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Weighted measure algebras and uniform norms

by

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Abstract. Let ω be a weight on an LCA group G. Let $M(G,\omega)$ consist of the Radon measures μ on G such that $\omega\mu$ is a regular complex Borel measure on G. It is proved that: (i) $M(G,\omega)$ is regular iff $M(G,\omega)$ has unique uniform norm property (UUNP) iff $L^1(G,\omega)$ has UUNP and G is discrete; (ii) $M(G,\omega)$ has a minimum uniform norm iff $L^1(G,\omega)$ has UUNP; (iii) $M_{00}(G,\omega)$ is regular iff $M_{00}(G,\omega)$ has UUNP iff $L^1(G,\omega)$ has UUNP, where $M_{00}(G,\omega) := \{\mu \in M(G,\omega) : \widehat{\mu} = 0 \text{ on } \Delta(M(G,\omega)) \setminus \Delta(L^1(G,\omega))\}$.

1. Introduction. A *uniform norm* on a Banach algebra $(\mathcal{A}, \|\cdot\|)$ is a (not necessarily complete) norm $|\cdot|$ on $\mathcal{A}$ satisfying the square property $|a^2| = |a|^2$ $(a \in \mathcal{A})$. A *minimum uniform norm* on $\mathcal{A}$ is a minimum norm among all uniform norms on $\mathcal{A}$. A Banach algebra $\mathcal{A}$ has the *Unique Uniform Norm Property (UUNP)* if $\mathcal{A}$ admits exactly one uniform norm [BhDe1]; in this case, the spectral radius $r(\cdot)$ is the only uniform norm on $\mathcal{A}$. The perspective of UUNP in Banach algebras is discussed in [BhDe1] and [BhDe2]. By [BhDe2, Theorem 4.1] and [BhDe3], the Beurling algebra $L^1(G,\omega)$ has UUNP if and only if $L^1(G,\omega)$ is regular. On the other hand, either $L^1(G,\omega)$ has exactly one uniform norm or it has infinitely many uniform norms [BhDe4].

The present note is aimed at investigating UUNP in weighted measure algebra $M(G,\omega)$. In what follows, we briefly discuss preliminaries to fix up the notations.

Throughout let G be an LCA group with the Haar measure λ. Let $\widehat{G}$ be the dual group of G. A *generalized character* on G is a continuous homomorphism $\alpha : G \to (\mathbb{C}\setminus\{0\}, \times)$. Let $H(G)$ denote the set of all generalized characters on G equipped with the compact-open topology. For $\alpha, \beta \in H(G)$, define $(\alpha+\beta)(s) = \alpha(s)\beta(s)$ $(s \in G)$. If G is compactly generated, then $H(G)$ is an LCA group [BhDe5]. Let $C_c(G)$ denote the set of all continuous

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functions on G having compact supports equipped with the inductive limit topology τ. Then $(C_c(G),\tau)$ is a commutative topological algebra with the convolution product. For $\alpha\in H(G)$, define

$$\varphi_\alpha(f)=\int_G f(s)\alpha(s)\,d\lambda(s)\qquad(f\in C_c(G)).$$

Then $\varphi_\alpha\in\Delta(C_c(G))$, the Gelfand space of $C_c(G)$. Let $T: H(G)\to\Delta(C_c(G))$, $\alpha\mapsto\varphi_\alpha$. If G is second countable and compactly generated, then T is a homeomorphism [BhDe5].

By a *weight* on G, we mean a continuous function $\omega: G\to(0,\infty)$ such that $\omega(s+t)\le\omega(s)\omega(t)$ $(s,t\in G)$. The associated *Beurling algebra* is the convolution Banach algebra $L^1(G,\omega)$ of all measurable functions f on G satisfying $\|f\|_\omega:=\int_G|f|\omega\,d\lambda<\infty$. An element $\alpha\in H(G)$ is an *ω-bounded generalized character* if $|\alpha(s)|\le\omega(s)$ $(s\in G)$; let $H(G,\omega)$ denote the set of all such elements of $H(G)$. Let T_1 be the map T restricted to $H(G,\omega)$. Then T_1 is a homeomorphism from $H(G,\omega)$ onto $\Delta(L^1(G,\omega))$ for any LCA group.

