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Spectral properties of quotients of Beurling-type submodules of the Hardy module over the unit ball

by

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Abstract. Let M be a Beurling-type submodule of $H^2(\mathbb{B}_d)$, the Hardy space over the unit ball $\mathbb{B}_d$ of $\mathbb{C}^d$, and let $N = H^2(\mathbb{B}_d)/M$ be the associated quotient module. We completely describe the spectrum and essential spectrum of N, and related index theory.

1. Introduction. Let $\mathbb{B}_d$ be the unit ball of $\mathbb{C}^d$, and let $H^2(\mathbb{B}_d)$ be the Hardy space over $\mathbb{B}_d$, which consists of the analytic functions in $\mathbb{B}_d$ satisfying

$$\sup_{0<r<1} \int_{\partial\mathbb{B}_d} |f(r\xi)|^2 \, d\sigma(\xi) < \infty,$$

where $d\sigma$ is the normalized Lebesgue measure on the unit sphere $\partial\mathbb{B}_d$. Let η be an *inner function* on the unit ball, that is, η is a nonconstant function in $H^\infty(\mathbb{B}_d)$ satisfying $|f^*(\zeta)| = 1$ a.e. $[\sigma]$ on $\partial\mathbb{B}_d$, where $f^*(\zeta) = \lim_{r\to 1} f(r\zeta)$. In the sixties, Rudin posed the existence problem: Do there exist inner functions in $H^\infty(\mathbb{B}_d)$ [Rud1]? This problem was affirmatively solved in 1982 by B. Aleksandrov [Rud2].

When $d = 1$, $\mathbb{B}_1 = U$ is the open unit disk, and $H^2(U)$ is the classical Hardy space over U. Let $N = H^2(U) \ominus \eta H^2(U)$ for a one-variable inner function η, and let $N_z = P_N M_z|N$ be the compression of the coordinate multiplication operator M_z to N; here P_N is the projection onto N. Then the Livšic–Moeller theorem [Nil, p. 62] states that the spectrum of N_z equals the zero set of η in the open unit disk, together with the points on the unit circle $\mathbb{T}$ to which η cannot be analytically continued from U, that is,

$$\sigma(N_z) = Z(\eta) \cup E,$$

where $Z(\eta)$ is the zero set of η in the unit disk and

$$E = \{\lambda \in \mathbb{T} \mid \text{there is a sequence } \{\xi_i\} \subset U \text{ such that } \lim_{i\to\infty} \xi_i = \lambda \text{ and } \lim_{i\to\infty} \eta(\xi_i) = 0\}.$$

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Moreover, if η is not a finite Blaschke product, then a result of Arveson [Arv1, Theorem 3.4.3] shows that the essential spectrum of N_z is exactly E. The proofs of these results rely heavily on the fact that each inner function η ($d = 1$) has the factorization $\eta = BS$, where B is a Blaschke product and S is a singular inner function. When $d > 1$, inner functions have no such factorization. In fact, the structure of inner functions in several variables is far from clear. In this paper, we will deal with the case $d > 1$.

Given a Hilbert space H, let $(T_1, \ldots, T_d)$ be a tuple of commuting bounded operators acting on H. Then one naturally makes H into a Hilbert module over the polynomial ring $\mathbb{C}[z_1, \ldots, z_d]$ (cf. [DP]) by setting

$$p \cdot \xi = p(T_1, \ldots, T_d)\xi, \quad p \in \mathbb{C}[z_1, \ldots, z_d],\ \xi \in H.$$

Now let $(M_{z_1}, \ldots, M_{z_d})$ be the coordinate multiplication operators acting on $H^2(\mathbb{B}_d)$. Then $H^2(\mathbb{B}_d)$ naturally becomes a Hilbert module over the polynomial ring $\mathbb{C}[z_1, \ldots, z_d]$, called the *Hardy module* on the unit ball $\mathbb{B}_d$. A closed subspace M of $H^2(\mathbb{B}_d)$ is called a *submodule* if $pM \subset M$ for any polynomial p. In the case of one variable, Beurling's theorem shows that each submodule has the form $M = \eta H^2(U)$ for some inner function η. However, in the case of several variables, there exist a lot of submodules that do not have the above form, for example, submodules of finite codimension [CG, DP]. Given an inner function η on $\mathbb{B}_d$, the submodule $M = \eta H^2(\mathbb{B}_d)$ is said to be a *Beurling-type submodule*. Given a submodule M of $H^2(\mathbb{B}_d)$, the quotient module $H^2(\mathbb{B}_d)/M$ is naturally identified with $M^\perp$, where the module action on $M^\perp$ is $p \cdot \xi = P_{M^\perp} p\xi$ for any polynomial p and $\xi \in M^\perp$.

