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Operator ideal properties of vector measures with finite variation

by

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Abstract. Given a vector measure m with values in a Banach space X, a desirable property (when available) of the associated Banach function space $L^1(m)$ of all m-integrable functions is that $L^1(m) = L^1(|m|)$, where $|m|$ is the $[0,\infty]$-valued variation measure of m. Closely connected to m is its X-valued integration map $I_m : f \mapsto \int f\,dm$ for $f \in L^1(m)$. Many traditional operators from analysis arise as integration maps in this way. A detailed study is made of the connection between the property $L^1(m) = L^1(|m|)$ and the membership of I_m in various classical operator ideals (e.g., the compact, p-summing, completely continuous operators). Depending on which operator ideal is under consideration, the geometric nature of the Banach space X may also play a crucial role. Of particular importance in this regard is whether or not X contains an isomorphic copy of the classical sequence space ℓ^1. The compact range property of X is also relevant.

1. Introduction. Let X be a Banach space (with closed unit ball $\mathbb{B}[X]$ and dual space X^*) and $m : \Sigma \to X$ be a *vector measure*, i.e., m is σ-additive on the σ-algebra Σ (of subsets of some non-empty set Ω). The *variation measure* $|m| : \Sigma \to [0,\infty]$ of m is defined analogously to that for scalar measures [14, Ch. I, Definition 1.4]. Then the classical space $L^1(|m|)$ delivers a certain collection of integrable functions associated with m. There are also others. Namely, a Σ-measurable function $f : \Omega \to \mathbb{C}$ is called *m-integrable* if

(I1) $\int_\Omega |f|\,d|\langle m, x^*\rangle| < \infty$ for all $x^* \in X^*$, and
(I2) for each $A \in \Sigma$ there is $\int_A f\,dm \in X$ satisfying

$$\Big\langle \int_A f\,dm, x^* \Big\rangle = \int_A f\,d\langle m, x^*\rangle, \qquad \forall x^* \in X^*$$

[25], [26]. Here, for each $x^* \in X^*$, the scalar measure $A \mapsto \langle m(A), x^*\rangle$, for $A \in \Sigma$, is denoted by $\langle m, x^*\rangle$. The space of all m-integrable functions is

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denoted by $L^1(m)$; it is identified with its quotient space modulo m-null functions, where an m-integrable function f is *m-null* if $\int_A f\,dm = 0$ for all $A \in \Sigma$. These two spaces of integrable functions associated with m are related via

$$L^1(|m|) \subseteq L^1(m) \tag{1.1}$$

[38, Lemma 3.14]. If a set $A \in \Sigma$ is called m-null whenever χ_A is an m-null function, then one can also form the Banach space $L^\infty(m)$ of all (equivalence classes of) bounded Σ-measurable functions, equipped with the essential sup-norm $\|\cdot\|_{L^\infty(m)}$ in the usual way, in which case $L^\infty(m) \subseteq L^1(m)$ [26, p. 161]. Of course, the family $\operatorname{sim}\Sigma$ of all Σ-simple functions is contained in $L^\infty(m)$. The space $L^1(m)$ is known to be complete with respect to the *lattice norm*

$$\|f\|_{L^1(m)} := \sup_{x^*\in\mathbb{B}[X^*]} \int_\Omega |f|\,d|\langle m, x^*\rangle|, \qquad \forall f \in L^1(m), \tag{1.2}$$

i.e., $L^1(m)$ is a (complex) Banach lattice, the space $\operatorname{sim}\Sigma$ is dense in $L^1(m)$, and the *integration map* $I_m : L^1(m) \to X$ defined by

$$I_m : f \mapsto \int_\Omega f\,dm, \qquad \forall f \in L^1(m), \tag{1.3}$$

is linear, continuous and has operator norm $\|I_m\|_{\mathrm{op}} = 1$ [38, p. 152]. Its restriction $I_{|m|} : L^1(|m|) \to X$ is also continuous because $|\langle m, x^*\rangle|(A) \le |m|(A)\|x^*\|$ for all $A \in \Sigma$ and $x^* \in X^*$ implies that

$$\|I_{|m|}(f)\|_X = \Big\|\int_\Omega f\,dm\Big\|_X \le \int_\Omega |f|\,d|m|, \qquad \forall f \in L^1(|m|). \tag{1.4}$$

The inclusion (1.1) may be proper, even if m has *finite variation*, i.e., $|m|(\Omega) < \infty$. Since $L^1(|m|)$ is a classical L^1-space, it is surely more tractable than $L^1(m)$ in general. So, there is some interest in the situation when (1.1) is actually an *equality*. For an arbitrary vector measure m, G. P. Curbera showed that $L^1(m) = L^1(|m|)$ iff the integration map I_m is positive 1-summing (also called cone absolutely summing) [8, Proposition 3.1]. It was recently shown that I_m is positive p-summing for some $1 \le p < \infty$ iff $L^1(m)$ is isomorphic to an AL-space [5, Theorem 2.7]. Combining this with [38, Lemma 3.14(iii)] it follows that $L^1(m) = L^1(|m|)$ iff I_m is positive p-summing for some $1 \le p < \infty$. However, given a specific vector measure m of finite variation, it is not always easy to identify the space $L^1(m)$ explicitly, although this is needed to test directly whether $L^1(m) = L^1(|m|)$ or whether I_m is positive p-summing. Moreover, even if equality in (1.1) is established, the positive p-summing nature of I_m alone reveals little about its possible finer structure, i.e., whether it is maybe weakly compact, or compact, or completely continuous, etc. Our aim is to provide new results and techniques which can be used in practise to decide when equality holds in (1.1) and which also allow for a finer analysis of I_m. We begin by formulating a general "Operator Ideal Principle" (cf. Proposition 1.1 below), which reduces the question of equality in (1.1) to determining solely whether or not m has finite variation. First some notation and terminology.

Given Banach spaces X and Y, let $\mathcal{L}(X,Y)$ be the Banach space of all continuous linear operators from X to Y. An *operator ideal* $\mathcal{A}$ is a method of assigning to every couple (X,Y) of Banach spaces a linear subspace $\mathcal{A}(X,Y) \subseteq \mathcal{L}(X,Y)$ which contains the finite rank operators and such that $R \circ S \circ T \in \mathcal{A}(W,Z)$ for every pair (W,Z) of Banach spaces and all choices of operators $T \in \mathcal{L}(W,X)$, $R \in \mathcal{L}(Y,Z)$ and $S \in \mathcal{A}(X,Y)$ [13, p. 131]. Some examples relevant for this paper are when $\mathcal{A}$ is all compact, or all p-summing, or all completely continuous operators. We point out that the collection of positive p-summing operators, which require a Banach lattice as domain space, does *not* form an operator ideal.

Let $\mathcal{A}$ be an operator ideal. A Banach space X is called *$\mathcal{A}$-variation admissible* if $|m|$ is a finite measure for every X-valued vector measure m whose integration map satisfies $I_m \in \mathcal{A}$. For instance, if $\mathcal{A}_c$ is the operator ideal of all compact operators, then *every* Banach space is $\mathcal{A}_c$-variation admissible [36, Theorem 4]. Or, if $\mathcal{A}_p$ is the operator ideal of all p-summing operators for some $1 \leq p < \infty$, then *every* Banach space is $\mathcal{A}_p$-variation admissible. This result was already presented in [39]; its complete proof will be given in Section 2.

The following result is a useful tool for establishing equality in (1.1).

PROPOSITION 1.1. *Let $\mathcal{A}$ be an operator ideal and X be any $\mathcal{A}$-variation admissible Banach space. Then $L^1(|m|) = L^1(m)$ for every X-valued vector measure m whose integration map satisfies $I_m \in \mathcal{A}$.*

The Banach sequence space ℓ^1 turns out to play a central role. Recall that a continuous linear map between Banach spaces is *completely continuous* if it maps weakly convergent sequences to norm convergent sequences. Such operators are also called Dunford–Pettis operators. Let $\mathcal{A}_{cc}$ denote the operator ideal consisting of all completely continuous operators [13, p. 49].

THEOREM 1.2. *Every Banach space X with an unconditional basis and not containing an isomorphic copy of ℓ^1 (briefly, $\ell^1 \not\hookrightarrow X$) is $\mathcal{A}_{cc}$-variation admissible. In particular, $L^1(m) = L^1(|m|)$ whenever m is an X-valued vector measure such that $I_m \in \mathcal{A}_{cc}$.*

According to a classical result of H. P. Rosenthal [2, p. 247], all non-reflexive, weakly sequentially complete Banach spaces X have the property that $\ell^1 \hookrightarrow X$. If m is any vector measure of infinite variation with values in such a space X (such measures always exist as X is infinite-dimensional), then necessarily $L^1(|m|) \subsetneq L^1(m)$. Restrict X further to come from the subclass of all infinite-dimensional Banach spaces with the *Schur property* (i.e., weakly convergent sequences are norm convergent or, equivalently, relatively weakly compact sets are relatively norm compact). Then, in addition to $L^1(|m|) \subsetneq L^1(m)$, we also automatically have $I_m \in \mathcal{A}_{cc}$. Examples of such vector measures m also exist with finite variation; see Section 4. These comments show that the requirement $\ell^1 \not\hookrightarrow X$ cannot be omitted from Theorem 1.2.

The class of Banach spaces covered by Theorem 1.2 includes all reflexive spaces with an unconditional basis, the sequence space c_0, and many more.

Recall that a Banach space X has the *weak Radon–Nikodým property* (briefly, WRNP) if, whenever (Ω, Σ, μ) is any complete probability space and $m : \Sigma \to X$ is any μ-continuous vector measure of finite variation, then m has a Pettis μ-integrable, X-valued density. Of relevance to this paper is that a Banach space X satisfies $\ell^1 \not\hookrightarrow X$ iff X^* has the WRNP [15, Theorem 6.8], [41, Corollary 7.3.8]. A Banach space X has the *compact range property* (briefly, CRP) if every X-valued vector measure of finite variation has relatively compact range [30, Definition 2]. If X has the WRNP, then X has the CRP [30, Proposition 4]. Also, in any weakly compactly generated Banach space (hence, in all separable spaces and in all reflexive spaces), the WRNP and the *Radon–Nikodým property* (briefly, RNP) are equivalent [29, Corollary 3]. The same is true in *arbitrary* Banach lattices [17, Theorem 5]. In particular, every reflexive Banach space has the CRP [14, p. 218]. Such spaces, even if separable, need *not* have an unconditional basis [27, p. 27].

The following "converse type" result should be compared with Theorem 1.2.

THEOREM 1.3. *For a Banach space X the following assertions are equivalent:*

(i) *X has the CRP.*
(ii) *Every X-valued vector measure m with $L^1(m) = L^1(|m|)$ satisfies $I_m \in \mathcal{A}_{cc}$.*

It is worth noting that the condition $L^1(m) = L^1(|m|)$ in part (ii) of Theorem 1.3 cannot be relaxed to the requirement that m has finite variation. Indeed, in Example 3.69 of [38] there is a vector measure m taking its values in the reflexive space ℓ^p, $1 < p < \infty$ (hence, has the CRP), such that m has finite variation and $L^1(|m|) \subsetneq L^1(m)$. Since its integration map $I_m : L^1(m) \to \ell^p$ is an isomorphism onto ℓ^p (and ℓ^p does not have the Schur property), I_m is surely *not* completely continuous. For a non-atomic example (in the reflexive space $L^r([0,1])$, $1 < r < \infty$) which exhibits the same features we refer to the $L^r([0,1])$-valued vector measure m_r induced by the classical Volterra kernel operator, namely $m_r(A)$ is the function in $L^r([0,1])$ given by $t \mapsto \int_0^t \chi_A(s)\,ds$ for $t \in [0,1]$ and every Borel set $A \in \mathcal{B}([0,1])$ [38, Example 3.26 & Proposition 3.52]. In this case I_{m_r} is not an isomorphism but still fails to be completely continuous.

Suppose that I_m is p-summing for some $1 \le p < \infty$ [13, Ch. 2], in which case $L^1(m) = L^1(|m|)$. Then I_m is also weakly compact and completely continuous [13, Theorem 2.17]. In general, neither the weak compactness of I_m alone implies that $L^1(m) = L^1(|m|)$ (cf. the Volterra measures $m_r : \mathcal{B}([0,1]) \to L^r([0,1])$ for $1 < r < \infty$, mentioned above), nor does complete continuity of I_m alone imply that $L^1(m) = L^1(|m|)$; consider ℓ^1-valued measures as discussed after Theorem 1.2. We construct several non-trivial examples of vector measures m (mostly arising in classical analysis) which show that I_m may belong to $\mathcal{A}_c \setminus \mathcal{A}_1$, or to $\mathcal{A}_1 \setminus \mathcal{A}_c$, or even to $(\bigcap_{p<r<\infty} \mathcal{A}_r) \setminus (\mathcal{A}_p \cup \mathcal{A}_c)$ for every $2 < p < \infty$.

2. Operator Ideal Principle and p-summing integration maps. Let $m : \Sigma \to X$ be a Banach-space-valued vector measure and let $L^0(\Sigma)$ denote the space of all $\mathbb{C}$-valued, Σ-measurable functions on Ω. Given $f \in L^1(m)$, its *indefinite integral* is the vector measure $m_f : \Sigma \to X$ given by

$$m_f : A \mapsto \int_A f\,dm, \qquad \forall A \in \Sigma; \tag{2.1}$$

the σ-additivity of m_f follows from (I1), (I2) and the Orlicz–Pettis Theorem [14, Ch. I, Corollary 4.4].

Lemma 2.1. *Let $m : \Sigma \to X$ be a Banach-space-valued vector measure.*

(i) *A function $f \in L^1(m)$ belongs to $L^1(|m|)$ iff its indefinite integral m_f has finite variation, in which case $\|f\|_{L^1(m)} \le \|f\|_{L^1(|m|)}$.*

(ii) *Let $f \in L^1(m)$. As an equality of vector spaces we have*

$$L^1(m_f) = \{g \in L^0(\Sigma) : gf \in L^1(m)\}. \tag{2.2}$$

Moreover, for each $g \in L^1(m_f)$,

$$\int_A g\,dm_f = \int_A gf\,dm, \qquad \forall A \in \Sigma, \tag{2.3}$$

and also

$$\|g\|_{L^1(m_f)} = \|gf\|_{L^1(m)}. \tag{2.4}$$

The multiplication operator $M_f : L^1(m_f) \to L^1(m)$ defined by $g \mapsto gf$ for $g \in L^1(m_f)$ is a linear isometry onto its range in $L^1(m)$ and

$$I_{m_f} = I_m \circ M_f. \tag{2.5}$$

Proof. (i) See Lemma 3.14(i) in [38].

(ii) To establish (2.2), let $g \in L^1(m_f)$. By Theorem 3.5 of [38] applied to m_f there exists a sequence $\{s_n\}_{n=1}^\infty \subseteq \operatorname{sim}\Sigma$ such that $s_n \to g$ pointwise and $\lim_{n\to\infty} \int_A s_n\,dm_f = \int_A g\,dm_f$ for $A \in \Sigma$. Since also $s_n f \to gf$ pointwise and $\int_A s_n\,dm_f = \int_A s_n f\,dm$ for $n \in \mathbb{N}$ and $A \in \Sigma$, again by Theorem 3.5 of [38], now applied to m, we can conclude that $gf \in L^1(m)$ and (2.3) holds. So, g belongs to the right side of (2.2).

