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Spectral synthesis and operator synthesis

by

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Abstract. Relations between spectral synthesis in the Fourier algebra $A(G)$ of a compact group G and the concept of operator synthesis due to Arveson have been studied in the literature. For an $A(G)$-submodule X of $\mathrm{VN}(G)$, X-synthesis in $A(G)$ has been introduced by E. Kaniuth and A. Lau and studied recently by the present authors. To any such X we associate a $V^\infty(G)$-submodule $\widehat{X}$ of $\mathcal{B}(L^2(G))$ (where $V^\infty(G)$ is the weak-$*$ Haagerup tensor product $L^\infty(G) \otimes_{w^*h} L^\infty(G)$), define the concept of $\widehat{X}$-operator synthesis and prove that a closed set E in G is of X-synthesis if and only if $E^* := \{(x,y) \in G \times G : xy^{-1} \in E\}$ is of $\widehat{X}$-operator synthesis.

Introduction. Arveson introduced and studied the concept of operator synthesis in [1]. He found that spectral synthesis on abelian groups is related to operator synthesis. Froelich [2] continued this study. Recently Spronk and Turowska [5] have investigated the relation between spectral synthesis in the Fourier algebra $A(G)$ of a compact group G and operator synthesis. Specifically, they considered the projective tensor product $T(G) = L^2(G) \widehat{\otimes} L^2(G)$ and the weak-$*$ Haagerup tensor product $V^\infty(G) = L^\infty(G) \otimes_{w^*h} L^\infty(G)$ and proved that a closed subset E of G is of spectral synthesis for $A(G)$ if and only if $E^* := \{(x,y) \in G \times G : xy^{-1} \in E\}$ is of operator synthesis.

In another direction, for an $A(G)$-submodule X of $\mathrm{VN}(G) = A(G)^*$, X-synthesis has recently been studied by Kaniuth and Lau [3] and Parthasarathy and Prakash [4].

In this paper we tie up these two threads. For a $V^\infty(G)$-submodule $\mathcal{M}$ of $\mathcal{B}(L^2(G)) = T(G)^*$, we define and characterise operator synthesis for $\mathcal{M}$ (Section 3). When $\mathcal{M} = \mathcal{B}(L^2(G))$, this reduces to operator synthesis of the earlier authors. With any $A(G)$-submodule X of $\mathrm{VN}(G)$, we associate a $V^\infty(G)$-submodule $\widehat{X}$ of $\mathcal{B}(L^2(G))$ and conversely, to any $V^\infty(G)$-submodule $\mathcal{M}$ of $\mathcal{B}(L^2(G))$ there corresponds an $A(G)$-submodule $\check{\mathcal{M}}$ of

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VN(G). Moreover, $\check{\widehat{X}} = X$ (Section 2). The main result (Theorem 4.6) states that a closed set E in G is of X-synthesis if and only if E^* is of $\widehat{X}$-operator synthesis. This is preceded, in Section 4, by a sequence of lemmas.

We begin with the required notations, definitions and results in the first section.

1. Preliminaries. For a compact group G, the *Fourier algebra* $A(G)$ is the space of continuous functions of the form $u(x) = \langle \lambda(x)f, g\rangle$, $x \in G$, where $f, g \in L^2(G)$ and λ is the left regular representation of G. VN(G) is the von Neumann algebra in $\mathcal{B}(L^2(G))$ generated by $\lambda(x)$, $x \in G$. $A(G)$ is a commutative, semisimple, regular Banach algebra with pointwise operations and with norm defined, for $u(\cdot) = \langle \lambda(\cdot)f, g\rangle \in A(G)$, by

$$\|u\|_A = \sup\{|\langle Tf, g\rangle| : T \in \mathrm{VN}(G),\ \|T\| \le 1\}.$$

VN(G) is the Banach space dual of $A(G)$ with the pairing given by $\langle T, u\rangle = \langle Tf, g\rangle$. Moreover, VN($G$) is an $A(G)$-module where the action is given as follows: for $T \in \mathrm{VN}(G)$ and $u \in A(G)$, $\langle u.T, v\rangle = \langle T, uv\rangle$, $v \in A(G)$. The *support* of $T \in \mathrm{VN}(G)$ is the closed set $\operatorname{supp} T = \{x \in G : u(x) \neq 0 \Rightarrow u.T \neq 0\}$.