This paper mainly considers spectral properties of quotients of Beurling-type submodules on the unit ball. Given an inner function η, let $N = H^2(\mathbb{B}_d) \ominus \eta H^2(\mathbb{B}_d)$ be the associated quotient module, and $N_z = (N_{z_1}, \ldots, N_{z_d})$ be the tuple of compression operators acting on N, where $N_{z_i} = P_N M_{z_i}|N$, and P_N is the projection onto N. We will determine the spectrum and essential spectrum of the quotient module, that is, the Taylor spectrum and essential spectrum of the tuple $N_z = (N_{z_1}, \ldots, N_{z_d})$. Determining the spectrum and essential spectrum of a general quotient module is very difficult since the problem is related to essential normality of the quotient module [Arv5, Arv6, Dou, GW]. But even for submodules generated by homogeneous polynomials, essential normality of submodules remains unknown [Arv5, Arv6, Dou]. The paper also concerns K-homology defined by quotient modules.

2. The Taylor spectrum of N_z. In this section, we will determine the spectrum of the quotient module N, i.e. the Taylor spectrum of $N_z = (N_{z_1}, \ldots, N_{z_d})$. First, let us recall the concept of the Taylor spectrum [Tay].

For $1 \le k \le d$, let

$$I_k = \{(i_1, \dots, i_k) \in \mathbb{N} \mid 1 \le i_1 < \dots < i_k \le d\}.$$

Let $\{e_i \mid 1 \le i \le d\}$ be an orthonormal basis for $\mathbb{C}^d$, and $\bigwedge^k \mathbb{C}^d$ be the kth exterior power of $\mathbb{C}^d$ with orthonormal basis $\{e_{i_1} \wedge \dots \wedge e_{i_k} \mid (i_1, \dots, i_k) \in I_k\}$. Let $\bigwedge \mathbb{C}^d = \bigoplus_{k=0}^d \bigwedge^k \mathbb{C}^d$, where $\bigwedge^0 \mathbb{C}^d = \mathbb{C}$. Then $\bigwedge \mathbb{C}^d$ is a Hilbert space with orthonormal basis $\{1\} \cup \{e_{i_1} \wedge \dots \wedge e_{i_k} \mid (i_1, \dots, i_k) \in I_k,\ 1 \le k \le d\}$. Define the canonical creation operators $C_1, \dots, C_d$ on $\bigwedge \mathbb{C}^d$ as follows:

$$C_i : \xi \mapsto e_i \wedge \xi, \qquad \xi \in \textstyle\bigwedge \mathbb{C}^d.$$

Let $T_1, \dots, T_d$ be a commuting d-tuple of operators on a Hilbert space H. Then the *Koszul complex* for the d-tuple $(T_1, \dots, T_d)$ is

$$0 \to \Omega^0 \to \Omega^1 \to \dots \to \Omega^d \to 0, \quad \text{where} \quad \Omega^k = H \otimes \textstyle\bigwedge^k \mathbb{C}^d$$

with cohomological boundary operator

$$B = T_1 \otimes C_1 + \dots + T_d \otimes C_d.$$

It is easy to see $B^2 = 0$. We denote the restriction of B to Ω^k by B_k, and hence $\operatorname{ran} B_{k-1} \subseteq \ker B_k$. Say that the tuple $(T_1, \dots, T_d)$ is *invertible* if $\operatorname{ran} B_{k-1} = \ker B_k$, and $(T_1, \dots, T_d)$ is *Fredholm* if

$$\dim(\ker B_k / \operatorname{ran} B_{k-1}) < \infty \quad \text{for } 1 \le k \le d.$$

When $T = (T_1, \dots, T_d)$ is Fredholm, define its *Fredholm index* as

$$\operatorname{ind}(T) = \sum_{k=1}^{d} (-1)^{k+1} \dim(\ker B_k / \operatorname{ran} B_{k-1}).$$

The *Taylor spectrum* and *essential spectrum* of $T = (T_1, \dots, T_d)$ are defined as

$$\sigma(T) = \{(\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d \mid (T_1 - \lambda_1, \dots, T_d - \lambda_d) \text{ is not invertible}\},$$
$$\sigma_{\mathrm{e}}(T) = \{(\lambda_1, \dots, \lambda_d) \in \mathbb{C}^d \mid (T_1 - \lambda_1, \dots, T_d - \lambda_d) \text{ is not Fredholm}\}.$$

We begin with a lemma which may be viewed as the H^∞ functional calculus of N_z. It is a generalization of the one-variable version in [Nil], and the proof is omitted.