Conversely, let g belong to the right side of (2.2). Choose a sequence $\{s_n\}_{n=1}^{\infty} \subseteq \operatorname{sim}\Sigma$ with $s_n \to g$ pointwise and $|s_n| \le |g|$ pointwise for each $n \in \mathbb{N}$. Then $|s_n f| \le |gf|$ pointwise for $n \in \mathbb{N}$, with $gf \in L^1(m)$. The Dominated Convergence Theorem for vector measures [38, Theorem 3.7(i)] then yields $s_n f \to gf$ in $L^1(m)$. In particular, $\int_A s_n f\,dm \to \int_A gf\,dm$ for $A \in \Sigma$ [38, p. 112]. Since $\int_A s_n f\,dm = \int_A s_n\,dm_f$ for $A \in \Sigma$ and $n \in \mathbb{N}$, we can conclude from Theorem 3.5 of [38] that $g \in L^1(m_f)$ and that (2.3) holds. So, (2.2) is valid. At the same time we have established (2.3).

The formula (2.4) follows from the identities

$$|\langle m_f, x^*\rangle|(A) = \int_A |f|\,d|\langle m, x^*\rangle|, \qquad \forall x^* \in X^*,\ A \in \Sigma$$

(cf. (I2) and (2.1)) together with the definition

$$\|g\|_{L^1(m_f)} := \sup_{x^* \in \mathbb{B}[X^*]} \int_\Omega |g|\,d|\langle m_f, x^*\rangle|.$$

Then (2.2) and (2.4) ensure that M_f is a linear isometry from $L^1(m_f)$ onto its range in $L^1(m)$. Finally, (2.5) follows routinely from the definitions involved. ∎

We can now establish the *Operator Ideal Principle.*

Proof of Proposition 1.1. Let $\mathcal{A}$, X and $m : \Sigma \to X$ be as in the statement of the result. Fix any $f \in L^1(m)$. Then the composition $I_{m_f} = I_m \circ M_f : L^1(m_f) \to X$ (cf. (2.5)) belongs to $\mathcal{A}$ because $I_m \in \mathcal{A}$ (by assumption). Since X is $\mathcal{A}$-variation admissible, we can conclude that m_f has finite variation, and hence, by Lemma 2.1(i), that $f \in L^1(|m|)$. This establishes that $L^1(m) \subseteq L^1(|m|)$ and, via (1.1), it follows that $L^1(m) = L^1(|m|)$. ∎

The following result, presented in [39], will now be established. The techniques in the proof are of independent interest.

Theorem 2.2. *Let $1 \le p < \infty$ and $\mathcal{A}_p$ be the operator ideal consisting of all p-summing operators. Then every Banach space is $\mathcal{A}_p$-variation admissible.*

Proof. Let X be any Banach space. Fix an X-valued vector measure m defined on a measurable space (Ω, Σ). Select any positive finite measure $\mu : \Sigma \to [0, \infty)$ which has the same null sets as m, written briefly as $m \simeq \mu$ [14, Ch. I, Corollary 2.6]. So, $L^\infty(\mu) = L^\infty(m)$ is continuously embedded, via the natural embedding, say α, into $L^1(m)$. The proof is in several steps.

STEP 1. *If $I_m \circ \alpha : L^\infty(\mu) \to X$ is q-summing for some $1 \le q < \infty$, then there is $g_q \in L^1(\mu)$ with $g_q(w) > 0$ for every $w \in \Omega$ such that, with continuous inclusions,*

$$L^\infty(\mu) \subseteq L^q(g_q\, d\mu) \subseteq L^1(m). \tag{2.6}$$

To verify this, observe that $L^\infty(\mu)$ is a (complex) AM-Banach lattice with unit χ_Ω. By the Kakutani Representation Theorem [40, Theorem 7.4 & p. 138], there is an isometric Banach lattice isomorphism β of $L^\infty(\mu)$ onto $C(K)$ with $\beta(\chi_\Omega) = \chi_K$, for some (extremely disconnected) compact Hausdorff space K. Since $\beta(|f|) = |\beta(f)|$, it follows that $\beta(|f|^q) = |\beta(f)|^q$ for each $f \in L^\infty(\mu)$. By assumption, the composition $I_m \circ \alpha \circ \beta^{-1} : C(K) \to X$ is q-summing so a factorization theorem of Pietsch [13, Corollary 2.15] implies that there exists a finite regular Borel measure $\lambda : \mathcal{B}(K) \to [0,\infty)$ satisfying

$$\|(I_m \circ \alpha \circ \beta^{-1})(\psi)\|_X \le \Big(\int_K |\psi|^q\, d\lambda\Big)^{1/q}, \quad \forall \psi \in C(K). \tag{2.7}$$

Let $\xi \in L^\infty(\mu)^*$ denote the positive linear functional corresponding to $\lambda \in M(K) = C(K)^*$ via β^*, i.e., $\xi := \beta^*(\lambda)$ where $\beta^* \in \mathcal{L}(C(K)^*, L^\infty(\mu)^*)$ is the dual operator of $\beta \in \mathcal{L}(L^\infty(\mu), C(K))$. Then (2.7) implies that

$$\|(I_m \circ \alpha)(f)\|_X \le (\langle |f|^q, \xi\rangle)^{1/q}, \quad \forall f \in L^\infty(\mu). \tag{2.8}$$

Define a finitely additive set function $\eta_\xi : A \mapsto \langle \chi_A, \xi\rangle = \int_K \beta(\chi_A)\, d\lambda \in [0,\infty)$ for $A \in \Sigma$. Whenever $A \in \Sigma$ is μ-null we have $\chi_A = 0$ in $L^\infty(\mu)$ and so $\beta(\chi_A) = 0$ in $C(K)$, i.e., $\eta_\xi(A) = 0$. So, there exist a positive finite measure $\eta_1 : \Sigma \to [0,\infty)$ and a *purely finitely additive* set function $\eta_2 : \Sigma \to [0,\infty)$ such that $\eta_\xi = \eta_1 + \eta_2$ on Σ [46, Theorem 1.23]; for the definition of a positive purely finitely additive set function see [46, Definition 1.13]. Apply [46, Theorem 1.22] to find a decreasing sequence $\{B(n)\}_{n=1}^\infty$ in Σ such that $\lim_{n\to\infty} \mu(B(n)) = 0$ and $\eta_2(B(n))) = \eta_2(\Omega)$ for all $n \in \mathbb{N}$. Consequently, with $A(n) := \Omega \setminus B(n)$ for $n \in \mathbb{N}$, the increasing sequence $\{A(n)\}_{n=1}^\infty$ in Σ has the property that $\Omega \setminus \bigcup_{n=1}^\infty A(n)$ is μ-null and $\eta_2(A(n)) = 0$ for each $n \in \mathbb{N}$. For each $A \in \Sigma$ which is μ-null it follows from $0 \le \eta_1(A) + \eta_2(A) = \eta_\xi(A) = 0$ that $\eta_1(A) = 0 = \eta_2(A)$. In particular, η_1 is absolutely continuous with respect to μ, and hence there is $0 \le h \in L^1(\mu)$ such that $\eta_1 = h\, d\mu$, i.e., $\eta_1(A) = \int_A h\, d\mu$ for $A \in \Sigma$. Via Theorem (20.33) of [20], the set functions η_1, η_2 correspond to positive linear functionals $\xi_1, \xi_2 \in L^\infty(\mu)^*$ such that $\eta_j(A) = \langle \chi_A, \xi_j\rangle$ for $A \in \Sigma$ and $j \in \{1,2\}$. Of course, $\xi = \xi_1 + \xi_2$, as $\eta_\xi = \eta_1 + \eta_2$ and $\operatorname{sim} \Sigma$ is dense in $L^\infty(\mu)$.

Fix $f \in L^\infty(\mu)$. The claim is that

$$\|(I_m \circ \alpha)(f)\|_X \le \Big(\int_\Omega |f|^q h\, d\mu\Big)^{1/q}. \tag{2.9}$$

In fact, given $n \in \mathbb{N}$, we have $\langle |f|^q \chi_{A(n)}, \xi_2 \rangle = 0$ because $\xi_2 \geq 0$ implies that

$$0 \leq \langle |f|^q \chi_{A(n)}, \xi_2 \rangle \leq \langle \| \, |f|^q \, \|_{L^\infty(\mu)} \chi_{A(n)}, \xi_2 \rangle \\ = \| \, |f| \, \|^q_{L^\infty(\mu)} \langle \chi_{A(n)}, \xi_2 \rangle = \|f\|^q_{L^\infty(\mu)} \eta_2(A(n)) = 0.$$

This and (2.8) with $f\chi_{A(n)}$ in place of f give, for each $n \in \mathbb{N}$,

$$\|(I_m \circ \alpha)(f\chi_{A(n)})\|^q_X \leq \langle |f|^q \chi_{A(n)}, \xi \rangle = \langle |f|^q \chi_{A(n)}, \xi_1 \rangle = \int_{A(n)} |f|^q h \, d\mu.$$

Since $f \in L^\infty(m) \subseteq L^1(m)$, we know that m_f (cf. (2.1)) is a vector measure and it is clearly absolutely continuous with respect to μ as $m \simeq \mu$. Hence,

$$\lim_{n\to\infty} \|(I_m \circ \alpha)(f - f\chi_{A(n)})\|_X = \lim_{n\to\infty} \|m_f(\Omega \setminus A(n))\|_X \\ = \lim_{n\to\infty} \|m_f(B(n))\|_X = 0.$$

This gives (2.9) because $\chi_{A(n)} \uparrow \chi_\Omega$ μ-a.e. and

$$\|(I_m \circ \alpha)(f)\|^q_X = \lim_{n\to\infty} \|(I_m \circ \alpha)(f\chi_{A(n)})\|^q_X \\ \leq \lim_{n\to\infty} \int_{A(n)} |f|^q h \, d\mu = \int_\Omega |f|^q h \, d\mu.$$

Define $g_q := h + \chi_\Omega \in L^1(\mu)^+$. It follows from (2.9) that $\|(I_m \circ \alpha)(f)\|_X \leq (\int_\Omega |f|^q g_q \, d\mu)^{1/q}$ for each $f \in L^\infty(\mu)$. In other words, $I_m \circ \alpha$ admits a continuous linear extension $T : L^q(g_q \, d\mu) \to X$ because $g_q \geq 1$ pointwise everywhere implies that $L^\infty(\mu) = L^\infty(g_q \, d\mu) \subseteq L^q(g_q \, d\mu)$ continuously. Moreover, the continuous inclusion $L^q(g_q \, d\mu) \subseteq L^1(m)$ holds by [38, Theorem 4.14] because $T(\chi_A) = m(A)$ for $A \in \Sigma$ and because $m \simeq g_q \, d\mu$.

Step 2. *If I_m is p-summing and $L^\infty(\mu) \subseteq L^q(g_q \, d\mu) \subseteq L^1(m)$ continuously for some $1 \leq q < \infty$ and some everywhere strictly positive function $g_q \in L^1(\mu)$, then $I_m \circ \alpha : L^\infty(\mu) \to X$ is $\max\{1, \frac{pq}{p+q}\}$-summing.*

To prove this, let $\alpha_1 : L^\infty(\mu) \to L^q(g_q \, d\mu)$ and $\alpha_2 : L^q(g_q \, d\mu) \to L^1(m)$ denote the corresponding natural embeddings. Then α_1 is q-summing [13, Example 2.9(d)]. So, [13, Theorem 2.22] gives that $I_m \circ \alpha_1$ is $\max\{1, \frac{pq}{p+q}\}$-summing, and hence so is $I_m \circ \alpha = (I_m \circ \alpha_1) \circ \alpha_2$.

Step 3. *If $I_m : L^1(m) \to X$ is p-summing, then there is $g_1 \in L^1(\mu)$ with $g_1 > 0$ pointwise everywhere such that $L^1(g_1 \, d\mu) \subseteq L^1(m)$ continuously.*

Indeed, since $I_m \circ \alpha$ is p-summing, Step 1 with $q := p$ gives $L^\infty(\mu) \subseteq L^p(g_p \, d\mu) \subseteq L^1(m)$ continuously for some $g_p \in L^1(\mu)$ with $g_p > 0$ pointwise everywhere. By Step 2 with $q := p$ we know that $I_m \circ \alpha$ is $\max\{1, p/2\}$-summing. Again Steps 1 and 2, now with $q := \max\{1, p/2\}$, give that $I_m \circ \alpha$ is $\max\{1, p/3\}$-summing. We can continue this process to conclude that $I_m \circ \alpha$

is $\max\{1, p/n\}$-summing for all $n \in \mathbb{N}$. Selecting $n \in \mathbb{N}$ with $n \geq p$ shows that $I_m \circ \alpha$ is 1-summing. Now apply Step 1 with $q := 1$ to obtain Step 3.

STEP 4. *If I_m is p-summing, then m has finite variation.*

By Step 3, we have the continuous inclusion $L^1(g_1\, d\mu) \subseteq L^1(m)$ for some $g_1 \in L^1(\mu)$ which is positive everywhere. With C denoting the operator norm of this continuous inclusion, we have

$$\begin{aligned}\|m(A)\|_X = \|I_m(\chi_A)\|_X &\leq \|\chi_A\|_{L^1(m)}\\ &\leq C \int_\Omega \chi_A g_1\, d\mu = C \int_A g_1\, d\mu, \qquad \forall A \in \Sigma,\end{aligned}$$

which implies that $|m|(\Omega) < \infty$. This establishes Step 4.

Since m is an arbitrary X-valued vector measure, Step 4 implies that X is $\mathcal{A}_p$-variation admissible. This completes the proof of Theorem 2.2. ■

COROLLARY 2.3. *Let m be a Banach-space-valued vector measure. Then the integration map satisfies $I_m \in \mathcal{A}_1$ iff $I_m \in \mathcal{A}_2$.*

Proof. Since $\mathcal{A}_1 \subseteq \mathcal{A}_2$ [13, Inclusion Theorem 2.8], we only need to consider the case when I_m is 2-summing. Suppose that m is X-valued. According to [13, Corollary 2.16], there is a probability measure μ and continuous linear maps $\alpha : L^1(m) \to L^2(\mu)$ and $\beta : L^2(\mu) \to X$ such that $I_m = \beta \circ \alpha$. But $L^1(m) = L^1(|m|)$ by Theorem 2.2 and so Grothendieck's Theorem [13, Theorem 3.4], [44, p. 202] implies that $\alpha : L^1(|m|) \to L^2(\mu)$ is 1-summing. Hence, also $I_m = \beta \circ \alpha \in \mathcal{A}_1$. ■

Proposition 1.1 and Theorem 2.2 show, for any vector measure m with $I_m \in \mathcal{A}_p$ for some $1 \leq p < \infty$, that $L^1(m) = L^1(|m|)$. In certain situations a converse is possible.

PROPOSITION 2.4. *Let $2 < p < \infty$ and m be an ℓ^p-valued vector measure satisfying $L^1(m) = L^1(|m|)$. Then $I_m \in \mathcal{A}_r$ for all $p < r < \infty$.*

Proof. By a result of P. Saphar, *every* operator from $\mathcal{L}(\ell^1, \ell^p)$ is r-summing for all $p < r < \infty$ (see [12, p. 321, Corollary], for example). By the local technique lemma for operator ideals [12, p. 301], the same statement holds if we replace ℓ^1 with any $\mathcal{L}_1$-space, in particular, by $L^1(|m|)$ (see also [38, Example 2.61(ii)]). Since $I_m \in \mathcal{L}(L^1(m), \ell^p) = \mathcal{L}(L^1(|m|), \ell^p)$, it follows that $I_m \in \mathcal{A}_r$ for all $p < r < \infty$. ■

From the proof, it is clear that the condition $L^1(m) = L^1(|m|)$ in the statement of Proposition 2.4 can be replaced with the requirement that $L^1(m)$ is an $\mathcal{L}_1$-space.

