Weak type estimates for certain
Calderón–Zygmund singular integral operators

by

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Abstract. We prove weak type $(1,1)$ estimates for a special class of Calderón–Zygmund homogeneous kernels represented as l^1 sums of "equidistributed" H^1 atoms on $\mathbb{S}^1$.

1. Introduction. Let $\Omega : \mathbb{S}^{n-1} \to \mathbb{C}$ be a complex-valued integrable function, with mean zero. We consider the Calderón–Zygmund operator defined by

$$T_\Omega f(x) = \lim_{\varepsilon \to 0} \int_{|y|>\varepsilon} \frac{\Omega(y/|y|)}{|y|^n} f(x-y) \, dy.$$

The L^p boundedness of T_Ω was proved by Calderón–Zygmund [1] under the optimal *size* condition

$$(1) \qquad \int_{\mathbb{S}^{n-1}} |\Omega(\theta)| \operatorname{Log}^+ |\Omega(\theta)| \, d\theta < \infty.$$

The question of weak type $(1,1)$ boundedness is more involved (the L^p case being an easy consequence). Hofmann [6] (when $\Omega \in L^q$, $q > 1$, $n = 2$) and Christ–Rubio de Francia [3] (when $\Omega \in L \operatorname{Log}^+(L)$, $n \leq 7$) proved weak type bounds for the operator T_Ω. It is worth mentioning that the ideas for these results originated in [2], where similar estimates for the corresponding maximal function were obtained. Finally, Seeger [7] proved a weak type $(1,1)$ result under condition (1) in all dimensions. While as we indicated, (1) is an optimal size condition, it is far from being necessary. In fact, Coifman and Weiss [4] and independently Connett [5] proved L^p boundedness of T_Ω under the more general assumption $\Omega \in H^1(\mathbb{S}^{n-1})$ (see also [8]). This condition reflects the size as well as the oscillation of the kernel. We will be concerned here about the following

PROBLEM. What is the most general assumption on Ω such that the operator T_Ω is of weak type $(1,1)$, i.e.

$$\|T_\Omega f\|_{L^{1,\infty}} \leq C\|f\|_{L^1}? \tag{2}$$

In particular does (2) hold if $\Omega \in H^1(\mathbb{S}^{n-1})$?

We answer this question affirmately for a special class of H^1 kernels on $\mathbb{S}^1$. Let

$$\Omega(\theta) = \sum_{\nu=1}^{N} \lambda_\nu a_\nu(\theta),$$

where $\lambda_\nu > 0$ and a_ν are $H^1(\mathbb{S}^1)$ atoms. The usual definition of $H^1(\mathbb{S}^1)$ requires that $\|\Omega\|_{H^1} \sim \sum \lambda_\nu$. We consider a special class of H^1 kernels by imposing an extra condition on the structure of the atoms, namely we want them to be of comparable size and have "almost" disjoint supports. To be precise, let I_ν be the support of a_ν and e_ν be the center of the arc I_ν. For simplicity we assign increasing numbers counterclockwise and assume that $e_1 = (1,0)$. We assume that there exists $l \in \mathbb{N}$ (which will be fixed throughout the argument) such that

(1) $|a_\nu(\theta)| \leq 2^l$,
(2) $|I_\nu| \sim 2^{-l}$,
(3) $|I_\nu \cap I_{\nu'}| \leq \frac{1}{2}\min\{|I_\nu|, |I_{\nu'}|\}$.

Condition (3) above can be rewritten as $|e_\nu - e_{\nu'}| \geq c_0|\nu - \nu'|2^{-l}$ and the constant c_0 will be implicitly present in most of our estimates. In particular $N \leq 2\pi 2^l/c_0$. We are now ready to state our main result.

THEOREM 1. *If* $\Omega = \sum_{\nu=1}^{N} \lambda_\nu a_\nu$ *as above, then*

$$\|T_\Omega f\|_{L^{1,\infty}} \leq C\|f\|_{L^1} \sum_\nu \lambda_\nu,$$

where the constant C is independent of N, l, f.

REMARKS. • $\lambda_\nu > 0$ is not a restriction of generality, because we can always incorporate modulus one numbers in the atoms.

• The $L \operatorname{Log}^+ L$ theory is largely inapplicable even for single H^1 atoms. The reason is that the proofs never make use of the oscillation properties of the kernel. In a sense, $T_\Omega f$ is a bilinear operator acting on Ω and f, thus we ought to decompose f taking into consideration the structure of Ω.

Let us introduce some notations and conventions. We always denote the Lebesgue measure of some set A by $|A|$. If Q is a rectangle, then Q^* will be used to denote the rectangle with the same center and three times longer sides. All the sums are with respect to every index present in the sums, unless otherwise indicated.

In Section 2 we present a variant of the standard Calderón–Zygmund decomposition needed for our purposes. In Section 3 we prove Theorem 1 basing on three lemmas dealing with L^2 and L^1 estimates for the operator. In the last two sections we demonstrate the technical lemmas used to prove Theorem 1.

2. Calderón–Zygmund decompositions. Fix $\lambda > 0$ and a function f. We will apply the standard decomposition $f = g + b$, where g always stands for a "good" part and b for a "bad" part. In our case, we shall need different decompositions in all directions $\{e_\nu\}_{\nu=1}^N$. This way, one gets N different decompositions, which later on will be used to analyze the operator T_Ω.