LEMMA 2.1. *Let*

$$\varphi(N_{z_1}, \dots, N_{z_d}) = P_N \varphi | N, \qquad \varphi(N_z^*) = P_+ \overline{\varphi}^t | N,$$

where $\varphi \in H^\infty$, $\varphi^t = \overline{\varphi(\overline{z}_1, \dots, \overline{z}_d)}$, $N_z^* = (N_{z_1}^*, \dots, N_{z_d}^*)$, *and* P_+ *is the projection from* $L^2(\partial \mathbb{B}_d)$ *onto* $H^2(\mathbb{B}_d)$. *Then*

(1) $\|\varphi(N_z)\| \le \|\varphi\|_\infty$;

(2) *the map* $\varphi \mapsto \varphi(N_z)$ *is linear, multiplicative, and*

$$P(N_z) = \sum_\alpha a_\alpha N_{z_1}^{\alpha_1} \cdots N_{z_d}^{\alpha_d}$$

for each $P = \sum_\alpha a_\alpha z^\alpha \in \mathbb{C}[z_1, \dots, z_d]$; *moreover,* $\varphi(N_z^*) = \varphi^t(N_z)^*$.

Proposition 2.2. $\sigma(N_z) \cap \mathbb{B}_d = Z(\eta)$, *where* $Z(\eta) = \{\lambda \in \mathbb{B}_d \mid \eta(\lambda) = 0\}$ *is the zero set of* η.

Proof. From [Rud1, p. 116], one can solve Gleason's problem in $H^\infty(\mathbb{B}_d)$, that is, given $f \in H^\infty(\mathbb{B}_d)$ and $\lambda \in \mathbb{B}_d$, there exist $g_1, \dots, g_d \in H^\infty(\mathbb{B}_d)$ such that

$$f(z) - f(\lambda) = \sum_{i=1}^d (z_i - \lambda_i) g_i.$$

Replacing f by η and using the H^∞ functional calculus for N_z, we have

$$\eta(N_z) - \eta(\lambda) = \sum_{i=1}^d (N_{z_i} - \lambda_i) g_i(N_z).$$

Lemma 2.1 implies $\eta(N_z) = 0$. Hence, if $\eta(\lambda) \neq 0$, then the tuple N_z is invertible relative to $(N_z)'$, the commutant algebra of the norm closed algebra generated by $N_{z_1}, \dots, N_{z_d}$. This means $\sigma'(N_z) \cap \mathbb{B}_d \subseteq Z(\eta)$, where $\sigma'(N_z)$ denotes the algebra spectrum relative to the commutant algebra $(N_z)'$ [Cur2]. By [Cur2, Lemma 4.5], $\sigma(N_z) \subseteq \sigma'(N_z)$. This gives

$$\sigma(N_z) \cap \mathbb{B}_d \subseteq Z(\eta).$$

To prove the reverse inclusion, let $\lambda \in Z(\eta)$. It suffices to show that

$$N \neq (N_{z_1} - \lambda_1)N + \cdots + (N_{z_d} - \lambda_d)N,$$

since this implies that $\lambda \in \sigma(N_z)$. Indeed, assume that

$$N = (N_{z_1} - \lambda_1)N + \cdots + (N_{z_d} - \lambda_d)N,$$

that is,

$$H^2(\mathbb{B}_d) \ominus \eta H^2(\mathbb{B}_d) = P_N\Big(\sum_{i=1}^d (M_{z_i} - \lambda_i)(H^2(\mathbb{B}_d) \ominus \eta H^2(\mathbb{B}_d))\Big).$$

Then, for each $g \in N = H^2(\mathbb{B}_d) \ominus \eta H^2(\mathbb{B}_d)$, there exist $f_1, \dots, f_d \in H^2(\mathbb{B}_d) \ominus \eta H^2(\mathbb{B}_d)$ and $h \in H^2(\mathbb{B}_d)$ such that

$$h = (z_1 - \lambda_1)f_1 + \cdots + (z_d - \lambda_d)f_d, \qquad g = P_N h.$$

Hence, there exists $g_1 \in H^2(\mathbb{B}_d)$ such that $h = \eta g_1 + g$. Thus, $g = h - \eta g_1$, and so

$$g(\lambda) = h(\lambda) - \eta(\lambda) g_1(\lambda) = 0$$

for each $g \in N = H^2(\mathbb{B}_d) \ominus \eta H^2(\mathbb{B}_d)$, which is obviously not true: if we decompose 1 as

$$1 = \eta f_1 + f_2,$$

where $f_1 \in H^2(\mathbb{B}_d)$ and $f_2 \in N$, then obviously $f_2(\lambda) \neq 0$. This contradiction completes the proof of Proposition 2.2.

