of A_0 in A_1, A_0^o, being different from A_1 (take for instance $A_0 = \ell_1$ and $A_1 = \ell_\infty$). By Lemma 3.2, $\bar{A}_{\infty;K} = A_0 + A_1 = A_1$. However, $\bar{A}_{\{\rho,\mu\},\infty;J} \subset A_0^o \neq A_1$. Indeed, take any $a \in \bar{A}_{\{\rho,\mu\},\infty;J}$ and let $a = \int_0^\infty u(t)\,dt/t$ be a J-representation with

$$\max\Big\{\sup_{0<t<1}(1-\log t)J(t,u(t)),\ \sup_{1\le t<\infty}\frac{1+\log t}{t}J(t,u(t))\Big\} \le 2\|a\|_{\bar{A}_{\{\rho,\mu\},\infty;J}}.$$

Then $\lim_{N\to\infty}\|a - \int_{1/N}^N u(t)\,dt/t\|_{A_1} = 0$ and $\int_{1/N}^N u(t)\,dt/t$ belongs to A_0 because

$$\int_{1/N}^N \|u(t)\|_{A_0}\frac{dt}{t} \le \int_{1/N}^1 \frac{J(t,u(t))}{1-\log t}(1-\log t)\frac{dt}{t} + \int_1^N \frac{t}{1+\log t}\,\frac{1+\log t}{t}J(t,u(t))\frac{dt}{t} \lesssim \|a\|_{\bar{A}_{\{\rho,\mu\},\infty;J}}.$$

Remark 6.4. More generally, the $(\infty;K)$-method does not admit a description as a J-space. Indeed, given any Banach couple $\bar{A} = (A_0, A_1)$, using Hölder's inequality, it is not hard to check that if $u(t)$ satisfies condition (6.2), then the integral $\int_0^\infty u(t)\,dt/t$ is convergent in $A_0 + A_1$. Moreover, if $t > 0$ and $w \in A_0 \cap A_1$ then $J(t,w;A_0,A_1) = J(t,w;A_0^o,A_1^o)$, because $A_0 \cap A_1 = A_0^o \cap A_1^o$ and the norms of A_j and A_j^o coincide for $j = 0,1$. These two facts imply that

$$(A_0,A_1)_{\{\rho,\mu\},q;J} = (A_0^o,A_1^o)_{\{\rho,\mu\},q;J}. \tag{6.5}$$

Equality (6.5) holds for any general J-method as considered in [5] because the assumptions on $J(t,u(t))$ still imply the convergence of $\int_0^\infty u(t)\,dt/t$ in $A_0 + A_1$ (see [5, p. 362]). Since for the couple (ℓ_1,ℓ_∞) we have

$$(\ell_1,\ell_\infty)_{\infty;K} = \ell_\infty \neq c_0 = (\ell_1,c_0)_{\infty;K} = (\ell_1^o,\ell_\infty^o)_{\infty;K},$$

we conclude that the $(\infty;K)$-method does not admit a description by means of the J-functional.

Corollary 6.5. *Let $\bar{A} = (A_0,A_1)$ be a Banach couple and let $1 \le q < \infty$. Then $A_0 \cap A_1$ is dense in $\bar{A}_{q;K}$.*

Proof. By Theorem 6.2, we can work with the norm $\|\cdot\|_{\bar{A}_{\{\rho,\mu\},q;J}}$. Let $a \in \bar{A}_{q;K}$ and take any $\varepsilon > 0$. We can find a J-representation $a = \int_0^\infty u(t)\,dt/t$ satisfying (6.2). Let $N \in \mathbb{N}$ be such that

$$\Big(\int_0^{1/N}(\rho(t)J(t,u(t)))^q\frac{dt}{t}\Big)^{1/q} + \Big(\int_N^\infty(\mu(t)J(t,u(t)))^q\frac{dt}{t}\Big)^{1/q} < \varepsilon.$$

Using Hölder's inequality and the continuity of the function $t^{-1}(1-\log t)^{-1}$ on $[1, 1/N]$ and of $t(1+\log t)^{-1}$ on $[1, N]$, we get

$$\int_{1/N}^{N} \|u(t)\|_{A_0\cap A_1} \frac{dt}{t} \le \int_{1/N}^{1} t^{-1} J(t,u(t)) \frac{dt}{t} + \int_{1}^{N} J(t,u(t)) \frac{dt}{t}$$
$$\le \Big(\int_{1/N}^{1} (\rho(t)J(t,u(t)))^q \frac{dt}{t}\Big)^{1/q} \Big(\int_{1/N}^{1} (t\rho(t))^{-q'} \frac{dt}{t}\Big)^{1/q'}$$
$$+ \Big(\int_{1}^{N} (\mu(t)J(t,u(t)))^q \frac{dt}{t}\Big)^{1/q} \Big(\int_{1}^{N} \mu(t)^{-q'} \frac{dt}{t}\Big)^{1/q'} < \infty.$$

Therefore, $w = \int_{1/N}^{N} u(t)\,dt/t$ belongs to $A_0 \cap A_1$. Since $a - w = \int_0^{1/N} u(t)\,dt/t + \int_N^\infty u(t)\,dt/t$, we obtain

$$\|a-w\|_{\bar{A}_{\{\rho,\mu\},q;J}} \le \Big(\int_{0}^{1/N} (\rho(t)J(t,u(t)))^q \frac{dt}{t}\Big)^{1/q}$$
$$+ \Big(\int_{N}^{\infty} (\mu(t)J(t,u(t)))^q \frac{dt}{t}\Big)^{1/q} < \varepsilon.$$

This shows the density of $A_0 \cap A_1$ in $\bar{A}_{q;K}$. ■

It follows from (6.5) and Theorem 6.2 that $(A_0^o, A_1^o)_{q;K} = (A_0, A_1)_{q;K}$ if $1 \le q < \infty$. Hence, as a direct consequence of Theorem 5.3, we derive the following.