For every $\nu \in \{1, \ldots, N\}$, let A_ν be the set of all rectangles with one long side parallel to e_ν with length δ and a shorter side of length $\delta 2^{-l}$. Define the uncentered directional maximal function in direction e_ν by

$$M_\nu f(x) = \sup_{x \in Q \in A_\nu} \frac{1}{|Q|} \int_Q |f(y)| \, dy.$$

Since the doubling constant for the family A_ν is independent of l, we have

$$(3) \qquad \|M_\nu f\|_{L^{1,\infty}} \leq C\|f\|_1,$$

where C is independent of l. Define an exceptional set $E_\nu = \{x : |M_\nu f(x)| > \lambda/\lambda_\nu\}$, $\nu = 1, \ldots, N$. Note that $|E_\nu| \leq \|f\|_1 \lambda_\nu/\lambda$ by (3). We proceed as follows:

• Represent E_ν as $\bigcup_i Q_{\nu,i}$, where $Q_{\nu,i} \in A_\nu$ are disjoint and $Q_{\nu,i}^* \cap E_\nu^c \neq \emptyset$ for every ν, i.e.

$$\int_{Q_{\nu,i}} |f(y)| \, dy \leq C \frac{\lambda}{\lambda_\nu} |Q_{\nu,i}|.$$

Define

$$b_{Q_{\nu,i}}(x) = \left(f(x) - \frac{1}{|Q_{\nu,i}|} \int_{Q_{\nu,i}} f(y) \, dy \right) \chi_{Q_{\nu,i}}(x),$$

$$g_{Q_{\nu,i}}(x) = \frac{1}{|Q_{\nu,i}|} \left(\int_{Q_{\nu,i}} f(y) \, dy \right) \chi_{Q_{\nu,i}}(x).$$

• Let $J_\nu = \{x : Mf(x) > \lambda/(\sum \lambda_\nu)\} \cup \bigcup_{k \neq \nu} E_k \setminus E_\nu$. Represent J_ν as $\bigcup_{Q_{\nu,i} \in A_\nu} Q_{\nu,i}$, where $\{Q_{\nu,i}\}$ are disjoint, and define $b_{Q_{\nu,i}}$ and $g_{Q_{\nu,i}}$ as above.
• For $K = \mathbb{R}^2 \setminus (\bigcup E_\nu \cup \{x : Mf(x) > \lambda/(\sum \lambda_\nu)\})$, set $g(x) = f(x)$ for $x \in K_\nu$ and $g(x) = 0$ otherwise.

Therefore for every $\nu \in \{1, \ldots, N\}$ we get a decomposition

$$f(x) = \sum_i b_{Q_{\nu,i}} + \sum_i g_{Q_{\nu,i}} + g = \sum_i b_{Q_{\nu,i}} + g_\nu + g$$

where

(1) $\int b_{Q_{\nu,i}} = 0$,
(2) $\int |b_{Q_{\nu,i}}| \le C(\lambda/\lambda_\nu)|Q_{\nu,i}|$, $\sum_i \|b_{Q_{\nu,i}}\|_1 \le C\|f\|_1$,
(3) $\sum |Q_{\nu,i}| \le \sum |E_\nu| + |\{x : Mf(x) > \lambda/(\sum \lambda_\nu)\}| \le C \sum \lambda_\nu \|f\|_1/\lambda$,
(4) $\|g_\nu\|_\infty \le \lambda/\lambda_\nu$, $\|g_\nu\|_1 \le \|f\|_1$,
(5) $\|g\|_\infty \le \lambda/(\sum \lambda_\nu)$, $\|g\|_1 \le \|f\|_1$.

Without loss of generality we may and do assume that the sides of the selected cubes are of dyadic sidelength. We will denote by $d(Q)$ the sidelength of the long side of Q.

3. Proof of Theorem 1. For fixed λ, f, we have

$$(4) \quad |\{x : |T_\Omega f(x)| > \lambda\}| \le |\{x : |T_\Omega g(x)| > \lambda/3\}|$$

$$+ \left| \left\{ x : \left| \sum_\nu \lambda_\nu \frac{a_\nu}{|x|^2} * g_\nu(x) \right| > \lambda/3 \right\} \right|$$

$$+ \left| \left\{ x : \left| \sum_\nu \sum_i \lambda_\nu \frac{a_\nu}{|x|^2} * b_{Q_{\nu,i}}(x) \right| > \lambda/3 \right\} \right|.$$

Clearly

$$|\{x : |T_\Omega g(x)| > \lambda/3\}| \le \frac{9}{\lambda^2} \|T_\Omega\|_2^2 \|g\|_2^2 \le C \frac{\sum \lambda_\nu}{\lambda} \|f\|_1,$$

which takes care of the first term. The second term is relatively easy to handle via L^2 estimates (see Lemma 1). The third term on the right hand side of (4) requires further considerations. Let $\beta : \mathbb{R}^+ \to \mathbb{R}$ be a smooth function such that

$$\text{supp}\,\beta \subset (1/2, 2) \quad \text{and} \quad \sum_{j \in \mathbb{Z}} \beta(2^{-j}t) = 1 \quad \text{for all } t > 0.$$

Then

$$K(x) = \frac{\Omega(x)}{|x|^2} = \sum_{\nu,j} \lambda_\nu a_{j,\nu}(x),$$

where $a_{j,\nu} = (a_\nu(x)/|x|^2)\beta(2^{-j}|x|)$. Introduce also smooth functions $\psi, \Phi : \mathbb{R} \to \mathbb{R}$ such that