In the next section, we will prove that $\sigma_{\mathrm{e}}(N_z) = \partial\mathbb{B}_d$. Since $\sigma_{\mathrm{e}}(N_z) \subseteq \sigma(N_z)$, it follows that $\sigma(N_z) \cap \partial\mathbb{B}_d = \partial\mathbb{B}_d$. Here, we want to give an elementary proof of this result.

PROPOSITION 2.3. $\sigma(N_z) \cap \partial\mathbb{B}_d = \partial\mathbb{B}_d$.

We need the following lemma which comes from [Rud2, Theorem 1.2].

LEMMA 2.4. *Suppose that*:

- Γ *is a nonempty open set in* $\partial\mathbb{B}_d$;
- $\{r_j\}$ *is a sequence satisfying* $0 \le r_j < 1$, $r_j \nearrow 1$ *as* $j \to \infty$;
- $f \in H^\infty(\mathbb{B}_d)$, f *is nonconstant*, $|f^*(\zeta)| = 1$ *a.e. on* Γ.

Then Γ *has a dense subset* H *such that the set*

$$\{f(r_j\zeta) \mid j = 1, 2, 3, \ldots\}$$

is dense in the unit disk U *for every* $\zeta \in H$.

For $f \in L^\infty(\partial\mathbb{B}_d)$, define the Toeplitz operator T_f with symbol f as

$$T_f : H^2(\mathbb{B}_d) \to H^2(\mathbb{B}_d), \quad T_f h = P_+ f h, \quad h \in H^2(\mathbb{B}_d),$$

where P_+ is the projection from $L^2(\partial\mathbb{B}_d)$ onto $H^2(\mathbb{B}_d)$.

Proof of Proposition 2.3. Applying Lemma 2.4 to $\partial\mathbb{B}_d$, with f replaced by the inner function η, we get a dense subset H of $\partial\mathbb{B}_d$ such that for each $\lambda \in H$, there is a sequence $\{\lambda_j\} \subseteq \mathbb{B}_d$ with $\lambda_j \to \lambda$ and $\eta(\lambda_j) \to 0$ as $j \to \infty$. Since $\sigma(N_z)$ is a compact set, it follows that $\sigma(N_z) \cap \partial\mathbb{B}_d$ is a closed subset of $\partial\mathbb{B}_d$. It is enough to prove that

$$H \subseteq \sigma(N_z) \cap \partial\mathbb{B}_d.$$

Indeed, assume there exists $\lambda_0 = (\lambda_{01}, \ldots, \lambda_{0d}) \in H$, but $\lambda_0 \notin \sigma(N_z)$. Set $N_z^* = (N_{z_1}^*, \ldots, N_{z_d}^*)$. It follows from [Cur1, Corollary 3.14] that

$$\sigma(N_z^*) = \{\overline{\lambda} \mid \lambda \in \sigma(N_z)\}.$$

Thus $\overline{\lambda}_0 \notin \sigma(N_z^*)$. Recall

$$\sigma_\pi(N_z^*) = \{\lambda \in \mathbb{C}^d \mid N_z^* - \lambda \text{ is not jointly bounded below}\}$$

(a tuple $(T_1, \ldots, T_d)$ on a Hilbert space X is said to be *jointly bounded below* if there exists $\varepsilon > 0$ such that $\sum_{i=1}^d \|T_i x\| \ge \varepsilon \|x\|$ for all $x \in X$.) Since $\sigma_\pi(N_z^*) \subseteq \sigma(N_z^*)$ (cf. [Cur2, p. 37]), this implies $\overline{\lambda}_0 \notin \sigma_\pi(N_z^*)$, that is, there exists $\varepsilon_0 > 0$ such that for all $f \in N$,

$$\sum_{i=1}^{d} \|(N_{z_i}^* - \overline{\lambda}_{0i})f\| = \sum_{i=1}^{d} \|(M_{z_i}^* - \overline{\lambda}_{0i})f\| \geq \varepsilon_0 \|f\|. \tag{2.1}$$

Now, we use (2.1) to prove that there exists $\delta > 0$ such that

$$\sum_{i=1}^{d} \|(M_{z_i}^* - \overline{\lambda}_{0i})g\| + \|T_{\overline{\eta}} g\| \geq \delta \|g\|, \quad \forall g \in H^2(\mathbb{B}_d). \tag{2.2}$$

For $g \in H^2(\mathbb{B}_d)$, write $g = h_1 + \eta h_2$, where $h_1 \in N$ and $h_2 \in H^2(\mathbb{B}_d)$. It is easy to see that $T_{\overline{\eta}} h_1 = 0$, and thus

$$T_{\overline{\eta}} g = T_{\overline{\eta}} h_1 + T_{\overline{\eta}} \eta h_2 = h_2.$$

