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A $C(K)$ Banach space which does not have the Schroeder–Bernstein property

by

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Abstract. We construct a totally disconnected compact Hausdorff space K_+ which has clopen subsets $K_+'' \subseteq K_+' \subseteq K_+$ such that K_+'' is homeomorphic to K_+ and hence $C(K_+'')$ is isometric as a Banach space to $C(K_+)$ but $C(K_+')$ is not isomorphic to $C(K_+)$. This gives two nonisomorphic Banach spaces (necessarily nonseparable) of the form $C(K)$ which are isomorphic to complemented subspaces of each other (even in the above strong isometric sense), providing a solution to the Schroeder–Bernstein problem for Banach spaces of the form $C(K)$. The subset K_+ is obtained as a particular compactification of the pairwise disjoint union of an appropriately chosen sequence $(K_{1,n} \cup K_{2,n})_{n\in\mathbb{N}}$ of Ks for which $C(K)$s have few operators. We have $K_+' = K_+ \setminus K_{1,0}$ and $K_+'' = K_+ \setminus (K_{1,0} \cup K_{2,0})$.

1. Introduction. If X, Y are Banach spaces, then $X \sim Y$ will mean that they are isomorphic, and $X \overset{c}{\hookrightarrow} Y$ that Y has a complemented subspace isomorphic to X. A. Pełczyński has proved (see [19], [2]) that if two Banach spaces X, Y satisfy $X \sim X^2$, $Y \sim Y^2$, then the following version of the Schroeder–Bernstein theorem holds for them:

$$\text{If } X \overset{c}{\hookrightarrow} Y \text{ and } Y \overset{c}{\hookrightarrow} X, \text{ then } X \sim Y.$$

The problem whether this holds in general, without demanding that X and Y are isomorphic to their squares, has been known as the *Schroeder–Bernstein problem for Banach spaces* ([2]). After several decades it was solved in the negative by T. Gowers in [11]. In this paper we give a quite different construction of a Banach space that solves this problem in the negative, which additionally is a classical Banach space of real valued continuous functions on a compact Hausdorff space with the supremum norm.

One of the main ingredients of Gowers' solution was the use of the Banach space X_{GM}, obtained by Gowers himself and Maurey ([12]), with few operators in the sense that each bounded, linear operator on X_{GM} is of the form $\lambda\,\mathrm{Id} + S$ where S is strictly singular and λ is a scalar. The space of [11]

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is an “exotic” completion of $c_{00}(X_{GM} \oplus X_{GM})$ where the norm in the sum is defined depending on the choice of a basic sequence in X_{GM}. Also the space X_{GM} has quite a complex definition of the norm. Other examples of Banach spaces which solve the Schroeder–Bernstein problem in the negative were also based on spaces like X_{GM} and were constructed by Gowers and Maurey [13], Galego ([6], [7], [8]) and Galego and Ferenczi [9].

By contrast, when we deal with Banach spaces of continuous functions on a compact Hausdorff space K, we have the simplest possible norm, the supremum norm, and nontrivial examples are obtained through nontrivial compact Ks.

In [15] we constructed Banach spaces of continuous functions with few operators in the sense that any operator on them is of the form $g\,\mathrm{Id} + S$ where $g \in C(K)$ and S is weakly compact (equivalently here, a strictly singular operator) or such that the adjoint of any operator is of the form $g\,\mathrm{Id} + S$ where g is a Borel bounded function and S is weakly compact on the dual.

The latter construction and its connected modification was used in [15] to obtain results analogous to some of the results of Gowers and Maurey but in the class of $C(K)$ Banach spaces, like examples of indecomposable Banach spaces or spaces nonisomorphic to their hyperplanes.

The former construction was obtained in [15] under the assumption of the continuum hypothesis but was later modified in [21] or [24] to avoid this assumption. Of course the former form of the operator is simpler and we will make use of $C(K)$ spaces with few operators in this sense. For our purpose the construction of [24] is the most convenient because its K is perfect, separable and totally disconnected, and the Boolean algebra $\mathrm{Clop}(K)$ of all clopen subsets of K has a dense subalgebra with many automorphisms. On the other hand by the results of [25] the approach where we deal with spaces with few adjoint operators in the above sense should work as well but would be formally more complicated.

The general plan of the construction follows [11] with X_{GM} replaced by a $C(K)$ with few operators. However, all this must be achieved in the $C(K)$-environment, that is, through an appropriate compactification of the topological disjoint sum of compact K_is rather than the definition of a norm. Note, also, that if K is infinite and metrizable then $C(K)$ is isomorphic to its square, so we must deal with nonseparable $C(K)$s. So, the realization of this general plan is quite far from [11]. Another complication is that our spaces have few operators in a different sense than those of [12]; this seems to cause that we have been unable to obtain spaces which are isomorphic to their cube but not isomorphic to their square as in [11]. Whether there could be such spaces of the form $C(K)$ remains an open problem.

The main idea is to define a Banach space X_+ as a $C(K_+)$ where K_+ is an appropriate compactification of the pairwise disjoint sum of clopen subsets $K_n = K_{1,n} \cup K_{2,n}$ for $n \in \mathbb{N}$ (see Section 3) such that all operators from $C(K_n)$ into itself are of the form $f\,\mathrm{Id} + S$ where $f \in C(K_n)$ and S is weakly compact and all K_n are pairwise homeomorphic. It follows that all operators from $C(K_{i,n})$ into $C(K_{3-i,m})$ are weakly compact, where $i = 1, 2$ and $n, m \in \mathbb{N}$.

We have $Y_+ = \{f \in X_+ : f|K_{1,0} = 0\} \sim C(K'_+)$ where $K'_+ = K_+ \setminus K_{1,0}$. It is clear that Y_+ is complemented in X_+ and the compactification is defined in such a way that $\{f \in X_+ : f|K_1 = 0\} \sim C(K''_+)$, where $K'_+ = K_+ \setminus (K_{1,0} \cup K_{2,0})$ is a subspace of Y_+ isomorphic to X_+ and clearly complemented in Y_+. Thus we only have to prove that X_+ and Y_+ are not isomorphic. The remaining sections are devoted to this purpose.

Although $C(K_{i,n})$ and $C(K_{3-i,m})$ are quite incomparable (there are only weakly compact and so strictly singular operators among them), we exploit the fact that the algebras of all clopen sets of $K_{1,n}$ and $K_{2,n}$ have isomorphic dense subalgebras. This allows us to compare $f_i \in K_{i,n}$ for $i = 1, 2$ using an isomorphism between these dense subalgebras. Finally the compactification K_+ is defined in such a way that each $f \in C(K_+)$ can be identified with a sequence of functions $f_n \in C(K_n)$ such that $f_n|K_{1,n}$ and $f_n|K_{2,n}$ get "closer" to each other (in a sense defined using the dense subalgebras and an isomorphism between them) when n tends to infinity. This construction and the relevant notation are described in Section 3.

The point of the proof that X_+ is not isomorphic to Y_+ is to show that any isomorphism T between these spaces would create infinitely many "bumps", i.e., $T(f_n)|K_{1,m(n)}$ and $T(f_n)|K_{2,m(n)}$ would not get closer to each other when n and some $m(n)$ tend to infinity. This would give $T(f)$ outside X_+, leading to a contradiction with the existence of such an isomorphism. At least these bumps are created when we look at the shift of the part of X_+ corresponding to $\bigcup_{n\in\mathbb{N}} K_{1,n}$, which should not be a well-defined operator on X_+. It turns out that all hypothetic isomorphisms from Y_+ to X_+ would share this behaviour and so there are none. This argument is the subject of the last section where we also prove a strengthening of the well-known fact that $\{x \in K : T^*(\delta_x)(\{x\}) \neq 0\}$ is at most countable if T is weakly compact (Lemma 6.2).

To be able to detect these bumps one approximates an operator $T : X_+ \to X_+$ by a matrix of multiplications by continuous functions. This approximation is developed in Section 5, which relies on Section 4 where we prove general properties of operators on X_+. Some of the ideas of Section 4, which we use for dealing with compactifications of disjoint K_is where each $C(K_i)$ has few operators, were developed in [17] and [16].

The next section gathers some classical results about $C(K)$s and some of their corollaries needed in the rest of the paper.

2. Some fundamental results on $C(K)$ spaces. In this section we recall some fundamental results of Bessaga, Grothendieck, Pełczyński and Rosenthal concerning the Banach spaces $C(K)$. We state them in the form most convenient for this paper, and we also give some simple corollaries.