(1) $\text{supp}\,\psi \subset (-1, 1)$ and $\psi(t) = 1$ for $t \in (-1/2, 1/2)$,
(2) $\text{supp}\,\Phi \subset (-2, -1/2) \cup (1/2, 2)$

and

$$(5) \quad \sum_{m \ge 0} \Phi(2^{-m}t) + \psi(t) = 1 \quad \text{for all } t \in \mathbb{R}.$$

Define $B_{j,\nu} = \sum_{d(Q_{\nu,i})=2^j} b_{Q_{\nu,i}}$. Thus the third term on the right of (4) is controlled by

$$(6) \quad \left|\left\{x : \left|\sum_\nu \sum_{Q_{\nu,i}} \lambda_\nu \frac{a_\nu}{|x|^2} * b_{Q_{\nu,i}}(x)\right| > \lambda/3\right\}\right|$$

$$\leq \left|\left\{x : \left|\sum_j \sum_{s>0} \sum_\nu \lambda_\nu a_{j,\nu} * B_{j-s,\nu}(x)\right| > \lambda/6\right\}\right|$$

$$+ \left|\left\{x : \left|\sum_j \sum_{s\leq 0} \sum_\nu \lambda_\nu a_{j,\nu} * B_{j-s,\nu}(x)\right| > \lambda/6\right\}\right|.$$

The support of the function in the second expression on the right hand side of (6) is in a multiple of the exceptional set $\bigcup_\nu E_\nu$ and therefore by the properties of the decomposition we obtain the desired estimate for it. For the first expression on the right hand side of (6) we will need a further decomposition of the kernel. Let P_ν^s be a function so that $\widehat{P_\nu^s}(\xi) = \psi(2^{l-s/4}\langle \xi/|\xi|, e_\nu\rangle)$. Convolution with P_ν^s essentially restricts the Fourier transform of the function to the part of the sphere "almost" orthogonal to e_ν, as hinted by the uncertainty principle.

We split the first term on the right of (6) in two parts as follows:

$$(7) \quad \left|\left\{x : \left|\sum_j \sum_{s>0} \sum_\nu \lambda_\nu (a_{j,\nu} * B_{j-s,\nu})(x)\right| > \lambda/6\right\}\right|$$

$$\leq \left|\left\{x : \left|\sum_j \sum_{s>0} \sum_\nu \lambda_\nu (a_{j,\nu} * P_\nu^s * B_{j-s,\nu})(x)\right| > \lambda/12\right\}\right|$$

$$+ \left|\left\{x : \left|\sum_j \sum_{s>0} \sum_\nu \lambda_\nu (a_{j,\nu} * (\delta - P_\nu^s) * B_{j-s,\nu})(x)\right| > \lambda/12\right\}\right|$$

It is convenient to handle (4) and the first term on the right of (7) using L^2 estimates and the second term on the right of (7) using an L^1 estimate. We need the following lemmas.

LEMMA 1. *Let $\{a_\nu\}$ be a collection of atoms of comparable size and almost disjoint supports as in the statement of Theorem 1. If $\|g_\nu\|_2^2 \leq \lambda\|f\|_1/\lambda_\nu$, then*

$$\left\|\sum_\nu \lambda_\nu \left(\frac{a_\nu(\cdot)}{|\cdot|^2} * g_\nu\right)\right\|_2^2 \leq C\lambda\left(\sum_\nu \lambda_\nu\right)\|f\|_1.$$

LEMMA 2. *Let $b_{Q_{\nu,i}}$ be the family of functions obtained in the Calderón–Zygmund decomposition and*

$$B_{j,\nu} = \sum_{Q_{\nu,i}:\, d(Q_{\nu,i})=2^j} b_{Q_{\nu,i}}.$$

Then for every $s > 0$, we have

$$\Big\| \sum_j \sum_\nu \lambda_\nu (a_{j,\nu} * P_\nu^s * B_{j-s,\nu}) \Big\|_2^2 \leq C 2^{-s/4} \lambda \|f\|_1 \sum \lambda_\nu.$$

Finally, we state the L^1 estimate needed to estimate the second term in (7).

LEMMA 3. *Let* $\{a_\nu\}$ *be a family of atoms as above. Then*

$$\|a_{j,\nu} * (\delta - P_\nu^s)\|_1 \leq C 2^{-s/8}.$$

The proof can be easily finished assuming the validity of these three lemmas. First, since $\|g_\nu\|_2^2 \leq \|g_\nu\|_\infty \|g_\nu\|_1 \leq \lambda \|f\|_1 / \lambda_\nu$, by Lemma 1 we have

$$\left| \left\{ x : \Big| \sum_\nu \lambda_\nu \Big(\frac{a_\nu(\cdot)}{|\cdot|^2} * g_\nu \Big)(x) \Big| > \lambda/3 \right\} \right| \leq \frac{9}{\lambda^2} \Big\| \sum_\nu \lambda_\nu \frac{a_\nu(\cdot)}{|\cdot|^2} * g_\nu \Big\|_2^2$$

$$\leq C \frac{\sum \lambda_\nu}{\lambda} \|f\|_1.$$

Next, Lemma 2 yields

$$\left| \left\{ x : \Big| \sum_j \sum_{s>0} \sum_\nu \lambda_\nu (a_{j,\nu} * P_\nu^s * B_{j-s,\nu})(x) \Big| > \lambda/12 \right\} \right|$$

$$\leq \frac{144}{\lambda^2} \Big(\sum_{s>0} \Big\| \sum_j \sum_\nu \lambda_\nu (a_{j,\nu} * P_\nu^s * B_{j-s,\nu}) \Big\|_2 \Big)^2$$

$$\leq \frac{C}{\lambda^2} \Big(\sum_{s>0} 2^{-s/8} \Big(\lambda \|f\|_1 \sum \lambda_\nu \Big)^{1/2} \Big)^2 \leq \frac{C}{\lambda} \|f\|_1 \sum \lambda_\nu.$$