We have

$$\begin{aligned} \sum_{i=1}^{d} \|(M_{z_i}^* - \overline{\lambda}_{0i})g\| + \|T_{\overline{\eta}} g\| \\ = \sum_{i=1}^{d} \|(M_{z_i}^* - \overline{\lambda}_{0i})h_1 + (M_{z_i}^* - \overline{\lambda}_{0i})\eta h_2\| + \|h_2\|. \end{aligned} \tag{2.3}$$

Let $A = \max_{1 \leq i \leq d} \|M_{z_i}^* - \overline{\lambda}_{0i}\| > 0$, and consider the following two cases:

Case 1. If $\|h_1\| \geq 2Ad\|h_2\|/\varepsilon_0$, then using (2.1), we obtain

$$\begin{aligned} &\sum_{i=1}^{d} (\|(M_{z_i}^* - \overline{\lambda}_{0i})h_1 + (M_{z_i}^* - \overline{\lambda}_{0i})\eta h_2\|) + \|h_2\| \\ &\qquad \geq \sum_{i=1}^{d} (\|(M_{z_i}^* - \overline{\lambda}_{0i})h_1\| - \|(M_{z_i}^* - \overline{\lambda}_{0i})\eta h_2\|) + \|h_2\| \\ &\qquad \geq \varepsilon_0 \|h_1\| - \frac{\varepsilon_0}{2} \|h_1\| + \|h_2\| = \frac{\varepsilon_0}{2} \|h_1\| + \|h_2\|. \end{aligned}$$

Noting that $\|g\|^2 = \|h_1\|^2 + \|h_2\|^2$, it is easy to see there exists $\delta_1 > 0$ independent of g such that

$$\sum_{i=1}^{d} \|(M_{z_i}^* - \overline{\lambda}_{0i})h_1 + (M_{z_i}^* - \overline{\lambda}_{0i})\eta h_2\| + \|h_2\| \geq \delta_1 \|g\|.$$

Case 2. If $\|h_1\| \leq 2Ad\|h_2\|/\varepsilon_0$, then

$$\begin{aligned} &\sum_{i=1}^{d} (\|(M_{z_i}^* - \overline{\lambda}_{0i})h_1 + (M_{z_i}^* - \overline{\lambda}_{0i})\eta h_2\|) + \|h_2\| \geq \|h_2\| \\ &\qquad \geq \|h_2\|/2 + \varepsilon_0 \|h_1\|/4Ad, \end{aligned}$$

and the same argument shows that there exists $\delta_2 > 0$ independent of g such that

$$\sum_{i=1}^{d} \|(M_{z_i}^* - \overline{\lambda}_{0i})h_1 + (M_{z_i}^* - \overline{\lambda}_{0i})\eta h_2\| + \|h_2\| \geq \delta_2\|g\|.$$

Letting $\delta = \min\{\delta_1, \delta_2\}$ proves that in both cases, the assumption (2.1) implies (2.2). Now, we show that (2.2) is not true, which implies that (2.1) is not either, and thus, $\overline{\lambda}_0 \in \sigma(N_z^*)$ and $\lambda_0 \in \sigma(N_z)$. Let us return to the beginning of the proof: there exists a sequence $\{\lambda_j\} \subseteq \mathbb{B}_d$ with $\lambda_j = (\lambda_{j1}, \ldots, \lambda_{jd})$ such that

$$\lambda_j \to \lambda_0, \quad \eta(\lambda_j) \to 0 \quad \text{as } j \to \infty.$$

Let K_λ be the reproducing kernel for $H^2(\mathbb{B}_d)$ and $k_\lambda = K_\lambda/\|K_\lambda\|$ the normalized reproducing kernel. Then

$$\begin{aligned}\sum_{i=1}^{d} \|(M_{z_i}^* - \overline{\lambda}_{0i})k_{\lambda_j}\| + \|T_{\overline{\eta}}k_{\lambda_j}\| &= \sum_{i=1}^{d} \|(\overline{\lambda}_{ji} - \overline{\lambda}_{0i})k_{\lambda_j}\| + \|\overline{\eta}(\lambda_j)k_{\lambda_j}\| \\ &= \sum_{i=1}^{d} |(\overline{\lambda}_{ji} - \overline{\lambda}_{0i})| + |\overline{\eta}(\lambda_j)| \to 0.\end{aligned}$$

This contradicts (2.2), and the proof of Proposition 2.3 is complete.

It is not difficult to see that $\sigma(N_z) \subseteq \overline{\mathbb{B}}_d$. Combining Propositions 2.2 and 2.3 gives the following.