THEOREM 2.1 ([4, Cor. 2]). *Let E be a Banach space and X be either l_∞ or c_0, and let $T : X \to E$ be a continuous linear operator. Then exactly one of the two possibilities holds:*

(1) $T(e_n) \to 0$.
(2) *There is an infinite subset $M \subseteq \mathbb{N}$ such that $T|X(M)$ is an isomorphism.*

THEOREM 2.2 ([22, Thm. 1.3]). *Let B be a Banach space and X an injective Banach space. Let $T : X \to B$ be an operator such that there exists a subspace A of X isomorphic to $c_0(\Gamma)$ with $T|A$ an isomorphism. Then there exists a subspace Y of X isomorphic to $l_\infty(\Gamma)$ with $T|Y$ an isomorphism.*

DEFINITION 2.3. Let K be a Hausdorff compact space. Suppose that (f_n) is a sequence in $C(K)$. We say that it *generates a copy* of c_0 if and only if it is a basic sequence which is equivalent to the standard basis of c_0.

DEFINITION 2.4. Let K be a Hausdorff compact space. We say that $f, g \in C(K)$ are *disjoint* if and only if $fg = 0$.

FACT 2.5. *Let K be a Hausdorff compact space and $\varepsilon > 0$. Suppose that (f_n) is a bounded pairwise disjoint sequence in $C(K)$ such that $\|f_n\| > \varepsilon > 0$ for each $n \in \mathbb{N}$. Then (f_n) generates a copy of c_0.*

FACT 2.6. *Let K be a Hausdorff compact space. Suppose that $(f_n)_{n\in\mathbb{N}}$ is a sequence in $C(K)$ that generates a copy of c_0 and suppose that $a_n \subseteq K$ are clopen sets such that $\|f_n\chi_{a_n}\| > \varepsilon$ for some $\varepsilon > 0$. Then there is an infinite $M \subseteq \mathbb{N}$ such that $(f_n\chi_{a_n})_{n\in M}$ generates a copy of c_0.*

Proof. Let $T : c_0 \to C(K)$ be an isomorphism such that $T(1_n) = f_n$. Note that for each $x \in K$ we have

$$\sum_{n\in\mathbb{N}} \alpha_n f_n \chi_{a_n}(x) = \sum_{n\in\mathbb{N}} \alpha_n^*(x) f_n(x),$$

where $\alpha_n^*(x) = \alpha_n$ if $x \in a_n$ and $\alpha_n^*(x) = 0$ otherwise, and so such an $(\alpha_n^*(x))_{n\in\mathbb{N}}$ belongs to c_0. For every $\varepsilon > 0$ and every $k_1 < k_2$ from $\mathbb{N}$ there is an $x \in K$ such that

$$\Big\| \sum_{n=k_1}^{k_2} \alpha_n f_n \chi_{a_n} \Big\| \le \Big\| \sum_{n=k_1}^{k_2} \alpha_n^*(x) f_n \Big\| + \varepsilon$$
$$\le \|T\| \, \|\alpha_{k_1}^*(x), \ldots, \alpha_{k_2}^*(x)\|_{c_0} + \varepsilon$$
$$\le \|T\| \, \|\alpha_{k_1}, \ldots, \alpha_{k_2}\|_{c_0} + \varepsilon.$$

So, if $(\alpha_n) \in c_0$, then $\|\alpha_{k_1}, \ldots, \alpha_{k_2}\|_{c_0}$ goes to zero if k_1 and k_2 go to infinity; hence we may conclude that $\sum_{n\in\mathbb{N}} \alpha_n f_n \chi_{a_n}$ converges uniformly and so the limit is continuous. A similar argument shows that $\|\sum_{n\in\mathbb{N}} \alpha_n f_n \chi_{a_n}\| \le \|T\| \, \|(\alpha_n)\|_{c_0}$. So we may conclude that

$$T'((\alpha_n)) = \sum_{n\in\mathbb{N}} \alpha_n f_n \chi_{a_n}$$

is a bounded linear operator defined on c_0 and into $C(K)$. Now we can use 2.1 since $\|T'(1_n)\| = \|f_n \chi_{a_n}\| > \varepsilon$.

Theorem 2.7 ([20, Thm. 1]). *A bounded linear operator $T : C(K) \to C(K)$ is weakly compact if and only if it is strictly singular.*

Theorem 2.8 ([3]). *A bounded linear operator $T : C(K) \to C(K)$ is weakly compact if and only if for every bounded pairwise disjoint sequence (f_n) in $C(K)$ we have $T(f_n) \to 0$.*

Theorem 2.9 ([22]). *Suppose that $\varepsilon > 0$ and for each $n, k \in \mathbb{N}$ we have non-negative $m_{nk} \in \mathbb{R}$ with $\sum_{n\in\mathbb{N}} m_{nk} < \varepsilon$ for each $k \in \mathbb{N}$ (i.e., the sums are uniformly bounded). Then for each $\delta > 0$ there is an infinite $M \subseteq \mathbb{N}$ such that*

$$\sum_{n\in M\setminus\{k\}} m_{nk} < \delta$$

for each $k \in M$.

Corollary 2.10. *Suppose $(f_n)_{n\in\mathbb{N}}$ is a bounded pairwise disjoint sequence in $C(K)$. Suppose $(\zeta_n)_{n\in\mathbb{N}}$ is a bounded sequence in $C^*(K)$ and let $\varepsilon > 0$. There is an infinite $M \subseteq \mathbb{N}$ such that whenever $M' \subseteq M$ and the supremum $f_{M'} = \sup_{n\in M'} f_n$ exists in $C(K)$, then $|\zeta_k(f_{M'}) - \zeta_k(f_k)| < \varepsilon$ if $k \in M'$ and $|\zeta_k(f_{M'})| < \varepsilon$ if $k \in M \setminus M'$.*

Proof. Let μ_k be the Radon measure on K corresponding to ζ_k. Define $m_{nk} = |\mu_k|(f_n)$. Apply the Rosenthal lemma 2.9 for $\delta = \varepsilon$ to obtain an infinite $M_1 \subseteq M$ such that $\sum_{n\in M_1\setminus\{k\}} |\zeta_k(f_n)| < \varepsilon$ if $k \in M_1$. Thus, it is enough to obtain an infinite $M \subseteq M_1$ such that whenever $M' \subseteq M$ and the supremum $f_{M'} = \sup_{n\in M'} f_n$ exists in $C(K)$, then for each $k \in \mathbb{N}$,

$$\sum_{n\in M'} \zeta_k(f_n) = \zeta_k(f_{M'}).$$

Note that

$$\sum_{n\in M'} \zeta_k(f_n) = \int \sum_{n\in M'} f_n \, d\mu_k,$$

and $\zeta_k(f_{M'}) = \int f_{M'} \, d\mu_k$ where $\sum_{n\in M'} f_n$ is taken pointwise and is possibly not in $C(K)$. Let $\{\mathbb{N}_\xi : \xi < \omega_1\}$ be a family of infinite sets of $\mathbb{N}$ whose pairwise intersections are finite. If none of them works as M, there is a $k \in \mathbb{N}$ and there are infinite $b_\xi \subseteq \mathbb{N}_\xi$ for uncountably many $\xi \in \omega_1$ such that

$$\int \Big(f_{b_\xi} - \sum_{n\in b_\xi} f_n \Big) \, d\mu_k \neq 0;$$

but this is impossible since the Borel functions which we integrate above are pairwise disjoint as shown in [15, Thm. 5.2]. ∎

THEOREM 2.11 ([1]). *Let X be a Banach space, $(x_n) \subseteq X$ be a basic sequence and $(x_n^*) \subseteq X^*$ a sequence biorthogonal to (x_n). If $(y_n) \subseteq X$ fulfills the condition*

$$\sum_{n\in\mathbb{N}} \|x_n - y_n\| \, \|x_n^*\| < \delta < 1,$$

then (y_n) is a basic sequence and (x_n) and (y_n) are equivalent. ∎

3. The construction and the notation. In this section we define our Banach spaces of the form $C(K)$ where K is a totally disconnected compact space. These spaces are defined as the Stone spaces of Boolean algebras, hence some basic knowledge of Boolean algebra and Stone duality is required. This is covered, for example, in the initial chapters of [14].

The Stone functor from the category of Boolean algebras and homomorphisms into the category of compact Hausdorff spaces with continuous functions will be denoted S. Thus $S(A)$ is the Stone space of the algebra A and $S(h) : S(B) \to S(A)$ is the continuous mapping induced by a homomorphism $h : A \to B$ of Boolean algebras and is given by $S(h)(x) = h^{-1}(x)$.

If a is an element of a Boolean algebra A, then the basic clopen set $\{u \in S(A) : a \in u\}$ of the Stone space of A corresponding to a will be denoted by $[a]$.

One can also embed $S(A)$ homeomorphically in $\{0,1\}^A$ taking $u \in S(A)$ to the point $x \in \{0,1\}^A$ such that $x(A) = 1$ if and only if $a \in u$. The Boolean algebra of clopen subsets of a compact space K is denoted by $\mathrm{Clop}(K)$. We will identify $S(\mathrm{Clop}(K))$ with K and $\mathrm{Clop}(S(A)$ with A. For more on Stone duality see Chapter 3 of [14].