COROLLARY 6.6. *Let $\bar{A} = (A_0, A_1)$ and $\bar{B} = (B_0, B_1)$ be Banach couples, let $1 \le q < \infty$, and let $T \in \mathcal{L}(\bar{A}, \bar{B})$. If $T : A_j \to B_j$ is compact for $j = 0, 1$, then $T : \bar{A}_{q;K} \to \bar{B}_{q;K}$ is also compact.*

7. Duality. This section is devoted to the duality relationships between limiting K- and J-spaces. Let $\bar{A} = (A_0, A_1)$ be a *regular* Banach couple, meaning that $A_0 \cap A_1$ is dense in A_j for $j = 0, 1$. Then the dual A_j^* of A_j can be identified with a subspace A_j' of $(A_0 \cap A_1)^*$, and (A_0', A_1') is a Banach couple too.

THEOREM 7.1. *Let $1 \le q \le \infty$, $1/q + 1/q' = 1$ and let $\bar{A} = (A_0, A_1)$ be a regular Banach couple. Then $(A_0, A_1)'_{q;K} = (A_0', A_1')_{q';J}$ with equivalent norms.*

Proof. By [4, Theorem 2.7.1], we know that $(A_0 + A_1)' = A_0' \cap A_1'$ and $(A_0 \cap A_1)' = A_0' + A_1'$. Hence, using Lemmata 4.2 and 3.2, we obtain

$$(A_0, A_1)'_{\infty;K} = (A_0 + A_1)' = A_0' \cap A_1' = (A_0', A_1')_{1;J}.$$

If $1 < q < \infty$, we derive from Lemmata 4.3, 3.5 and [9, Theorem 8.2] that

$$\begin{aligned}(A_0, A_1)'_{q;K} &= (A_0 \cap A_1, A_0 + A_1)'_{1,q;K} = (A'_0 + A'_1, A'_0 \cap A'_1)_{0,q';J} \\ &= (A'_0, A'_1)_{q';J}.\end{aligned}$$

The remaining case $q = 1$ can be treated similarly because the arguments in [9, Theorem 8.2] also work for $q = 1$. ■

THEOREM 7.2. *Let $1 \le q < \infty$, $1/q + 1/q' = 1$ and let $\bar{A} = (A_0, A_1)$ be a regular Banach couple. Then $(A_0, A_1)'_{q;J} = (A'_0, A'_1)_{q';K}$ with equivalent norms.*

Proof. The case $q = 1$ follows again by Lemmata 4.2 and 3.2 and [4, Theorem 2.7.1], namely

$$(A_0, A_1)'_{1;J} = (A_0 \cap A_1)' = A'_0 + A'_1 = (A'_0, A'_1)_{\infty;K}.$$

For $1 < q < \infty$, by Lemmata 4.3 and 3.5 and [9, Theorem 8.1], we derive

$$\begin{aligned}(A_0, A_1)'_{q;J} &= (A_0 \cap A_1, A_0 + A_1)'_{0,q;J} = (A'_0 + A'_1, A'_0 \cap A'_1)_{1,q';K} \\ &= (A'_0, A'_1)_{q';K}. \ ■\end{aligned}$$

In order to study the dual of the J-space when $q = \infty$, let $(A_0, A_1)_{c_0;J}$ be the collection of all $a \in A_0 + A_1$ for which there exists a sequence $(u_m)_{m\in\mathbb{Z}} \subset A_0 \cap A_1$ such that $a = \sum_{m\in\mathbb{Z}} u_m$ (convergence in $A_0 + A_1$) and

$$\max(1, 2^{-m}) J(2^m, u_m) \xrightarrow[m\to\pm\infty]{} 0. \tag{7.1}$$

We put

$$\|a\|_{\bar{A}_{c_0;J}} = \inf_{a=\sum u_m} \Big[\sup_{m\in\mathbb{Z}} \max(1, 2^{-m}) J(2^m, u_m)\Big].$$

LEMMA 7.3. *Let $\bar{A} = (A_0, A_1)$ be a Banach couple and let $(A_0, A_1)^o_{\infty;J}$ be the closure of $A_0 \cap A_1$ in $(A_0, A_1)_{\infty;J}$. Then $(A_0, A_1)_{c_0;J} = (A_0, A_1)^o_{\infty;J}$ with equivalence of norms.*