For a closed set $E \subseteq G$, define

$$I_A(E) := \{u \in A(G) : u(x) = 0\ \forall x \in E\},$$
$$j_A(E) := \{u \in A(G) : u \text{ vanishes in a neighbourhood of } E\},$$
$$J_A(E) := \overline{j_A(E)}.$$

$I_A(E)$ is the largest ideal of $A(G)$ with zero set E and $j_A(E)$ is the smallest such ideal. The set E is called a set of *spectral synthesis* (or a *spectral set*) if there is a unique closed ideal of $A(G)$ with zero set E. Thus E is a set of synthesis if and only if $I_A(E) = J_A(E)$. For an $A(G)$-submodule X of VN(G), E is called a set of X-*synthesis* (or an X-*spectral set*) if $\langle T, u\rangle = 0$ for all $u \in I_A(E)$ and $T \in X$ with $\operatorname{supp} T \subseteq E$ (see [3]). It is proved in [4] that E is a set of X-synthesis if and only if $I_A^X(E) = J_A^X(E)$ where

$$I_A^X(E) = \{u \in A(G) : \langle T, u\rangle = 0 \text{ for every } T \in X \cap I_A(E)^\perp\},$$
$$J_A^X(E) = \{u \in A(G) : \langle T, u\rangle = 0 \text{ for every } T \in X \cap J_A(E)^\perp\}.$$

Observe that $I_A^X(E) = I_A(E)$ and $J_A^X(E) = J_A(E)$ when $X = \mathrm{VN}(G)$.

The concept of operator synthesis was introduced by Arveson [1], who also initiated the study of its relations with spectral synthesis. This study has been continued by Froelich [2] and Spronk and Turowska [5]. To describe the setting, let $T(G)$ be the projective tensor product $L^2(G) \mathbin{\widehat{\otimes}} L^2(G)$. The Banach space dual $T(G)^*$ of $T(G)$ is identified with $\mathcal{B}(L^2(G))$ where the pairing satisfies the relation $\langle S, f \otimes g\rangle = \langle Sf, \overline{g}\rangle$ for $S \in \mathcal{B}(L^2(G))$ and $f, g \in L^2(G)$. Arveson [1] has shown that an element $\omega = \sum_{n=1}^{\infty} f_n \otimes g_n \in T(G)$

may be considered as a function $\omega(x,y)=\sum_{n=1}^{\infty} f_n(y)g_n(x)$ for marginally almost all $(x,y)\in G\times G$ (see [1], [5]). For such an ω, $\operatorname{supp}\omega=\{(x,y)\in G\times G:\omega(x,y)\neq 0\}$ is defined up to marginally null sets. A *marginally null set* is a subset of a set of the form $E\times G\cup G\times F$ where E,F have measure zero. For a closed set $F\subseteq G\times G$, define

$$\begin{aligned}\Phi(F)&=\{\omega\in T(G):\operatorname{supp}\omega\cap F=\emptyset\},\\ \psi(F)&=\{\omega\in T(G):\operatorname{supp}\omega\cap U=\emptyset \text{ for some open } U\supseteq F\},\\ \Psi(F)&=\overline{\psi(F)}.\end{aligned}$$

If $\Phi(F)=\Psi(F)$, then F is called a set of *operator synthesis* (or is said to be *synthetic*). (Arveson [1] considered these concepts in a very general set up, but we consider only the case of the product normalised Haar measure $m\times m$ on $G\times G$ as in Spronk and Turowska [5].)

Spronk and Turowska proved that a closed subset E of G is a set of synthesis if and only if $E^*:=\{(x,y)\in G\times G: xy^{-1}\in E\}$ is a set of operator synthesis. Note that $E^*=\theta^{-1}(E)$ where $\theta:G\times G\to G$ is the continuous surjection defined by $\theta(x,y)=xy^{-1}$. Crucial to the proof of this result is the embedding $\widetilde{N}$ of $A(G)$ in $T(G)$. This is analogous to the embedding N of $A(G)$ in the Varopoulos algebra $V(G)$ ([6], [5]) and is given by $\widetilde{N}u(x,y)=u(xy^{-1})$ for marginally almost all $(x,y)\in G\times G$. It is proved in [5] that $\widetilde{N}$ is an isometry whose range is the set of functions in $T(G)$ which are G-invariant. More precisely, if for $\omega\in T(G)$ and $t\in G$, $t.\omega(x,y)=\omega(xt,yt)$ for marginally almost all (x,y) in $G\times G$, then

$$\widetilde{N}(A(G))=T_{\mathrm{inv}}(G):=\{\omega\in T(G):t.\omega=\omega \text{ for all } t\in G\}.$$

$T_{\mathrm{inv}}(G)$ is complemented in $T(G)$ and a projection $\widetilde{P}$ of $T(G)$ onto $T_{\mathrm{inv}}(G)$ is given by $\widetilde{P}\omega=\int_G t.\omega\,dt$. Further, $\widetilde{Q}$ defined by $\widetilde{Q}\omega(x)=\int_G\omega(xt,t)\,dt$ gives a contraction $T(G)\to A(G)$ such that $\widetilde{Q}(\widetilde{N}u)=u$ for all $u\in A(G)$.