2. Weighted measure algebras and UUNP. A *Radon measure* on G is a continuous linear functional on $C_c(G)$. Let $M_{\mathrm{loc}}(G)$ denote the linear space of all Radon measures on G (see [Da]). Let $L_{\mathrm{loc}}(G)$ denote the space of all locally integrable, measurable functions on G. Then clearly $L_{\mathrm{loc}}(G)\subseteq M_{\mathrm{loc}}(G)$. Let $M(G)$ be the convolution Banach algebra of all complex regular Borel measures (necessarily finite) on G with the total variation norm $\|\cdot\|$. For a weight ω on G, define

$$\begin{aligned}M(G,\omega)&=\{\mu\in M_{\mathrm{loc}}(G):\omega\mu\in M(G)\},\\ C_0(G,1/\omega)&=\{f\in C(G): f/\omega\in C_0(G)\},\\ C_b(G,1/\omega)&=\{f\in C(G): f/\omega\in C_b(G)\}.\end{aligned}$$

All are Banach spaces; the first one with norm $\|\mu\|_\omega:=\|\omega\mu\|$ and the others with norm $\|f\|_{\omega^{-1},\infty}:=\|\omega^{-1}f\|_\infty$. It is clear that every $\mu\in M(G,\omega)$ is a continuous linear functional on $C_b(G,1/\omega)$ by $\langle f,\mu\rangle=\int_G f/\omega\,d\nu$, where $\nu=\omega\mu\in M(G)$. For $\mu\in M(G,\omega)$ and $f\in C_b(G,1/\omega)$, define

$$T_f^\mu: G\to\mathbb{C}\quad\text{as}\quad T_f^\mu(s)=\langle\tau_{-s}f,\mu\rangle\quad(s\in G).$$

Then $T_f^\mu\in C_b(G,1/\omega)$ and $\|T_f^\mu\|_{\omega^{-1},\infty}\le\|\mu\|_\omega\|f\|_{\omega^{-1},\infty}$. For $\mu,\nu\in M(G,\omega)$, the convolution product $\mu*\nu$ is defined as

$$\langle f,\mu*\nu\rangle=\langle T_f^\mu,\nu\rangle\qquad(f\in C_b(G,1/\omega)).$$

Then it is a routine verification that $\mu*\nu\in M(G,\omega)$, $\mu*\nu=\nu*\mu$ and $\|\mu*\nu\|_\omega\le\|\mu\|_\omega\|\nu\|_\omega$. Thus $M(G,\omega)$ is a commutative Banach algebra. The pointmass measure δ_0 is the identity of $M(G,\omega)$. The Beurling algebra

$L^1(G,\omega)$ is a closed ideal in $M(G,\omega)$; and by [Gh, Lemma 2.3], $M(G,\omega)$ is exactly the multiplier algebra of $L^1(G,\omega)$.

Let $\mu \in M_{\mathrm{loc}}(G)$. Define $D_{\mathcal{L}\mu} = \{\alpha \in H(G) : \int_G |\alpha(s)|\, d|\mu|(s) < \infty\}$. When $D_{\mathcal{L}\mu} \neq \emptyset$, $\mathcal{L}\mu$ (also denoted by $\widehat{\mu}$), defined as

$$(\mathcal{L}\mu)(\alpha) = \widehat{\mu}(\alpha) = \int_G \alpha(s)\, d\mu(s) \qquad (\alpha \in D_{\mathcal{L}\mu}),$$

is the *Laplace transform* of μ. The following introduces weighted analogues of the classical transforms of harmonic analysis, thereby providing important tools for abelian weighted harmonic analysis.