REMARK 2.5. The method of proof of Theorem 2.2 relies on three classical results, namely the Kakutani Representation Theorem, the Yosida–Hewitt Decomposition Theorem and a factorization theorem of Pietsch. We point out that this argument can be adapted to provide a completely different proof of the result mentioned in Section 1, namely that $L^1(m) = L^1(|m|)$ iff the integration map I_m is positive p-summing for some $1 \le p < \infty$ [5]. The proof given in [5] is based on p-concavity arguments.

Concerning a proof via the methods of this paper recall, for $1 \le p < \infty$ and X a Banach space, that an X-valued, continuous linear operator T defined on a (complex) Banach lattice W is called *positive p-summing* if there exists $C > 0$ such that

$$\Big(\sum_{j=1}^{n} \|Tw_j\|_X^p\Big)^{1/p} \le C \sup_{w^* \in \mathbb{B}[W^*]} \Big(\sum_{j=1}^{n} |\langle w_j, w^*\rangle|^p\Big)^{1/p}$$

whenever $\{w_j\}_{j=1}^n$ is a finite set of *positive* elements in W and $n \in \mathbb{N}$ [4, Definition 1]. For such an operator $T : W \to X$ and for any W-valued *positive* operator S defined on a (complex) Banach lattice, the composition $T \circ S$ is again positive p-summing [3, Proposition 1(d)]. Fix an X-valued vector measure m defined on a measurable space (Ω, Σ) and let $\mu : \Sigma \to [0, \infty)$ be a finite measure such that $m \simeq \mu$. The natural embedding of $L^\infty(\mu) = L^\infty(m)$ into $L^1(m)$ is denoted by α. When referring to Steps 1 to 4 we mean those in the proof of Theorem 2.2.

Step 1′. *If $I_m \circ \alpha : L^\infty(\mu) \to X$ is positive q-summing for some $1 \le q < \infty$, then there is $g_q \in L^1(\mu)$ with $g_q(w) > 0$ for every $w \in \Omega$ such that* (2.6) *holds with continuous inclusions.*

To establish Step 1′ let $\beta : L^\infty(\mu) \to C(K)$ be the isometric Banach lattice isomorphism as in the proof of Step 1. Then $\beta^{-1} : C(K) \to L^\infty(\mu)$ is a positive operator and hence, as noted above, $(I_m \circ \alpha) \circ \beta^{-1} : C(K) \to X$ is then positive q-summing. According to [4, Proposition 3] we see that $I_m \circ \alpha \circ \beta^{-1}$ is actually q-summing, and hence so is $I_m \circ \alpha = (I_m \circ \alpha \circ \beta^{-1}) \circ \beta$ as β is a positive operator. Thus, Step 1 can be applied to obtain (2.6) with continuous inclusions.

Step 2′. *If I_m is positive p-summing and* (2.6) *holds continuously for some $1 \le q < \infty$ and some everywhere strictly positive function $g_q \in L^1(\mu)$, then $I_m \circ \alpha : L^\infty(\mu) \to X$ is positive* $\max\{1, \frac{pq}{p+q}\}$*-summing.*

To see this, let $\alpha_1 : L^\infty(\mu) \to L^q(g_q\, d\mu)$ and $\alpha_2 : L^q(g_q\, d\mu) \to L^1(m)$ denote the respective embeddings determined by (2.6). As noted in the proof of Step 2, α_1 is q-summing, and hence so is $\alpha := \alpha_2 \circ \alpha_1$. An examination of the proof of the Composition Theorem 2.22 (and Lemma 2.23) in [13] shows that it can be adapted to show that the composition $I_m \circ \alpha$ is *positive* $\max\{1, \frac{pq}{p+q}\}$-summing.

Step 3′. *If I_m is positive p-summing, then there is $g_1 \in L^1(\mu)$ with $g_1 > 0$ pointwise everywhere such that $L^1(g_1\, d\mu) \subseteq L^1(m)$ continuously.*

The proof of Step 3′ is similar to that of Step 3, by applying Steps 1′ and 2′ repeatedly.

Step 4′. *If I_m is positive p-summing, then m has finite variation.*

The continuous inclusion $L^1(g_1\,d\mu) \subseteq L^1(m)$, guaranteed by Step 3′, establishes Step 4′ as in the proof of Step 4.

It remains to show that $L^1(m) = L^1(|m|)$ whenever I_m is positive p-summing for some $1 \le p < \infty$. Since the positive p-summing operators do not form an operator ideal, we cannot appeal to Proposition 1.1. Now, by Lemma 2.1(i) we have $L^1(|m|) \subseteq L^1(m)$. To prove the reverse inclusion, let $f \in L^1(m)$. Select non-negative functions $f^{(j)} \in L^1(m)$ for $j = 1, \ldots, 4$ such that $f = f^{(1)} - f^{(2)} + i(f^{(3)} - f^{(4)})$. For j fixed the continuous multiplication operator $M_{f^{(j)}} : L^1(m_{f^{(j)}}) \to L^1(m)$, as given in Lemma 2.1(ii), is positive, and hence the composition $I_{m_{f^{(j)}}} = I_m \circ M_{f^{(j)}}$ is positive p-summing. Via Step 4′, with $m_{f^{(j)}}$ in place of m, it follows that $m_{f^{(j)}}$ has finite variation. Since $\|m_f(A)\|_X \le \sum_{j=1}^4 \|m_{f^{(j)}}(A)\|_X$ for $A \in \Sigma$, it follows that m_f also has finite variation. So, $f \in L^1(|m|)$ (cf. Lemma 2.1(i)). Thus, $L^1(m) \subseteq L^1(|m|)$. This establishes that $L^1(m) = L^1(|m|)$.

As already noted in Section 1, the operator ideal $\mathcal{A}_c$ has the property that *every* Banach space is $\mathcal{A}_c$-admissible [36, Theorem 4]. According to Theorem 2.2, the same is true for the operator ideal $\mathcal{A}_p$, $1 \le p < \infty$. We now show that there exist vector measures m for which $I_m \in \mathcal{A}_c \setminus \mathcal{A}_p$ and others for which $I_m \in \mathcal{A}_p \setminus \mathcal{A}_c$.

Example 2.6. Let G be any infinite compact abelian group with normalized *Haar measure* μ. For each $1 \le p < \infty$ and each regular, complex Borel measure $\lambda \in M(G)$, the linear operator $C_\lambda^{(p)}$ of *convolution* with λ belongs to $\mathcal{L}(L^p(G))$ where, for $f \in L^p(G) := L^p(\mu)$, we have

$$C_\lambda^{(p)}(f) : x \mapsto \int_G f(x-y)\,d\lambda(y) \qquad \mu\text{-a.e. } x \in G.$$

Indeed, $\|C_\lambda^{(p)}\|_{\mathrm{op}} \le |\lambda|(G)$. The operator $C_\lambda^{(p)}$ induces the vector measure $m_\lambda^{(p)} : \mathcal{B}(G) \to L^p(G)$ defined by

$$m_\lambda^{(p)} : A \mapsto C_\lambda^{(p)}(\chi_A) = \chi_A * \lambda, \qquad \forall A \in \mathcal{B}(G).$$

It is known that $m_\lambda^{(p)} \simeq \mu$ (provided $\lambda \ne 0$) and, with continuous (natural) inclusions, that $L^p(G) \subseteq L^1(m_\lambda^{(p)}) \subseteq L^1(G)$. Moreover, the integration map $I_{m_\lambda^{(p)}} : L^1(m_\lambda^{(p)}) \to L^p(G)$ is also given by convolution, i.e.,

$$I_{m_\lambda^{(p)}}(f) = f * \lambda, \qquad \forall f \in L^1(m_\lambda^{(p)}). \tag{2.10}$$

For $\lambda \ll \mu$, i.e., there exists $g \in L^1(G)$ such that $\lambda(A) = \int_A g\,d\mu$ for $A \in \mathcal{B}(G)$, we write $g\,d\mu$ for λ and $C_g^{(p)}$ (resp. $m_g^{(p)}$) for $C_\lambda^{(p)}$ (resp. $m_\lambda^{(p)}$). All of the above claims can be found in [38, Ch. 7, §7.4], for example. The following characterization occurs in Theorem 7.67 of [38].

FACT. *Let $1 < p < \infty$ and $\lambda \in M(G) \setminus \{0\}$. The following assertions are equivalent:*

(i) *There exists $g \in L^p(G)$ such that $\lambda = g\,d\mu$.*
(ii) *The integration map $I_{m_\lambda^{(p)}} : L^1(m_\lambda^{(p)}) \to L^p(G)$ is compact.*
(iii) $L^1(m_\lambda^{(p)}) = L^1(|m_\lambda^{(p)}|) = L^1(G)$.

Concerning 1-summing operators we require the following result.

LEMMA 2.7. *Let $1 \le p < 2$ and $g \in L^p(G)$. Then $L^1(m_g^{(p)}) = L^1(G)$, and the integration map $I_{m_g^{(p)}} : L^1(m_g^{(p)}) \to L^p(G)$ is 1-summing iff $g \in L^2(G)$.*

Proof. In view of (2.10) and the above Fact we see that $L^1(m_g^{(p)}) = L^1(G)$ and that $I_{m_g^{(p)}}$ is precisely the bounded operator $C_g^{(1,p)} : L^1(G) \to L^p(G)$ of convolution with g.

Suppose that $g \in L^2(G)$. Let $J^{(2,p)} : L^2(G) \to L^p(G)$ denote the natural inclusion and $C_g^{(1,2)} : L^1(G) \to L^2(G)$ be the bounded operator of convolution with g. Since $C_g^{(1,2)}$ is necessarily 1-summing [13, Theorem 3.4], so is the composition $I_{m_g^{(p)}} = C_g^{(1,p)} = J^{(2,p)} \circ C_g^{(1,2)}$.

Conversely, suppose that $C_g^{(1,p)}$ is 1-summing. Since $g \in L^p(G)$, the set $S(\widehat{g}) := \{\gamma \in \Gamma : \widehat{g}(\gamma) \neq 0\}$ is countable, where Γ is the dual group of G and $\widehat{g}$ is the Fourier transform of g, i.e., $\widehat{g}(\gamma) := \int_G \overline{(x,\gamma)} g(x)\,d\mu(x)$ for $\gamma \in \Gamma$, with (x,γ) denoting the value of the *character* γ at $x \in G$. The *trigonometric monomial* $x \mapsto (x,\gamma)$ on G is denoted by $(\cdot,\gamma)$. Since $\sum_{\gamma\in S(\widehat{g})} |\langle(\cdot,\gamma),\varphi\rangle|^2 = \sum_{\gamma\in S(\widehat{g})} |(\overline{\varphi})^\frown(\gamma)|^2 < \infty$ for each $\varphi \in L^\infty(G) = L^1(G)^* \subseteq L^2(G)$, the sequence $\{(\cdot,\gamma) : \gamma \in S(\widehat{g})\} \subseteq L^1(G)$ is weakly 2-summable in $L^1(G)$ (cf. [13, p. 32] for the definition). But $C_g^{(1,p)}$ is also 2-summing [13, Inclusion Theorem 2.8], and hence $C_g^{(1,p)}$ maps $\{(\cdot,\gamma) : \gamma \in S(\widehat{g})\}$ to a norm 2-summable sequence in $L^p(G)$ [13, Proposition 2.1], i.e.,

$$\sum_{\gamma\in S(\widehat{g})} |\widehat{g}(\gamma)|^2 = \sum_{\gamma\in S(\widehat{g})} \|\widehat{g}(\gamma)(\cdot,\gamma)\|_{L^p(G)}^2 = \sum_{\gamma\in S(\widehat{g})} \|C_g^{(1,p)}((\cdot,\gamma))\|_{L^p(G)}^2 < \infty.$$

Hence, $\widehat{g} \in \ell^2(\Gamma)$, i.e., $g \in L^2(G)$. ∎

It follows from the above Fact and Lemma 2.7 that, for every $1 < p < 2$ and $g \in L^p(G) \setminus L^2(G)$ (such functions g exist as μ is *non-atomic* [38, Lemma 7.97]), the integration map $I_{m_g^{(p)}}$ is compact, but fails to be 1-summing.

Consider now $p := \infty$, i.e., $X = L^\infty(G)$, and the *finitely additive* $L^\infty(G)$-valued set function $m_\lambda^{(\infty)} : A \mapsto C_\lambda^{(\infty)}(\chi_A)$ for $A \in \mathcal{B}(G)$, where $C_\lambda^{(\infty)} \in \mathcal{L}(L^\infty(G))$ is again the operator of convolution with $\lambda \in M(G)$. Then $m_\lambda^{(\infty)}$ is *norm* σ-additive (i.e., a vector measure) iff $\lambda = g\,d\mu$ for some $g \in L^1(G)$ [34, Theorem 1], in which case the integration map $I_{m_g^{(\infty)}}$ is compact iff $g \in C(G)$ [34, Corollary 3]. On the other hand, $I_{m_g^{(\infty)}}$ is 1-summing iff $\widehat{g} \in \ell^1(\Gamma)$ [34, Proposition 4], i.e., iff $g \in A(G)$ in the notation of Example 3.5(ii) below. So, $I_{m_g^{(\infty)}}$ is compact but fails to be 1-summing whenever $g \in C(G) \setminus A(G)$. That the inclusion $A(G) \subsetneq C(G)$ is always *proper* is known [19, Theorem (37.4)]. ■

In the proof of Lemma 2.7, it was observed that the integration map $I_{m_g^{(p)}} : L^1(G) \to L^p(G)$ is the continuous convolution operator $C_g^{(1,p)} : f \mapsto f * g$ from $L^1(G)$ into $L^p(G)$. A detailed study of those convolution operators belonging to $\mathcal{L}(L^p(G))$ and to $\mathcal{L}(L^1(G), L^p(G))$, for $1 \le p \le \infty$, and which are 1-*summing* appears in [37]. For the latter case, i.e., for $\mathcal{L}(L^1(G), L^p(G))$, such convolution operators always arise as the integration map of an $L^p(G)$-valued vector measure.

We now present some examples of m with $I_m \in \mathcal{A}_1 \setminus \mathcal{A}_c$.