In this paper, we define an analogue, for operator synthesis, of the concept of X-synthesis and obtain a relation between X-synthesis and this concept.

2. $V^\infty(G)$-submodules of $\mathcal{B}(L^2(G))$. We need the Banach algebra $V^\infty(G)$ that is the weak-$*$ Haagerup tensor product $L^\infty(G)\otimes_{w^*h}L^\infty(G)$. This is defined as follows. Consider $\Im(L^2(G))\otimes_h\Im(L^2(G))$, the Haagerup tensor product of trace class operators on $L^2(G)$. The dual of this space is, by definition, the weak-$*$ Haagerup tensor product $\mathcal{B}(L^2(G))\otimes_{w^*h}\mathcal{B}(L^2(G))$. The weak-$*$ closure of $L^\infty(G)\otimes L^\infty(G)$ in this space is defined as $L^\infty(G)\otimes_{w^*h}L^\infty(G)$. (Note that $L^\infty(G)$ is here considered as an algebra of operators on $L^2(G)$.) Another description: $L^\infty(G)\otimes_{w^*h}L^\infty(G)=(L^1(G)\otimes_h L^1(G))^*$. But the description of $V^\infty(G)$ that is useful for our purposes is the following. Every element of $V^\infty(G)$ can be considered as a function (up to a marginally null set) on $G \times G$ of the form $w = \sum_{n=1}^\infty \varphi_n \otimes \psi_n$ where φ_n, ψ_n are in $L^\infty(G)$ and the series is weak-$*$ convergent. Moreover,

$$\|w\|_{V^\infty} = \inf\Big\{\Big\|\sum |\varphi_n|^2\Big\|_\infty^{1/2} \Big\|\sum |\psi_n|^2\Big\|_\infty^{1/2} : w = \sum \varphi_n \otimes \psi_n\Big\}$$

with the series $\sum |\varphi_n|^2$ and $\sum |\psi_n|^2$ converging in the weak-$*$ topology. Spronk and Turowska [5] have also proved that $V^\infty(G)$ is the algebra of multipliers of $T(G)$. In other words,

$$V^\infty(G) = \{w : w \text{ is a complex function on } G \times G \text{ and}$$
$$m_w : \omega \mapsto w.\omega \text{ is a bounded linear map on } T(G)\}.$$

with $\|w\|_{V^\infty(G)} = \|m_w\|$. Two functions w and w' in $V^\infty(G)$ are identified if they differ on a marginally null set. It is shown that the map J which identifies a function in $V^\infty(G)$ as an element of $T(G)$ is a contractive injection. More precisely, the map $J : V^\infty(G) \to T(G)$ is defined by $Jw = m_w\, 1 \otimes 1$.

$V^\infty(G)$ acts on $\mathcal{B}(L^2(G)) = T(G)^*$ via the dual action: for $w \in V^\infty(G)$ and $S \in \mathcal{B}(L^2(G))$,

$$\langle w.S, \omega\rangle = \langle S, m_w\omega\rangle, \qquad \omega \in T(G).$$

Now, given an $A(G)$-submodule X of $\mathrm{VN}(G)$, we define

$$\widehat{X} = \{S \in \mathcal{B}(L^2(G)) : w.S \circ \widetilde{N} \in X \text{ for all } w \in V^\infty(G)\}.$$

It is clearly a $V^\infty(G)$-submodule of $\mathcal{B}(L^2(G))$. In the opposite direction, with any $V^\infty(G)$-submodule $\mathcal{M}$ of $\mathcal{B}(L^2(G))$ we associate an $A(G)$-submodule $\check{\mathcal{M}}$ of $\mathrm{VN}(G)$ by defining

$$\check{\mathcal{M}} = \{T \in \mathrm{VN}(G) : u.T \circ \widetilde{N}^{-1} \circ \widetilde{P} \in \mathcal{M} \text{ for all } u \in A(G)\}.$$

We have the following nice-looking result on these correspondences.