Proof. Let $a \in (A_0, A_1)_{c_0;J}$. Choose $(u_m) \subseteq A_0 \cap A_1$ with $a = \sum_{m\in\mathbb{Z}} u_m$ and satisfying (7.1). Given any $\varepsilon > 0$, there exists $M \in \mathbb{N}$ such that if $|m| \ge M$ then $\sup(1, 2^{-m}) J(2^m, u_m) < \varepsilon/2$. Let $w = \sum_{|m|\le M} u_m \in A_0 \cap A_1$. Since $a - w$ can be represented in $\bar{A}_{\infty;J}$ by means of the function

$$v(t) = \begin{cases} u_m/\log 2 & \text{if } |m| > M, \\ 0 & \text{otherwise,}\end{cases}$$

we obtain

$$\|a - w\|_{\bar{A}_{\infty;J}} \lesssim \sup_{m>M} J(2^m, u_m) + \sup_{m<-M} 2^{-m} J(2^m, u_m) \le \varepsilon.$$

This implies that $a \in (A_0, A_1)^o_{\infty;J}$.

Conversely, pick $a \in (A_0, A_1)^o_{\bar{A}_{\infty;J}}$. Then we can find $v_1 \in A_0 \cap A_1$ such that $\|a - v_1\|_{\bar{A}_{\infty;J}} < 1/2$. Now, $a - v_1 \in (A_0, A_1)^o_{\bar{A}_{\infty;J}}$. Hence, there exists

$v_2 \in A_0 \cap A_1$ such that $\|a - v_1 - v_2\|_{\bar{A}_{\infty;J}} < 1/2^2$. Repeating this process, for each $k \in \mathbb{N}$ we can find $v_k \in A_0 \cap A_1$ such that

$$\Big\|a - \sum_{n=1}^{k} v_n\Big\|_{\bar{A}_{\infty;J}} < \frac{1}{2^k}.$$

Hence, $a = \sum_{n\in\mathbb{N}} v_n$ in $(A_0, A_1)_{\infty;J}$. It follows that $(\sum_{n=1}^{k} v_n)_{k\in\mathbb{N}}$ is a Cauchy sequence in $(A_0, A_1)_{c_0;J}$. Since this space is complete (see [5]), we derive that $(\sum_{n=1}^{k} v_n)_{k\in\mathbb{N}}$ is convergent in $(A_0, A_1)_{c_0;J}$. Its limit should be the same as in $(A_0, A_1)_{\infty;J}$. Consequently, $a \in (A_0, A_1)_{c_0;J}$. ∎

THEOREM 7.4. *We have* $((A_0, A_1)^o_{\infty;J})' = (A_0', A_1')_{1;K}$ *with equivalence of norms.*

Proof. With the help of Lemma 7.3, we can proceed similarly to [9, Theorem 8.1]. Namely, put

$$G_m = \begin{cases} A_0 \cap A_1 \ \text{ normed by } J(2^m, \cdot) & \text{if } m \in \mathbb{N}, \\ A_0 \cap A_1 \ \text{ normed by } \|\cdot\|_{A_0\cap A_1} & \text{if } m = 0, \\ A_0 \cap A_1 \ \text{ normed by } 2^{-m}J(2^m, \cdot) & \text{if } -m \in \mathbb{N}. \end{cases}$$

Let $W = c_0(G_m)_{m\in\mathbb{Z}}$ and put

$$M = \Big\{(w_m) \in W : \sum_{m\in\mathbb{Z}} w_m = 0 \text{ (convergence in } A_0 + A_1)\Big\}.$$

As usual, let

$$M^\perp = \{\tilde{f} \in W^* : \tilde{f}(w_m) = 0 \text{ for each } (w_m) \in M\}.$$

The space $(A_0, A_1)^o_{\infty;J} = (A_0, A_1)_{c_0;J}$ coincides with W/M with equivalent norms. Therefore,

$$((A_0, A_1)^o_{\infty;J})^* = (W/M)^* = M^\perp.$$

Let us identify $M^\perp$. Put

$$F_m = \begin{cases} A_0' + A_1' \ \text{ normed by } 2^{-m}K(2^m, \cdot; A_0', A_1') & \text{if } m \in \mathbb{N}, \\ A_0' + A_1' \ \text{ normed by } \|\cdot\|_{A_0'+A_1'} & \text{if } m = 0, \\ A_0' + A_1' \ \text{ normed by } K(2^m, \cdot; A_0', A_1') & \text{if } -m \in \mathbb{N}. \end{cases}$$

For each $m \in \mathbb{Z}$, we have $G_m' = F_{-m}$ with equal norms. Hence $W^* = \ell_1(F_{-m})$. This means that functionals $\tilde{f} \in W^*$ are given by sequences $(f_{-m}) \in \ell_1(F_{-m})$ with

$$\tilde{f}(w_m) = \sum_{m\in\mathbb{Z}} f_{-m}(w_m) \quad \text{and} \quad \|\tilde{f}\|_{W^*} = \sum_{m\in\mathbb{Z}} \|f_{-m}\|_{F_{-m}}.$$

We claim that if $\tilde{f} \in M^\perp$, then $f_n = f_m$ for all $n, m \in \mathbb{Z}$. Indeed, if there is $a \in A_0 \cap A_1$ such that $f_n(a) \neq f_m(a)$, then for the sequence $w = (w_k) \in W$

defined by $w_k = a$ if $k = -n$, $w_k = -a$ if $k = -m$ and $w_k = 0$ for the other $k \in \mathbb{Z}$, we have $\tilde{f}(w_k) = f_n(a) - f_m(a) \neq 0$, but $w \in M$.