Proposition 2.1. *Let $\mu, \nu \in M_{\mathrm{loc}}(G)$.*

(i) *$D_{\mathcal{L}\mu} \neq \emptyset$ iff $\mu \in M(G,\omega)$ for some weight ω on G.*
(ii) *$D_{\mathcal{L}\mu} \cap D_{\mathcal{L}\nu} \neq \emptyset$ iff $\mu, \nu \in M(G,\omega)$ for some weight ω on G. In this case, $\mu * \nu \in M(G,\omega)$ and $D_{\mathcal{L}\mu*\nu} \neq \emptyset$.*
(iii) *If $\alpha \in D_{\mathcal{L}\mu}$, then $\alpha + \widehat{G} \subseteq D_{\mathcal{L}\mu}$.*
(iv) *If $\alpha \in D_{\mathcal{L}\mu}$ and if $\mathcal{L}\mu = 0$ on $\alpha + \widehat{G}$, then $\mu = 0$.*
(v) *Let ω be a weight on G and let $\mu \in M(G,\omega)$. Then $H(G,\omega) \subseteq D_{\mathcal{L}\mu}$; the restriction of $\mathcal{L}\mu$ to $H(G,\omega)$ is the* generalized Fourier–Stieltjes transform *of μ.*
(vi) *Let ω be a weight on G and let $f \in L^1(G,\omega)$. Then $H(G,\omega) \subseteq D_{\mathcal{L}f}$; the restriction of $\mathcal{L}f$ to $H(G,\omega)$ is the* generalized Fourier transform *of f.*
(vii) *Let ω be a weight on G such that $\omega \geq 1$ on G and let $\mu \in M(G,\omega)$. Then $\widehat{G} \subseteq D_{\mathcal{L}\mu}$; the restriction of $\mathcal{L}\mu$ to $\widehat{G}$ is the* Fourier–Stieltjes transform *of μ.*
(viii) *Let ω be a weight on G with $\omega \geq 1$ on G and let $f \in L^1(G,\omega)$. Then $\widehat{G} \subseteq D_{\mathcal{L}f}$; the restriction of $\mathcal{L}f$ to $\widehat{G}$ is the* Fourier transform *of f.*

Proof. (i) Let $\alpha \in D_{\mathcal{L}\mu}$. Define $\omega(s) = |\alpha(s)|$ $(s \in G)$. Then $\mu \in M(G,\omega)$. Conversely, let $\mu \in M(G,\omega)$ for some weight ω on G. By [BhDe3], $H(G,\omega) \neq \emptyset$. Let $\beta \in H(G,\omega)$. Then $\int_G |\beta(s)|\, d|\mu|(s) \leq \int_G \omega(s)\, d|\mu|(s) < \infty$. Hence $H(G,\omega) \subseteq D_{\mathcal{L}\mu}$. In particular, $D_{\mathcal{L}\mu} \neq \emptyset$.

(ii) Assume that $D_{\mathcal{L}\mu} \cap D_{\mathcal{L}\nu} \neq \emptyset$. Choose $\alpha \in D_{\mathcal{L}\mu} \cap D_{\mathcal{L}\nu}$. Define $\omega(s) = |\alpha(s)|$ $(s \in G)$. Then $\mu, \nu \in M(G,\omega)$. Since $M(G,\omega)$ is a Banach algebra, $\mu * \nu \in M(G,\omega)$. By (i) above, $D_{\mathcal{L}\mu*\nu} \neq \emptyset$. Conversely, let $\mu, \nu \in M(G,\omega)$ for some weight ω on G. Then, as in the proof of (i), $\emptyset \neq H(G,\omega) \subseteq D_{\mathcal{L}\mu} \cap D_{\mathcal{L}\nu}$.

(iii) For $\alpha \in D_{\mathcal{L}\mu}$ and $\theta \in \widehat{G}$, $\int_G |(\alpha+\theta)(s)|\, d|\mu|(s) = \int_G |\alpha(s)|\, d|\mu|(s) < \infty$. Hence $\alpha + \widehat{G} \subseteq D_{\mathcal{L}\mu}$.

(iv) Let $\alpha \in D_{\mathcal{L}\mu}$ be such that $\mathcal{L}\mu = 0$ on $\alpha + \widehat{G}$. Since $\alpha \in D_{\mathcal{L}\mu}$, $\alpha\mu \in M(G)$. Now $\mathcal{L}(\alpha\mu)(\theta) = \mathcal{L}(\mu)(\alpha + \theta) = 0$ $(\theta \in \widehat{G})$. Hence $\alpha\mu = 0$. Since $\alpha(s) \neq 0$ for any $s \in G$, $\mu = 0$.