LEMMA 3.1. *There is an infinite Boolean algebra A whose Stone space K is a union $K_{1,*} \cup K_{2,*}$ with $K_{1,*}$ and $K_{2,*}$ disjoint and clopen which satisfies the following:*

(1) *K is a separable* (*compact totally disconnected*) *perfect space.*

(2) *Every operator on* $C(K)$ *is of the form* $T = f\,\mathrm{Id} + S$ *where* $f \in C(K)$ *and* S *is weakly compact.*
(3) $C(K)$ *contains no copy of* l_∞.
(4) *There is a dense subalgebra* $B \subseteq A$ *such that* $K_{1,*}, K_{2,*} \in B$ *and there is an automorphism* $j : B \to B$ *such that* $j(K_{3-i,*}) = K_{i,*}$ *for* $i = 1, 2$ *and* $j^2 = \mathrm{Id}_B$.
(5) *Suppose that* (a_n) *is a sequence of clopen and pairwise disjoint subsets of* K. *There is an infinite* $M \subseteq \mathbb{N}$ *such that the supremum* $\sup_{n\in M} a_n = a$ *exists in* A.

Proof. Let K be the Stone space of the algebra A obtained in Chapter 3 of [24]. By [24, 3.6.1–3.6.3] and the fact that K has property (K') of [24, 3.3.1] we obtain (1) without perfectness, (2) and (5). (3) follows from [15, 2.4] because every space which contains l_∞ must have all hyperplanes isomorphic to itself. So, we are left with showing that there is a dense subalgebra B and its automorphism j as above.

B will be a free Boolean algebra with 2^ω independent generators. Namely, the algebra A includes the algebra B of clopen subsets of $\{0,1\}^{2^\omega}$ and is included in the algebra of regular open subsets of $\{0,1\}^{2^\omega}$, hence B is dense in A (see [24, p. 57]). As B is a free Boolean algebra, it is clear that there is an automorphism j and $K_{1,*}$, $K_{2,*}$ as required, and that K is prefect. ∎

Remark 3.2. In the literature there are several constructions of spaces $C(K)$ where all operators are of the form $T = f\,\mathrm{Id} + S$ where $f \in C(K)$ and S is weakly compact. Some of these constructions are obtained using special set-theoretic assumptions as in Section 6 of [15], or in [5], and have some additional properties. Other constructions are obtained without any additional set-theoretic assumptions, like in [21] or [24]. The spaces of Section 6 of [15] or of [5] have countable subalgebras B satisfying (4) of the above lemma. The space of [21] is only presented in the connected version and it is unclear to us if its totally disconnected version would have some natural dense subalgebras with appropriate automorphisms. The separability of K from [24] is not used in any argument in our paper but it shows that a $C(K)$ solution to the Schroeder–Bernstein problem could be a subspace of l_∞. We conjecture that also the space of Section 3 of [15] obtained without any special set-theoretic assumptions can be used to obtain the main result of this paper; the results of [25] support this conjecture, but it seems that it would complicate the details. Thus it seems that the space of [24] is the most optimal for our purpose.

Let A, B, K, $K_{1,*}$, $K_{2,*}$, j be as in 3.1, moreover let

- $C = \{b \in B : j(b) = b\} = \{b \cup j(b) : b \in B,\ [b] \subseteq K_{1,*}\} = \{j(b) \cup b : b \in B,\ [b] \subseteq K_{2,*}\}$,

- L be the Stone space of C,
- $\tau = S(\subseteq) : K \to L$ be the canonical surjection, where $\subseteq : C \to A$ is the inclusion.
- $Z \equiv C(L)$ considered as a subspace of $C(K)$, i.e., $Z = \{f \circ \tau : f \in C(L)\}$.

REMARK 3.3. Note that τ identifies, among others, the pairs x, y of points of $K_{1,*}$ and $K_{2,*}$ respectively such that $S(j)(x \cap B) = y \cap B$. So in particular, for each $x \in K_{i,*}$ there is $y \in K_{3-i,*}$ such that $\tau(x) = \tau(y)$.

As K is separable, we may assume that A is a subalgebra of $\wp(\mathbb{N})$. For $n \in \mathbb{N}$ let

- $\mathbb{N}_n$ be a copy of $\mathbb{N}$, with $\mathbb{N}_m, \mathbb{N}_n$ disjoint for $n \neq m$,
- A_n be the copy of A in $\wp(\mathbb{N}_n)$,
- K_n be the Stone space of A_n, with $\mathbb{N}_n, \mathbb{N}_m$ disjoint for $n \neq m$,
- $K_{1,n}$ and $K_{2,n}$ be the copies of $K_{1,*}$ and $K_{2,*}$ in K_n,
- $K^{-n} = \bigcup_{i \leq n} K_i$,
- B_n be the copy of B in $A_n \subseteq \wp(\mathbb{N}_n)$,
- C_n be the copy of C in $A_n \subseteq \wp(\mathbb{N}_n)$,
- L_n be the Stone space of C_n,
- $\tau_n = S(\subseteq_n) : K_n \to L_n$ be the canonical surjection, where $\subseteq_n : C_n \to A_n$ is the inclusion,
- j_n be the copy of j on B_n,
- $i_{n,*} : K \to K_n$, $i_{*,n} : K_n \to K$, $i_{m,n} : K_n \to K_m$ be homeomorphisms preserving the above mentioned objects respectively,
- $h_{n,*} : L \to L_n$, $h_{*,n} : L_n \to L$, $h_{m,n} : L_n \to L_m$ be homeomorphisms.

We will consider the following Boolean algebras:

- $D_\infty = \{a \subseteq \bigcup_{n\in\mathbb{N}} \mathbb{N}_n : \forall n \in \mathbb{N}\ \ a \cap \mathbb{N}_n \in A_n\}$,
- $D_0 = \{a \in D_\infty : \exists m \in \mathbb{N}\ (\forall n > m\ \ a \cap \mathbb{N}_n = \emptyset) \text{ or } (\forall n > m\ \ a \cap \mathbb{N}_n = \mathbb{N}_n)\}$,
- $D_+ = \{a \in D_\infty : \exists m \in \mathbb{N}\ \forall n > m\ \ a \cap \mathbb{N}_n \in C_n\}$.

In other words, D_∞ is the product algebra of A_ns, D_0 is the direct sum algebra of A_ns, and finally D_+ is the algebra of those elements of D_∞ whose coordinates eventually belong to C_n.

Let

- K_∞, K^0, K_+ be the Stone spaces of D_∞, D_0, and D_+ respectively.

And finally, let

- $X_\infty = C(K_\infty)$, $X_+ = C(K_+)$, $Z_n \equiv C(L_n)$ considered as a subspace of $C(K_n)$, i.e., $Z_n = \{f \circ \tau_n : f \in C(L_n)\}$.
- $X_0 = C_0(K^0) = \{f \in C(K^0) : f(\infty) = 0\}$, where ∞ is the only ultrafilter of D_0 which does not contain any $\mathbb{N}_n$.
- $K^{+n} = K_+ \setminus K^{-n}$.

We will also need the following notation for some natural projections and inclusions:

- $P_n : C(K_+) \to C(K_n)$, restriction to K_n,
- $I_n : C(K_n) \to C(K_+)$, extension from K_n by zero,
- $P_{-n} : C(K_+) \to C(K^{-n})$, restriction to K^{-n},
- $I_{-n} : C(K^{-n}) \to C(K_+)$, extension from K^{-n} by zero,
- $P_{+n} : C(K_+) \to C(K^{+n})$, restriction to K^{+n},
- $I_{+n} : C(K^{+n}) \to C(K_+)$, extension from K^{+n} by zero.

The Stone–Weierstrass theorem which implies that the functions which assume only finitely many values on clopen sets from the algebra A are dense in $C(K)$ where K is the Stone space of A, gives an idea about which functions belong to the spaces X_0, X_+, X_∞. In particular we have the following three descriptions:

LEMMA 3.4.
$$X_\infty \equiv \Big\{(f_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} C(K_n) : (\|f_n\|)_{n\in\mathbb{N}} \textit{ is bounded}\Big\}.$$
The norm is the supremum norm.

LEMMA 3.5.
$$X_0 \equiv \Big\{(f_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} C(K_n) : \lim_{n\to\infty} \|f_n\| = 0\Big\}.$$
The norm is the supremum norm.

For $f \in C(K_n)$ denote by $d_n(f)$ the distance between f and the subspace $Z_n \subseteq C(K_n)$. Then we have:

LEMMA 3.6.
$$X_+ \equiv \Big\{(f_n)_{n\in\mathbb{N}} \in \prod_{n\in\mathbb{N}} C(K_n) : (\|f_n\|)_{n\in\mathbb{N}} \textit{ is bounded and } \lim_{n\to\infty} d_n(f_n) = 0\Big\}.$$
The norm is the supremum norm.

We will often denote elements of the above spaces using the above representations, i.e., as a sequence (f_n). Then, under the appropriate identification, we have $X_0 \subseteq X_+ \subseteq X_\infty$, which will be used as well.