EXAMPLE 2.8. (i) Let (Ω, Σ, μ) be a finite positive measure space for which there exists an infinite partition $\{A(n)\}_{n=1}^\infty \subseteq \Sigma$ of Ω with $\mu(A(n)) > 0$ for $n \in \mathbb{N}$. Let X be any infinite-dimensional Hilbert space. According to Lemma 2.5 in [36] (see also its proof), there exists a sequence $\{x_n\}_{n=1}^\infty \subseteq X$ of unit vectors which is *not* a relatively compact subset of X and a *finite* set $\mathcal{F} \subseteq X^*$ such that

$$1 = \|x_n\|_X \le \sum_{x^* \in \mathcal{F}} |\langle x_n, x^* \rangle|, \qquad \forall n \in \mathbb{N}. \tag{2.11}$$

Define $H : \Omega \to X$ by $H(w) := \sum_{n=1}^\infty \chi_{A(n)}(w) x_n$ for $w \in \Omega$, in which case H is *strongly measurable* (its range is separable) and bounded, as $\|H(w)\| = \sum_{n=1}^\infty \chi_{A(n)}(w) = 1$ for $w \in \Omega$. Hence, H is *Bochner* μ*-integrable* and so the vector measure $m : \Sigma \to X$ defined by $m(A) := \int_A H\,d\mu$ for $A \in \Sigma$ has finite variation. Indeed, $|m| = \mu$ since $|m|(A) = \int_A \|H(w)\|\,d\mu(w)$ for $A \in \Sigma$ [14, Ch. II, Theorem 1.4]. For each $x^* \in \mathcal{F}$ and $A \in \Sigma$ we have

$$|\langle m, x^* \rangle|(A) = \int_A |\langle H(w), x^* \rangle|\,d\mu(w) = \sum_{n=1}^\infty |\langle x_n, x^* \rangle| \mu(A \cap A(n)),$$

and hence

$$\sum_{x^*\in\mathcal{F}} |\langle m, x^*\rangle|(A) = \sum_{n=1}^{\infty} \mu(A\cap A(n)) \sum_{x^*\in\mathcal{F}} |\langle x_n, x^*\rangle| = \sum_{n=1}^{\infty} \alpha_n \mu(A\cap A(n)),$$

where $\alpha_n := \sum_{x^*\in\mathcal{F}} |\langle x_n, x^*\rangle| \geq 1$ for all $n\in\mathbb{N}$ (cf. (2.11)). Accordingly,

$$|m|(A) = \mu(A) = \sum_{n=1}^{\infty} \mu(A\cap A(n)) \leq \sum_{n=1}^{\infty} \alpha_n \mu(A\cap A(n)) = \sum_{x^*\in\mathcal{F}} |\langle m, x^*\rangle|(A)$$

for each $A\in\Sigma$, and so $L^1(m) = L^1(|m|) = L^1(\mu)$ [36, Lemma 2.6(i)].

Observe that $\{\frac{1}{\mu(A(n))}\chi_{A(n)}\}_{n=1}^{\infty} \subseteq \mathbb{B}[L^1(\mu)] = \mathbb{B}[L^1(|m|)]$ with $\mathbb{B}[L^1(|m|)] \subseteq \mathbb{B}[L^1(m)]$ (see Lemma 2.1(i)). Since

$$I_m\bigg(\frac{1}{\mu(A(n))}\chi_{A(n)}\bigg) = \frac{1}{\mu(A(n))} \int_{A(n)} H\,d\mu = x_n, \quad \forall n\in\mathbb{N},$$

and $\{x_n\}_{n=1}^{\infty}$ is not relatively compact, it follows that the integration map I_m *fails* to be compact. However, since $L^1(m) = L^1(\mu)$ with equivalence of norms, $I_m : L^1(m) \to X$ is surely 1-summing by Grothendieck's Theorem [44, p. 202]. So, $I_m \in \mathcal{A}_1 \setminus \mathcal{A}_c$.

(ii) Let the measure space (Ω,Σ,μ) and the partition $\{A(n)\}_{n=1}^{\infty}$ of Ω be as in (i) above. Define a vector measure $m:\Sigma\to\ell^1$ by

$$m(A) := \mu(A)f_1 + \sum_{n=1}^{\infty} \mu(A\cap A(n))f_{n+1}, \quad \forall A\in\Sigma, \tag{2.12}$$

where $\{f_n\}_{n=1}^{\infty}$ is the canonical basis of ℓ^1, and observe that m is precisely the vector measure of Example 3.7 in [33]. According to Proposition 3.5 of [33] we have $L^1(m) = L^1(\mu)$, in which case $L^1(m) = L^1(|m|)$ with equivalent norms [38, Lemma 3.14]. As noted in Example 3.7 of [33] the integration map $I_m : L^1(m) \to \ell^1$ is *not* compact. Since ℓ^1 has the Schur property, I_m also fails to be weakly compact, and hence is *not* 1-summing [13, Theorem 2.17].

For $2\leq p<\infty$, let $j^{(1,p)} : \ell^1 \to \ell^p$ be the natural embedding. Then $m_p := j^{(1,p)} \circ m$ is an ℓ^p-valued vector measure on Σ. It is clear from (2.12) that m, m_p and μ all have the same null sets. Since $j^{(1,p)}$ is injective, it follows that $L^1(m) \subseteq L^1(m_p)$ [38, Lemma 3.27]. Suppose that $f\in L^1(m_p)$. For $e_1^* := (1,0,0,\ldots) \in (\ell^p)^*$ we have $\int_\Omega |f|\,d\mu = \int_\Omega |f|\,d|\langle m_p, e_1^*\rangle| < \infty$, and so $f\in L^1(\mu) = L^1(m)$. Accordingly, $L^1(m) = L^1(m_p) = L^1(\mu)$ with equivalence of norms, for each $2\leq p<\infty$.

Now, $I_{m_2} : L^1(m_2) \to \ell^2$ is 1-summing by Grothendieck's Theorem (see (i) above). Let $j^{(2,p)} : \ell^2 \to \ell^p$ be the natural inclusion. Then $I_{m_p} = j^{(2,p)} \circ I_{m_2}$, and so I_{m_p} is 1-summing for every $2\leq p<\infty$.

Observe that $\{\frac{1}{\mu(A(n))}\chi_{A(n)}\}_{n=1}^{\infty} \subseteq \mathbb{B}[L^1(\mu)]$, and so $\{\frac{1}{\mu(A(n))}\chi_{A(n)}\}_{n=1}^{\infty}$ is also contained in a multiple of $\mathbb{B}[L^1(m_p)]$. It follows from the formula

$$I_{m_p}(f) = \Big(\int_{\Omega} f\, d\mu,\ \int_{A(1)} f\, d\mu,\ \int_{A(2)} f\, d\mu, \dots \Big), \qquad \forall f \in L^1(m_p),$$

that $I_{m_p}\big(\frac{1}{\mu(A(n))}\chi_{A(n)}\big) = e_1 + e_{n+1}$ for $n \in \mathbb{N}$, where $\{e_n\}_{n=1}^{\infty}$ is the canonical basis of ℓ^p. Since $\{e_1 + e_n\}_{n=1}^{\infty}$ is not relatively compact in ℓ^p, we conclude that I_{m_p} is *not* compact, for each $2 \le p < \infty$.

EXAMPLE 2.9. Let (Ω, Σ, μ) be a finite positive measure space and $\{A(n)\}_{n=1}^{\infty}$ be a partition of Ω as in Example 2.8(i). Fix any $2 < p < \infty$. We exhibit a vector measure $m : \Sigma \to \ell^p$ such that

$$I_m \in \Big(\bigcap_{p<r<\infty} \mathcal{A}_r \Big) \setminus (\mathcal{A}_p \cup \mathcal{A}_c)\,. \tag{2.13}$$

The canonical unit basis of ℓ^1 (resp. ℓ^p) is denoted by $\{f_n\}_{n=1}^{\infty}$ (resp. $\{e_n\}_{n=1}^{\infty}$). Select any operator $T \in \mathcal{L}(\ell^1, \ell^p)$ which is *not* p-summing [23, Lemma 4.1]; a concrete construction of such a T is presented in [24, §4]. Since ℓ^p is a separable Banach space, there is a surjective operator $Q \in \mathcal{L}(\ell^1, \ell^p)$ [27, p. 108]. In particular, Q is not compact. By the *lifting property* of ℓ^1 [27, Proposition 2.f.7], we have $T = Q \circ S$ for some $S \in \mathcal{L}(\ell^1)$. So, Q is not p-summing as T is not p-summing. On the other hand, Q is r-summing for all $p < r < \infty$ because *every* operator in $\mathcal{L}(\ell^1, \ell^p)$ is r-summing for such r [12, Corollary 24.6]. The surjective linear operator $P : L^1(\mu) \to \ell^1$ given by $P(h) := \sum_{n=1}^{\infty} (\int_{A(n)} h\, d\mu) f_n$ for $h \in L^1(\mu)$ is continuous, and hence $Q \circ P \in \mathcal{L}(L^1(\mu), \ell^p)$ is r-summing for all $p < r < \infty$. But, being a surjection, $Q \circ P$ is not compact. Also, $Q \circ P$ is not p-summing because, with $J \in \mathcal{L}(\ell^1, L^1(\mu))$ denoting the injection $\varphi \mapsto \sum_{n=1}^{\infty} (\varphi(n)/\mu(A(n)))\chi_{A(n)}$ for $\varphi \in \ell^1$, we have $Q = Q \circ (P \circ J) = (Q \circ P) \circ J$ as $P \circ J$ is the identity operator in $\mathcal{L}(\ell^1)$.

Let $R \in \mathcal{L}(\ell^p)$ denote the forward shift operator, i.e., $R(\sum_{n=1}^{\infty} a_n e_n) := \sum_{n=1}^{\infty} a_n e_{n+1}$ for $\sum_{n=1}^{\infty} a_n e_n \in \ell^p$. Since $R \circ Q \circ P$ is continuous and $L^1(\mu)$ has σ-order continuous norm, it follows that $m : \Sigma \to \ell^p$ defined by

$$m(A) := \mu(A) e_1 + (R \circ Q \circ P)(\chi_A), \qquad \forall A \in \Sigma,$$

is a vector measure. Observing that the range of $R \circ Q \circ P$ lies in $\overline{\mathrm{span}}(\{e_n\}_{n=2}^{\infty}) \subseteq \ell^p$, it follows that $m \simeq \mu$. Moreover, $L^1(\mu) \subseteq L^1(m)$ as

$$\int_A f\, dm = \Big(\int_A f\, d\mu \Big) e_1 + (R \circ Q \circ P)(f\chi_A), \qquad \forall f \in L^1(\mu),\ A \in \Sigma.$$

Let e_1^* denote the continuous linear functional $\sum_{n=1}^{\infty} \psi(n) e_n \mapsto \psi(1)$ on ℓ^p. Then $L^1(m) = L^1(\langle m, e_1^* \rangle) = L^1(\mu)$. Hence, $L^1(m) = L^1(\mu)$ as isomorphic Banach spaces with

$$I_m(f) = I_\mu(f) e_1 + (R \circ Q \circ P)(f), \qquad \forall f \in L^1(m).$$

Since $f \mapsto I_\mu(f) e_1$ is a rank-1 operator in $\mathcal{L}(L^1(m), \ell^p)$ and R is a linear isometry onto its (closed) range $\overline{\mathrm{span}}(\{e_n\}_{n=2}^{\infty}) \subseteq \ell^p$, it follows that I_m is

r-summing for each $p < r < \infty$, but I_m is neither p-summing nor is it compact. That is, (2.13) holds.

To conclude this section, let us point out that there exist vector measures m satisfying $L^1(m) = L^1(|m|)$ and I_m is neither compact nor 1-summing.

For $1 < r < \infty$, consider the *Volterra vector measure* $m_r : \mathcal{B}([0,1]) \to L^r([0,1])$ (see Section 1). Then m_r has finite variation with $L^1(|m_r|) \subsetneq L^1(m_r)$ [38, Example 3.26]. Moreover, I_{m_r} is *not* compact [38, p. 154]. Since I_{m_r} fails to be completely continuous [38, Proposition 3.52], it *fails* to be p-summing for every $1 \le p < \infty$ [13, Theorem 2.17]. Consider now $r \in \{1, \infty\}$, in which case the Volterra measure m_r is defined by the same formula as for $1 < r < \infty$ given in Section 1. According to Example 3.26 in [38] we have $L^1(m_1) = L^1(|m_1|)$ and $L^1(|m_\infty|) = L^1(m_\infty)$, which implies that $I_{m_r} \in \mathcal{A}_{cc}$. Indeed, the range $m_r(\mathcal{B}([0,1]))$ of m_r is relatively compact because the classical Volterra integral operator V_r from $L^r([0,1])$ into itself is compact [38, pp. 113–114]. Since $L^1(m_r) = L^1(|m_r|)$, the complete continuity of $I_{m_r} = V_r$ follows from [38, Corollary 2.42]. However, since both I_{m_1} and I_{m_∞} fail to be weakly compact [38, Example 3.49(iv)], they also *fail* to be compact and *fail* to be p-summing for every $1 \le p < \infty$ [13, Theorem 2.17].

Or, let $0 < \alpha < 1$ and consider the *Sobolev vector measure* $m : \mathcal{B}([0,1]) \to L^\infty([0,1])$ defined by

$$m(A) : t \mapsto \int_t^1 \chi_A(s) s^{-\alpha}\, ds, \qquad \forall t \in [0,1],\ A \in \mathcal{B}([0,1]). \tag{2.14}$$

Then $L^1(m) = L^1(|m|)$ [11, Proposition 2.1], and the integration map $I_m : L^1(m) \to L^\infty([0,1])$ fails to be weakly compact [11, Proposition 2.2]. By an argument as for the Volterra measures with $r \in \{1, \infty\}$ we can conclude that I_m is *not* compact and *not* p-summing, for every $1 \le p < \infty$. Suppose that X is any *rearrangement invariant* Banach function space on $[0,1]$ which is not isomorphic to $L^\infty([0,1])$. Since $L^\infty([0,1])$ imbeds continuously into X, the same formula (2.14) specifies an X-valued vector measure, denoted by m_X, which has finite variation [11, Proposition 3.1], and whose integration map $I_{m_X} : L^1(m_X) \to X$ is *not* compact [11, Proposition 3.6]. If X happens to be a Lorentz Λ_φ-space with φ an increasing concave function on $[0,1]$ such that $\varphi(0) = 0$ (and with the norm $\|f\|_{\Lambda_\varphi} := \int_0^1 f^*(s)\, d\varphi(s)$, where f^* is the decreasing rearrangement of f), then actually $L^1(m_{\Lambda_\varphi}) = L^1(|m_{\Lambda_\varphi}|)$ and $I_{m_{\Lambda_\varphi}}$ *fails* to be weakly compact [11, Proposition 3.8 & Corollary 4.3]. Arguing as above we can conclude that $I_{m_{\Lambda_\varphi}}$ is *not* p-summing, for every $1 \le p < \infty$.

Finally, consider the vector measure $m : \Sigma \to \ell^1$ occurring in (2.12) in Example 2.8(ii), where it was observed that $L^1(m) = L^1(|m|)$ and $I_m : L^1(m) \to \ell^1$ is not weakly compact. Hence, I_m is *not* compact and *not* p-summing, for every $1 \le p < \infty$.

3. Proof of Theorem 1.2. The proof of Theorem 1.2 proceeds via a series of lemmata. We begin with a result from the realm of Banach space theory. Sometimes we will express sequences of scalars (i.e., elements of $\mathbb{C}^{\mathbb{N}}$) as functions defined on $\mathbb{N}$. Vectors from the finite-dimensional space $\mathbb{R}^N$ are denoted by $(\xi_n)_{n=1}^N$.