(v) For $\beta \in H(G,\omega)$, $\int_G |\beta(s)|\,d|\mu|(s) \le \int_G \omega(s)\,d|\mu|(s) < \infty$.

(vi) This can be proved as (v).

(vii) Let $\theta \in \widehat{G}$. Then $\theta \in H(G)$ and $\int_G |\theta(s)|\,d|\mu|(s) \le \int_G \omega(s)\,d|\mu|(s) < \infty$. Hence $\widehat{G} \subseteq D_{\mathcal{L}\mu}$.

(viii) This can be proved as in (vii). ∎

Proposition 2.2. *Let ω be a weight on G. Then there exists a weight $\widetilde{\omega} \ge 1$ on G such that $M(G,\omega)$ is isometrically isomorphic to $M(G,\widetilde{\omega})$.*

Proof. Since $H(G,\omega)$ is non-empty, choose $\alpha \in H(G,\omega)$. Define $\widetilde{\omega}(s) = \omega(s)/|\alpha(s)|$ $(s \in G)$. Then $\widetilde{\omega}$ is a weight and $\widetilde{\omega} \ge 1$ on G. Now define $T : M(G,\omega) \to M(G,\widetilde{\omega})$ as $T(\mu) = \alpha\mu$. Then, for $\mu \in M(G,\omega)$,

$$\|T(\mu)\|_{\widetilde{\omega}} = \|\alpha\mu\|_{\widetilde{\omega}} = \int_G \frac{\omega(s)}{|\alpha(s)|}\,d|\alpha\mu|(s) = \int_G \omega(s)\,d|\mu|(s) = \|\mu\|_\omega.$$

It is easy to see that T is an algebra isomorphism. ∎

Proposition 2.3. *Let ω be a weight on G. Then the following are equivalent*:

(i) *$M(G,\omega)$ is regular*;
(ii) *$M(G,\omega)$ has UUNP*;
(iii) *$L^1(G,\omega)$ has UUNP and G is discrete.*

Proof. By Proposition 2.2, we can assume that $\omega \ge 1$ on G.

(i)⇒(ii). This is true for all semisimple, commutative Banach algebras.

(ii)⇒(iii). Since $M(G,\omega)$ is a dense subalgebra of $M(G)$, $M(G)$ has UUNP due to Theorem 2.4 in [BhDe2]. Hence, by [BhDe2, p. 233], G must be discrete. In this case, $L^1(G,\omega) = M(G,\omega)$ has UUNP.

(iii)⇒(i). Since G is discrete, $M(G,\omega) = L^1(G,\omega)$. By [BhDe2, Theorem 4.1] and [BhDe3], $L^1(G,\omega)$ is regular. ∎

The following is the main result of the paper. It compares with the result that $L^1(G,\omega)$ has a minimum uniform norm if and only if $L^1(G,\omega)$ has UUNP [BhDe4, Theorem 1].

Theorem 2.4. *$M(G,\omega)$ has a minimum uniform norm if and only if $L^1(G,\omega)$ has UUNP.*

Proof. By Proposition 2.2, we can assume that $\omega \ge 1$ on G. Assume that $L^1(G,\omega)$ has UUNP. Then it is regular due to [BhDe2, Theorem 4.1]. Since $M(G,\omega)$ is the multiplier algebra of $L^1(G,\omega)$ and $\varDelta(L^1(G,\omega)) \cong H(G,\omega)$ is a set of uniqueness for $M(G,\omega)$ by Proposition 2.1, it follows from [BhDe2, Corollary 6.3] that $|\mu|_\infty := \sup\{|\widehat{\mu}(\alpha)| : \alpha \in H(G,\omega)\}$ is the minimum uniform norm on $M(G,\omega)$.

Conversely, assume that $M(G,\omega)$ has a minimum uniform norm, say $|\cdot|_0$. Define $F = \{\alpha \in H(G,\omega) : \alpha \text{ is } |\cdot|_0\text{-continuous}\}$. Then $|f|_0 = \sup\{|\varphi_\alpha(f)| : \alpha \in F\} := |f|_F$ $(f \in L^1(G,\omega))$. We break up the proof into three steps.