DEFINITION 3.7. For $x, y \in K_n$ we write $x \bowtie y$ if and only if $\tau_n(x) = \tau_n(y)$.

LEMMA 3.8. *Suppose that $f \in C(K_n)$. Then*
$$d_n(f) = \frac{\sup\{|f(x) - f(y)| : x, y \in K_n,\ x \bowtie y\}}{2}.$$

Proof. Let s be the supremum from the lemma. If $g \in Z_n$, then clearly $g(x) = g(y)$ for any x, y such that $x \bowtie y$, so we obtain $d_n(f) \geq s/2$.

Now let $f \in C(K_n)$ and $\delta > s$. It is possible to obtain a clopen finite partition $U_1, \ldots, U_k$ of L_n such that $\mathrm{diam}(f[\tau_n^{-1}[U_m]]) < \delta$ for each $m = 1, \ldots, k$. This follows from the fact that $\mathrm{diam}(f[\tau_n^{-1}[\{t\}]]) \le s$ for each $t \in L_n$ and the continuity of f, as well as from the compactness of the spaces involved. Now define $g \in Z_n$ as a function which is constant on each $\tau_n^{-1}[U_m]$ with value the arithmetic mean of the extrema of the values of $f[\phi^{-1}[U_m]]$. This way $\|f - g\| \le \delta/2$ and hence $d_n(f) \le s/2$. ∎

COROLLARY 3.9. *Suppose that $g \in Z_n$ and $f \in C(K_n)$ with $\|f - g\| < \varepsilon$. Suppose that $x, y \in K_n$ and $x \bowtie y$. Then $|f(x) - f(y)| < 2\varepsilon$.*

Proof. Follows directly from the previous lemma. ∎

COROLLARY 3.10. *Suppose that $(f_n)_{n\in\mathbb{N}} \in X_+$ and $x_n, y_n \in K_n$ with $x_n \bowtie y_n$. Then $|f_n(x_n) - f_n(y_n)| \to 0$ as $n \to \infty$.*

Proof. Follows directly from the previous corollary and from Lemma 3.6. ∎

Finally, we will need the following notation:

$$Y_0 = \{(f_n) \in X_0 : f|K_{1,0} = 0\}, \qquad Y_+ = \{(f_n) \in X_+ : f|K_{1,0} = 0\}.$$

LEMMA 3.11. *Y_+ is complemented in X_+, and Y_+ contains a complemented subspace isometric to X_+. Both projections are of norm one.*

Proof. Clearly

$$X_+ = C(K_+) \equiv C(K_{1,0}) \oplus C(K_{2,0}) \oplus \{(f_n) \in X_+ : f_1 = 0\}$$

and $\{(f_n) \in X_+ : f_1 = 0\}$ is isometric to X_+ while $C(K_{2,0}) \oplus \{(f_n) \in X_+ : f_1 = 0\}$ is isometric to Y_+. ∎

The rest of this paper is devoted to the proof that Y_+ is not isomorphic to X_+.

4. Operators on X_+.

First, we will define two operators which will serve to illustrate several phenomena in Proposition 4.2 and Remarks 4.8 and 5.8.

DEFINITION 4.1. Fix a pairwise disjoint sequence (g_n) in $C(K)$ such that $\|g_n\| = 1$ and fix a dense countable subset $\{x_n : n \in \mathbb{N}\}$ of K and for each $n \in \mathbb{N}$ fix $\theta_n \in C^*(K)$ such that $\|\theta_n\| = 1$ and $\theta_n|Z_n = 0$. Define

$$\Lambda : C(K) \to X_+ \quad \text{by} \quad \Lambda(f)|K_n = f(x_n)\chi_{K_n} \quad \text{for } f \in C(K),$$

$$\Theta : X_+ \to C(K) \quad \text{by} \quad \Theta((f_n)) = \sum_{n\in\mathbb{N}} \theta_n(f_n) g_n.$$

PROPOSITION 4.2. *Θ and Λ are well-defined bounded linear operators. The image of Θ is isomorphic to c_0, in particular X_+ is not a Grothendieck space, and so it has a complemented copy of c_0.*

Proof. The only nontrivial part of the first statement is that $\Theta((f_n))$ is always in $C(K)$ for any $(f_n) \in X^+$. It is enough to note that $\theta_n(f_n) \to 0$. But for each $n \in \mathbb{N}$ there is a $z_n \in Z_n$ such that $\|f_n - z_n\| \to 0$. Then using the fact that $\theta_n|Z_n = 0$ we have

$$|\theta_n(f_n)| = |\theta_n(f_n - z_n) + \theta_n(z_n)| \le \|\theta_n\| \, \|f_n - z_n\| \to 0.$$

So, the last statement follows as well (see [23, 5.1, 5.3]). ∎

Note that $X = \{(f_n) \in X_+ : \forall n \in \mathbb{N} \;\; f_n|K_n \text{ is constant}\}$ is a complemented copy of l_∞ in X_+. Thus using it and the operators Θ and Λ we may construct many nontrivial operators from X^+ into X^+, which have nothing to do with any multiplications. It was already noted in [17] that nevertheless dealing with this kind of spaces we have strong tools (see for example 4.7 below) to analyze all the operators.

Lemma 4.3. *Suppose that $T : C(K) \to X_+$ is a bounded linear operator. Let $(f_m)_{m\in\mathbb{N}}$ be a bounded pairwise disjoint sequence in $C(K)$. Then*

$$\forall \varepsilon \; \exists k \; \forall n > k \; \forall m \qquad d_n(T(f_m)|K_n) \le \varepsilon.$$

Proof. We may assume that the norms of f_m are bounded by 1. Suppose that the lemma is false. Let $\varepsilon > 0$ be such that for every $k \in \mathbb{N}$ there are $n > k$ and $m_k \in \mathbb{N}$ such that

$$d_n(T(f_{m_k})|K_n) > \varepsilon.$$

Hence we can choose increasing n_ks such that

$$d_{n_k}(T(f_{m_k})|K_{n_k}) > \varepsilon.$$

Moreover as $T(f_{m_k}) \in X_+$, by 3.6 the m_ks assume infinitely many values and so we may assume that

$$d_{n_k}(T(f_{m_k})|K_{n_k}) > \varepsilon$$

and m_ks are increasing.

By 3.8 there are points $x_k, y_k \in K_{n_k}$ such that $x_k \bowtie y_k$ and for each $k \in \mathbb{N}$,

$$|T(f_{m_k})(x_k) - T(f_{m_k)}(y_k)| > 2\varepsilon.$$

Let

$$\mu_k = T^*(\delta_{x_k} - \delta_{y_k}),$$

so that $|\int f_{m_k} \, d\mu_k| > 2\varepsilon$. Since K is totally disconnected we can choose pairwise disjoint clopen $[a_{m_k}]$ included in the support of f_{m_k} such that

$$|\mu_k([a_{m_k}])| > \varepsilon.$$

Now we use 2.10 for the pairwise disjoint bounded sequence $(\chi_{[a_{m_k}]})_{k\in\mathbb{N}}$, $\varepsilon/2$ and the functionals equal to the measures μ_k and then 3.1(5) to obtain an infinite $M \subseteq \mathbb{N}$ such that the supremum $a = \sup_{k\in M} a_{m_k}$ exists in $\mathrm{Clop}(K)$ and $|\mu_k(a) - \mu_k(a_{m_k})| < \varepsilon/2$ for each $k \in M$. However this implies that

$|T(\chi_{[a]})(x_k) - T(\chi_{[a]})(y_k)| > \varepsilon/2$ for each $k \in M$, which by 3.10 implies that $T(\chi_{[a]})$ is not in X_+, a contradiction. ∎

DEFINITION 4.4. Let $f \in X_+$. We say that f is (C_n)-*simple* if for every $n \in \mathbb{N}$ the function $f|K_n$ assumes only finitely many values on clopen sets from C_n.

LEMMA 4.5. *Suppose that* (f_m) *is a sequence of elements of* X_+ *such that:*

(1) f_m *is* (C_n)*-simple for each* $m \in \mathbb{N}$,
(2) $f_m|K_{-m} = 0$,
(3) (f_m) *generates a copy of* c_0.

Suppose $T : X_+ \to C(K)$ *is a bounded linear operator. Then* $\|T(f_m)\| \to 0$.

Proof. We will find an injective subspace of X_+ which contains all the f_ms and then assuming that the lemma is false we will apply 2.1 and 2.2 to obtain a contradiction with 3.1(3).

Let E_n be the Boolean algebra of subsets of K_n generated by the preimages of all possible sets under the functions f_m where $m \in \mathbb{N}$ (there are finitely many such preimages). Note that it is finite, as almost all the functions f_m are zero on K_n and the remaining functions are (C_n)-simple. Consider the Boolean algebra

$$E = \{a \in D_\infty : \forall n \in \mathbb{N}\ \ a \cap \mathbb{N}_n \in E_n\},$$

and note that it is a complete Boolean algebra, and hence the Banach space V of all continuous functions which are in the closure of finite linear combinations of characteristic functions of elements of E is included in X_+, is injective and $f_m \in V$ for each $m \in \mathbb{N}$.