2.1. Proposition. *If X is an $A(G)$-submodule of* $\mathrm{VN}(G)$, *then* $\check{\widehat{X}} = X$.

Proof. Let $T \in X$. Suppose $u \in A(G)$ and let $S = u.T \circ \widetilde{N}^{-1} \circ \widetilde{P}$. We claim that $S \in \widehat{X}$. Indeed, for $w \in V^\infty(G)$ and $v \in A(G)$,

$$\begin{aligned}\langle w.S \circ \widetilde{N}, v\rangle &= \langle w.S, \widetilde{N}v\rangle = \langle S, m_w\widetilde{N}v\rangle = \langle u.T \circ \widetilde{N}^{-1} \circ \widetilde{P}, m_w\widetilde{N}v\rangle\\ &= \langle u.T \circ \widetilde{N}^{-1}, \widetilde{P}m_w\widetilde{N}v\rangle = \langle u.T \circ \widetilde{N}^{-1}, \widetilde{N}(u_1v)\rangle\\ &= \langle u_1u.T, v\rangle, \qquad u_1 = \widetilde{N}^{-1}(\widetilde{P}(Jw)).\end{aligned}$$

Thus $w.S \circ \widetilde{N} = u_1u.T \in X$. So $S = u.T \circ \widetilde{N}^{-1} \circ \widetilde{P} \in \widehat{X}$ for all $u \in A(G)$. This means that $T \in \check{\widehat{X}}$, by definition.

Conversely, suppose $T \in \check{\widehat{X}}$. Then $u.T \circ \widetilde{N}^{-1} \circ \widetilde{P} \in \widehat{X}$ for all $u \in A(G)$ and so $w.(u.T \circ \widetilde{N}^{-1} \circ \widetilde{P}) \circ \widetilde{N} \in X$ for all $w \in V^\infty(G)$. For $u, v \in A(G)$ and

$w \in V^\infty(G)$, we have

$$\begin{aligned}\langle w.(u.T \circ \widetilde{N}^{-1} \circ \widetilde{P}) \circ \widetilde{N}, v\rangle &= \langle w.(u.T \circ \widetilde{N}^{-1} \circ \widetilde{P}), \widetilde{N}v\rangle \\ &= \langle u.T \circ \widetilde{N}^{-1} \circ \widetilde{P}, m_w\widetilde{N}v\rangle = \langle u_1 u.T, v\rangle\end{aligned}$$

as before. Thus $u_1 u.T = w.(u.T \circ \widetilde{N}^{-1} \circ \widetilde{P}) \circ \widetilde{N} \in X$. Taking $u = 1$ and $w = 1 \otimes 1$ we get $u_1 = 1$ and $T \in X$. ∎

3. Operator synthesis for $V^\infty(G)$-modules. For an operator $S \in \mathcal{B}(L^2(G))$, the *operator support* of S is defined by $\operatorname{supp}_{\mathrm{op}} S = \{(x,y) \in G \times G$: for neighbourhoods U of x and V of y there are $f, g \in L^2(G)$ with $\operatorname{supp} f \subset V$, $\operatorname{supp} g \subset U$ and $\langle Sf, g\rangle \neq 0\}$. Here $\operatorname{supp} f = \{x \in G : f(x) \neq 0\}$. It is known that $\operatorname{supp}_{\mathrm{op}} S$ is a closed set in $G \times G$.

3.1. Definition. Let $\mathcal{M}$ be a $V^\infty(G)$-submodule of $\mathcal{B}(L^2(G))$. A closed set $F \subseteq G \times G$ is said to be a *set of operator synthesis* for $\mathcal{M}$ (or an $\mathcal{M}$-*synthetic set* for short) if $\langle S, \omega\rangle = 0$ for $S \in \mathcal{M}$ with $\operatorname{supp}_{\mathrm{op}} S \subseteq F$ and $\omega \in \Phi(F)$. Observe that F is of operator synthesis if and only if it is $\mathcal{B}(L^2(G))$-synthetic.

To get a reformulation of this concept we define the following subsets of $T(G)$: If $\mathcal{M}$ is a $V^\infty(G)$-submodule of $\mathcal{B}(L^2(G))$, let

$$\Phi^{\mathcal{M}}(F) = \{\omega \in T(G) : \langle S, \omega\rangle = 0 \text{ for any } S \in \mathcal{M} \cap \Phi(F)^\perp\},$$
$$\Psi^{\mathcal{M}}(F) = \{\omega \in T(G) : \langle S, \omega\rangle = 0 \text{ for any } S \in \mathcal{M} \cap \Psi(F)^\perp\}.$$

Note that these are closed $V^\infty(G)$-submodules of $T(G)$.