Conversely, let $f \in (A_0 \cap A_1)'$ with $(\dots, f, f, f, \dots) \in W^*$. We claim that the functional $\tilde{f}$ defined by this constant sequence belongs to $M^\perp$. Indeed, take any $(w_m) \in M$ and let us show that $\tilde{f}(w_m) = \sum_{m\in\mathbb{Z}} f(w_m) = 0$. Since $(\dots, f, f, f, \dots) \in W^* = \ell_1(F_{-m})$, we derive that $(\dots, f, f, f, \dots)$ belongs to $(A_0', A_1')_{1;K}$. Using the J-representation of this space given by Theorem 6.2, we can find $(g_j) \subset A_0' \cap A_1'$ such that $f = \sum_{j\in\mathbb{Z}} g_j$ (convergence in $A_0' + A_1'$) and

$$\Big\| f - \sum_{j=-N}^{M} g_j \Big\|_{(A_0', A_1')_{1;K}} \to 0 \quad \text{as } M, N \to \infty.$$

Hence, given any $\varepsilon > 0$, there is $L \in \mathbb{N}$ such that

$$\begin{aligned} \Big\| f - \sum_{|j|\le L} g_j \Big\|_{(A_0', A_1')_{1;K}} & \\ &= \Big\| \Big(\dots, f - \sum_{|j|\le L} g_j, f - \sum_{|j|\le L} g_j, f - \sum_{|j|\le L} g_j, \dots \Big) \Big\|_{W^*} \\ &< \frac{\varepsilon}{2\|(w_m)\|_W}. \end{aligned}$$

Let $g = \sum_{|j|\le L} g_j$. Then $g \in A_0' \cap A_1' = (A_0 + A_1)'$. Since $\sum_{m\in\mathbb{Z}} w_m = 0$ in $A_0 + A_1$, we can find $N \in \mathbb{N}$ such that for any $m \ge N$ we have

$$\Big| g\Big(\sum_{k=-m}^{m} w_k \Big) \Big| < \frac{\varepsilon}{2}.$$

Therefore, for each $m \ge N$, we derive that

$$\begin{aligned} \Big| \sum_{k=-m}^{m} f(w_k) \Big| &= \Big| \sum_{k=-m}^{m} f(w_k) - g\Big(\sum_{k=-m}^{m} w_k \Big) + g\Big(\sum_{k=-m}^{m} w_k \Big) \Big| \\ &\le \|(\dots, f-g, f-g, f-g, \dots)\|_{W^*} \|(w_n)\|_W \\ &\quad + \Big| g\Big(\sum_{k=-m}^{m} w_k \Big) \Big| < \frac{\varepsilon}{2\|(w_m)\|_W} \|(w_m)\|_W + \frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

This shows that $\tilde{f} \in M^\perp$.

Consequently, $((A_0, A_1)^o_{\infty;J})'$ consists of all $f \in (A_0 \cap A_1)' = A_0' + A_1'$ such that $(\min(1, 2^{-n}) K(2^n, f; A_0', A_1')) \in \ell_1$. This establishes that $((A_0, A_1)^o_{\infty;J})' = (A_0', A_1')_{1;K}$. ∎

8. Examples. Let (Ω, μ) be a σ-finite measure space. Given any measurable function f which is finite almost everywhere, the *non-increasing re-*

arrangement of f is defined by

$$f^*(t) = \inf\{s > 0 : \mu(\{x \in \Omega : |f(x)| > s\}) \le t\}.$$

We write $f^{**}(t) = t^{-1} \int_0^t f^*(s)\, ds$.

Let $1 \le p, q \le \infty$ and $b \in \mathbb{R}$. The Lorentz–Zygmund space $L_{p,q}(\log L)_b(\Omega)$ is defined to be the collection of all (equivalence classes of) measurable functions f on Ω such that the functional

$$\|f\|_{L_{p,q}(\log L)_b} = \Big(\int_0^\infty (t^{1/p}(1 + |\log t|)^b f^*(t))^q \, \frac{dt}{t} \Big)^{1/q}$$

is finite. The space $L_{(p,q)}(\log L)_b(\Omega)$ is defined similarly but replacing f^* by f^{**}. According to [19, Lemma 3.4.39], $L_{p,q}(\log L)_b(\Omega) = L_{(p,q)}(\log L)_b(\Omega)$ provided that $1 < p \le \infty$, $1 \le q \le \infty$ and $b \in \mathbb{R}$. Note that if $q = p$ then $L_{p,p}(\log L)_b(\Omega)$ is the Zygmund space $L_p(\log L)_b(\Omega)$. If in addition $b = 0$, then $L_{p,p}(\log L)_0(\Omega)$ is the Lebesgue space $L_p(\Omega)$.