Now, if $\|T(f_m)\| \not\to 0$, then by 2.1 we may assume that T is an isomorphism on the copy of c_0 generated by (f_m). Finally apply 2.2 to conclude that $C(K_n)$ contains a copy of l_∞, which contradicts 3.1(3). ∎

LEMMA 4.6. *Suppose that* $S : C(K) \to X_+$ *and* $T : X_+ \to C(K)$ *are bounded linear operators and* (f_m) *is a pairwise disjoint sequence in* $C(K)$ *which generates a copy of* c_0 *such that for every* $n \in \mathbb{N}$ *we have*

$$\|T \circ I_{-n} \circ P_{-n} \circ S(f_m)\| \to 0$$

as $m \to \infty$. *Then* $\|T(S(f_m))\| \to 0$.

Proof. Assume that $\|T(S(f_m))\| \not\to 0$. Then $\|S(f_m)\| \not\to 0$ and so by 2.1 we may assume that $(S(f_m))_{m\in\mathbb{N}}$ generates a copy of c_0.

Since $\|T \circ I_{-n} \circ P_{-n} \circ S(f_m)\| \to 0$ and $\|T(S(f_m))\| \not\to 0$, for every $n \in \mathbb{N}$ we may choose $m_n \in \mathbb{N}$ such that the $\|T \circ I_{+n} \circ P_{+n} \circ S(f_{m_n})\|$ are separated from 0 and hence so also are the $P_{+n}(S(f_{m_n}))$.

Applying 2.6 we may choose a strictly increasing sequence (n_k) such that $n_k > k$ and $(P_{+n_k} \circ S(f_{m_{n_k}}))$ generates a copy of c_0.

Now we will look for some basic sequence equivalent to $(P_{+n_k}(S(f_{m_{n_k}}))$. Let $\rho > 1$ be a bound for the norms of a sequence biorthogonal to $(P_{+n_k}(S(f_{m_{n_k}})))$.

By 4.3 we can thin out the sequence $(n_k) \subseteq \mathbb{N}$ so that

$$|d_n(P_n(P_{+n_k}(S(f_{m_{n_k}}))))| < \frac{1}{2^{k+2}\rho}$$

for all $n, k \in \mathbb{N}$.

By the above and the Weierstrass–Stone theorem we can find (g_k)s which are (C_n)-simple, satisfy $P_{-n}(g_n) = 0$ for each $n \in \mathbb{N}$ and

$$\|g_k - I_{+n_k} \circ P_{+n_k} \circ S(f_{m_{n_k}})\| < \frac{1}{2^{k+2}\rho},$$

which means by the Bessaga–Pełczyński criterion 2.11 that (g_k) generates a copy of c_0. Thus $\|T(g_k)\| \to 0$ by 4.5. But also $\|T(g_k - I_{+n_k} \circ P_{+n_k} \circ S(f_{m_{n_k}})\| \to 0$, which means that $T \circ I_{+n_k} \circ P_{+n_k} \circ S(f_{m_{n_k}}) \to 0$, contrary to the choice of m_ns. ∎

Corollary 4.7. *Suppose that $S : C(K) \to X_+$ and $T : X_+ \to C(K)$ are bounded linear operators such that for every $n \in \mathbb{N}$ either*

(1) *$P_{-n} \circ S$ is weakly compact, or*
(2) *$T \circ I_{-n}$ is weakly compact.*

Then $T \circ S : C(K) \to C(K)$ is weakly compact.

Proof. Let (f_m) be a bounded pairwise disjoint sequence in $C(K)$. We will use 2.8, and so we need to prove that $(T \circ S)(f_m) \to 0$. By 2.1 we may assume that (f_m) generates a copy of c_0. By hypothesis $T \circ I_{-n} \circ P_{-n} \circ S$ is weakly compact for each $n \in \mathbb{N}$ and so by 2.8, $T \circ I_{-n} \circ P_{-n} \circ S(f_m) \to 0$ as $m \to \infty$ and $n \in \mathbb{N}$ is fixed. So, we may apply 4.6 to conclude that $(T \circ S)(f_m) \to 0$ and so $T \circ S$ is weakly compact. ∎

Remark 4.8. Suppose that $\Lambda : C(K) \to X^+$ and $\Theta : X^+ \to C(K)$ are as in 4.1. Note that $P_{-n} \circ \Lambda$ and $\Theta \circ P_{-n}$ are finite-dimensional and so weakly compact for all $n \in \mathbb{N}$ but neither Λ nor Θ are weakly compact. This shows that the composition cannot be replaced by a single operator in the above results. Note that $\Theta \circ \Lambda = 0$.

5. The matrix of multiplications of an operator. The matrix of multiplications discussed in this section is a crucial tool in the analysis of operators on the space X_+.

LEMMA 5.1. *Suppose that K' and K'' are homeomorphic perfect compact spaces and $i : K'' \to K'$ is a homeomorphism. Let $\phi, \psi \in C(K')$ and let $S : C(K') \to C(K'')$ be a weakly compact operator. Suppose that for every $f \in C(K')$ we have*

$$(\phi f) \circ i = (\psi f) \circ i + S(f).$$

Then $S = 0$ and $\phi = \psi$.

Proof. Suppose that $\phi \neq \psi$ and choose a clopen $U \subseteq K'$ and $\varepsilon > 0$ such that $|(\phi - \psi)|U| > \varepsilon$. Let (f_n) be pairwise disjoint functions of norm one whose supports are included in U. They exist since K' is perfect. Note that $\|(\phi - \psi)f_n\| \nrightarrow 0$ and so $\|[(\phi - \psi)f_n] \circ i\| \nrightarrow 0$ since i is onto K'. This contradicts the fact that $S(f_n) \to 0$ by 2.8. Now S must be zero as well. ∎

Recall the definition of $i_{n,m}$ from Section 3. Suppose that $T : X_+ \to X_+$ is an operator. By 3.1(2) for each $n, m \in \mathbb{N}$ there are continuous functions $\phi^T_{m,n} \in C(K_n)$ and weakly compact operators $S^T_{m,n} : C(K_n) \to C(K_m)$ such that for each $f \in C(K_n)$,

$$T(f)|K_m = [\phi^T_{m,n} f] \circ i_{n,m} + S^T_{m,n}(f).$$

We will skip the superscript T if it is clear from the context. In this section we analyze the matrix $(\phi^T_{m,n})_{m,n\in\mathbb{N}}$ for an operator T on X_+. Note that Lemma 5.1 implies that such a decomposition of an operator is unique.

LEMMA 5.2. *Let $T : X_+ \to X_+$ be a bounded linear operator. For each $n \in \mathbb{N}$ fixed, the sequence $(\phi^T_{m,n})_{m\in\mathbb{N}}$ converges to 0.*

Proof. Fix $n \in \mathbb{N}$. If the lemma is false, there are points $x_m \in K_n$ and an $\varepsilon > 0$ such that $|\phi_{m,n}(x_m)| > \varepsilon$ for all m from an infinite set $M_1 \subseteq \mathbb{N}$. We may assume that $x_m \in K_{1,n}$ for all $m \in \mathbb{N}$ or $x_m \in K_{2,n}$ for all $m \in \mathbb{N}$; say $x_m \in K_{1,n}$, the other case is analogous. By the continuity of $\phi_{m,n}$ and the fact that K_n is perfect we can choose pairwise disjoint open sets $U_m \subseteq K_{1,n}$ such that $|\phi_{m,n}(x)| > \varepsilon$ for every $x \in U_m$.