STEP I: $|\cdot|_0$ *is a minimum uniform norm on* $L^1(G,\omega)$. Let $|\cdot|$ be any uniform norm on $L^1(G,\omega)$. Define

$$|\mu|_{\mathrm{op}} = \sup\{|f * \mu| : f \in L^1(G,\omega) \text{ and } |f| \le 1\} \quad (\mu \in M(G,\omega)).$$

Since $L^1(G,\omega)$ has a bounded approximate identity, $|\cdot|_{\mathrm{op}}$ is a uniform norm on $M(G,\omega)$ and it is identical to $|\cdot|$ on $L^1(G,\omega)$. Since $|\cdot|_0$ is a minimum uniform norm on $M(G,\omega)$, $|\cdot|_0 \le |\cdot|_{\mathrm{op}}$ on $M(G,\omega)$ and hence $|\cdot|_0 \le |\cdot|_{\mathrm{op}} = |\cdot|$ on $L^1(G,\omega)$. Thus $|\cdot|_0$ is a minimum uniform norm on $L^1(G,\omega)$.

STEP II: $F = \widehat{G} + F$. Fix $\theta \in \widehat{G}$. Define $|f|_{\theta+F} = |\theta f|_F$ on $L^1(G,\omega)$, where $\theta + F := \{\theta + \alpha : \alpha \in F\}$. Then $|\cdot|_{\theta+F}$ is a uniform norm on $L^1(G,\omega)$. Since $|\cdot|_F$ $(= |\cdot|_0)$ is the minimum uniform norm on $L^1(G,\omega)$, we have

$$(1) \qquad |f|_F \le |f|_{\theta+F} \quad (f \in L^1(G,\omega)).$$

This holds for each $\theta \in \widehat{G}$. So $|f|_{\theta+F} \le |\theta f|_{\overline{\theta}+F} = |f|_F$. Hence $\overline{\theta} + F \subseteq F$ by the definition of F and $|\cdot|_F = |\cdot|_0$. Thus it follows that $F = \widehat{G} + F$.

STEP III: $F = H(G,\omega)$. Suppose this is not the case. Then there exist positive generalized characters $\alpha \in F$ and $\beta \in H(G,\omega) \setminus F$ because $F = \widehat{G} + F$. Choose $t \in G$ such that $\beta(t) < \alpha(t)$. Let U be an open neighbourhood of t such that its closure is compact and $\beta(s) < \alpha(s)$ $(s \in U)$. Take $f = \chi_U$. Then $f \in L^1(G,\omega)$ because ω is continuous. Now

$$\begin{aligned}
|f|_{\beta+\widehat{G}} &= \sup\{|\varphi_{\beta+\theta}(f)| : \theta \in \widehat{G}\} \\
&= \sup\Big\{\Big|\int_G f(s)\beta(s)\theta(s)\,d\lambda(s)\Big| : \theta \in \widehat{G}\Big\} \\
&\le \sup\Big\{\int_U \beta(s)|\theta(s)|\,d\lambda(s) : \theta \in \widehat{G}\Big\} \\
&= \int_U \beta(s)\,d\lambda(s) < \int_U \alpha(s)\,d\lambda(s) \\
&\le \sup\{|\varphi_{\alpha+\theta}(f)| : \theta \in \widehat{G}\} = |f|_{\alpha+\widehat{G}} \le |f|_F.
\end{aligned}$$

Thus $|\cdot|_{\beta+\widehat{G}}$ is a uniform norm and $|f|_{\beta+\widehat{G}} < |f|_F$, which is a contradiction because the latter is the minimum uniform norm on $L^1(G,\omega)$. Thus $F = H(G,\omega)$.