THEOREM 8.1. *Let (Ω, μ) be a σ-finite measure space.*

(i) *If $1 < q \le \infty$ then*

$$(L_\infty(\Omega), L_1(\Omega))_{q;J} = L_{(\infty,q)}(\log L)_{-1}(\Omega) \cap L_{(1,q)}(\log L)_{-1}(\Omega).$$

(ii) *If $1 \le q < \infty$ then*

$$(L_\infty(\Omega), L_1(\Omega))_{q;K} = \Big\{ f : \|f\| = \Big(\int_0^\infty \min(1,t) f^{**}(t)^q \, \frac{dt}{t} \Big)^{1/q} < \infty \Big\}.$$

Proof. It is well-known (see [4] or [31]) that

$$K(t, f; L_\infty(\Omega), L_1(\Omega)) = f^{**}(1/t). \tag{8.1}$$

According to (4.4), we obtain

$$\begin{aligned} \|f\|_{(L_\infty,L_1)_{q;J}} &\sim \Big(\int_0^1 \Big[\frac{f^{**}(1/t)}{t(1-\log t)} \Big]^q \frac{dt}{t} \Big)^{1/q} + \Big(\int_1^\infty \Big[\frac{f^{**}(1/t)}{1+\log t} \Big]^q \frac{dt}{t} \Big)^{1/q} \\ &= \Big(\int_0^1 \Big[\frac{f^{**}(t)}{1-\log t} \Big]^q \frac{dt}{t} \Big)^{1/q} + \Big(\int_1^\infty \Big[\frac{t f^{**}(t)}{1+\log t} \Big]^q \frac{dt}{t} \Big)^{1/q}. \end{aligned}$$

Now we study each of these two terms. As $f^{**}(t)$ is non-increasing, we get

$$\begin{aligned} \Big(\int_1^\infty \Big(\frac{f^{**}(t)}{1+\log t} \Big)^q \frac{dt}{t} \Big)^{1/q} &\le f^{**}(1) \Big(\int_1^\infty (1+\log t)^{-q} \, \frac{dt}{t} \Big)^{1/q} \\ &\lesssim f^{**}(1) \Big(\int_0^1 (1-\log t)^{-q} \, \frac{dt}{t} \Big)^{1/q} \\ &\le \Big(\int_0^1 \Big(\frac{f^{**}(t)}{1-\log t} \Big)^q \frac{dt}{t} \Big)^{1/q}. \end{aligned}$$

Hence,

$$\Big(\int_0^1 \Big(\frac{f^{**}(t)}{1-\log t}\Big)^q \frac{dt}{t}\Big)^{1/q} \sim \Big(\int_0^\infty \Big(\frac{f^{**}(t)}{1+|\log t|}\Big)^q \frac{dt}{t}\Big)^{1/q} = \|f\|_{L_{(\infty,q)}(\log L)_{-1}}.$$

For the second term, we observe that

$$\begin{aligned}\Big(\int_0^1 \Big(\frac{tf^{**}(t)}{1-\log t}\Big)^q \frac{dt}{t}\Big)^{1/q} &\le \Big(\int_0^1 f^*(s)\,ds\Big)\Big(\int_0^1 (1-\log t)^{-q} \frac{dt}{t}\Big)^{1/q} \\ &\lesssim \Big(\int_0^1 f^*(s)\,ds\Big)\Big(\int_1^\infty (1+\log t)^{-q} \frac{dt}{t}\Big)^{1/q} \\ &\le \Big(\int_1^\infty \Big(\frac{tf^{**}(t)}{1+\log t}\Big)^q \frac{dt}{t}\Big)^{1/q}.\end{aligned}$$

So,

$$\Big(\int_1^\infty \Big(\frac{tf^{**}(t)}{1+\log t}\Big)^q \frac{dt}{t}\Big)^{1/q} \sim \|f\|_{L_{(1,q)}(\log L)_{-1}}.$$

This yields (i). Formula (ii) follows by inserting (8.1) in the interpolation norm. Namely,

$$\begin{aligned}\|f\|_{(L_\infty,L_1)_{q;K}} &\sim \Big(\int_0^1 f^{**}(1/t)^q \frac{dt}{t} + \int_1^\infty \Big(\frac{f^{**}(1/t)}{t}\Big)^q \frac{dt}{t}\Big)^{1/q} \\ &\sim \Big(\int_0^1 (tf^{**}(t))^q \frac{dt}{t} + \int_1^\infty f^{**}(t)^q \frac{dt}{t}\Big)^{1/q} \\ &= \Big(\int_0^\infty \min(1,t) f^{**}(t)^q \frac{dt}{t}\Big)^{1/q}. \ \blacksquare\end{aligned}$$

Let now ω be a *weight* on Ω, that is, a positive measurable function on Ω. As usual, we put

$$L_q(\omega) = \{f : \|f\|_{L_q(\omega)} = \|\omega f\|_{L_q} < \infty\}.$$

THEOREM 8.2. *Let (Ω,μ) be a σ-finite measure space, let $1 \le q \le \infty$, $1/q + 1/q' = 1$ and let ω_0, ω_1 be weights on Ω.*

(i) *We have*

$$(L_q(\omega_0), L_q(\omega_1))_{q;K} = L_q(\omega_K) \quad \textit{with equivalence of norms,}$$

where

$$\omega_K(x) = \min(\omega_0(x),\omega_1(x))\Big(1 + \Big|\log\frac{\omega_0(x)}{\omega_1(x)}\Big|\Big)^{1/q}.$$