THEOREM 2.5. *$\sigma(N_z) = Z(\eta) \cup \partial\mathbb{B}_d$.*

3. The essential spectrum of N_z. First, we prove the following proposition.

PROPOSITION 3.1. *The quotient module N is essentially normal, that is, the commutators $[N_{z_i}^*, N_{z_j}]$ are compact for all $1 \leq i, j \leq d$.*

Proof. For any $f \in L^\infty(\partial\mathbb{B}_d)$, it is easy to verify that $T_f T_{z_i} - T_{z_i} T_f$ is compact for $i = 1, \ldots, d$. Since η is an inner function, this means

$$P_N = I - T_\eta T_\eta^* = I - T_\eta T_{\overline{\eta}}. \tag{3.1}$$

Since

$$\begin{aligned}N_{z_i}^* N_{z_j} - N_{z_j} N_{z_i}^* &= P_N T_{\overline{z}_i} P_N T_{z_j} P_N - P_N T_{z_j} P_N T_{\overline{z}_i} P_N \\ &= P_N (T_{\overline{z}_i} P_N T_{z_j} - T_{z_j} P_N T_{\overline{z}_i}) P_N,\end{aligned}$$

inserting (3.1) and using the fact that T_{z_i} essentially commutes with each Toeplitz operator for $i = 1, \ldots, d$, we conclude that $[N_{z_i}^*, N_{z_j}]$ is compact for $1 \leq i, j \leq d$.

We now determine the essential spectrum $\sigma_{\mathrm{e}}(N_z)$.

LEMMA 3.2. $\sigma_{\mathrm{e}}(N_z) \subseteq \partial\mathbb{B}_d$.

Proof. Set $M = \eta H^2(\mathbb{B}_d)$. Since $z_i M \subseteq M$, we have $M_{z_i}^* N \subseteq N$ for $i = 1, \dots, d$. This means

$$\sum_{i=1}^{d} N_{z_i} N_{z_i}^* = \sum_{i=1}^{d} P_N M_{z_i} P_N M_{z_i}^* | N = P_N \sum_{i=1}^{d} M_{z_i} M_{z_i}^* | N.$$

Since $\sum_{i=1}^{d} M_{z_i} M_{z_i}^* - I$ is compact, this implies

$$\sum_{i=1}^{d} N_{z_i} N_{z_i}^* = I_N + K, \tag{3.2}$$

where K is a compact operator. By Proposition 3.1, we know the tuple $N_z = \{N_{z_1}, \dots, N_{z_d}\}$ is essentially normal. Combining [Cur1, Cor. 3.10] and (3.2) gives $\sigma_{\mathrm{e}}(N_z) \subseteq \partial\mathbb{B}_d$.

LEMMA 3.3. $\partial\mathbb{B}_d \subseteq \sigma_{\mathrm{e}}(N_z)$.

Proof. As in the proof of Proposition 2.3, we get a dense subset H of $\partial\mathbb{B}_d$ such that for each $\lambda_0 \in H$, there is a sequence $\{\lambda_j\} \subseteq \mathbb{B}_d$ such that $\lambda_j \to \lambda_0$ and $\eta(\lambda_j) \to 0$ as $j \to \infty$. Since $\sigma_{\mathrm{e}}(N_z)$ is a closed subset of $\partial\mathbb{B}_d$, it suffices to prove that $H \subseteq \sigma_{\mathrm{e}}(N_z)$.

Suppose otherwise, that is, there is a point $\lambda_0 = (\lambda_{01}, \dots, \lambda_{0d}) \in H$ such that

$$N_z - \lambda_0 = (N_{z_1} - \lambda_{01}, \dots, N_{z_d} - \lambda_{0d})$$

is Fredholm. Since $[N_{z_i}, N_{z_i}^*]$ is compact for $1 \le i, j \le d$, and

$$[N_{z_i} - \lambda_{0i}, N_{z_i}^* - \overline{\lambda}_{0i}] = [N_{z_i}, N_{z_i}^*],$$

this ensures that the tuple $(N_{z_1} - \lambda_{01}, \dots, N_{z_d} - \lambda_{0d})$ is essentially normal. By [Cur2, Corollary 3.9], $(N_{z_1} - \lambda_{01}, \dots, N_{z_d} - \lambda_{0d})$ is Fredholm if and only if

$$\sum_{i=1}^{d} (N_{z_i} - \lambda_{0i})(N_{z_i}^* - \overline{\lambda}_{0i})$$

is Fredholm. Since this last operator is positive, there exist an invertible positive operator A and a compact operator K such that

$$\sum_{i=1}^{d} (N_{z_i} - \lambda_{0i})(N_{z_i}^* - \overline{\lambda}_{0i}) = A + K. \tag{3.3}$$