Finally, by Lemma 3 we have

$$\left| \left\{ x : \Big| \sum_j \sum_{s>0} \sum_\nu \lambda_\nu (a_{j,\nu} * (\delta - P_\nu^s) * B_{j-s,\nu})(x) \Big| > \lambda/12 \right\} \right|$$

$$\leq \frac{C}{\lambda} \sum_j \sum_{s>0} \sum_\nu \lambda_\nu \|a_{j,\nu} * (\delta - P_\nu^s)\|_1 \|B_{j-s,\nu}\|_1$$

$$\leq \frac{C}{\lambda} \sum_{s>0} 2^{-s/8} \sum_j \sum_\nu \lambda_\nu \|B_{j-s,\nu}\|_1 \leq \frac{C}{\lambda} \|f\|_1 \sum_\nu \lambda_\nu.$$

This concludes the proof of the theorem based on the lemmas.

4. L^2 estimates. Consider the disjoint decomposition of $\mathbb{R}^2$ into the cones

$$A_\nu = \{ \xi \in \mathbb{R}^2 : (\nu - 1) 2^{-l} \leq |\langle e_1, \xi/|\xi| \rangle| \leq \nu 2^{-l} \}, \quad \nu = 1, \ldots, 2^l.$$

We will need the following claim.

Claim 1. *If a_ν is an $H^1(\mathbb{S}^1)$ atom supported in I_ν with $\|a_\nu\|_\infty \leq 2^l$, $|I_\nu| = 2^{-l}$, then for every $\xi \in A_{\nu'}$, $\nu' \neq \nu$, we have*

$$|\mathcal{F}(a_\nu(\cdot)/|\cdot|^2)(\xi)| \leq C/(1 + |\nu - \nu'|),$$

where $\mathcal{F}$ denotes the Fourier transform.

Proof. According to the explicit formula for the Fourier transform of homogeneous kernels (see [10], p. 39), we have

$$\mathcal{F}(a_\nu(\cdot)/|\cdot|^2)(\xi) = \int\limits_{\mathbb{S}^1} a_\nu(\theta) \left[\ln \left(\frac{1}{|\langle \theta, \xi/|\xi|\rangle|} \right) + \frac{\pi i}{2} \operatorname{sign}\langle \theta, \xi \rangle \right] d\theta.$$

The above formula makes sense at least in the case $a_\nu \in H^1(\mathbb{S}^1)$ and the estimate by a constant is immediate (via the H^1-BMO duality). In particular, the L^2 norm of the corresponding convolution operator is uniformly bounded. Since $\int a_\nu = 0$, it is easy to see that if $\xi \in A_{\nu'}$, $\nu' \neq \nu$, then

$$|\mathcal{F}(a_\nu(\cdot)/|\cdot|^2)(\xi)| \leq \left| \int\limits_{\mathbb{S}^1} a_\nu(\theta) \left[\ln \left(\frac{1}{|\langle \theta, \xi/|\xi|\rangle|} \right) - \ln \left(\frac{1}{|\langle e_\nu, \xi/|\xi|\rangle|} \right) \right] d\theta \right|$$

$$\leq \int |a_\nu(\theta)| \ln \left(1 + \frac{\langle \theta - e_\nu, \xi/|\xi|\rangle}{\langle e_\nu, \xi/|\xi|\rangle} \right) d\theta \leq \frac{C}{1 + |\nu - \nu'|},$$

since $|e_\nu - \theta| \leq 2^{-l}$ and $|\langle e_\nu, \xi/|\xi|\rangle| \sim |\nu - \nu'|2^{-l}$. ∎

We are now prepared for the proof of Lemma 1.

Proof of Lemma 1. We use the function P_ν^0 defined by

$$\widehat{P_\nu^0}(\xi) = \psi(2^l \langle \xi/|\xi|, e_\nu \rangle).$$

Clearly

(8) $$\left\| \sum_\nu \lambda_\nu \frac{a_\nu(\cdot)}{|\cdot|^2} * g_\nu \right\|_2^2 \leq 2 \left\| \sum_\nu \lambda_\nu \frac{a_\nu(\cdot)}{|\cdot|^2} * P_\nu^0 * g_\nu \right\|_2^2$$

$$+ 2 \left\| \sum_\nu \lambda_\nu \frac{a_\nu(\cdot)}{|\cdot|^2} * (\delta - P_\nu^0) * g_\nu \right\|_2^2.$$