3.2. Lemma. *Let $\mathcal{M}$ be a $V^\infty(G)$-submodule of $\mathcal{B}(L^2(G))$. A closed subset F of $G \times G$ is $\mathcal{M}$-synthetic if and only if $\Phi^{\mathcal{M}}(F) = \Psi^{\mathcal{M}}(F)$.*

Proof. Suppose F is $\mathcal{M}$-synthetic. Let $S \in \mathcal{M} \cap \Psi(F)^\perp$. Then $\operatorname{supp}_{\mathrm{op}} S \subseteq F$ and $\langle S, \omega'\rangle = 0$ for $\omega' \in \Phi(F)$, since F is $\mathcal{M}$-synthetic. Thus, if $\omega \in \Phi^{\mathcal{M}}(F)$, then $\langle S, \omega\rangle = 0$, so $\omega \in \Psi^{\mathcal{M}}(F)$. The inclusion $\Psi^{\mathcal{M}}(F) \subseteq \Phi^{\mathcal{M}}(F)$ being trivial, one part of the lemma is proved.

For the converse, suppose $\Phi^{\mathcal{M}}(F) = \Psi^{\mathcal{M}}(F)$. If $S \in \mathcal{M}$ and $\operatorname{supp}_{\mathrm{op}} S \subseteq F$, then $S \in \mathcal{M} \cap \Psi(F)^\perp$. Thus $\langle S, \omega\rangle = 0$ for $\omega \in \Psi^{\mathcal{M}}(F) = \Phi^{\mathcal{M}}(F)$. In particular, $\langle S, \omega\rangle = 0$ for $\omega \in \Phi(F)$. Hence F is $\mathcal{M}$-synthetic. ∎

4. Spectral synthesis in $A(G)$ and operator synthesis in $T(G)$. Let X be an $A(G)$-submodule of $\mathrm{VN}(G)$ and let $\widehat{X}$ be the associated $V^\infty(G)$-submodule of $\mathcal{B}(L^2(G))$ as in Section 2. In this section we prove the main result that if E is an X-spectral set, then $E^* = \{(x,y) : xy^{-1} \in E\}$ is $\widehat{X}$-synthetic.

We begin with a couple of lemmas. The first of these identifies the images of the ideals $I_A(E)$ and $J_A(E)$ under the map $\widetilde{N}$. The analogous result for the isometric imbedding N of $A(G)$ in the Varopoulos algebra $V(G)$ is due to Spronk and Turowska [5].

4.1. Lemma. *Let E be a closed subset of G.*

(i) $u \in I_A(E) \Leftrightarrow \widetilde{N}u \in \Phi(E^*)$.
(ii) $u \in J_A(E) \Leftrightarrow \widetilde{N}u \in \Psi(E^*)$.

Proof. (i) is obvious. The proof that $u \in J_A(E)$ implies $\widetilde{N}u \in \Psi(E^*)$ is essentially contained in the proof of [5, Theorem 4.6]. Conversely, suppose that $u \in A(G)$ and $\widetilde{N}u \in \Psi(E^*)$. Then there is a sequence $\{\omega_n\}$ in $\psi(E^*)$ such that $\omega_n \to \widetilde{N}u$, hence $\widetilde{Q}\omega_n \to \widetilde{Q}\widetilde{N}u = u$. We complete the proof by showing that $\widetilde{Q}\omega_n \in j_A(E)$ for all n. Now,

$$\begin{aligned} x \notin \theta(\operatorname{supp}\omega_n) &\Rightarrow (xt,t) \notin \operatorname{supp}\omega_n \ \forall t \in G \\ &\Rightarrow \omega_n(xt,t) = 0 \ \forall t \in G \\ &\Rightarrow \widetilde{Q}\omega_n(x) = 0. \end{aligned}$$

Thus $W_n = \{x \in G : \widetilde{Q}\omega_n(x) \neq 0\} \subseteq \theta(\operatorname{supp}\omega_n)$. But $\omega_n \in \psi(E^*)$ implies that there is a neighbourhood U_n of E^* such that $\operatorname{supp}\omega_n \subseteq U_n^{\mathrm{c}}$. This means that $W_n \subseteq \theta(U_n^{\mathrm{c}})$. So $\operatorname{supp}\widetilde{Q}(\omega_n) = \overline{W_n} \subseteq \overline{\theta(U_n^{\mathrm{c}})} = \theta(U_n^{\mathrm{c}})$ and $\widetilde{Q}\omega_n \in j_A(E)$. ∎

The following result is [5, Theorem 4.6].