(ii) *For the* $(q;J)$*-method, we have*

$$(L_q(\omega_0), L_q(\omega_1))_{q;J} = L_q(\omega_J) \quad \text{with equivalence of norms,}$$

where

$$\omega_J(x) = \max(\omega_0(x), \omega_1(x))\left(1 + \left|\log \frac{\omega_0(x)}{\omega_1(x)}\right|\right)^{-1/q'}.$$

Proof. It is easy to check that $L_q(\omega_0) \cap L_q(\omega_1) = L_q(\max(\omega_0, \omega_1))$ and $L_q(\omega_0) + L_q(\omega_1) = L_q(\min(\omega_0, \omega_1))$. Hence, by Lemma 3.5,

$$(L_q(\omega_0), L_q(\omega_1))_{q;K} = (L_q(\max(\omega_0, \omega_1)), L_q(\min(\omega_0, \omega_1)))_{q;K}.$$

Now (i) follows from the corresponding result for the ordered case (see [9, Theorem 7.4]). The proof of (ii) is similar but using now Lemma 4.3 and [9, Theorem 4.8]. ∎

Next we show a consequence of this result for interpolation of a certain class of Sobolev spaces. We write $\mathcal{S}(\mathbb{R}^d)$ and $\mathcal{S}'(\mathbb{R}^d)$ for the Schwartz space of all rapidly decreasing infinitely differentiable functions on $\mathbb{R}^d$, and the space of tempered distributions on $\mathbb{R}^d$, respectively. The symbol $\mathcal{F}$ stands for the Fourier transform and $\mathcal{F}^{-1}$ for the inverse Fourier transform. For $s \in \mathbb{R}$, we denote by $H^s = H_2^s(\mathbb{R}^d)$ the set of all $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{H^s} = \left\|(1 + \|x\|_{\mathbb{R}^d}^2)^{s/2}\mathcal{F}f\right\|_{L_2(\mathbb{R}^d)} < \infty.$$

More generally, if φ is a *temperate weight function* in the sense of [27, Definition 10.1.1], we put (see [27, 30])

$$H^\varphi = \{f \in \mathcal{S}'(\mathbb{R}^d) : \|f\|_{H^\varphi} = \|\varphi(x)\mathcal{F}f\|_{L_2(\mathbb{R}^d)} < \infty\}.$$

As a direct consequence of Theorem 8.2 and the interpolation property of the $(q;K)$- and $(q;J)$-methods, we obtain the following.

Corollary 8.3. *Let* $-\infty < s_1 < s_0 < \infty$. *Put*

$$\varphi_j(x) = (1 + \|x\|_{\mathbb{R}^d}^2)^{s_j/2}\left(1 + \frac{1}{2}(s_0 - s_1)\log(1 + \|x\|_{\mathbb{R}^d}^2)\right)^{(-1)^{j+1}/2}$$

where $j = 0, 1$. *Then*

$$(H^{s_0}, H^{s_1})_{2;K} = H^{\varphi_1} \quad \text{and} \quad (H^{s_0}, H^{s_1})_{2;J} = H^{\varphi_0}$$

with equivalence of norms.

Next let $(\phi_n)_{n=0}^\infty \subset \mathcal{S}(\mathbb{R}^d)$ have the following properties:

- $\operatorname{supp}\phi_0 \subset \{x \in \mathbb{R}^d : \|x\|_{\mathbb{R}^d} \le 2\}$,
- $\operatorname{supp}\phi_n \subset \{x \in \mathbb{R}^d : 2^{n-1} \le \|x\|_{\mathbb{R}^d} \le 2^{n+1}\}$, $n \in \mathbb{N}$,
- $\sup_{x\in\mathbb{R}^d} |D^\alpha\phi_n(x)| \le c_\alpha 2^{-n|\alpha|}$, $n \in \mathbb{N} \cup \{0\}$, $\alpha \in (\mathbb{N} \cup \{0\})^d$,
- $\sum_{n=0}^\infty \phi_n(x) = 1$, $x \in \mathbb{R}^d$.

For $s \in \mathbb{R}$ and $1 \le p, q \le \infty$, the Besov space $B^s_{p,q}$ consists of all those $f \in \mathcal{S}'(\mathbb{R}^d)$ such that

$$\|f\|_{B^s_{p.q}} = \Big(\sum_{n=0}^{\infty} \big(2^{sn}\|\mathcal{F}^{-1}(\phi_n \mathcal{F} f)\|_{L_p(\mathbb{R}^d)}\big)^q\Big)^{1/q} < \infty.$$

The spaces $B^{s,b}_{p,q}$, where $b \in \mathbb{R}$, are defined similarly but replacing the role of t^s by $t^s(1 + |\log t|)^b$ in the above definition. That is,

$$\|f\|_{B^{s,b}_{p.q}} = \Big(\sum_{n=0}^{\infty} \big(2^{sn}(1+n)^b\|\mathcal{F}^{-1}(\phi_n \mathcal{F} f)\|_{L_p(\mathbb{R}^d)}\big)^q\Big)^{1/q} < \infty.$$

The spaces $B^{s,b}_{p,q}$ are a special case of Besov spaces of generalized smoothness, which were considered in [8, 14] among other papers. They are of interest in fractal analysis and the related spectral theory (see [33, 34] and the references given there).