For $m \in \mathbb{N}$ fixed let $\alpha_{k,m} : K_n \to \mathbb{R}$ be pairwise disjoint characteristic functions whose supports $V_{k,m}$ are included in U_m. Now we use the characterization 2.8 of weakly compact operators on $C(K)$ spaces, concluding for every $m \in M_1$ that $\|S_{m,n}(\alpha_{k,m})\| \to 0$ as $k \to \infty$. So, for each $m \in M_1$ we can choose $k(m) \in \mathbb{N}$ such that

$$\|S_{m,n}(\alpha_{k(m),m})\| < \varepsilon/5.$$

The definitions of $S_{m,n}$ and the behaviour of $\phi_{n,m}$ on U_m imply that for $x \in i_{m,n}[V_{k(m),m}]$ we have

$$|T(\alpha_{k(m),m})(x)| > 4\varepsilon/5$$

and for all $x \in K_{2,m}$,

$$|T(\alpha_{k(m),m})(x)| < \varepsilon/5$$

for each $m \in M_1$. For $m \in M_1$ pick $y_m \in i_{n,m}[V_{k(m),m}]$ and $z_m \in K_{2,m}$ such that $y_m \bowtie z_m$ (see 3.3). Consider $\zeta_m = T^*(\delta_{y_m})|K_n$ and $\theta_m = T^*(\delta_{z_m})|K_n$ (restrictions of measures on K_+ to K_n) as functionals on $C(K_n)$. Now apply 2.9 twice, obtaining an infinite $M_2 \subseteq M_1$ such that whenever $M \subseteq M_2$ is an infinite set such that the supremum $\sup_{m\in M} \alpha_{k(m),m} = \alpha$ exists then

$$|\zeta_m(\alpha) - \zeta_m(\alpha_{k(m),m})| < \varepsilon/5, \quad |\theta_m(\alpha) - \theta_m(\alpha_{k(m),m})| < \varepsilon/5,$$

for $m \in M$. Since $|\zeta_m(\alpha_{k(m),m})| > 4\varepsilon/5$ and $|\theta_m(\alpha_{k(m),m})| < \varepsilon/5$, in terms of T we find that for $m \in M$ we have

$$|T(\alpha)(z_m)| > 3\varepsilon/5, \quad |T(\alpha)(y_m)| < 2\varepsilon/5.$$

Of course the above suprema α exist in $C(K_n)$ by 3.1(5). As $T(\alpha)$ is in X_+ and $y_m \bowtie z_m$, this contradicts 3.10 and completes the proof of the lemma. ∎

LEMMA 5.3. *Let $T : X_+ \to X_+$ be a bounded linear operator. Suppose that $m \in \mathbb{N}$ and $F \subseteq \mathbb{N}$ is a finite set such that $\| \sum_{n\in F} |\phi^T_{m,n} \circ i_{n,m}| \| > \varepsilon$. Then there is $f \in C(\bigcup_{n\in F} K_n)$ which is (C_n)-simple and of norm one such that $\|T(f)|K_m\| > \varepsilon$.*

Proof. Let $x \in K_m$ be such that $\sum_{n\in F} |\phi^T_{m,n} \circ i_{n,m}(x)| > \varepsilon$. Either $x \in K_{1,m}$ or $x \in K_{2,m}$; say $x \in K_{1,m}$, the other case is analogous. By the continuity of $\phi_{m,n}$ and the fact that K_n is perfect we can find a clopen $U \subseteq K_{1,m}$ such that $\sum_{n\in F} |\phi_{m,n} \circ i_{n,m}(x)| > \varepsilon + \varepsilon'$ for every $x \in U$ and some $\varepsilon' > 0$ and such that no $\phi_{m,n}$ for $n \in F$ changes its sign on $i_{n,m}[U]$ (if $\phi_{m,n}(x) = 0$ we may remove n from F). Let $V_l \subseteq U \subseteq K_{1,m}$ for $l \in \mathbb{N}$ be pairwise disjoint. By the density of B_m in $\mathrm{Clop}(K_{1,m})$ we may choose all V_ls to be in B_m. Let

$$\alpha_{n,l} = \delta_n \chi_{i_{n,m}[V_l]}$$

where $\delta_n = \pm 1$ and its sign is the same as $\phi_{m,n}$ on $i_{n,m}[U]$ so

$$\sum_{n\in F} (\phi_{m,n}\alpha_{n,l})(i_{n,m}(x)) > \varepsilon + \varepsilon'$$

for each $x \in V_l$ and each $l \in \mathbb{N}$.

Let $W_l = j[V_l] \subseteq K_{2,m}$ (for the definition of j see 3.1) and let $\beta_{n,l} = \delta_n \chi_{i_{n,m}[W_l]}$. Then, for each $l \in \mathbb{N}$,

$$f_l = \sum_{n\in F} (\alpha_{n,l} + \beta_{n,l})$$

is (C_n)-simple, $f_l \in C(\bigcup_{n\in F} K_n)$ and

$$\sum_{n\in F} (\phi_{m,n} f_l)(i_{n,m}(x)) > \varepsilon + \varepsilon'$$

for each $x \in V_l$. Moreover f_ls are pairwise disjoint.

Now we use the characterization 2.8 of weakly compact operators on $C(K)$ spaces to deduce that $\|\sum_{n\in F} S_{m,n}(\alpha_{n,l}+\beta_{n,l})\| \to 0$ as $l\to\infty$. So, there is an $l_0\in\mathbb{N}$ such that

$$\Big\|\sum_{n\in F} S_{m,n}(\alpha_{n,l_0}+\beta_{n,l_0})\Big\| < \varepsilon'.$$

Consequently, for $x\in V_l$ we have

$$|T(f_{l_0})(x)| \ge \Big|\sum_{n\in F}(\phi_{m,n}f_{l_0})(i_{n,m}(x))\Big| - \Big\|\sum_{n\in F} S_{m,n}(\alpha_{n,l_0}+\beta_{n,l_0})\Big\| \ge \varepsilon,$$

and hence $\|T(f)|K_m\| > \varepsilon$ for $f = f_{l_0}$. ∎

LEMMA 5.4. *Let $T: X_+\to X_+$ be a bounded linear operator. For each $m\in\mathbb{N}$ fixed, the series $\sum_{n\in\mathbb{N}}|\phi^T_{m,n}\circ i_{n,m}|$ converges uniformly.*

Proof. Fix $m\in\mathbb{N}$. If the series does not converge uniformly, then there is $\varepsilon>0$, strictly increasing $n_k\in\mathbb{N}$ and $x_k\in K_m$ for $k\in\mathbb{N}$ such that

$$\sum_{n_k\le n<n_{k+1}} |\phi_{m,n}(i_{n,m}(x_k))| > \varepsilon.$$

We may assume that $k<n_k$.

By 5.3 we can find $f_k\in C(\bigcup_{n_k\le n<n_{k+1}} K_n)$ which are (C_n)-simple and of norm one such that $\|T(f_k)\|>\varepsilon$. As the supports of f_ks are disjoint, they generate a copy of c_0 by 2.5. This contradicts 4.5. ∎

LEMMA 5.5. *Let $T: X_+\to X_+$ be a bounded linear operator and $f=(f_n)\in X_+$. Then the series $\sum_{n\in\mathbb{N}}[\phi^T_{m,n}f_n]\circ i_{n,m}$ converges uniformly to a function $g\in C(K_m)$ satisfying $\|g\|\le\|T\|\,\|f\|$.*

Proof. Note that for each $x\in K_m$,

$$\begin{aligned}\Big|\sum_{n_k\le n<n_{k+1}}[\phi_{m,n}f_n](i_{n,m}(x))\Big| &\le \|f\|\Big(\sum_{n_k\le n<n_{k+1}}|\phi_{m,n}(i_{n,m}(x))|\Big)\\ &\le \|f\|\Big\|\sum_{n_k\le n<n_{k+1}}|\phi_{m,n}\circ i_{n,m}|\Big\|,\end{aligned}$$

and so by Lemma 5.4 the partial sums of $\sum_{n\in\mathbb{N}}[\phi^T_{m,n}f_n]\circ i_{n,m}$ satisfy the uniform Cauchy condition, hence $\sum_{m\in\mathbb{N}}[\phi^T_{n,m}f_n]\circ i_{n,m}$ is uniformly convergent. Call the limit $g\in C(K_m)$.

For the proof of the estimate, note that if $\|g\|>\|T\|\,\|f\|$ for some $f\in X_+$, then by the above inequalities $\sum_{1\le n\le n_1}|\phi_{n,m}(i_{n,m}(x))|>\|T\|$ for all $x\in K_m$; but this would give an $f'\in C(\bigcup_{1\le n\le n_1}K_n)$ of norm one such that $\|T(f')\|>\|T\|$ by 5.3, a contradiction. ∎

LEMMA 5.6. *Let $T : X_+ \to X_+$ be a bounded linear operator. There is a bounded linear operator $\Pi^T : X_0 \to X_0$ which satisfies*

$$\Pi^T(f)|K_m = [\phi^T_{m,n} f] \circ i_{n,m}$$

for $f \in C(K_n)$ and whose norm is not greater than $\|T\|$.

Proof. Let $f \in X_0$ and $f_n = f|K_n$. Fix $n \in \mathbb{N}$. By 5.5 for every $m \in \mathbb{N}$ the series $\sum_{n\in\mathbb{N}}[\phi_{m,n} f_n] \circ i_{n,m}$ converges uniformly and its norm does not exceed $\|T\|\,\|f\|$. So for $f = (f_n) \in X_0$ we can define $\Pi^T(f)$ by requiring

$$\Pi^T(f)|K_m = \Big[\sum_{n\in\mathbb{N}} \phi^T_{m,n} f_n\Big] \circ i_{n,m},$$

which agrees with the condition of the lemma for $f \in C(K_n)$. It is a bounded operator into X_∞ whose norm is not greater than $\|T\|$ by 5.4. However, its restriction to any $C(K_n)$ is into X_0 by Lemma 5.2, so on X_0, it is into X_0. ∎

LEMMA 5.7. *Let $T, R : X_+ \to X_+$ be any operators and $m, n \in \mathbb{N}$. The operators from $C(K_n)$ into $C(K_m)$ given by*

- $P_m(\Pi^T - T)RI_n$,
- $P_m(\Pi^R - R)TI_n$,
- $P_m(\Pi^T - T)(\Pi^R - R)I_n$,
- $P_m(\Pi^R - R)(\Pi^T - T)I_n$,
- $P_m R(\Pi^T - T)I_n$,
- $P_m T(\Pi^R - R)I_n$,

are all weakly compact.