By Step III, $|f|_0 = |f|_F = |f|_{H(G,\omega)} = r(f)$ $(f \in L^1(G,\omega))$, where $r(\cdot)$ denotes the spectral radius on $L^1(G,\omega)$. Thus the spectral radius is the only uniform norm on $L^1(G,\omega)$, and hence $L^1(G,\omega)$ has UUNP. ∎

EXAMPLE 2.5. For $\alpha \geq 0$, define $\omega_\alpha(s) = (1+|s|)^\alpha$ $(s \in \mathbb{R})$. Then the Beurling algebra $L^1(\mathbb{R}, \omega_\alpha)$ has UUNP. Thus $M(\mathbb{R}, \omega_\alpha)$ has a minimum uniform norm, namely $\mu \mapsto |\widehat{\mu}|_\infty$, but $M(\mathbb{R}, \omega_\alpha)$ fails to have UUNP. ∎

Let

$$\begin{aligned} M_0(G,\omega) &:= \{\mu \in M(G,\omega) : \widehat{\mu} \in C_0(\Delta(L^1(G,\omega)))\} \\ &= \{\mu \in M(G,\omega) : \widehat{\mu} \in C_0(H(G,\omega))\} \end{aligned}$$

and

$$\begin{aligned} M_{00}(G,\omega) &:= \{\mu \in M(G,\omega) : \widehat{\mu} = 0 \text{ on } \Delta(M(G,\omega)) \setminus \Delta(L^1(G,\omega))\} \\ &= \{\mu \in M(G,\omega) : \widehat{\mu} = 0 \text{ on } \Delta(M(G,\omega)) \setminus H(G,\omega)\} \\ &= \{\mu \in M(G,\omega) : \widehat{\mu} = 0 \text{ on } h(L^1(G,\omega))\}, \end{aligned}$$

where $h(L^1(G,\omega))$ is the hull of $L^1(G,\omega)$ in $M(G,\omega)$. Then both are closed ideals in $M(G,\omega)$ and $L^1(G,\omega) \subseteq M_{00}(G,\omega) \subseteq M_0(G,\omega)$. These are weighted measure algebra analogues of $M_0(G)$ and $M_{00}(G)$ considered in [LN, p. 376]. Our feeling is that $M_{00}(G,\omega)$ is close in spirit to $L^1(G,\omega)$, whereas $M_0(G,\omega)$ is close to $M(G,\omega)$. The following supports this. It also compares with Proposition 2.3.

PROPOSITION 2.6. *Let ω be a weight on G. Then the following are equivalent*:

(i) $M_{00}(G,\omega)$ *is regular*;
(ii) $M_{00}(G,\omega)$ *has UUNP*;
(iii) $L^1(G,\omega)$ *has UUNP.*

Proof. (i)⇒(ii). This is true for all semisimple, commutative, Banach algebras.

(ii)⇒(iii). Let $M_{00}(G,\omega)$ have UUNP. Let $|\cdot|$ be a uniform norm on $L^1(G,\omega)$. Define

$$|\mu|_{\mathrm{op}} = \sup\{|f * \mu| : f \in L^1(G,\omega) \text{ and } |f| \leq 1\} \quad (\mu \in M_{00}(G,\omega)).$$

Then $|\cdot|_{\mathrm{op}}$ is a uniform norm on $M_{00}(G,\omega)$ and it is identical to $|\cdot|$ on $L^1(G,\omega)$. Since $M_{00}(G,\omega)$ has UUNP, $|\cdot|_{\mathrm{op}}$ is identical to the spectral radius of $M_{00}(G,\omega)$. Since $L^1(G,\omega)$ is an ideal in $M_{00}(G,\omega)$, $|\cdot| = |\cdot|_{\mathrm{op}}$ is identical to the spectral radius on $L^1(G,\omega)$. Thus $L^1(G,\omega)$ has UUNP.

(iii)⇒(i). Assume that $L^1(G,\omega)$ has UUNP. By [BhDe2, Theorem 4.1], $L^1(G,\omega)$ is regular. Since $L^1(G,\omega)) \subseteq M_{00}(G,\omega)$ and since $\Delta(L^1(G,\omega)) = \Delta(M_{00}(G,\omega))$, $M_{00}(G,\omega)$ is regular. ∎

CONJECTURE. Motivated by the fact that $M_0(G)$ fails to have UUNP [BhDe2, p. 234], we conjecture that $M_0(G,\omega)$ has UUNP if and only if $L^1(G,\omega)$ has UUNP and G is discrete.

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