Theorem 8.4. *Let* $-\infty < s_1 < s_0 < \infty$, $1 \le p, q \le \infty$ *and* $1/q + 1/q' = 1$. *Then*

$$(B^{s_0}_{p,q}, B^{s_1}_{p,q})_{q;K} = B^{s_1,1/q}_{p,q} \quad \textit{and} \quad (B^{s_0}_{p,q}, B^{s_1}_{p,q})_{q;J} = B^{s_0,-1/q'}_{p,q}$$

with equivalence of norms.

Proof. It is shown in [4, Theorem 6.4.3] and [31, Theorem 2.3.2] that $B^{s_j}_{p,q}$ is a retract of $\ell_q(2^{ns_j} L_p)$ for $j = 0, 1$. Moreover, by Remark 3.3 and [9, p. 2352], we have

$$(\ell_q(2^{ns_0} L_p), \ell_q(2^{ns_1} L_p))_{q;K} = \ell_q((1+n)^{1/q} 2^{ns_1} L_p).$$

These two results yield the formula for the limiting K-method. The proof for the J-case has the same structure, but using now [9, Corollary 3.6]. ∎

We finish the paper with an application of limiting methods to Fourier coefficients. Let $\Omega = [0, 2\pi]$ with the Lebesgue measure and, given $f \in L_1([0, 2\pi])$, we write (c_m) for its Fourier coefficients, defined by

$$c_m = \hat{f}(m) = \frac{1}{2\pi} \int_0^{2\pi} f(x) e^{-imx}\, dx, \qquad m \in \mathbb{Z}.$$

We designate by (c^*_m) the decreasing rearrangement of the sequence $(|c_m|)$, given by $c^*_1 = \max\{|c_m| : m \in \mathbb{Z}\} = |c_{m_1}|$, $c^*_2 = \max\{|c_m| : m \in \mathbb{Z}\backslash\{m_1\}\} = |c_{m_2}|$ and so on.

Theorem 8.5. *If* $f \in L_2(\log L)_{-1/2}$, *then* $\sum_{n=1}^{\infty}(1 + \log n)^{-1}(c^*_n)^2 < \infty$.

Proof. Let $F(f) = (\hat{f}(m))$ be the operator assigning to each function f the sequence of its Fourier coefficients. As is well-known, the restrictions $F :$

$L_2([0,2\pi]) \to \ell_2$ and $F : L_1([0,2\pi]) \to \ell_\infty$ are bounded. Hence, interpolating by the $(2;J)$-method, we deduce that

$$F : (L_2([0,2\pi]), L_1([0,2\pi]))_{2;J} \to (\ell_2, \ell_\infty)_{2;J}$$

is also bounded. Now we proceed to identify these spaces. Since $L_2([0,2\pi]) = (L_\infty([0,2\pi]), L_1([0,2\pi]))_{1/2,2}$, it follows from [9, Theorem 4.6] and (8.1) that

$$\begin{aligned}\|f\|_{(L_2,L_1)_{2;J}} &\sim \Big(\int_1^\infty (t^{-1/2}(1+\log t)^{-1/2} f^{**}(1/t))^2 \,\frac{dt}{t}\Big)^{1/2} \\ &\sim \Big(\int_0^{2\pi} (t^{1/2}(1+|\log t|)^{-1/2} f^{**}(t))^2 \,\frac{dt}{t}\Big)^{1/2} \sim \|f\|_{L_2(\log L)_{-1/2}},\end{aligned}$$

where we have used [19, Lemma 3.4.39] in the last equivalence. As for the sequence space, since $K(t,\xi;\ell_1,\ell_\infty) \sim \sum_{j=1}^{[t]} \xi_j^*$ (see [31, p. 126]), where $[t]$ is the largest integer less than or equal to t, using again [9, Theorem 4.6] we obtain

$$\begin{aligned}\|\xi\|_{(\ell_2,\ell_\infty)_{2;J}} &\sim \Big(\sum_{n=1}^\infty \int_n^{n+1} \Big(t^{-1/2}(1+\log t)^{-1/2} \sum_{j=1}^{[t]} \xi_j^*\Big)^2 \frac{dt}{t}\Big)^{1/2} \\ &\sim \Big(\sum_{n=1}^\infty \Big(n^{-1/2}(1+\log n)^{-1/2} \sum_{j=1}^{n} \xi_j^*\Big)^2 n^{-1}\Big)^{1/2} \\ &\ge \Big(\sum_{n=1}^\infty (1+\log n)^{-1} (\xi_n^*)^2\Big)^{1/2}.\end{aligned}$$

This yields the result. ■

Other results on Fourier coefficients can be found in [2, 25].

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References

[1] I. Ahmed, D. E. Edmunds, W. D. Evans and G. E. Karadzhov, *Reiteration theorems for the K-interpolation method in limiting cases*, Math. Nachr. 284 (2011), 421–442.

[2] C. Bennett and K. Rudnick, *On Lorentz–Zygmund spaces*, Dissertationes Math. 175 (1980), 72 pp.