Recalling that k_λ is the normalized reproducing kernel of $H^2(\mathbb{B}_d)$, we have

$$P_N k_\lambda = (I - M_\eta M_\eta^*) k_\lambda = k_\lambda - \overline{\eta(\lambda)} \eta k_\lambda. \tag{3.4}$$

Assume $\lambda = (\lambda_1, \dots, \lambda_d)$. Then

$$\begin{aligned}(3.5)\quad & \sum_{i=1}^{d}(\langle (N_{z_i} - \lambda_{0i})(N_{z_i}^* - \overline{\lambda}_{0i})P_N k_\lambda, P_N k_\lambda\rangle)^{1/2} \\ &= \sum_{i=1}^{d} \|M_{z_i-\lambda_{0i}}^* P_N k_\lambda\| = \sum_{i=1}^{d} \|M_{z_i-\lambda_{0i}}^*(k_\lambda - \overline{\eta(\lambda)}\eta k_\lambda)\| \\ &= \sum_{i=1}^{d} \|\overline{\lambda_i - \lambda_{0i}}\, k_\lambda - \overline{\eta(\lambda)} M_{z_i-\lambda_{0i}}^* \eta k_\lambda\| \\ &\le \sum_{i=1}^{d}(|\lambda_i - \lambda_{0i}| + |\overline{\eta(\lambda)}|\ \|\, M_{z_i-\lambda_{0i}}^* \eta k_\lambda\|).\end{aligned}$$

Replacing λ in (3.5) with $\lambda_j = (\lambda_{j1}, \dots, \lambda_{jd})$ and noting $\|\eta k_\lambda\| = 1$ and $\eta(\lambda_j) \to 0$, we see that the last two terms in (3.5) converge to 0 as $j \to \infty$. Thus we have

$$(3.6)\quad \sum_{i=1}^{d}\langle (N_{z_i} - \lambda_{0i})(N_{z_i}^* - \overline{\lambda}_{0i})P_N k_{\lambda_j}, P_N k_{\lambda_j}\rangle \to 0 \quad \text{as } j \to \infty.$$

On the other hand,

$$\begin{aligned}&\langle (A+K)P_N k_\lambda, P_N k_\lambda\rangle \\ &\quad = \langle A(k_\lambda - \overline{\eta(\lambda)}\eta k_\lambda), k_\lambda - \overline{\eta(\lambda)}\eta k_\lambda\rangle + \langle K(k_\lambda - \overline{\eta(\lambda)}\eta k_\lambda), k_\lambda - \overline{\eta(\lambda)}\eta k_\lambda\rangle.\end{aligned}$$

We compute the two terms on the right of the above equality:

$$\begin{aligned}(3.7)\quad & \langle A(k_\lambda - \overline{\eta(\lambda)}\eta k_\lambda), k_\lambda - \overline{\eta(\lambda)}\eta k_\lambda\rangle \\ &= \langle Ak_\lambda, k_\lambda\rangle - \overline{\eta(\lambda)}\langle A\eta k_\lambda, k_\lambda\rangle - \eta(\lambda)\langle Ak_\lambda, \eta k_\lambda\rangle + |\eta(\lambda)|^2\langle A\eta k_\lambda, \eta k_\lambda\rangle.\end{aligned}$$

Note that A is an invertible positive operator, hence so is $\sqrt{A}$. Thus $\sqrt{A}$ is bounded below, i.e. there is $c > 0$ such that $\|\sqrt{A}\, f\| \ge c\|f\|$ for any $f \in N$. Therefore,

$$\langle Ak_\lambda, k_\lambda\rangle = \|\sqrt{A}\, k_\lambda\|^2 \ge c^2.$$

With λ replaced by λ_j, we conclude that the other three terms of the last equality in (3.7) converge to 0 as $j \to \infty$. So we have

$$(3.8)\quad \langle AP_N k_{\lambda_j}, P_N k_{\lambda_j}\rangle \ge c^2.$$

Moreover, since $P_N k_{\lambda_j} \xrightarrow{w} 0$, and K is compact, we have

$$(3.9)\quad \langle KP_N k_{\lambda_j}, P_N k_{\lambda_j}\rangle \to 0 \quad \text{as } j \to \infty.$$

Combining (3.3), (3.6), (3.8), and (3.9), we get a contradiction. Therefore, $\partial\mathbb{B}_d \subseteq \sigma_{\mathrm{e}}(N_z)$.

From Lemmas 3.2 and 3.3, we have

THEOREM 3.4. $\sigma_{\mathrm{e}}(N_z) = \partial\mathbb{B}_d$.