LEMMA 3.1. *Let X be a Banach space, $K_n \subseteq \mathbb{B}[X]$ for $n \in \mathbb{N}$ be a sequence of non-empty compact sets, and $\delta > 0$ be such that*

$$\delta \sum_{n=1}^{N} |a_n| \le \sup \Big\{ \Big\| \sum_{n=1}^{N} a_n x_n \Big\|_X : x_n \in K_n,\ n = 1, \dots, N \Big\} \tag{3.1}$$

for all choices of $N \in \mathbb{N}$ and $\{a_n : n = 1, \dots, N\} \subseteq \mathbb{C}$. Then there exists $x_0^* \in \mathbb{B}[X^*]$ such that

$$\limsup_{n\to\infty} \big(\sup_{x\in K_n} |\langle x, x_0^*\rangle| \big) > 0. \tag{3.2}$$

Proof. Define a set of real sequences by

$$W := \{(\varepsilon_n |\langle x_n, x^*\rangle|)_{n=1}^{\infty} \in \mathbb{R}^{\mathbb{N}} : x^* \in \mathbb{B}[X^*],\ \varepsilon_n \in \{-1, 1\},\ x_n \in K_n,\ \forall n \in \mathbb{N}\}.$$

Then $W \subseteq \mathbb{B}[\ell^\infty]$ and the convex hull $\mathrm{co}(W)$ of W in ℓ^∞ also consists of real sequences belonging to $\mathbb{B}[\ell^\infty]$. Clearly $\mathrm{co}(W) \neq \emptyset$.

STEP 1. *For each $N \in \mathbb{N}$, there exists $\varphi_N \in \mathrm{co}(W)$ satisfying*

$$\varphi_N(n) > \delta/2, \quad \forall n = 1, \dots, N.$$

To see this, fix $N \in \mathbb{N}$, define an $\mathbb{R}$-linear map $\Phi_N : \ell^\infty \to \mathbb{R}^N$ by $\Phi_N(\psi) := (\mathrm{Re}(\psi(n)))_{n=1}^N \in \mathbb{R}^N$ for $\psi \in \ell^\infty$, and set $U_N := \Phi(\mathrm{co}(W)) \subseteq \mathbb{R}^N$. Let

$$V_N := \{(\xi_n)_{n=1}^N \in \mathbb{R}^N : \xi_n > \delta/2,\ \forall n = 1, \dots, N\}.$$

Now, suppose that the conclusion of Step 1 is *not valid*, i.e., for every $\varphi \in \mathrm{co}(W)$ we have $\varphi(n) \le \delta/2$ for some $n \in \{1, \dots, N\}$, depending on φ. Then $U_N \cap V_N = \emptyset$. Since $U_N \neq \emptyset$ is convex and $V_N \neq \emptyset$ is convex and open, there exist $(a_n)_{n=1}^N \in (\mathbb{R}^N)^*$ and $r \in \mathbb{R}$ such that

$$\sum_{n=1}^{N} a_n \xi_n \le r < \sum_{n=1}^{N} a_n \eta_n, \quad \forall (\xi_n)_{n=1}^N \in U_N,\ (\eta_n)_{n=1}^N \in V_N \tag{3.3}$$

[22, First Separation Theorem, p. 130]. We claim that also

$$\sum_{n=1}^{N} |a_n| \xi_n \le r, \quad \forall (\xi_n)_{n=1}^N \in U_N. \tag{3.4}$$

In fact, fix $(\xi_n)_{n=1}^N \in U_N$ and select $\varphi \in \mathrm{co}(W)$ satisfying $\varphi(n) = \xi_n$ for $n = 1, \dots, N$. Choose $j_0 \in \mathbb{N}$, $b_k \in [0,1]$ with $\sum_{k=1}^{j_0} b_k = 1$, and $\varphi_k \in W$ for $k = 1, \dots, j_0$ such that $\varphi = \sum_{k=1}^{j_0} b_k \varphi_k$. Furthermore, there exist $\varepsilon_n \in$

$\{-1,1\}$ for $n=1,\dots,N$ satisfying $\varepsilon_n a_n = |a_n|$. For each $k=1,\dots,j_0$ define a function ψ_k on $\mathbb{N}$ by

$$\psi_k(i) := \begin{cases} \varepsilon_i \varphi_k(i) & \text{for } i \in \{1,\dots,N\}, \\ \varphi_k(i) & \text{for } i > N. \end{cases}$$

It is routine to check from the definition of W that $\{\psi_1,\dots,\psi_{j_0}\} \subseteq W$. Moreover, direct calculation yields $\sum_{n=1}^N |a_n|\xi_n = \sum_{n=1}^N |a_n|(\sum_{k=1}^{j_0} b_k\psi_k)(n)$. Since $\psi := \sum_{k=1}^{j_0} b_k\psi_k \in \mathrm{co}(W)$, it follows from the left inequality in (3.3), applied to $(\psi(n))_{n=1}^N \in U_N$, that $\sum_{n=1}^N |a_n|\xi_n \le r$. This establishes (3.4).

Next we claim that

$$\sup\Big\{\Big\|\sum_{n=1}^N a_n x_n\Big\|_X : x_n \in K_n,\ n=1,\dots,N\Big\} \le r. \tag{3.5}$$

Indeed, fix any choice of $x_n \in K_n$ for $n=1,\dots,N$. For each $x^* \in \mathbb{B}[X^*]$ it follows from the fact that $(|\langle x_n, x^*\rangle|)_{n=1}^N \in U_N$ and (3.4) that $\sum_{n=1}^N |a_n| \cdot |\langle x_n, x^*\rangle| \le r$. Accordingly,

$$\Big\|\sum_{n=1}^N a_n x_n\Big\|_X \le \sup_{x^*\in\mathbb{B}[X^*]} \sum_{n=1}^N |a_n|\cdot|\langle x_n, x^*\rangle| \le r.$$

Since $x_n \in K_n$ for $n=1,\dots,N$ are arbitrary, this establishes (3.5).

It follows from (3.1) and (3.5) that $\delta\sum_{n=1}^N |a_n| \le r$. On the other hand, since $(3\delta/4)(1,\dots,1) \in V_N$, from the right inequality in (3.3) we have $r < \sum_{n=1}^N 3\delta a_n/4 < \delta\sum_{n=1}^N |a_n|$, which contradicts $\delta\sum_{n=1}^N |a_n| \le r$. Hence, Step 1 is established.

STEP 2. *Let $\overline{W}^{\sigma^*}$ denote the closure of W in ℓ^∞ with respect to the weak-$*$ topology $\sigma(\ell^\infty,\ell^1)$. Then $(\overline{W}^{\sigma^*}) \setminus c_0$ is non-empty, where c_0 is considered as a closed subspace of ℓ^∞.*

To establish Step 2, first observe that $\mathbb{B}[\ell^\infty]$ is compact for the weak-$*$ topology [28, Theorem 2.6.18]. Recalling that $W \subseteq \mathbb{B}[\ell^\infty]$ it follows that $\overline{W}^{\sigma^*} \subseteq \mathbb{B}[\ell^\infty]$, and hence $\overline{W}^{\sigma^*}$ is also weak-$*$ compact. Proceeding by contradiction, suppose that $(\overline{W}^{\sigma^*}) \setminus c_0 = \emptyset$, i.e., $\overline{W}^{\sigma^*} \subseteq c_0$. Since the weak topology $\sigma(c_0,\ell^1)$ on c_0 is that induced by the weak-$*$ topology of ℓ^∞, it follows that $\overline{W}^{\sigma^*}$ is weakly compact in c_0, and hence so is its closed convex hull $\overline{\mathrm{co}}(\overline{W}^{\sigma^*})$ [28, Theorem 2.8.14]. Choose a sequence $\{\varphi_N\}_{N=1}^\infty \subseteq \mathrm{co}(W) \subseteq \overline{\mathrm{co}}(\overline{W}^{\sigma^*})$ according to Step 1, which then admits a subsequence converging weakly in c_0 (hence also pointwise on $\mathbb{N}$) to some element $\varphi \in c_0$ [28, Theorem 2.8.6]. It follows that $\varphi(n) \ge \delta/2$ for each $n \in \mathbb{N}$ because $\varphi_N(n) > \delta/2$ for $1 \le n \le N$ whenever $N \in \mathbb{N}$. This is impossible as $\varphi \in c_0$. So, we must have $(\overline{W}^{\sigma^*}) \setminus c_0 \ne \emptyset$.

STEP 3. *There exist $c > 0$, a vector $x_0^* \in \mathbb{B}[X^*]$ and an infinite subset $\Delta \subseteq \mathbb{N}$ such that*

$$\sup_{x \in K_n} |\langle x, x_0^* \rangle| \geq c, \qquad \forall n \in \Delta.$$

To see this, use Step 2 to select $\psi \in \overline{W}^{\sigma^*} \setminus c_0$. Since the Banach space ℓ^1 is separable, its dual unit ball $\mathbb{B}[\ell^\infty]$ is *metrizable* for the weak-$*$ topology [28, Corollary 2.6.20]. Recalling that $\overline{W}^{\sigma^*} \subseteq \mathbb{B}[\ell^\infty]$ enables us to choose a sequence $\{\psi_j\}_{j=1}^\infty \subseteq W$ which converges weak-$*$ to ψ. As $\psi \in \ell^\infty \setminus c_0$, there is $c > 0$ such that $\Delta := \{n \in \mathbb{N} : |\psi(n)| > c\}$ is an infinite subset of $\mathbb{N}$. By the definition of W, given $j \in \mathbb{N}$ there exist $\varepsilon_{j,n} \in \{-1, 1\}$, $x_{j,n} \in K_n$ and $x_j^* \in \mathbb{B}[X^*]$ such that

$$\psi_j(n) = \varepsilon_{j,n} |\langle x_{j,n}, x_j^* \rangle|, \qquad \forall n \in \mathbb{N}.$$

Now, the closed subspace $Y := \overline{\operatorname{span}}(\bigcup_{n=1}^\infty K_n)$ of X is separable because each set K_n for $n \in \mathbb{N}$ is compact; this follows routinely from [28, Theorem 1.12.15]. Accordingly, $\mathbb{B}[Y^*]$ is compact and metrizable for the weak-$*$ topology $\sigma(Y^*, Y)$ [28, Theorem 2.6.18 & Corollary 2.6.20]. Since the restrictions $y_j^* := x_j^*|_Y$ for $j \in \mathbb{N}$ belong to $\mathbb{B}[Y^*]$, there is a subsequence $\{y_{j(k)}^*\}_{k=1}^\infty$ of $\{y_j^*\}_{j=1}^\infty$ which admits a weak-$*$ limit $y_0^* \in \mathbb{B}[Y^*]$. In particular, $\lim_{k\to\infty} \langle y, y_{j(k)}^* \rangle = \langle y, y_0^* \rangle$ for all $y \in Y$. We claim that

$$\sup_{y \in K_n} |\langle y, y_0^* \rangle| \geq c, \qquad \forall n \in \Delta. \tag{3.6}$$

Indeed, fix $n \in \Delta$. Then, K_n being also compact in Y, the bounded sequence $\{y_{j(k)}^*\}_{k=1}^\infty \subseteq \mathbb{B}[Y^*]$, which converges pointwise on Y to y_0^*, also converges *uniformly* over K_n to y_0^* [22, Banach Steinhaus Theorem, p. 220]. In other words, the seminorm

$$p_n(y^*) := \sup_{y \in K_n} |\langle y, y^* \rangle|, \qquad \forall y^* \in Y^*,$$

satisfies $\lim_{k\to\infty} p_n(y_{j(k)}^* - y_0^*) = 0$. On the other hand, since the subsequence $\{\psi_{j(k)}\}_{k=1}^\infty$ converges weak-$*$ to ψ, it follows that $\lim_{k\to\infty} |\psi_{j(k)}(n)| = |\psi(n)| > c$ (as $n \in \Delta$). Choose $k_0 \in \mathbb{N}$ such that $|\psi_{j(k)}(n)| > c$ for all $k \geq k_0$. For such k we have, as $x_{j(k),n} \in K_n \subseteq Y$,

$$p_n(y_{j(k)}^*) \geq |\langle x_{j(k),n}, y_{j(k)}^* \rangle| = |\langle x_{j(k),n}, x_{j(k)}^* \rangle| = |\psi_{j(k)}(n)| > 0.$$

This implies that

$$\sup_{y \in K_n} |\langle y, y_0^* \rangle| = p_n(y_0^*) = \lim_{k\to\infty} p_n(y_{j(k)}^*) \geq c.$$

Since $n \in \Delta$ is arbitrary, (3.6) holds.

Now, let $x_0^* \in \mathbb{B}[X^*]$ be any continuous linear extension of $y_0^* \in \mathbb{B}[Y^*]$ to X [28, Theorem 1.9.6]. Then (3.6) establishes Step 3.

The proof of Lemma 3.1 is thereby complete, as Step 3 means precisely that (3.2) holds. ∎

We now require further preparatory results from vector measure theory.

LEMMA 3.2. *Let m be a Banach-space-valued vector measure. For each $f \in L^1(m)$, the subset of $L^1(m)$ given by*

$$f\mathbb{B}[L^\infty(m)] := \{f\psi : \psi \in \mathbb{B}[L^\infty(m)]\} \tag{3.7}$$

is convex and weakly compact.

Proof. Convexity is clear.

The boundedness of $f\mathbb{B}[L^\infty(m)]$ follows from the inequality (cf. (1.2))

$$\|f\psi\|_{L^1(m)} \le \|\psi\|_{L^\infty(m)}\|f\|_{L^1(m)}, \quad \forall \psi \in L^\infty(m). \tag{3.8}$$

Next we show that $f\mathbb{B}[L^\infty(m)]$ is weakly closed in $L^1(m)$; by convexity it suffices to establish its norm-closedness. So, let $\{f\psi_n\}_{n=1}^\infty \subseteq f\mathbb{B}[L^\infty(m)]$ converge in $L^1(m)$ to $g \in L^1(m)$. In view of [38, Proposition 2.2(ii) & Theorem 3.7(iii)] there is a subsequence $\{f\psi_{n(k)}\}_{k=1}^\infty$ such that $f\psi_{n(k)} \to g$ pointwise m-a.e. as $k \to \infty$. Since $|f\psi_{n(k)}| \le \|\psi_{n(k)}\|_{L^\infty(m)}|f| \le |f|$ (m-a.e.) for each $k \in \mathbb{N}$, it follows that $|g| \le |f|$ (m-a.e.). Define the measurable set $A := \{w : f(w) \ne 0\}$, so $f\chi_A = f$, and the function $h \in \mathbb{B}[L^\infty(m)]$ by setting $h := (g/f)\chi_A$. Since

$$\|f\psi_n - fh\|_{L^1(m)} = \|f\psi_n\chi_A - g\chi_A\|_{L^1(m)} \le \|\chi_A\|_{L^\infty(m)}\|f\psi_n - g\|_{L^1(m)}$$

for each $n \in \mathbb{N}$, we conclude that $g = fh$ and so $g \in f\mathbb{B}[L^\infty(m)]$. This shows that $f\mathbb{B}[L^\infty(m)]$ is closed in $L^1(m)$ and hence, as noted, its weak closedness follows.

Let μ be a finite positive measure on Σ satisfying $m \simeq \mu$. It follows from (3.8) and the fact that $\chi_\Omega \in \mathbb{B}[L^\infty(m)]$ that

$$\|f\chi_A\|_{L^1(m)} = \sup\{\|f\chi_A\psi\|_{L^1(m)} : \psi \in \mathbb{B}[L^\infty(m)]\}, \quad \forall A \in \Sigma.$$

But $L^1(m)$ is a σ-order continuous Banach function space (relative to (Ω, Σ, μ)) [38, p. 23 & Theorem 3.7(iii)], and so Lemma 2.37(ii) of [38] yields

$$\lim_{\mu(A)\to 0} \sup_{\psi \in \mathbb{B}[L^\infty(m)]} \|f\psi\chi_A\|_{L^1(m)} = \lim_{\mu(A)\to 0} \|f\chi_A\|_{L^1(m)} = 0, \tag{3.9}$$

i.e., the bounded subset $f\mathbb{B}[L^\infty(m)] \subseteq L^1(m)$ is *uniformly μ-absolutely continuous* [38, p. 56]. It then follows that $f\mathbb{B}[L^\infty(m)]$ is a relatively weakly compact subset of $L^1(m)$ [38, Proposition 2.39(ii)]. Since $f\mathbb{B}[L^\infty(m)]$ is weakly closed, it is actually a weakly compact subset of $L^1(m)$. ∎

LEMMA 3.3. *Let (Ω, Σ) be a measurable space and $m : \Sigma \to X$ be a Banach-space-valued vector measure with relatively compact range. Fix any $f \in L^1(m)$.*

(i) $I_m(f\mathbb{B}[L^\infty(m)])$ *is a compact subset of* X.