Proof. Note that for each $k \in \mathbb{N}$ the operators

$$[P_m(\Pi^T - T)]I_{-k} = \sum_{l\le k} S^T_{m,l},$$

$$[P_m(\Pi^R - R)]I_{-k} = \sum_{l\le k} S^R_{m,l},$$

$$P_{-k}[(\Pi^T - T)I_n] = \sum_{l\le k} S^T_{l,n},$$

$$P_{-k}[(\Pi^R - R)I_n] = \sum_{l\le k} S^R_{l,n}$$

are all weakly compact because the right hand sides are finite sums of weakly compact operators. So by 4.7 we may conclude the proof. ∎

REMARK 5.8. If $\Lambda : C(K) \to X^+$ and $\Theta : X^+ \to C(K)$ are as in 4.1 then they decompose so that all $\phi^\Lambda_{n,*}$ and $\phi^\Theta_{*,m}$ are zero. This proves that the approximation of a single operator T by Π^T is not always modulo a weakly compact operator.

Corollary 5.9. *Suppose that* $T, R : X_+ \to X_+$ *are operators such that* $R\circ T = 0_{C(K_{1,0})} \oplus \mathrm{Id}_{Y_+}$ *and* $T\circ R = \mathrm{Id}_{X_+}$. *Then* $\Pi^R\Pi^T = 0_{C(K_{1,0})} \oplus \mathrm{Id}_{Y_0}$ *and* $\Pi^T\Pi^R = \mathrm{Id}_{X_0}$.

Proof. Note that

$$\begin{aligned}\Pi^T\Pi^R &= (\Pi^T - T + T)(\Pi^R - R + R)\\ &= (\Pi^T - T)(\Pi^R - R) + (\Pi^T - T)R + T(\Pi^R - R) + TR\end{aligned}$$

and

$$\begin{aligned}\Pi^R\Pi^T &= (\Pi^R - R + R)(\Pi^T - T + T)\\ &= (\Pi^R - R)(\Pi^T - T) + (\Pi^R - R)T + R(\Pi^T - T) + RT.\end{aligned}$$

So by Lemma 5.7 we have

$$P_m\Pi^T\Pi^R I_n = S'_{m,n} + P_m I_n$$

for all $m, n \in \mathbb{N}$ where $S'_{m,n}$s are all weakly compact, and

$$P_m\Pi^R\Pi^T I_n = S''_{m,n} + P_m I_n$$

for all $n, m \in \mathbb{N}\setminus\{0\}$ where $S''_{m,n}$s are all weakly compact. Moreover

$$P_0\Pi^R\Pi^T I_0 = S''_{0,0} + P_0 0_{C(K_{1,0})} \oplus \mathrm{Id}_{Y_+} I_0 = S''_{0,0} + 0_{C(K_{1,0})} \oplus \mathrm{Id}_{C(K_{2,0})}.$$

Of course $P_m I_n$ is zero or the identity operator depending on whether $m \neq n$ or not, but in both cases it is multiplication by a continuous function; also multiplying by 0 on $K_{1,0}$ and by 1 on $K_{2,0}$ is multiplication by a continuous function, so by 5.1 all the operators $S'_{m,n}$ and $S''_{m,n}$ are zero operators, concluding the proof. ∎

Corollary 5.10. *Suppose that* $T, R : X_+ \to X_+$ *are operators such that* $R\circ T = 0_{C(K_{1,0})} \oplus \mathrm{Id}_{Y_+}$ *and* $T\circ R = \mathrm{Id}_{X_+}$. *Then we have the following:*

- $\sum_{k\in\mathbb{N}}(\phi^R_{m,k}\circ i_{k,n})\phi^T_{k,n}(x) = 0$ *for distinct* $n, m \in \mathbb{N}$ *and all* $x \in K_n$,
- $\sum_{k\in\mathbb{N}}(\phi^R_{n,k}\circ i_{k,n})\phi^T_{k,n}(x) = 1$ *for all* $(n, x) \in (\mathbb{N}\setminus\{0\} \times K_+ \setminus K_0) \cup (\{0\} \times K_{2,0})$,
- $\sum_{k\in\mathbb{N}}(\phi^R_{0,k}\circ i_{k,0})\phi^T_{k,0}(x) = 0$ *for all* $x \in K_{1,0}$,
- $\sum_{k\in\mathbb{N}}(\phi^T_{m,k}\circ i_{k,n})\phi^R_{k,n}(x) = 0$ *for distinct* $n, m \in \mathbb{N}$ *and all* $x \in K_n$,
- $\sum_{k\in\mathbb{N}}(\phi^T_{n,k}\circ i_{k,n})\phi^R_{k,n}(x) = 1$ *for all* $n \in \mathbb{N}$ *and* $x \in K_n$.

Lemma 5.11. *Let* $x \in K$ *and* $x_n = i_{n,*}(x)$ *and* $A = \{x_n : n \in \mathbb{N}\}$. *Let* $T : X_+ \to X_+$ *be a bounded linear operator. There is a bounded linear operator* $\Pi^T_x : c_0(A) \to c_0(A)$ *which satisfies*

$$\Pi^T_x(1_{x_n})(x_m) = \phi^T_{m,n}(x_n),$$

and whose norm is not greater than $\|T\|$.

Proof. Note that if $\vec{1}_n$ denotes the constant function on K_n whose value is one, then $\Pi^T(\vec{1}_n)(x_m) = \phi^T_{m,n}(x_n)$. Since the range of Π^T is X_0 we may conclude that $(\phi^T_{m,n}(x_n))_{m\in\mathbb{N}}$ belongs to $c_0(A)$ and its norm is not greater than $\|T\|$ by 5.6. So the above formula well defines $\Pi^T_x : c_0(A) \to c_0(A)$. ■

LEMMA 5.12. *Suppose that $T, R : X_+ \to X_+$ are operators such that $R\circ T = 0_{C(K_{1,0})}\oplus \mathrm{Id}_{Y_+}$ and $T\circ R = \mathrm{Id}_{X_+}$. Let $x \in K_{1,*}$, $y \in K_{2,*}$, $x_n = i_{n,*}(x)$ $y_n = i_{n,*}(y)$, $A = \{x_n : n \in \mathbb{N}\}$, $A' = \{x_n : n \in \mathbb{N}\setminus\{0\}\}$, $B = \{y_n : n \in \mathbb{N}\}$. Let*

$$\Pi^T_{x,y} = (\Pi^T_x, \Pi^T_y) : c_0(A\cup B) \to c_0(A\cup B),$$
$$\Pi^R_{x,y} = (\Pi^R_x, \Pi^R_y) : c_0(A\cup B) \to c_0(A\cup B).$$

Then $\Pi^R_{x,y} \circ \Pi^T_{x,y} = 0_{x_0} \oplus \mathrm{Id}_{c_0(A'\cup B)}$ and $\Pi^T_{x,y}\circ \Pi^R_{x,y} = \mathrm{Id}_{c_0(A\cup B)}$.

Proof. Apply Corollary 5.10 pointwise. ■

6. Detecting shifts in isomorphisms. If I is a set and $f \in c_0(I)$, by the *support* of f we mean $\{i : f(i) \neq 0\}$.

LEMMA 6.1. *Let $A = \{x_n : n \in \mathbb{N}\}$ and $B = \{y_n : n \in \mathbb{N}\}$. Suppose that $T : c_0(A\cup B) \to c_0(A\cup B)$ is an operator such that $T(1_{x_0}) = 0$, $T|(\{f \in c_0(A) : f(1_{x_0}) = 0\})$ is an isomorphism onto $c_0(A)$ and $T_2 = T|c_0(B)$ is an isomorphism onto $c_0(B)$. Then there are $f_k \in c_0(A\cup B)$ with pairwise disjoint and finite supports and there is an $\varepsilon > 0$ for which*

$$f_k(x_n) = f_k(y_n)$$

for every $n \in \mathbb{N}$, and for some distinct m_ks we have

$$|T(f_k)(x_{m_k}) - T(f_k)(y_{m_k})| > \varepsilon.$$

Proof. Let $T_1 = T|c_0(A)$. Consider $T_3 : c_0(B) \to c_0(B)$ so that $T_3(1_{\{y_n\}})(y_m) = T_1(1_{\{x_n\}})(x_m)$ for every $m, n \in \mathbb{N}$. That is, we consider a copy of T_1 copied from $c_0(A)$ on $c_0(B)$.