By support considerations, $\widehat{P_\nu^0}\widehat{P_{\nu'}^0} \neq 0$ only if $|\nu - \nu'| \leq 100/c_0$. Therefore by Plancherel's theorem and Claim 1,

$$\left\| \sum_\nu \lambda_\nu \frac{a_\nu(\cdot)}{|\cdot|^2} * P_\nu^0 * g_\nu \right\|_2^2 \leq C \sum_\nu \lambda_\nu^2 \left\| \frac{a_\nu(\cdot)}{|\cdot|^2} * g_\nu \right\|_2^2$$

$$\leq C \sum_\nu \lambda_\nu^2 \|g_\nu\|_2^2 \leq C \left(\sum_\nu \lambda_\nu \right) \lambda \|f\|_1.$$

For the second term on the right (8) we use Claim 1, Plancherel's theorem, and Cauchy–Schwarz to obtain

$$\left\| \sum_\nu \lambda_\nu \frac{a_\nu(\cdot)}{|\cdot|^2} * (\delta - P_\nu^0) * g_\nu \right\|_2^2$$

$$\leq \sum_\nu \sum_{\nu'} \lambda_\nu \lambda_{\nu'} \int |\widehat{g}_\nu||\widehat{g}_{\nu'}| \left| \frac{\widehat{a_\nu(\cdot)}}{|\cdot|^2} \right| \left| \frac{\widehat{a_{\nu'}(\cdot)}}{|\cdot|^2} \right| |1 - \widehat{P_\nu^0}||1 - \widehat{P_{\nu'}^0}| \, d\xi$$

$$\leq C \sum_{\nu,\nu',\nu''} \lambda_\nu \lambda_{\nu'} \int_{A_{\nu''}} \frac{|\widehat{g}_\nu|}{1 + |\nu'' - \nu'|} \cdot \frac{|\widehat{g}_{\nu'}|}{1 + |\nu'' - \nu|} \, d\xi$$

$$\leq C \sum_{\nu,\nu''} \sum_{\nu'} \lambda_\nu^2 \int_{A_{\nu''}} \frac{|\widehat{g}_\nu|^2}{(1 + |\nu'' - \nu'|)^2} \, d\xi$$

$$+ C \sum_{\nu,\nu''} \sum_\nu \lambda_{\nu'}^2 \int_{A_{\nu''}} \frac{|\widehat{g}_{\nu'}|^2}{(1 + |\nu'' - \nu|)^2} \, d\xi$$

$$\leq C \sum_\nu \lambda_\nu^2 \|g_\nu\|_2^2 \leq \left(\sum \lambda_\nu \right) \lambda \|f\|_1. \quad \blacksquare$$

We turn to the estimate concerning the first term on the right of (8). We use the TT^* tools elaborated in [3], but note that no cancellation of the atoms a_ν is needed in the proof, whereas the cancellation of $b_{Q_{\nu,i}}$ is heavily exploited.

Proof of Lemma 2. Observe that since $\{e_\nu\}$ are "equidistributed", each ξ is contained in no more than $C2^{s/4}$ sets of the form supp $\widehat{P_\nu^s}$. By Plancherel's identity,

$$\left\| \sum_j \sum_\nu \lambda_\nu a_{j,\nu} * P_\nu^s * B_{j-s,\nu} \right\|_2^2 = \left\| \sum_j \sum_\nu \lambda_\nu \widehat{a}_{j,\nu} \widehat{P_\nu^s} \widehat{B}_{j-s,\nu} \right\|_2^2$$

$$\leq \sum_\nu \lambda_\nu^2 2^{s/4} \left\| \sum_j a_{j,\nu} * B_{j-s,\nu} \right\|_2^2.$$

Therefore it suffices to prove the following inequality:

$$(9) \qquad \left\| \sum_j a_{j,\nu} * B_{j-s,\nu} \right\|_2^2 \leq C 2^{-s/2} \frac{\lambda}{\lambda_\nu} \sum_j \|B_{j-s,\nu}\|_1$$

for every $\nu = 1, \ldots, N$.

Without loss of generality $e_\nu = e_1 = (1,0)$ and for simplicity we drop dependence on ν from our notation. All the rectangles that appear below

will be assumed to be in the family A_1. We have

$$\Big\| \sum_j a_j * B_{j-s} \Big\|_2^2 \leq 2 \Big| \sum_i \sum_{i \leq j} \langle a_j * B_{j-s}, a_i * B_{i-s} \rangle \Big|$$

$$\leq 2 \sum_j \|B_{j-s}\|_1 \sum_{i \leq j} \|\widetilde{a}_j * a_i * B_{i-s}\|_\infty.$$

Thus it suffices to prove $\sum_{i \leq j} \|\widetilde{a}_j * a_i * B_{i-s}\|_\infty \leq C 2^{-s/2} \lambda / \lambda_1$. By dilation, this is equivalent to the case $j = 0$, $i \leq 0$. Write $(\widetilde{a}_0 * a_i)(y) = \langle L_i^y a, \widetilde{a} \rangle$, where

$$(10) \qquad L_i^y a(\theta) = \int a\left(\frac{y - t\theta}{|y - t\theta|} \right) \frac{\beta(|y - t\theta|)}{|y - t\theta|^2} \cdot \frac{\beta(2^{-i}t)}{t} \, dt.$$

We see that $\widetilde{a}_0 * a_i * B_{i-s}(0) = \langle T_{i,s}a, \widetilde{a} \rangle$, where

$$T_{i,s}a(\theta) = \int B_{i-s}(-y) L_i^y a(\theta) \, dy = \sum_{Q : d(Q) = 2^{i-s}} \int b_Q(-y) L_i^y a(\theta) \, dy.$$

Since $\|\widetilde{a}\|_\infty \leq 2^l$ and $|I| = |\text{supp }\widetilde{a}| \leq 2^{-l}$, it suffices to show that

$$(11) \qquad \sum_{i \leq 0} \sup_{\theta \in I} |T_{i,s}a(\theta)| \leq C 2^{-s/2} \lambda / \lambda_1,$$

since the general case for $\widetilde{a}_0 * a_i * B_{i-s}(x)$ follows by translation invariance.