The next proposition comes from a discussion with R. Yang.

PROPOSITION 3.5. *The C^*-algebra $C^*(N_z)$ generated by $\{N_{z_1}, \dots, N_{z_d}, I\}$ is irreducible, that is, if Q is a projection on N and $QN_{z_i} = N_{z_i}Q$ for $i = 1, \dots, d$, then $Q = 0$ or $Q = I$.*

Proof. From the proof of Proposition 2.2 in [GZ], there exist polynomials p_i, q_i such that

$$1 \otimes 1 = \sum_{i=1}^{m} M_{p_i} M_{q_i}^* \quad \text{on } H^2(\mathbb{B}_d). \tag{3.10}$$

Set $e = P_N 1$. By (3.10), we have

$$e \otimes e = P_N(1 \otimes 1)P_N = \sum_{i=1}^{m} P_N M_{p_i} M_{q_i}^* P_N = \sum_{i=1}^{m} N_{p_i} N_{q_i}^*,$$

where, for a polynomial p, $N_p = P_N M_p | N = p(N_{z_i})$. If Q is a projection as in the hypothesis, then $QN_q = N_q Q$ and $QN_q^* = N_q^* Q$ for any polynomial q. Thus, $Q(e \otimes e)$=$(e \otimes e)Q$, which implies

$$Qe \otimes e = e \otimes Qe.$$

Making both sides of the above equality act on e, we have

$$Qe = \frac{\|Qe\|^2}{\|e\|^2}\, e. \tag{3.11}$$

CASE 1. If $Qe \neq 0$, then

$$Qe = Q^2 e = \frac{\|Qe\|^4}{\|e\|^4}\, e = \frac{\|Qe\|^2}{\|e\|^2}\, e.$$

This implies $\|Qe\| = \|e\|$, hence $Qe = e$. Now for any $f \in N$, there exists a sequence of polynomials q_n such that $q_n \to f$ as $n \to \infty$. Writing $1 = e + \xi$ with $\xi \in \eta H^2(\mathbb{B}_d)$, we have

$$q_n = q_n(e + \xi) \to f \quad \text{as } n \to \infty.$$

Hence,

$$P_N(q_n e + q_n \xi) = P_N q_n e \to f,$$

that is, $N_{q_n} e \to f$. Therefore, $QN_{q_n} e \to Qf$. Moreover, since

$$QN_{q_n} e = N_{q_n} Qe = N_{q_n} e \to f,$$

we obtain $Qf = f$, which shows $Q = I$.

CASE 2. If $Qe = 0$, the same reasoning gives $Q = 0$, and the proof is complete.

Since $C^*(N_z)$ is irreducible, and $C^*(N_z) \cap \mathcal{K}(N) \neq \emptyset$ from (3.2), where $\mathcal{K}(N)$ denotes the compact operator ideal on N, this implies $C^*(N_z) \supseteq \mathcal{K}(N)$

by [Arv4]. Since the tuple N_z is essentially normal, we have the following exact sequence:

$$0 \to \mathcal{K}(N) \hookrightarrow C^*(N_z) \xrightarrow{\pi} C(\partial\mathbb{B}_d) \to 0, \tag{3.12}$$

where π is the unital $*$-homomorphism given by $\pi(N_{z_i}) = Z_i$.

From [BDF], the above exact sequence gives an extension of $\mathcal{K}(N)$ by $C(\partial\mathbb{B}_d)$.

PROPOSITION 3.6. *The short exact sequence* (3.12) *is split*, *that is*, *there exists a $*$-homomorphism $\sigma : C(\partial\mathbb{B}_d) \to C^*(N_z)$ such that $\pi\sigma = I$.*

Proof. It follows from Theorem 3.4 that the tuple $(N_{z_1}, \ldots, N_{z_d})$ is Fredholm. Moreover, by [GRS, Corollary 3.5] the tuple has Fredholm index 0. Since $\mathrm{Ext}(\partial\mathbb{B}_d) = K_1(\partial\mathbb{B}_d) = \mathbb{Z}$, the short exact sequence (3.12) defines a zero element in $K_1(\partial\mathbb{B}_d)$. Indeed, this K-homology element is determined by the index of the tuple $(N_{z_1}, \ldots, N_{z_d})$ (cf. [BDF, Guo2]) and thus is 0. So the extension defined by (3.12) is a trivial element in $\mathrm{Ext}(\partial\mathbb{B}_d)$. This means that the short exact sequence (3.12) is split.

REMARK. In the cases $d = 2, 3$, the extensions defined by quotients of submodules generated by homogeneous polynomials are not split [GW].

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