4.2. Corollary. *A closed subset E of G is of spectral synthesis for $A(G)$ if and only if E^* is of operator synthesis.*

Proof. This is immediate from Lemma 4.1. The only new feature here is the different proof of the part that if E^* is operator synthetic then E is of synthesis. (This is a consequence of Lemma 4.1(i) and the implication $\widetilde{N}u \in \Psi(E^*) \Rightarrow u \in J_A(E)$ in 4.1(ii).) The rest is as in [5].

We use Lemma 4.1 to prove the following more general version, which is a crucial ingredient in the proof of the main result (Theorem 4.6 below), of which 4.2 is a special case.

4.3. Lemma. *Let E be a closed set in G and let $u \in A(G)$. Let X be an $A(G)$-submodule of* $\mathrm{VN}(G)$ *and let $\widehat{X}$ be the associated $V^\infty(G)$-submodule of $\mathcal{B}(L^2(G))$. Then*

(i) $u \in I_A^X(E) \Leftrightarrow \widetilde{N}u \in \Phi^{\widehat{X}}(E^*)$.
(ii) $u \in J_A^X(E) \Leftrightarrow \widetilde{N}u \in \Psi^{\widehat{X}}(E^*)$.

Proof. (i) First, let $u \in I_A^X(E)$ and $S \in \widehat{X} \cap \Phi(E^*)^\perp$. To show that $\widetilde{N}u \in \Phi^{\widehat{X}}(E^*)$, we need to prove that $\langle S, \widetilde{N}u\rangle = 0$. We first claim that $S \circ \widetilde{N} \in X \cap I_A(E)^\perp$. To see this, observe that if $u' \in I_A(E)$ then $\widetilde{N}u' \in \Phi(E^*)$

by Lemma 4.1, and so $\langle S\circ\widetilde{N},u'\rangle = \langle S,\widetilde{N}u'\rangle = 0$ as $S\in\Phi(E^*)^{\perp}$. Thus, $S\circ\widetilde{N}\in I_A(E)^{\perp}$, and, of course, $S\circ\widetilde{N}\in X$ by definition of $\widehat{X}$. This proves the claim and consequently $0=\langle S\circ\widetilde{N},u\rangle = \langle S,\widetilde{N}u\rangle$.

Conversely, let $u\in A(G)$, $\widetilde{N}u\in\Phi^{\widehat{X}}(E^*)$ and $T\in X\cap I_A(E)^{\perp}$. Then $S:=T\circ\widetilde{N}^{-1}\circ\widetilde{P}\in T(G)^*=\mathcal{B}(L^2(G))$. We prove that $S\in\widehat{X}\cap\Phi(E^*)^{\perp}$. For $w\in V^{\infty}(G)$ and $u\in A(G)$, $wS\circ\widetilde{N}=u_1T$, where $u_1=\widetilde{N}^{-1}(\widetilde{P}(Jw))$, as in the proof of Proposition 2.1, and so $S\in\widehat{X}$. Moreover, if $\omega\in\Phi(E^*)$, then clearly $\widetilde{P}\omega\in\Phi(E^*)$ and $\langle S,\omega\rangle = \langle T\circ\widetilde{N}^{-1}\circ\widetilde{P},\omega\rangle = \langle T\circ\widetilde{N}^{-1},\widetilde{P}\omega\rangle = \langle T\circ\widetilde{N}^{-1},\widetilde{N}u_2\rangle$ where $\widetilde{N}u_2=\widetilde{P}\omega\in\Phi(E^*)$, so $u_2\in I_A(E)$ by Lemma 4.1(i). Thus $\langle S,\omega\rangle=\langle T,u_2\rangle=0$. Hence we have shown that $S\in\widehat{X}\cap\Phi(E^*)^{\perp}$. This implies that

$$\begin{aligned}0&=\langle S,\widetilde{N}u\rangle=\langle T\circ\widetilde{N}^{-1}\circ\widetilde{P},\widetilde{N}u\rangle=\langle T\circ\widetilde{N}^{-1},\widetilde{P}(\widetilde{N}u)\rangle\\&=\langle T\circ\widetilde{N}^{-1},\widetilde{N}u\rangle=\langle T,u\rangle.\end{aligned}$$

Since $T\in X\cap I_A(E)^{\perp}$ is arbitrary, this shows that $u\in I_A^X(E)$ and (i) is proved.