(ii) *There exists* $\psi_f \in \mathbb{B}[L^\infty(m)]$ *satisfying* $\|f\|_{L^1(m)} = \|I_m(f\psi_f)\|_X$.

Proof. (i) Let $\mu : \Sigma \to [0,\infty)$ be a scalar measure satisfying $m \simeq \mu$. As shown in the proof of Lemma 3.2, the set $f\mathbb{B}[L^\infty(m)]$ is bounded and uniformly μ-absolutely continuous in $L^1(m)$. This implies that its image $I_m(f\mathbb{B}[L^\infty(m)])$ is relatively compact in X because $\{I_m(\chi_A) : A \in \Sigma\} = m(\Sigma)$ is relatively compact (by assumption); see Proposition 2.41 of [38] with $T := I_m$ there. On the other hand, $f\mathbb{B}[L^\infty(m)]$ is weakly compact in $L^1(m)$ (see Lemma 3.2). Since $I_m : L^1(m) \to X$ is also continuous when both $L^1(m)$ and X are equipped with their respective weak topologies [28, Theorem 2.5.11], it follows that $I_m(f\mathbb{B}[L^\infty(m)])$ is weakly compact in X and, in particular, norm-closed. Being relatively norm-compact in X, it is actually norm-compact.

(ii) The restriction of $\|\cdot\|_X : X \to [0,\infty)$ to the compact subset $I_m(f\mathbb{B}[L^\infty(m)]) \subseteq X$ attains its maximum at $f\psi_f$ for some $\psi_f \in \mathbb{B}[L^\infty(m)]$, i.e.,

$$\|I_m(f\psi_f)\|_X = \sup\{\|I_m(f\psi)\|_X : \psi \in \mathbb{B}[L^\infty(m)]\} = \|f\|_{L^1(m)},$$

where the second equality is known (see the identity (3.60) on p. 132 of [38]). ∎

We recall some facts about a Banach space $(X, \|\cdot\|_X)$ with an *unconditional basis*, say $\{e_n\}_{n=1}^\infty$ [1, Section 3.1], [28, Section 4.2]. Let $\{e_n^*\}_{n=1}^\infty \subseteq X^*$ denote the biorthogonal coordinate functionals associated with $\{e_n\}_{n=1}^\infty$, i.e., $x = \sum_{n=1}^\infty \langle x, e_n^*\rangle e_n$ for $x \in X$, with $\langle e_k, e_n^*\rangle = \delta_{k,n}$ for $k, n \in \mathbb{N}$ [28, Section 4.1 & Corollary 4.1.16]. Define

$$\text{(3.10)} \qquad |\!|\!| x |\!|\!|_X := \sup\Big\{\Big\|\sum_{n=1}^\infty c_n\langle x, e_n^*\rangle e_n\Big\|_X : (c_n)_{n=1}^\infty \in \mathbb{B}[\ell^\infty]\Big\}, \qquad \forall x \in X.$$

Then $|\!|\!| e_n |\!|\!|_X = \|e_n\|_X$ for $n \in \mathbb{N}$. The function $|\!|\!| \cdot |\!|\!|_X : X \to [0,\infty)$ is a norm on X equivalent to $\|\cdot\|_X$, and $|\!|\!| x |\!|\!| \le |\!|\!| y |\!|\!|$ whenever $x, y \in X$ satisfy $|\langle x, e_n^*\rangle| \le |\langle y, e_n^*\rangle|$ for all $n \in \mathbb{N}$ [13, p. 344], [28, pp. 373–375]. It follows from the definition of $|\!|\!| \cdot |\!|\!|_X$ that

$$\text{(3.11)} \qquad \Big|\!\Big|\!\Big| \sum_{n=1}^\infty \langle x, e_n^*\rangle e_n \Big|\!\Big|\!\Big|_X = \Big|\!\Big|\!\Big| \sum_{n=1}^\infty |\langle x, e_n^*\rangle| e_n \Big|\!\Big|\!\Big|_X, \qquad \forall x \in X.$$

Moreover, for arbitrary choices of $\varepsilon_n \in \{0,1\}$, for $n \in \mathbb{N}$, we have

$$\text{(3.12)} \qquad \Big|\!\Big|\!\Big| \sum_{n=1}^\infty \varepsilon_n\langle x, e_n^*\rangle e_n \Big|\!\Big|\!\Big|_X \le \Big|\!\Big|\!\Big| \sum_{n=1}^\infty \langle x, e_n^*\rangle e_n \Big|\!\Big|\!\Big|_X, \qquad \forall x \in X.$$

Note that $\{(|\!|\!| e_n |\!|\!|_X)^{-1} e_n\}_{n=1}^\infty$ is a *normalized* unconditional basis for $(X, |\!|\!| \cdot |\!|\!|_X)$ [28, Corollary 4.2.13]. Henceforth, it is assumed that the norm of X is chosen to be $\|\!|\cdot|\!\|_X$ and that $\{e_n\}_{n=1}^\infty$ is a normalized unconditional basis relative to $\|\!|\cdot|\!\|_X$.

Fix $k \in \mathbb{N}$ and consider the kth natural projection

$$P_k : x \mapsto \sum_{n=1}^{k} \langle x, e_n^* \rangle e_n, \qquad \forall x \in X,$$

necessarily continuous [28, Theorem 4.1.15], of X onto the finite-dimensional subspace $\operatorname{span}(\{e_n\}_{n=1}^k) \subseteq X$. Since P_k is a non-zero projection we always have $\|P_k\|_{\mathrm{op}} \geq 1$, whereas (3.12) then implies that actually $\|P_k\|_{\mathrm{op}} = 1$. By a similar argument, also $\|Q_k\|_{\mathrm{op}} = 1$ where $Q_k := I - P_k$, i.e.,

$$Q_k : x \mapsto \sum_{n=k+1}^{\infty} \langle x, e_n^* \rangle e_n, \qquad \forall x \in X,$$

is the natural projection of X onto its closed subspace $\overline{\operatorname{span}}(\{e_n\}_{n=k+1}^\infty)$.

Lemma 3.4. *Let $(X, \|\cdot\|_X)$ be a Banach space with a normalized unconditional basis $\{e_n\}_{n=1}^\infty$. Equip X with the equivalent norm $\|\!|\cdot|\!\|_X$ given by (3.10) and let P_k, Q_k, for $k \in \mathbb{N}$, be the natural projections associated with $\{e_n\}_{n=1}^\infty$. Let m be any X-valued vector measure, defined on a measurable space (Ω, Σ), whose range is relatively compact in X and which has infinite variation.*

(i) *There exist a strictly increasing sequence $\{k(j)\}_{j=1}^\infty$ in $\mathbb{N}$ and a sequence $\{A(j)\}_{j=1}^\infty \subseteq \Sigma$ of non-m-null sets such that*

$$\sup_{x \in K_j} \|\!|P_{k(j-1)}(x)|\!\|_X \leq \frac{1}{2^j} \quad \textit{and} \quad \sup_{x \in K_j} \|\!|Q_{k(j)}(x)|\!\|_X \leq \frac{1}{2^j}, \qquad \forall j \in \mathbb{N}, \tag{3.13}$$

with $k(0) := 0$ and $P_{k(0)} := 0$, where the compact sets $K_j \neq \emptyset$ are given by

$$K_j := \Big\{ \int_\Omega f_j \psi \, dm : \psi \in \mathbb{B}[L^\infty(m)] \Big\} = I_m\left(f_j \mathbb{B}[L^\infty(m)]\right), \qquad \forall j \in \mathbb{N}, \tag{3.14}$$

with the corresponding non-negative functions $\{f_j\}_{j=1}^\infty \subseteq L^1(m)$ defined by

$$f_j := \left(\|\chi_{A(j)}\|_{L^1(m)}\right)^{-1} \chi_{A(j)}, \qquad \forall j \in \mathbb{N}. \tag{3.15}$$

(ii) *There exists a sequence $\{\psi_j\}_{j=1}^\infty \subseteq \mathbb{B}[L^\infty(m)]$ satisfying*

$$\|\!|I_m(f_j \psi_j)|\!\|_X = \|f_j\|_{L^1(m)} = 1, \qquad \forall j \in \mathbb{N}, \tag{3.16}$$

and also

$$\|\!|I_m(f_j \psi_j) - I_m(f_q \psi_q)|\!\|_X \geq 1/4, \qquad \forall j, q \in \mathbb{N} \textit{ with } j \neq q. \tag{3.17}$$

Proof. (i) Let $\mu : \Sigma \to [0, \infty)$ be a scalar measure satisfying $m \simeq \mu$. Set $A(1) := \Omega$ and then define f_1 by (3.15). The subset $K_1 := I_m(f_1 \mathbb{B}[L^\infty(m)])$ $\subseteq X$ is compact by Lemma 3.3(i). Since $\sup_{k \in \mathbb{N}} \|Q_k\|_{\mathrm{op}} = 1 < \infty$ and

$\|Q_k(x)\|_X \to 0$ as $k \to \infty$, for every $x \in X$, it follows from the Banach–Steinhaus Theorem [22, p. 220], that $Q_k \to 0$ *uniformly* over the compact set K_1, i.e., $\sup_{x\in K_1} \|Q_k(x)\|_X \to 0$ as $k \to \infty$. So, choose $k(1) \in \mathbb{N}$ such that $\sup_{x\in K_1} \|Q_{k(1)}(x)\|_X \le 1/2$. Then (3.13) holds with $j := 1$ as $P_{k(0)} = 0$.

Now assume, for some fixed $N \in \mathbb{N}$, that (3.13) holds for each $j = 1, \dots, N$. Since $\|e_n\|_X = 1$, for $n \in \mathbb{N}$ and for each $x \in X$ we have $\langle P_{k(N)}(x), e_n^*\rangle = \langle x, e_n^*\rangle$ if $1 \le n \le k(N)$ and 0 otherwise, it follows that

$$\|P_{k(N)}(x)\|_X = \Big\| \sum_{n=1}^{k(N)} \langle P_{k(N)}(x), e_n^*\rangle e_n \Big\|_X \le \sum_{n=1}^{k(N)} |\langle P_{k(N)}(x), e_n^*\rangle| \tag{3.18}$$

for each $x \in X$. Let $0 \le \varphi \in L^1(\mu)$ be the Radon–Nikodým derivative of the non-negative scalar measure $\sum_{n=1}^{k(N)} |\langle m, e_n^* \circ P_{k(N)}\rangle|$ with respect to μ. Then there exists a set $A(N+1) \in \Sigma$ such that

$$2^{N+1} \int\limits_{A(N+1)} \varphi\, d\mu < \|m(A(N+1))\|_X \le \|\chi_{A(N+1)}\|_{L^1(m)}. \tag{3.19}$$

Indeed, if the first inequality failed to hold for some $A(N+1) \in \Sigma$, then $\|m(A)\|_X \le 2^{N+1} \int_A \varphi\, d\mu$ for all $A \in \Sigma$, which contradicts $|m|(\Omega) = \infty$. The inequality $\|m(A)\|_X \le \|\chi_A\|_{L^1(m)}$ always holds for every $A \in \Sigma$ [38, (3.21), p. 112]. So, with $A(N+1)$ satisfying (3.19) we can define $f_{N+1} \in L^1(m)$ by (3.15).

Given $\psi \in \mathbb{B}[L^\infty(m)]$, it follows from (3.18) with $x := I_m(f_{N+1}\psi)$ there, the definition of f_{N+1} (cf. (3.15)), and (3.19) that

$$\begin{aligned}
\|(P_{k(N)} \circ I_m)(f_{N+1}\psi)\|_X &\le \sum_{n=1}^{k(N)} |\langle P_{k(N)}(I_m(f_{N+1}\psi)), e_n^*\rangle| \\
&= \sum_{n=1}^{k(N)} \Big|\Big\langle \int\limits_\Omega f_{N+1}\psi\, dm, e_n^* \circ P_{k(N)} \Big\rangle\Big| \le \sum_{n=1}^{k(N)} \int\limits_\Omega f_{N+1}|\psi|\, d|\langle m, e_n^* \circ P_{k(N)}\rangle| \\
&= \int\limits_\Omega f_{N+1}|\psi|\varphi\, d\mu \le \Big(\int\limits_{A(N+1)} \varphi\, d\mu\Big)/\|\chi_{A(N+1)}\|_{L^1(m)} \le 2^{-(N+1)}.
\end{aligned}$$

So, with $K_{N+1} := I_m(f_{N+1}\mathbb{B}[L^\infty(m)])$ we have shown that

$$\sup_{x\in K_{N+1}} \|P_{k(N)}(x)\|_X \le \frac{1}{2^{N+1}}.$$

Next, since $K_{N+1} \subseteq X$ is compact (cf. Lemma 3.3(i)), we can repeat the argument used to produce $k(1)$ to find $k(N+1) \in \mathbb{N}$ with $k(N+1) > k(N)$ such that

$$\sup_{x\in K_{N+1}} \|Q_{k(N+1)}(x)\|_X \le \frac{1}{2^{N+1}}.$$

Accordingly, (3.13) holds for all $j = 1, \dots, N+1$.

(ii) Given $j \in \mathbb{N}$, apply Lemma 3.3(ii) to find $\psi_j \in \mathbb{B}[L^\infty(m)]$ satisfying the first equality in (3.16). The second equality in (3.16) is clear from (3.15).

To verify (3.17), let $j < q$. For ease of notation set $x_j := I_m(f_j\psi_j) \in K_j$ and $x_q := I_m(f_q\psi_q) \in K_q$. Then it follows from (3.13), from the identities $\|P_{k(j)}\|_{\mathrm{op}} = 1$, $P_{k(j)} + Q_{k(j)} = I$ and $P_{k(j)} = P_{k(j)} \circ P_{k(q-1)}$ (as $j < q$ implies that $k(j) \le k(q-1)$), and from (3.16) which yields $\|x_j\|_X = \|f_j\|_{L^1(m)} = 1$, that

$$\begin{aligned}\|x_j - x_q\|_X &\ge \|P_{k(j)}(x_j - x_q)\|_X = \|x_j - Q_{k(j)}(x_j) - P_{k(j)}(x_q)\|_X \\ &\ge \|x_j\|_X - \|Q_{k(j)}(x_j)\| - \|P_{k(j)} \circ P_{k(q-1)}(x_q)\|_X \\ &\ge 1 - \frac{1}{2^j} - \|P_{k(q-1)}(x_q)\|_X \ge 1 - \frac{1}{2^j} - \frac{1}{2^q} \ge \frac{1}{4}.\end{aligned}$$

This is precisely (3.17). ∎

Let us see that the hypotheses on X and m as required in Lemma 3.4 arise in many interesting settings.