[3] C. Bennett and R. Sharpley, *Interpolation of Operators*, Academic Press, Boston, 1988.

[4] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin, 1976.

[5] Yu. A. Brudnyĭ and N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces*, Vol. 1, North-Holland, Amsterdam, 1991.

[6] P. L. Butzer and H. Berens, *Semi-Groups of Operators and Approximation*, Springer, New York, 1967.

[7] F. Cobos, *Interpolation theory and compactness*, in: Function Spaces, Inequalities and Interpolation (Paseky, 2009), J. Lukeš and L. Pick (eds.), Matfyzpress, Prague, 2009, 31–75.

[8] F. Cobos and D. L. Fernandez, *Hardy–Sobolev spaces and Besov spaces with a function parameter*, in: Function Spaces and Applications, Lecture Notes in Math. 1302, Springer, Berlin, 1988, 158–170.

[9] F. Cobos, L. M. Fernández-Cabrera, T. Kühn and T. Ullrich, *On an extreme class of real interpolation spaces*, J. Funct. Anal. 256 (2009), 2321–2366.

[10] F. Cobos, L. M. Fernández-Cabrera and A. Martínez, *Measure of non-compactness of operators interpolated by limiting real methods*, in: Operator Theory Adv. Appl. 219, Springer, Basel, 2012, 37–54.

[11] F. Cobos, L. M. Fernández-Cabrera and M. Mastyło, *Abstract limit J-spaces*, J. London Math. Soc. 82 (2010), 501–525.

[12] F. Cobos, L. M. Fernández-Cabrera and P. Silvestre, *New limiting real interpolation methods and their connection with the methods associated to the unit square*, Math. Nachr., DOI 10.1002/mana.201100076.

[13] F. Cobos, L. M. Fernández-Cabrera and P. Silvestre, *Limiting J-spaces for general couples*, Z. Anal. Anwend. 32 (2013), 83–101.

[14] F. Cobos and T. Kühn, *Approximation and entropy numbers in Besov spaces of generalized smoothness*, J. Approx. Theory 160 (2009), 56–70.

[15] F. Cobos and T. Kühn, *Equivalence of K- and J-methods for limiting real interpolation spaces*, J. Funct. Anal. 261 (2011), 3696–3722.

[16] F. Cobos, T. Kühn and T. Schonbek, *One-sided compactness results for Aronszajn–Gagliardo functors*, J. Funct. Anal. 106 (1992), 274–313.

[17] F. Cobos and J. Peetre, *Interpolation of compact operators: the multidimensional case*, Proc. London Math. Soc. 63 (1991), 371–400.

[18] M. Cwikel, *Real and complex interpolation and extrapolation of compact operators*, Duke Math. J. 65 (1992), 333–343.

[19] D. E. Edmunds and W. D. Evans, *Hardy Operators, Function Spaces and Embeddings*, Springer, Berlin, 2004.

[20] W. D. Evans and B. Opic, *Real interpolation with logarithmic functors and reiteration*, Canad. J. Math. 52 (2000), 920–960.

[21] W. D. Evans, B. Opic and L. Pick, *Real interpolation with logarithmic functors*, J. Inequal. Appl. 7 (2002), 187–269.

[22] D. L. Fernandez, *Interpolation of 2^n Banach spaces*, Studia Math. 45 (1979), 175–201.

[23] L. M. Fernández-Cabrera and A. Martínez, *Interpolation methods defined by means of polygons and compact operators*, Proc. Edinburgh Math. Soc. 50 (2007), 653–671.

[24] A. Gogatishvili, B. Opic and W. Trebels, *Limiting reiteration for real interpolation with slowly varying functions*, Math. Nachr. 278 (2005), 86–107.

[25] M. E. Gomez and M. Milman, *Extrapolation spaces and almost-everywhere convergence of singular integrals*, J. London Math. Soc. 34 (1986), 305–316.

[26] J. Gustavsson, *A function parameter in connection with interpolation of Banach spaces*, Math. Scand. 42 (1978), 289–305.

[27] L. Hörmander, *The Analysis of Linear Partial Differential Operators II*, Springer, Berlin, 1983.
[28] A. Kufner, L. Maligranda and L.-E. Persson, *The Hardy Inequality. About its History and Some Related Results*, Vydavatelský Servis, Pilsen, 2007.
[29] L. Maligranda, *The K-functional for symmetric spaces*, in: Interpolation Spaces and Allied Topics in Analysis, Lecture Notes in Math. 1070, Springer, Berlin, 1984, 170–182.
[30] V. A. Mikhailets and A. A. Murach, *Interpolation with a function parameter and refined scale of spaces*, Methods Funct. Anal. Topology 14 (2008), 81–100.
[31] H. Triebel, *Interpolation Theory, Function Spaces, Differential Operators*, North-Holland, Amsterdam, 1978.
[32] H. Triebel, *Theory of Function Spaces II*, Birkhäuser, Basel, 1992.
[33] H. Triebel, *The Structure of Functions*, Birkhäuser, Basel, 2001.
[34] H. Triebel, *Theory of Function Spaces III*, Birkhäuser, Basel, 2006.
[35] A. C. Zaanen, *Integration*, North-Holland, Amsterdam, 1967.

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