(ii) Suppose $u\in J_A^X(E)$. To show $\widetilde{N}u\in\Psi^{\widehat{X}}(E^*)$, let $S\in\widehat{X}\cap\Psi(E^*)^{\perp}$. Then $S\circ\widetilde{N}\in X$. We show $S\circ\widetilde{N}\in J_A(E)^{\perp}$ as well. For, if $u'\in J_A(E)$, then $\widetilde{N}u'\in\Psi(E^*)$ by Lemma 4.1(ii) and $\langle S\circ\widetilde{N},u'\rangle=\langle S,\widetilde{N}u'\rangle=0$ as $S\in\Psi(E^*)^{\perp}$. Thus $S\circ\widetilde{N}\in X\cap J_A(E)^{\perp}$ and so $\langle S\circ\widetilde{N},u\rangle=\langle S,\widetilde{N}u\rangle=0$. Hence $\widetilde{N}u\in\Psi^{\widehat{X}}(E^*)$.

Conversely, suppose $u\in A(G)$ and $\widetilde{N}u\in\Psi^{\widehat{X}}(E^*)$. To show $u\in J_A^X(E)$, let $T\in X\cap J_A(E)^{\perp}$. Then $\operatorname{supp}T\subseteq E$ and $S=T\circ\widetilde{N}^{-1}\circ\widetilde{P}\in T(G)^*=\mathcal{B}(L^2(G))$. Now, for $f,g\in L^2(G)$,

$$\begin{aligned}\langle Sf,g\rangle&=\langle S,f\otimes\overline{g}\rangle=\langle T\widetilde{N}^{-1}\widetilde{P},f\otimes\overline{g}\rangle\\&=\langle T\circ\widetilde{N}^{-1},\widetilde{P}(f\otimes\overline{g})\rangle=\langle T\circ\widetilde{N}^{-1},\widetilde{N}u_1\rangle\quad\text{where } u_1(x)=\langle\lambda(x)f,g\rangle\\&=\langle T,u_1\rangle=\langle Tf,g\rangle.\end{aligned}$$

In other words, $S=T$ as operators on $L^2(G)$. Thus $\operatorname{supp}_{\mathrm{op}}S=\operatorname{supp}_{\mathrm{op}}T$. But by a result of Spronk and Turowska $\operatorname{supp}_{\mathrm{op}}T=(\operatorname{supp}_{\mathrm{VN}}T)^*\subseteq E^*$. This shows that $\operatorname{supp}_{\mathrm{op}}S\subseteq E^*$ and $S\in\Psi(E^*)^{\perp}$. Moreover, for $w\in V^{\infty}(G)$, $w.S\circ\widetilde{N}=u_1T\in X$ as before and so $S\in\widehat{X}$. Thus $S\in\widehat{X}\cap\Psi(E^*)^{\perp}$ and so $0=\langle S,\widetilde{N}u\rangle=\langle T\circ\widetilde{N}^{-1}\circ\widetilde{P},\widetilde{N}u\rangle=\langle T,u\rangle$. The proof is complete. ∎

Now, it has been observed in [5] that the group G acts on $T(G)$ by isometries as follows: for $t\in G$ and $\omega\in T(G)$, $t.\omega(x,y)=\omega(xt,yt)$ for marginally almost all $(x,y)\in G\times G$. This action, in turn, gives rise to an action of $L^1(G)$ on $T(G)$: for $f \in L^1(G)$ and $\omega \in T(G)$,

$$f.\omega = \int_G f(t)t.\omega \, dt$$

This way $T(G)$ becomes an essential $L^1(G)$-module. We also have $T(G)^* = \mathcal{B}(L^2(G))$ as an $L^1(G)$-module with the dual action.

4.4. Lemma. *Let $E \subseteq G$ be closed. Then $\Phi(E^*)$ is an $L^1(G)$-submodule of $T(G)$.*

Proof. This is clear from the definitions of $\Phi(E^*)$ and the $L^1(G)$-action on $T(G)$. ■

4.5. Lemma. *Let $E \subseteq G$ be a closed set, let X be an $A(G)$-submodule of $\mathrm{VN}(G)$ and let $\widehat{X}$ be the associated $V^\infty(G)$-submodule of $\mathcal{B}(L^2(G))$. Then the following are $L^1(G)$-submodules of $\mathcal{B}(L^2(G))$.*

(i) $\Phi(E^*)^\perp$,
(ii) $\widehat{X}$,
(iii) $\Phi^{\widehat{X}}(E^*)$.

Proof. (i) is immediate from Lemma 4.4 since the $L^1(G)$-action on $\mathcal{B}(L^2(G))$ is the dual action.