Example 3.5. (i) Let $X = \ell^p$ for $1 \le p < 2$, in which case X has an unconditional basis. Moreover, by a result of H. P. Rosenthal, *every* X-valued vector measure has relatively compact range (see Lemma 3.53(v) in [38] and its proof). So, for every X-valued vector measure m of infinite variation, all the hypotheses of Lemma 3.4 are satisfied. We point out that in every infinite-dimensional Banach space there always exist vector measures m of infinite variation, which can even be chosen to be either purely atomic or non-atomic: consider the vector measure m_f constructed in the proof of Proposition 4.4 below.

Or, let X be any infinite-dimensional Banach space with the Schur property. Then the range of *every* X-valued measure, being relatively weakly compact, is also relatively compact. If, in addition, X has an unconditional basis, then again for every X-valued vector measure m of infinite variation all the hypotheses of Lemma 3.4 are fulfilled. It is worth noting that such spaces X exist besides ℓ^1. Indeed, for any sequence $\{p_n\}_{n=1}^\infty \subseteq (1,\infty)$ define

$$\ell^{(p_n)} := \Big\{(x_n)_{n=1}^\infty \in \mathbb{R}^{\mathbb{N}} : \sum_{n=1}^\infty |tx_n|^{p_n} < \infty \text{ for some } t > 0\Big\},$$

equipped with the norm

$$\|(x_n)_{n=1}^\infty\| := \inf\Big\{t > 0 : \sum_{n=1}^\infty |x_n/t|^{p_n} \le 1\Big\}, \quad \forall (x_n)_{n=1}^\infty \in \ell^{(p_n)}.$$

Then $\ell^{(p_n)}$ is a (real) Banach lattice [7, §2], [18], [42], [43]. Moreover, the closed ideal $\ell_a^{(p_n)}$ consisting of all absolutely continuous elements of $\ell^{(p_n)}$ is precisely

$$\ell_a^{(p_n)} = \Big\{(x_n)_{n=1}^{\infty} \in \ell^{(p_n)} : \sum_{n=1}^{\infty} |tx_n|^{p_n} < \infty,\ \forall t \geq 0\Big\}$$

[42, p. 485]. A result of I. Halperin and H. Nakano states that $\ell^{(p_n)}$ has the Schur property iff $\lim_{n\to\infty} p_n = 1$ [18]; see also [43, pp. 1–3]. In this case, also the closed subspace $\ell_a^{(p_n)}$ has the Schur property. Since spaces with the Schur property are hereditarily ℓ^1 [43, p. 4], i.e., every infinite-dimensional closed subspace contains another closed subspace isomorphic to ℓ^1, it follows that $\ell_a^{(p_n)}$ cannot contain a copy of c_0 [2, Theorem 14.21]. Accordingly, Theorem 3.5 of [45] with $M_n(s) := s^{p_n}$ for $s \in [0,\infty)$ and $n \in \mathbb{N}$ (in which case the space $c\{M_n\}$ given there is precisely $\ell_a^{(p_n)}$) shows that $\ell^{(p_n)} = \ell_a^{(p_n)}$ and that the canonical unit vectors form an unconditional basis of $\ell^{(p_n)}$. It follows that $\ell^{(p_n)}$ has the Schur property and possesses an unconditional basis whenever $\lim_{n\to\infty} p_n = 1$. If, in addition, we have $\lim_{n\to\infty} p_n/(p_n-1)\ln(n) = 0$, then $\ell^{(p_n)}$ is *not* isomorphic to ℓ^1 [42, Lemma 4]. For instance, $p_n := 1 + (\ln(n))^{-1/2}$ for $n \geq 2$ satisfies this condition. Actually, this same choice of $\{p_n\}_{n=2}^{\infty}$ also satisfies the condition $1/p_{2n} - 1/p_n \leq a/\ln(n)$ for $n \geq 2$ (with $a = 1$), and hence the canonical unit vectors are actually the *only* unconditional basis (up to equivalence) in $\ell^{(p_n)}$ [7, Theorem 5.8].

(ii) In the notation of Example 2.6, let G be any infinite compact abelian group with dual group Γ and normalized Haar measure μ. Recall the classical Banach algebra $A(G) := \{f \in L^1(G) : \widehat{f} \in \ell^1(\Gamma)\}$ under convolution and equipped with the norm

$$\|f\|_{A(G)} := \|\widehat{f}\|_{\ell^1(\Gamma)} := \sum_{\gamma\in\Gamma} |\widehat{f}(\gamma)|, \qquad \forall f \in A(G).$$

According to [19, Corollary 34.7], the Fourier transform map $f \mapsto \widehat{f}$ is an isometric isomorphism of $(A(G), \|\cdot\|_{A(G)})$ onto $(\ell^1(\Gamma), \|\cdot\|_{\ell^1(\Gamma)})$, and hence $A(G)$ has the Schur property. Moreover, if G is also *metrizable*, then Γ is countable and so the characters $\{(\cdot,\gamma) : \gamma \in \Gamma\}$ form an unconditional basis for $A(G)$. Let $\varphi \in L^2(G)$. Since $L^2(G) * L^2(G) = A(G)$ [19, Corollary 34.16], we can define a finitely additive set function $m_\varphi : \mathcal{B}(G) \to A(G)$ by $m_\varphi : A \mapsto \chi_A * \varphi$ for $A \in \mathcal{B}(G)$. It turns out that m_φ is actually σ-additive [35, Proposition 2.3 & Corollary 3.4]. As $A(G)$ has the Schur property, m_φ necessarily has relatively compact range. However, m_φ has finite variation iff $\varphi \in A(G)$ [35, Theorem 3.8], i.e., m_φ has infinite variation whenever $\varphi \in L^2(G) \setminus A(G)$.

(iii) Let G be as in (ii) above (and metrizable). Recall that μ is necessarily non-atomic (cf. Example 2.6). Since G is a Polish space, for each

$1 < p < \infty$ the Banach space $L^p(G)$ is isometrically isomorphic to $L^p([0,1])$ [1, p. 125]. But $L^p([0,1])$ has an unconditional basis [1, Theorem 6.1.6], and hence so does $L^p(G)$ [28, Proposition 4.2.14]. For each measure $\lambda \in M(G)$, let $m_\lambda^{(p)} : \mathcal{B}(G) \to L^p(G)$ be the vector measure defined in Example 2.6. If $M_0(G) := \{\nu \in M(G) : \widehat{\nu} \in c_0(\Gamma)\}$, where $\widehat{\nu} : \Gamma \to \mathbb{C}$ is the Fourier–Stieltjes transform of ν, i.e., $\widehat{\nu}(\gamma) := \int_G \overline{(x,\gamma)}\, d\nu(x)$ for $\gamma \in \Gamma$, then it is known that the vector measure $m_\lambda^{(p)}$ has relatively compact range in $L^p(G)$ iff $\lambda \in M_0(G)$ [38, Proposition 7.58]. On the other hand, $m_\lambda^{(p)}$ has finite variation iff there exists $h \in L^p(G)$ such that $\lambda(A) = \int_A h\, d\mu$ for $A \in \mathcal{B}(G)$ [38, Theorem 7.67], i.e., $m_\lambda^{(p)}$ has infinite variation whenever $\lambda \in M_0(G) \setminus L^p(G)$.

More generally, let $1 < p \le 2$ and $\psi \in \ell^\infty(\Gamma)$ be any *Fourier p-multiplier* for G, i.e., there exists an operator $T_\psi^{(p)} \in \mathcal{L}(L^p(G))$, necessarily commuting with all translation operators, such that $(T_\psi^{(p)} f)\widehat{\ } = \psi \widehat{f}$ for all $f \in L^p(G)$. The convolution operators $C_\lambda^{(p)}$ $(= T_{\widehat{\lambda}}^{(p)})$ for $\lambda \in M(G)$ form a *proper* subclass of the Fourier p-multiplier operators. For each Fourier p-multiplier $\psi \neq 0$, the set function $m_\psi^{(p)} : A \mapsto T_\psi^{(p)}(\chi_A)$ for $A \in \mathcal{B}(G)$ is a vector measure with $m_\psi^{(p)} \simeq \mu$ [32, Proposition 2.2]. It is known that $m_\psi^{(p)}$ has relatively compact range in $L^p(G)$ precisely when $\psi \in c_0(\Gamma)$ (for $\psi := \widehat{\lambda}$ with $\lambda \in M(G)$ this corresponds to $\lambda \in M_0(G)$) [32, Proposition 2.3], whereas $m_\psi^{(p)}$ has finite variation iff $\psi = \widehat{\lambda}$ for some $\lambda \in L^p(G)$, [32, Proposition 2.8]. For the circle group $G = \mathbb{T}$, we note (for every $1 < p \le 2$) that there exist Fourier p-multipliers $\psi \in c_0(\mathbb{Z})$ which are *not* of the form $\widehat{\lambda}$ for any $\lambda \in M_0(\mathbb{T})$ [32, Remark 2.6(ii)]. In particular, such a p-multiplier ψ cannot be the Fourier–Stieltjes transform of any function from $L^p(\mathbb{T})$.

PROPOSITION 3.6. *Let X be a Banach space with an unconditional basis. If there exists an X-valued vector measure m having infinite variation and satisfying $I_m \in \mathcal{A}_{cc}$, then $\ell^1 \hookrightarrow X$.*

Proof. Let $\{e_n\}_{n=1}^\infty$ be a normalized unconditional basis of X and equip X with the norm $\|\!| \cdot |\!\|_X$ as given by (3.10). Since $I_m \in \mathcal{A}_{cc}$, the range of m is a relatively compact subset of X [38, p. 153]. Let the sequence of non-empty compact sets $\{K_j\}_{j=1}^\infty \subseteq X$ be given by (3.14), the functions $\{f_j\}_{j=1}^\infty \subseteq L^1(m)$ be given by (3.15), and the sequence $\{\psi_j\}_{j=1}^\infty \subseteq \mathbb{B}[L^\infty(m)]$ be as in Lemma 3.4(ii).

STEP 1. *There exists a strictly increasing sequence $\{j(n)\}_{n=1}^\infty \subseteq \mathbb{N}$ such that $\{f_{j(n)}\psi_{j(n)}\}_{n=1}^\infty$ is a basic sequence in $L^1(m)$ which is equivalent to the* *canonical basis of* ℓ^1*. In particular, there exists* $\delta > 0$ *such that*

$$\delta \sum_{n=1}^{N} |a_n| \le \Big\| \sum_{n=1}^{N} a_n f_{j(n)} \psi_{j(n)} \Big\|_{L^1(m)} \tag{3.20}$$

for all choices of $N \in \mathbb{N}$ *and* $\{a_n : n = 1, \ldots, N\} \subseteq \mathbb{C}$.

To see this, first observe that (3.8) and (3.16) imply that $\|f_j \psi_j\|_{L^1(m)} \le 1$ for all $j \in \mathbb{N}$. Moreover, (3.17) shows that $\{f_j \psi_j\}_{j=1}^{\infty} \subseteq L^1(m)$ cannot contain any weak Cauchy subsequences because the completely continuous operator I_m maps such subsequences of $L^1(m)$ to norm-convergent sequences in X. So, a result of H. P. Rosenthal [2, Theorem 14.24] establishes Step 1.

Step 2. *With* $\{j(n)\}_{n=1}^{\infty} \subseteq \mathbb{N}$ *as in Step* 1*, there exists* $x_0^* \in \mathbb{B}[X^*]$ *such that*

$$\varepsilon := \limsup_{n\to\infty} \big(\sup_{x \in K_{j(n)}} |\langle x, x_0^* \rangle| \big) > 0. \tag{3.21}$$

Indeed, it follows from $\|I_m\|_{\mathrm{op}} = 1$, the definition of $K_{j(n)}$ (cf. (3.14)), and (3.16) that $K_{j(n)} \subseteq \mathbb{B}[X]$ for $n \in \mathbb{N}$. Fix $(a_n)_{n=1}^{\infty} \in \mathbb{C}^{\mathbb{N}}$. Given $N \in \mathbb{N}$, Lemma 3.3(ii) with the m-integrable function $\sum_{n=1}^{N} a_n f_{j(n)} \psi_{j(n)}$ in place of f guarantees the existence of $\psi \in \mathbb{B}[L^\infty(m)]$ satisfying

$$\Big\| \sum_{n=1}^{N} a_n f_{j(n)} \psi_{j(n)} \Big\|_{L^1(m)} = \Big\| I_m \Big(\sum_{n=1}^{N} a_n f_{j(n)} \psi_{j(n)} \psi \Big) \Big\|_X. \tag{3.22}$$

Since $\{\psi_{j(n)} \psi\}_{n=1}^{\infty} \subseteq \mathbb{B}[L^\infty(m)]$, we have $I_m(f_{j(n)} \psi_{j(n)} \psi) \in K_{j(n)}$ for $n = 1, \ldots, N$. It then follows from (3.20) and (3.22) that

$$\begin{aligned}\delta \sum_{n=1}^{N} |a_n| &\le \Big\| \sum_{n=1}^{N} a_n I_m(f_{j(n)} \psi_{j(n)} \psi) \Big\|_X \\ &\le \sup \Big\{ \Big\| \sum_{n=1}^{N} a_n x_n \Big\|_X : x_n \in K_{j(n)},\ n = 1, \ldots, N \Big\}.\end{aligned}$$

This shows that (3.1) in the statement of Lemma 3.1 holds for the sequence $\{K_{j(n)}\}_{n=1}^{\infty}$ of non-empty compact sets. Hence, (3.2) yields (3.22), i.e., Step 2 is valid.

Step 3. *Let* $R_n := P_{k(j(n))} - P_{k(j(n)-1)}$ *for* $n \in \mathbb{N}$*, with* $\{j(n)\}_{n=1}^{\infty}$ *as in Step* 1*, and let* $x_0^* \in \mathbb{B}[X^*]$ *satisfy* (3.21)*. Then there exist an infinite subset* $\Delta \subseteq \mathbb{N}$ *and vectors* $y_n \in R_n(K_{j(n)})$ *for* $n \in \mathbb{N}$ *such that*

$$\varepsilon/3 < |\langle y_n, x_0^* \rangle|, \qquad \forall n \in \Delta.$$

To see this first observe, for each $n \in \mathbb{N}$, that

$$R_n(x) = \sum_{i=k(j(n)-1)+1}^{k(j(n))} \langle x, e_i^* \rangle e_i, \quad \forall x \in X, \tag{3.23}$$

and hence via (3.12) it follows that $\|R_n\|_{\mathrm{op}} = 1$. Apply Step 2 to obtain an infinite subset $\Delta_0 \subseteq \mathbb{N}$ such that

$$\varepsilon/2 < \sup_{x \in K_{j(n)}} |\langle x, x_0^* \rangle|, \quad \forall n \in \Delta_0.$$

Fix $n \in \Delta_0$. Select $x_n \in K_{j(n)}$ such that $|\langle x_n, x_0^* \rangle| > \varepsilon/2$ and observe that

$$x_n = P_{k(j(n)-1)}(x_n) + R_n(x_n) + Q_{k(j(n))}(x_n).$$

So, apply (3.13) with $j(n)$ in place of j to obtain

$$\begin{aligned} |\langle x_n - R_n(x_n), x_0^* \rangle| &\le \|x_n - R_n(x_n)\|_X \\ &\le \|P_{k(j(n)-1)}(x_n)\|_X + \|Q_{k(j(n))}(x_n)\|_X \\ &\le \frac{1}{2^{j(n)}} + \frac{1}{2^{j(n)}} = \frac{1}{2^{j(n)-1}}, \end{aligned}$$

which implies (as $x_n \in K_{j(n)}$) that

$$\begin{aligned} |\langle R_n(x_n), x_0^* \rangle| &\ge |\langle x_n, x_0^* \rangle| - |\langle x_n - R_n(x_n), x_0^* \rangle| \\ &\ge |\langle x_n, x_0^* \rangle| - \frac{1}{2^{j(n)-1}} > \frac{\varepsilon}{2} - \frac{1}{2^{j(n)-1}}. \end{aligned}$$

In view of this inequality, which is valid for each $n \in \Delta_0$, there is an infinite subset $\Delta \subseteq \Delta_0$ such that

$$\varepsilon/3 < |\langle R_n(x_n), x_0^* \rangle|, \quad \forall n \in \Delta. \tag{3.24}$$

So, with $y_n := R_n(x_n)$ for $n \in \Delta$, we have established Step 3.