To prove (11), note first that $\text{supp}_y L_i^y a(\theta) \subseteq \{y : 1 \leq |y| \leq 2, \ |y/|y| - e_1| \leq 2^{1-l}\}$. Set $\delta = 2^{-l-s/2}$. Consider two cases.

In the first case, $1 \leq |y| \leq 2$, $|y/|y| - e_1| \leq \delta$. Split the rectangles accordingly in two groups:

$$\mathcal{C}_i' = \{Q \in A_1 : d(Q) = 2^{i-s}, \ Q \cap \{y : 1 \leq |y| \leq 2, \ |y/|y| - e_1| \leq \delta\} \neq \emptyset\},$$
$$\mathcal{C}_i'' = A_1 \setminus \mathcal{C}_i' \cap \{Q : d(Q) = 2^{i-s}\}.$$

Since $\|L_i^y a\|_\infty \leq C 2^l$ we have

$$\sum_{i \leq 0} \sup_\theta \Big| \sum_{Q \in \mathcal{C}_i'} \int b_Q(-y) L_i^y a(\theta) \, dy \Big| \leq \sum_{i \leq 0} \sum_{Q \in \mathcal{C}_i'} \|b_Q\|_1 \|L_i^y a\|_\infty$$

$$\leq C 2^l \frac{\lambda}{\lambda_1} \sum_{i \leq 0} \sum_{Q \in \mathcal{C}_i'} |Q| \leq C 2^l \frac{\lambda}{\lambda_1} \delta$$

$$= C \frac{\lambda}{\lambda_1} 2^{-s/2},$$

which gives the estimate in the first case.

To treat the second case, we need the following

CLAIM 2. *For every $\theta \in \operatorname{supp} a$, $1 \leq |y| \leq 2$, $\delta \leq |y/|y| - e_1| \leq 2^{-l}$, we have*

$$\left| \frac{\partial}{\partial \theta} L_i^y a(\theta) \right| \leq C2^{l-i}, \qquad \left| \frac{\partial}{\partial \theta^\perp} L_i^y a(\theta) \right| \leq C2^{l-i}\delta^{-1} = C2^{2l+s/2-i},$$

where $\theta^\perp \in \mathbb{S}^1$, $\langle \theta^\perp, \theta \rangle = 0$, and $\partial/\partial\theta$ is the directional derivative in direction θ.

Proof. By rotation invariance, it suffices to check the Claim for $\theta = e_1$, i.e.

(12) $$\left| \frac{\partial}{\partial y_1} L_i^y a(e_1) \right| \leq C2^{l-i}, \qquad \left| \frac{\partial}{\partial y_2} L_i^y a(e_1) \right| \leq C2^{l-i}\delta^{-1}.$$

Note that since $|y/|y| - e_1| > \delta$, we have $|y_2| > c\delta$. Following [3], we make the change of variables $\omega = (y - te_1)/|y - te_1|$ and differentiate (10) with respect to y_1. Introduce also the inverse Jacobi factor $J(\omega, y) = |d\omega/dt| = |y_2|/|y - te_1|^2$. We have

$$\frac{\partial}{\partial y_1} L_i^y a(e_1) = \int a(\omega) \frac{\partial}{\partial y_1} \left(\frac{\beta(|y - t(\omega)e_1|)}{|y - t(\omega)e_1|^2} \cdot \frac{\beta(2^{-i}t(\omega))}{t(\omega)} \right) \frac{1}{J(\omega, y)} \, d\omega$$

$$- \int a(\omega) \frac{\beta(|y - t(\omega)e_1|)}{|y - t(\omega)e_1|^2} \cdot \frac{\beta(2^{-i}t(\omega))}{t(\omega)} \cdot \frac{\partial J}{\partial y_1} \cdot \frac{1}{J^2(\omega, y)} \, d\omega.$$

The first term on the right is clearly bounded by 2^{l-i} (by changing coordinates back to t), for the second observe that $|\partial J/\partial y_1|/|J| \leq C/|y - te_1| \leq C$, since $|y - te_1| \sim 1$. This implies the bound 2^l for the second term above.