(ii) Let $f \in L^1(G)$ and $S \in \widehat{X}$. We have to show that $f.S \in \widehat{X}$, i.e. $w.(f.S)\circ\widetilde{N} \in X$ for any $w \in V^\infty(G)$. So let $w \in V^\infty(G)$. Then for $u \in A(G)$,

$$\begin{aligned}\langle w.(f.S) \circ \widetilde{N}, u\rangle &= \langle w.(f.S), \widetilde{N}u\rangle = \langle f.S, m_w(\widetilde{N}u)\rangle = \langle S, f.m_w(\widetilde{N}u)\rangle \\ &= \langle S, m_{f.w}\widetilde{N}u\rangle = \langle (f.w).S, \widetilde{N}u\rangle = \langle (f.w).S \circ \widetilde{N}, u\rangle.\end{aligned}$$

Thus, $w.(f.S) \circ \widetilde{N} = (f.w).S \circ \widetilde{N} \in X$, and (ii) is proved.

(iii) This follows at once from (i) and (ii) because of the definition of $\Phi^{\widehat{X}}(E^*)$. ■

We are now ready for the main theorem.

4.6. Theorem. *Let X be an $A(G)$-submodule of $\mathrm{VN}(G)$ and let $\widehat{X}$ be the corresponding $V^\infty(G)$-submodule of $\mathcal{B}(L^2(G))$. Then a closed set $E \subseteq G$ is an X-spectral set for $A(G)$ if and only if E^* is $\widehat{X}$-synthetic for $T(G)$.*

Proof. One part is immediate from Lemma 4.3. If E^* is $\widehat{X}$-synthetic, then $\Phi^{\widehat{X}}(E^*) = \Psi^{\widehat{X}}(E^*)$ and so, by Lemma 4.3, $I_A^X(E) = J_A^X(E)$ whence E is X-spectral.

For the converse, suppose E is X-spectral. To show that E^* is $\widehat{X}$-synthetic, we need only prove that $\Phi^{\widehat{X}}(E^*) \subseteq \Psi^{\widehat{X}}(E^*)$ by Lemma 3.2. In view of our lemmas, the proof is similar to that of the case $X = \mathrm{VN}(G)$ given in [5]. For the sake of completeness, here is a brief sketch.

We have to show that $\omega \in \Phi^{\widehat{X}}(E^*)$ implies $\omega \in \Psi^{\widehat{X}}(E^*)$. Consider first the case $\omega \in \Phi^{\widehat{X}}(E^*) \cap T_{\mathrm{inv}}(G)$. In this case $\omega = \widetilde{N}u$ with $u \in I_A^X(E)$ (see the proof of [5, Theorem 4.6]). Thus, by assumption $u \in I_A^X(E) = J_A^X(E)$, so $\omega = \widetilde{N}u \in \Psi^{\widehat{X}}(E^*)$ by Lemma 4.3 and the result is proved in this case.

Now consider an arbitrary $\omega \in \Phi^{\widehat{X}}(E^*)$. For $\pi \in \widehat{G}$ and the matrix coefficients u_{ij}^π corresponding to π, $\omega_{ij}^\pi = u_{ij}^\pi.\omega \in \Phi^{\widehat{X}}(E^*)$ by Lemma 4.5. If $\widetilde{\omega}_{ij}^\pi = \sum_k m_{u_{ik}^\pi \otimes 1}\omega_{kj}^\pi$, then $\widetilde{\omega}_{ij}^\pi \in \Phi^{\widehat{X}}(E^*) \cap T_{\mathrm{inv}}(G)$ since $\Phi^{\widehat{X}}(E^*)$ is a $V^\infty(G)$-submodule. Thus $\widetilde{\omega}_{ij}^\pi \in \Psi^{\widehat{X}}(E^*)$. But $\omega_{ij}^\pi = \sum_k m_{\check{u}_{ik}^\pi \otimes 1}\widetilde{\omega}_{kj}^\pi \in \Psi^{\widehat{X}}(E^*)$. $L^1(G)$ has an approximate identity $\{u_\alpha\}$ with $u_\alpha \in \mathrm{span}\{u_{ij}^\pi : i,j = 1,\ldots,d_\pi,\ \pi \in \widehat{G}\}$ for all α and $u_\alpha.\omega \in \mathrm{span}\{\omega_{ij}^\pi : i,j = 1,\ldots,d_\pi,\ \pi \in \widehat{G}\} \subseteq \Psi^{\widehat{X}}(E^*)$, so $\omega = \lim u_\alpha.\omega \in \Psi^{\widehat{X}}(E^*)$. ■

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