CLAIM. *$T_3 - T_2 : c_0(B) \to c_0(B)$ is not weakly compact.*

Proof of the Claim. If it is, then it is strictly singular by 2.7, and so we can apply Fredholm theory (see [18]). Namely, the Fredholm index of $T = (T_1, T_2) : c_0(A\cup B) \to c_0(A\cup B)$ is equal to the Fredholm index of $(T_1, T_2) + (0, T_3 - T_2) = (T_1, T_3)$, which must be an even integer since T_1 and T_3 are copies of the same operator. However, T_2 is an isomorphism and T_1 is onto and has kernel of dimension one, and so $T = (T_1, T_2)$ must have odd Fredholm index, a contradiction which completes the proof of the claim.

Now use 2.8 to find pairwise disjoint $e_k \in c_0(B)$ such that there are m_ks such that

$$|[(T_3 - T_2)(e_k)](y_{m_k})| > \varepsilon$$

for some $\varepsilon > 0$. Since the operator is bounded, and the norms of sums of e_ks are bounded as well since they are disjoint, we may assume that all m_ks are distinct.

Now define $f_k \in c_0(A \cup B)$ by $f_k|B = e_k$ and $f_k(x_n) = e_k(y_n)$ for each $n \in \mathbb{N}$. So we have $f_k(x_n) = f_k(y_n)$ for all $k, n \in \mathbb{N}$. Also

$$T(f_k)(x_{m_k}) = T_1(f_k|A)(x_{m_k}) = T_3(e_k)(y_{m_k}), \quad T(f_k)(y_{m_k}) = T_2(e_k)(y_{m_k}),$$

so we obtain

$$|T(f_k)(x_{m_k}) - T(f_k)(y_{m_k})| > \varepsilon$$

as required. ∎

LEMMA 6.2. *Suppose that $n, m \in \mathbb{N}$ and that $S : C(K_n) \to C(K_m)$ is a weakly compact operator. Let $\tau_n : K_n \to L_n$ be the canonical surjection as in Section 3. Then for each $j \in \mathbb{N}$, the set*

$$\Gamma(S, j) = \{t \in L_n : \exists x \in \tau_n^{-1}[\{t\}] \ \ |S^*(\delta_{i_{m,n}(x)})(\tau_n^{-1}[\{t\}])| > 1/j\}$$

is finite.

Proof. If $\Gamma(S, j) \subseteq L_n$ is infinite, we can choose a discrete sequence $\{t_l : l \in \mathbb{N}\} \subseteq \Gamma(S, j)$. Let $x_l \in K_n$ be such that $\tau_n(x_l) = t_l$ and

$$|S^*(\delta_{i_{m,n}(x_l)})(\tau_n^{-1}[\{t_l\}])| > 1/j.$$

Recall that $[a]$ denotes the basic clopen set of the Stone space of a Boolean algebra which is determined by its element a. Now use the fact that

$$\tau^{-1}[\{t\}] = \bigcap\{\tau^{-1}[[a]] : t \in [a],\, a \in C_n\}$$

and the family whose intersection appears above is directed to conclude that for each $l \in \mathbb{N}$ there is an $a_l \in C_n$ such that $t_l \in [a_l]$ and

$$|S^*(\delta_{i_{m,n}(x_l)})(\tau_n^{-1}[[a_l]])| > 1/j.$$

Moreover, as $\{t_l : l \in \mathbb{N}\}$ is discrete we may assume that a_ls are pairwise disjoint. This means that

$$|S(\chi_{\tau_n^{-1}[[a_l]]})(i_{m,n}(x_l))| > 1/j$$

for each $l \in \mathbb{N}$, which contradicts 2.8 since a_ls are pairwise disjoint. ∎

THEOREM 6.3. *Y_+ and X_+ are not isomorphic.*

Proof. Suppose $T_1 : Y_+ \to X_+$ and $R_1 : X_+ \to Y_+$ are mutually inverse isomorphisms. Define $T, R : X_+ \to X_+$ by $R = R_1$ and $T = 0_{C(K_{1,0})} \oplus T_1$. We have $R \circ T = 0_{C(K_{1,0})} \oplus \mathrm{Id}_{Y_+}$ and $T \circ R = \mathrm{Id}_{X_+}$.

By 5.12, $\Pi^T_{x,y}$ satisfies the hypothesis of 6.1 To use it successfully we still need to make an appropriate choice of x and y. Let

$$\Gamma = \{t \in L : \exists n, m, j \in \mathbb{N} \ \ h_{n,*}(t) \in \Gamma(S_{m,n}, j)\}.$$

By 6.2, $\varGamma$ is countable where $S_{m,n} = S^T_{m,n}$. Pick $t \in L \setminus \varGamma$ and $x \in K_{1,*}$, $y \in K_{2,*}$ such that $\tau(x) = \tau(y) = t$ (see 3.3). As usual let $x_n = i_{n,*}(x)$, $y_n = i_{n,*}(y)$, and $t_n = h_{n,*}(t)$.

Now use 6.1 to obtain $\alpha_k \in c_0(A \cup B)$ with pairwise disjoint and finite supports F_k and an $\varepsilon > 0$ such that $\alpha_k(x_n) = \alpha_k(y_n)$ for every $n \in \mathbb{N}$ and for some distinct m_ks,

$$|\varPi^T_{x,y}(\alpha_k)(x_{m_k}) - \varPi^T_{x,y}(\alpha_k)(y_{m_k})| > \varepsilon.$$

For $n \in F_k$ we will find $a_n \in C_n$ such that the operator T will have approximately the same behavior on the vectors

$$f_k = \sum_{n \in F_k} \alpha_k(x_n)\chi_{\tau_n^{-1}[[a_n]]} \in X_0 \quad \text{ as } \varPi^T_{x,y} \text{ on } \alpha_k\text{s}.$$

Note that for each z_n satisfying $\tau(z_n) = t_n$ (in particular $z_n = x_n, y_n$), by the choice of t outside $\varGamma$, the measure $S^*_{m,n}(\delta_{i_{m,n}(z_n)})$ of the set $\tau_n^{-1}[\{t_n\}]$ is zero. So, using the fact that $\tau_n^{-1}[\{t_n\}] = \bigcap\{\tau_n^{-1}[[a]] : t_n \in [a],\ a \in C_n\}$ for each $n \in F_k$ we can find $a_n \in C_n$ such that $t_n \in [a_n]$ and

$$|S^*_{m_k,n}(\delta_{x_{m_k}})(\tau_n^{-1}[[a_n]])|, |S^*_{m_k,n}(\delta_{y_{m_k}})(\tau_n^{-1}[[a_n]])| < \frac{\varepsilon}{3|F_k|\,|\alpha_k(x_n)|},$$

so, for each $k \in \mathbb{N}$ and $n \in F_k$ we have

$$|\alpha_k(x_n)S_{m_k,n}(\chi_{\tau_n^{-1}[[a_n]]})(x_{m_k})|, |\alpha_k(x_n)S_{m_k,n}(\chi_{\tau_n^{-1}[[a_n]]})(y_{m_k})| < \frac{\varepsilon}{3|F_k|},$$

and hence

$$\Big|S_{m_k,n}\Big(\sum_{n \in F_k} \alpha_k(x_n)\chi_{\tau_n^{-1}[[a_n]]}(x_{m_k})\Big)\Big|, \Big|S_{m_k,n}\Big(\sum_{n \in F_k} \alpha_k(x_n)\chi_{\tau_n^{-1}[[a_n]]}(y_{m_k})\Big)\Big| < \frac{\varepsilon}{3}$$

and finally

$$|S_{m_k,n}(f_k)(x_{m_k})|,\ |S_{m_k,n}(f_k)(y_{m_k})| < \varepsilon/3$$

for each $k \in \mathbb{N}$ and each $n \in F_k$. On the other hand,

$$|\varPi^T(f_k)(x_{m_k}) - \varPi^T(f_k)(y_{m_k})| = |\varPi^T_{x,y}(f_k)(x_{m_k}) - \varPi^T_{x,y}(f_k)(y_{m_k})| > \varepsilon,$$

hence for each k we have

$$|T(f_k)(x_{m_k}) - T(f_k)(y_{m_k})| > \varepsilon/3$$

where m_ks are distinct and f_ks are bounded (C_n)-simple and have disjoint supports.

Now note that for any infinite $M \subseteq \mathbb{N}$ the supremum f_M of $\{f_k : k \in \mathbb{N}\}$ exists in X_+.

Let $\zeta_k = T^*(\delta_{x_{m_k}} - \delta_{y_{m_k}})$. In particular, $|\zeta_k(f_k)| > \varepsilon/3$. Apply 2.10 to obtain an infinite M such that $|\zeta_k(f_M)| > \varepsilon/4$ for each $k \in M$. But this means that

$$|T(f_M)(x_{m_k}) - T(f_M)(y_{m_k})| > \varepsilon/4$$

for each $k \in M$. This finally contradicts 3.10 since $x_{m_k} \bowtie y_{m_k}$. ∎

Corollary 6.4. *X_+ and Y_+ are not isomorphic but each is isomorphic to a complemented subspace of the other.*

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