The second estimate in (12) follows by the same argument, once we observe that $|\partial J/\partial y_2|/|J| \leq C|y - te_1|^4/|y_2| \leq C/\delta$. ∎

We estimate the remaining terms as follows:

$$\left| \sum_{Q \in \mathcal{C}_i''} \int b_Q(-y)(L_i^y a)(\theta) \, dy \right| \leq \sum_{Q \in \mathcal{C}_i''} \int |b_Q(-y)| \cdot |L_i^y a(\theta) - L_i^{y_Q} a(\theta)| \, dy$$

$$\leq \sum_{Q \in \mathcal{C}_i''} \int |b_Q(-y)| \cdot |L_i^y a(\theta) - L_i^{y_\theta} a(\theta)| \, dy$$

$$+ \sum_{Q \in \mathcal{C}_i''} \int |b_Q(-y)| \cdot |L_i^{y_\theta} a(\theta) - L_i^{y_Q} a(\theta)| \, dy,$$

where y_Q is the center of Q and y_θ is such that $y_\theta - y$ is parallel to θ, and $y_\theta - y_Q$ is parallel to $\theta^\perp$ and $|y_\theta - y_Q| \leq 2^{i-s-l}$, $|y - y_\theta| \leq 2^{i-s}$. This is possible, because $|\theta - e_1| \leq 2^{-l}$. Thus, according to Claim 2 and the previous

inequality, we have

$$\left| \sum_{i\leq 0} \sum_{Q\in\mathcal{C}_i''} \int b_Q(-y) L_i^y a(\theta)\, dy \right|$$

$$\leq \sum_{i\leq 0} \sum_{Q\in\mathcal{C}_i''} \|b_Q\|_1 \left(\sup_y \left| \frac{\partial}{\partial\theta} L_i^y a(\theta) \right| |y - y_\theta| + \sup_y \left| \frac{\partial}{\partial\theta^\perp} L_i^y a(\theta) \right| |y_\theta - y_Q| \right)$$

$$\leq \lambda \sum_{i\leq 0} \sum_{Q\in\mathcal{C}_i''} |Q|(2^{i+l-s} + 2^{i-s/2+l})/\lambda_1 \leq C 2^{-s/2} \lambda/\lambda_1. \ \blacksquare$$

5. L^1 estimates. In this section we prove Lemma 3. The methods are similar to those in [7], although we need one more decomposition in the angular variable. In what follows, the mean value zero property of the atoms will be crucial for us.

Proof of Lemma 3. Without loss of generality, it suffices to consider the case $e_\nu = (1,0)$. Using the construction of the function Φ in (5), we write

$$a_{j,1} - P_1^s * a_{j,1} = \sum_{m=1}^{l-s/4+10} \mathcal{F}^{-1}(\Phi(2^{l-s/4-m}\langle \xi/|\xi|, e_1\rangle)) * a_{j,1},$$

where $\mathcal{F}^{-1}$ is the inverse Fourier transform. Introduce also a function $\gamma \in C_0^\infty$ with $\operatorname{supp}\gamma \subseteq (1/2, 2)$ such that $\sum \gamma^2(2^{-k}\xi) = 1$. Define $\widehat{L}_k(\xi) = \gamma(2^{-k}\xi)$. Thus

$$a_{j,1} - P_1^s * a_{j,1} = \sum_{k,m} \mathcal{F}^{-1}[c_{j,k}^{m,s}] * L_k,$$

where

$$c_{j,k}^{m,s}(\xi) = \gamma(2^{-k}\xi)\Phi(2^{l-s/4-m}\langle \xi/|\xi|, e_1\rangle)\widehat{a}_{j,1}(\xi).$$

Next, we claim that

(13) $$\|L_k * \mathcal{F}^{-1}(\Phi(2^{l-s/4-m}\langle \xi/|\xi|, e_1\rangle))\|_1 \leq C$$

uniformly with respect to m, k, s. To show this, observe that the following Sobolev embedding formula holds:

$$\|\widehat{h}\|_1 \leq C \sum_{|\alpha|\leq 2} \|\partial^\alpha h_Q\|_2,$$

where $h_Q(x) := h(Qx)$ and Q is a nonsingular linear transformation. Consider then a linear transformation $A_{k,m}$ such that

$$\xi = A_{k,m}\eta = (2^{k-l+s/4+m}\eta_1, 2^k\eta_2).$$

To prove (13), it suffices to demonstrate

$$\|\partial_\eta^\alpha[\widehat{L}_k \Phi(2^{l-s/4-m}\langle \cdot/|\cdot|, e_1\rangle)(A_{k,m}\eta)]\|_2 \leq C_\alpha$$

for every multiindex $\alpha \in \mathbb{N}^2$. It is not difficult to see that this is the case (see Lemma 2.2 in [7] for a similar calculation). Thus by (13) and since $\int a_1(\theta) \, d\theta = 0$, we have

$$\|\mathcal{F}^{-1}[c_{j,k}^{m,s}] * L_k\|_1 \leq C\|L_k * a_{j,1}\|_1$$

$$= C \int \left| \int L_k(x-y) a_1(y) \beta(2^{-j}|y|)|y|^{-2} \, dy \right| dx$$

$$\leq C \int \int \int \sup_\theta |L_k(x-r\theta) - L_k(x-re_1)| \cdot |a_1(\theta)| \beta(2^{-j}r) r^{-1} \, dr \, d\theta \, dx$$

$$\leq C 2^{j-l} \|\nabla L_k\|_1 \leq C 2^{k+j-l}.$$

For large k, we need a more refined estimate. Again by Sobolev embedding we have

$$\|\mathcal{F}^{-1}[\Phi(2^{l-s/4-m}\langle \xi/|\xi|, e_1\rangle) \gamma^2(2^{-k}\xi)\widehat{a}_{j,1}]\|_1$$

$$\leq \sum_{|\alpha| \leq 2} \left\| \frac{\partial}{\partial \eta^\alpha} [\Phi(2^{l-s/4-m}\langle \cdot/|\cdot|, e_1\rangle) \gamma^2(2^{-k}\cdot)\widehat{a}_{j,1}(\cdot)(A_{k,m}\eta)] \right\|_2.$$

Observe that the function has compact support $\subset \{\eta : |\eta| \leq 10\}$ and therefore the derivatives of Φ and γ with respect to η are bounded by a universal constant. Furthermore, the L^2 norm is dominated by the L^∞ norm. Notice also that if $\Phi(2^{l-s/4-m}\langle \xi/|\xi|, e_1\rangle) \gamma(2^{-k}\xi) \neq 0$, then $|\langle \xi/|\xi|, e_1\rangle| \sim 2^{m+s/4-l}$, $|\xi| \sim 2^k$ and thus $|\langle \xi, \theta\rangle| \sim 2^{m+k+s/4-l}$ for all $\theta \in \operatorname{supp} a_1$, since $m, s > 0$. Having made these observations, we proceed to bound the corresponding derivatives of $\widehat{a}_{j,1}$. We have

$$\widehat{a}_{j,1}(A_{k,m}\eta) = \int a_1(\theta) \int r^{-1} \beta(2^{-j}r) \exp(-ir(\theta_1 2^{k-l+s/4+m}\eta_1 + \theta_2 2^k \eta_2)) \, dr \, d\theta.$$

After differentiating α times in η and integrating by parts three times (in r) we get

$$\frac{\partial}{\partial \eta^\alpha} \widehat{a}_{j,1}(A_{k,m}\eta) = \int a_1(\theta) \frac{1}{(-i\langle \theta, A_{k,m}\eta\rangle)^3} \int \partial_r^3 (r^{-1}\beta(2^{-j}r))(-ir)^{|\alpha|}$$

$$\times (\theta_1 2^{k+m-l+s/4})^{\alpha_1} (\theta_2 2^k)^{\alpha_2} \exp(-ir\langle \theta, A_{k,m}\eta\rangle) \, dr \, d\theta.$$

Thus, taking into account that $|\langle \theta, A_{k,m}\eta\rangle| \sim 2^{m+k+s/4-l}$ and $|\theta_2| \leq 2^{-l}$ we obtain

$$\left\| \frac{\partial}{\partial \eta^\alpha} \widehat{a}_{j,1}(A_{k,m}\eta) \right\|_\infty \leq \frac{C}{2^{3(k+j-l+m+s/4)}} (2^{j+k+m-l+s/4} + 2^{k+j-l})^{|\alpha|}.$$

Since $|\alpha| \leq 2$, we finally arrive at

$$\|\mathcal{F}^{-1}[\Phi(2^{l-s/4-m}\langle \xi/|\xi|, e_1\rangle) \gamma^2(2^{-k}\xi)\widehat{a}_{j,1}]\|_1 \leq \frac{C}{2^{k+j-l+m+s/4}}.$$

Thus, the needed estimate for large k is proved. We now have to combine

the two estimates to obtain the final statement of Lemma 3. We have

$$\Big\| \sum_{k,m} \mathcal{F}^{-1}[c_{j,k}^{m,s}] * L_k \Big\|_1 \leq \sum_{m=1}^{l-s/4+10} \sum_{k:\, k+j-l \leq -s/8-m/2} \big\| \mathcal{F}^{-1}[c_{j,k}^{m,s}] * L_k \big\|_1$$

$$+ \sum_{m=1}^{l-s/4+10} \sum_{k:\, k+j-l \geq -s/8-m/2} \big\| \mathcal{F}^{-1}[c_{j,k}^{m,s}] * L_k \big\|_1$$

$$\leq C \sum_{m=1}^{l-s/4+10} 2^{-s/8-m/2} \leq C 2^{-s/8}. \quad \blacksquare$$

References

[1]   A. P. Calderón and A. Zygmund, *On singular integrals*, Amer. J. Math. 78 (1956), 289–309.

[2]   M. Christ, *Weak type* $(1,1)$ *bounds for rough operators*, Ann. of Math. 128 (1988), 19–42.

[3]   M. Christ and J.-L. Rubio de Francia, *Weak type* $(1,1)$ *bounds for rough operators, II*, Invent. Math. 93 (1988), 225–237.

[4]   R. R. Coifman and G. Weiss, *Extensions of Hardy spaces and their use in analysis*, Bull. Amer. Math. Soc. 83 (1977), 569–645.

[5]   W. C. Connett, *Singular integrals near L^1*, in: Proc. Sympos. Pure Math. 35, S. Wainger and G. Weiss (eds.), Part I, Amer. Math. Soc., 1979, 163–165.

[6]   S. Hofmann, *Weak* $(1,1)$ *boundedness of singular integrals with nonsmooth kernels*, Proc. Amer. Math. Soc. 103 (1988), 260–264.

[7]   A. Seeger, *Singular integral operators with rough convolution kernels*, J. Amer. Math. Soc. 9 (1996), 95–105.

[8]   A. Stefanov, *Characterizations of H^1 and applications to singular integral operators*, Illinois J. Math. 44 (2000), 574–592.

[9]   E. M. Stein, *Harmonic Analysis*: *Real-Variable Methods, Orthogonality, and Oscillatory Integrals*, Princeton Univ. Press, Princeton, NJ, 1993.

[10] —, *Singular Integrals and Differentiability Properties of Functions*, Princeton Univ. Press, Princeton, NJ, 1970.

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