STEP 4. *Let Δ be as in Step 3 and $\{n(q) : q \in \mathbb{N}\}$ be an enumeration of Δ with $\{n(q)\}_{q=1}^\infty$ a strictly increasing sequence in $\mathbb{N}$. Then $\{y_{n(q)}\}_{q=1}^\infty$ is a basic sequence in X which is equivalent to the canonical basis of ℓ^1.*

Indeed, since (3.23) holds for $n(q)$ in place of n with $y_{n(q)} = R_{n(q)}(x_{n(q)}) \ne 0$ (because of (3.24)), the vectors $y_{n(q)}$ for $q \in \mathbb{N}$ form a *block basic sequence* taken from $\{e_n\}_{n=1}^\infty$ [28, Definition 4.3.15]. In particular, $\{y_{n(q)}\}_{q=1}^\infty$ is an unconditional basic sequence in X [28, p. 398, Ex. 4.39], i.e., $\{y_{n(q)}\}_{q=1}^\infty$ is an unconditional basis for the closed subspace $Y := \overline{\mathrm{span}}(\{y_{n(q)}\}_{q=1}^\infty)$ of X. So, there exist positive constants α, β and a norm $\|\cdot\|$ in Y satisfying

$$\alpha \|y\| \le \|y\|_X \le \beta \|y\|, \quad \forall y \in Y,$$

and with the property that

$$\Big\| \sum_{q=1}^\infty c_q y_{n(q)} \Big\| = \Big\| \sum_{q=1}^\infty |c_q| y_{n(q)} \Big\|, \quad \forall y = \sum_{q=1}^\infty c_q y_{n(q)} \in Y. \tag{3.25}$$

Given $N \in \mathbb{N}$ and $\{a_q\}_{q=1}^{N} \subseteq \mathbb{C}$ we claim that

$$\frac{\varepsilon}{3}\sum_{q=1}^{N}|a_q| \le \frac{\beta}{\alpha}\Big\|\Big\|\sum_{q=1}^{N} a_q y_{n(q)}\Big\|\Big\|_X \le \frac{\beta}{\alpha}\sum_{q=1}^{N}|a_q|. \tag{3.26}$$

In fact, for each $q = 1,\dots,N$, we have $\langle y_{n(q)}, x_0^*\rangle \neq 0$ (cf. (3.24)) and so we can define $b_q := |\langle y_{n(q)}, x_0^*\rangle|/\langle y_{n(q)}, x_0^*\rangle$, in which case $|b_q| = 1$ and $|\langle y_{n(q)}, x_0^*\rangle| = \langle b_q y_{n(q)}, x_0^*\rangle$. It then follows from (3.25) and Step 3 that

$$\begin{aligned}\frac{\varepsilon}{3}\sum_{q=1}^{N}|a_q| &\le \sum_{q=1}^{N}|a_q|\cdot|\langle y_{n(q)}, x_0^*\rangle| = \sum_{q=1}^{N}|a_q|\langle b_q y_{n(q)}, x_0^*\rangle \\ &= \Big\langle \sum_{q=1}^{N}|a_q| b_q y_{n(q)}, x_0^*\Big\rangle \le \beta\,\Big|\sum_{q=1}^{N}|a_q| b_q y_{n(q)}\Big| = \beta\,\Big|\sum_{q=1}^{N}|a_q|\cdot|b_q| y_{n(q)}\Big| \\ &= \beta\,\Big|\sum_{q=1}^{N} a_q y_{n(q)}\Big| \le \frac{\beta}{\alpha}\Big\|\Big\|\sum_{q=1}^{N} a_q y_{n(q)}\Big\|\Big\|_X.\end{aligned}$$

So, the first inequality in (3.26) is valid. The second inequality in (3.26) is a consequence of $|||y_{n(q)}|||_X = |||R_{n(q)}(x_{n(q)})|||_X \le \|R_{n(q)}\|_{\mathrm{op}}|||x_{n(q)}|||_X$ together with $\|R_{n(q)}\|_{\mathrm{op}} = 1$ and $|||x_{n(q)}|||_X \le 1$ as $x_{n(q)} \in K_{j(n(q))} \subseteq \mathbb{B}[X]$, for $q \in \mathbb{N}$.

Step 4 is now immediate from (3.26) (see also [28, Theorem 4.3.6]) as $|||y_{n(q)}|||_X \le 1$ for $q \in \mathbb{N}$.

Finally, Step 4 implies that $\ell^1 \hookrightarrow X$ [28, Theorem 4.3.17], which completes the proof of Proposition 3.6. ■

Proof of Theorem 1.2. Let X be a Banach space with an unconditional basis and such that $\ell^1 \not\hookrightarrow X$. If m is any X-valued vector measure with $I_m \in \mathcal{A}_{cc}$, then Proposition 3.6 implies that m must have finite variation, i.e., X is $\mathcal{A}_{cc}$-variation admissible. By Proposition 1.1 we have $L^1(m) = L^1(|m|)$. ■

As an application, for each $1 < r < \infty$ consider the Volterra vector measure $m_r : \mathcal{B}([0,1]) \to L^r([0,1])$ (see Section 2). As the Banach space $L^r([0,1])$ is reflexive, we surely have $\ell^1 \not\hookrightarrow L^r([0,1])$. Moreover, it was observed in Example 3.5(iii) that $L^r([0,1])$ has an unconditional basis. Since m_r has finite variation *but* $L^1(|m_r|) \subsetneq L^1(m_r)$ (cf. Section 2), Theorem 1.2 implies that the integration map I_{m_r} fails to be completely continuous. An alternative proof (rather non-trivial) of this fact can be found in [38, pp. 154–157].

We end this section with a

QUESTION. Does there exist a Banach space X with $\ell^1 \not\hookrightarrow X$ such that X is not $\mathcal{A}_{cc}$-variation admissible? Of course, X could not have an unconditional basis.

4. Theorem 1.3 and related results. In this final section we establish Theorem 1.3 and present some related results and relevant examples.

Proof of Theorem 1.3. (i)⇒(ii). Let X be a Banach space with the CRP and $m : \Sigma \to X$ be any vector measure satisfying $L^1(m) = L^1(|m|)$. Then m has finite variation, and hence its range $m(\Sigma)$ is relatively compact in X. Since $\{I_m(\chi_A) : A \in \Sigma\} = m(\Sigma)$, it follows that $I_m \in \mathcal{A}_{cc}$ [38, Corollary 2.42].

(ii)⇒(i). Proceeding via a contrapositive argument suppose that X fails the CRP, in which case there exists a vector measure $\nu : \Sigma \to X$ with *finite variation* such that $\nu(\Sigma)$ is *not* relatively compact in X.

Fix $u \in X \setminus \{0\}$ with $\|u\|_X = 1$ and choose $x^* \in X^*$ such that $\langle u, x^*\rangle = 1$. Then $X = \mathbb{C}u \oplus Y$ with $Y := \mathrm{Ker}(x^*)$. Let P be any continuous projection of X onto Y, in which case $\eta := P \circ \nu$ is a Y-valued vector measure on Σ whose range $\eta(\Sigma)$ is not relatively compact. Since

$$\|\eta(A)\|_Y \le \|P\|_{\mathrm{op}}\|\nu(A)\|_X \le \|P\|_{\mathrm{op}}|\nu|(A), \quad \forall A \in \Sigma,$$

it is clear that η has finite variation and satisfies $|\eta|(A) \le \|P\|_{\mathrm{op}}|\nu|(A)$ for $A \in \Sigma$. Define the vector measure $m : \Sigma \to X$ by

$$m(A) := |\eta|(A)u + \eta(A), \quad \forall A \in \Sigma. \tag{4.1}$$

Then $\|m(A)\|_X \le 2|\eta|(A)$ for $A \in \Sigma$ implies that m has finite variation and satisfies $|m|(A) \le 2|\eta|(A)$ for $A \in \Sigma$. Moreover,

$$\langle m, x^*\rangle(A) = |\eta|(A)\langle u, x^*\rangle + \langle \eta(A), x^*\rangle = |\eta|(A), \quad \forall A \in \Sigma,$$

as $\eta(\Sigma) \subseteq Y$. Accordingly, $|m| \le 2|\eta| = 2|\langle m, x^*\rangle|$ setwise on Σ and so x^* is a *Rybakov functional* for m [14, Ch. IX, §2], [38, p. 108]. In particular, $L^1(m) = L^1(|m|)$ [38, Corollary 3.19(i)].

In view of the fact that $I_m \in \mathcal{A}_{cc}$ iff $\{I_m(\chi_A) : A \in \Sigma\} = m(\Sigma) \subseteq X$ is relatively compact [38, Corollary 2.42], it remains to check that $m(\Sigma)$ is *not* relatively compact in X. But $\|\,|\eta|(A)u\|_X \le |\eta|(\Omega)$ for all $A \in \Sigma$ and so $\{|\eta|(A)u : A \in \Sigma\}$ is contained in a compact subset of the 1-dimensional space $\mathbb{C}u$. It then follows from (4.1) and the fact that $\eta(\Sigma)$ is not relatively compact in Y that $m(\Sigma)$ indeed fails to be relatively compact in X. ∎

REMARK 4.1. A Banach space X has the CRP iff $\mathcal{L}(L^1([0,1]), X) \subseteq \mathcal{A}_{cc}$. This is stated in [41, Ch. 7]; a proof can be found in [16].

A local version of Theorem 1.3 is also available for an individual vector measure. Given an X-valued vector measure m let X_m denote the closed subspace of X generated by the range of m. Since the simple functions are dense in $L^1(m)$, it follows that X_m is also the closure in X of the range $I_m(L^1(m))$ of I_m. Observe that X_m is weakly compactly generated [14, Ch. I, Corollary 2.7], and hence X_m has the WRNP iff it has the RNP (cf. Section 1).

Proposition 4.2. *Let X be a Banach space and $m : \Sigma \to X$ be a vector measure such that $L^1(m) = L^1(|m|)$. If X_m has the* CRP, *then $I_m \in \mathcal{A}_{cc}$.*

Proof. Let $\widetilde{m} : \Sigma \to X_m$ be the vector measure $\widetilde{m} : A \mapsto m(A)$ for $A \in \Sigma$, and $j : X_m \to X$ be the identity imbedding. It follows from [38, Theorem 3.5] that $L^1(m) = L^1(\widetilde{m})$. Since $|\widetilde{m}| = |m|$ is a finite measure, we can apply Theorem 1.3 to $\widetilde{m}$ in X_m to conclude that $I_{\widetilde{m}} \in \mathcal{A}_{cc}$. Hence, also $I_m = j \circ I_{\widetilde{m}} \in \mathcal{A}_{cc}$. ∎

Remark 4.3. (i) If $\ell^1 \not\hookrightarrow X$ and m is any X^*-valued vector measure satisfying $L^1(m) = L^1(|m|)$, then $I_m \in \mathcal{A}_{cc}$. This follows from Theorem 1.3 and the fact that X^* has the CRP (cf. Section 1).

(ii) Let m be any *purely atomic* vector measure with *finite variation.* If $L^1(m) = L^1(|m|)$, then $I_m \in \mathcal{A}_{cc}$. Indeed, m necessarily has compact range [21, Theorem 10], and so [38, Corollary 2.42] implies that $I_m \in \mathcal{A}_{cc}$.

The converse is false. To see this, let X be any infinite-dimensional Banach space with the Schur property. Then $I_m \in \mathcal{A}_{cc}$ for *every* X-valued vector measure m. On the other hand, Proposition 4.4 below shows that there always exists a purely atomic, X-valued vector measure m with finite variation such that $L^1(|m|) \subsetneq L^1(m)$. Many examples of Schur spaces (and their properties) occur in [43]; see also the references. Every Banach lattice with the Schur property has the RNP [43, Theorem 5]. Of course, Banach spaces with the Schur property always have the CRP.

Proposition 4.4. *In every infinite-dimensional Banach space there exists a vector measure m with finite variation such that $L^1(|m|) \subsetneq L^1(m)$. Moreover, m can be chosen to be purely atomic or to be non-atomic.*

Proof. Let X be an infinite-dimensional Banach space and $\sum_{n=1}^{\infty} x_n$ be any unconditionally convergent series in X which is not absolutely convergent [13, Theorem 1.2]. Let $\mu : \Sigma \to [0, \infty)$ be any measure for which there exists a sequence $\{A(n)\}_{n=1}^{\infty} \subseteq \Sigma$ of pairwise disjoint sets with $\mu(A(n)) > 0$ for $n \in \mathbb{N}$. Define $m : \Sigma \to X$ by

$$m(A) := \sum_{n=1}^{\infty} \mu(A \cap A(n)) x_n, \qquad \forall A \in \Sigma.$$

The Vitali–Hahn–Saks Theorem [14, Ch. I, Corollary 5.6] ensures that m is σ-additive. Moreover, $\|m(A)\|_X \le (\sup_{n\in\mathbb{N}} \|x_n\|_X) \cdot \mu(A)$ for $A \in \Sigma$, so that m has finite variation. Now, the function $f := \sum_{n=1}^{\infty} \frac{1}{\mu(A(n))} \chi_{A(n)}$ is m-integrable with

$$\int_A f\, dm = \sum_{n=1}^{\infty} \frac{\mu(A \cap A(n))}{\mu(A(n))} \cdot x_n, \qquad \forall A \in \Sigma.$$

However, in the notation of (2.1), we have

$$|m_f|(\Omega) \geq \sum_{n=1}^{\infty} |m_f|(A(n)) \geq \sum_{n=1}^{\infty} \|m_f(A(n))\|_X = \sum_{n=1}^{\infty} \|x_n\|_X = \infty,$$

and hence $f \in L^1(m) \setminus L^1(|m|)$ (see Lemma 2.1(i)).

Finally, m is purely atomic (resp. non-atomic) iff μ is purely atomic (resp. non-atomic). ∎

Recall that a Banach-space-valued vector measure m is called *σ-decomposable* if there exist (countably) infinitely many pairwise disjoint, non-m-null sets [38, p. 129].

Proposition 4.5. *Let m be any Banach-space-valued vector measure.*

(i) *If $I_m \in \mathcal{A}_{cc}$, then $c_0 \not\hookrightarrow L^1(m)$.*
(ii) *If m is σ-decomposable and $I_m \in \mathcal{A}_{cc}$, then $\ell^1 \hookrightarrow L^1(m)$.*

Proof. (i) According to [6, Theorem 3.6] and [38, Proposition 3.38(I)], $I_m \in \mathcal{A}_{cc}$ implies that the Banach lattice $L^1(m)$ is weakly sequentially complete. Hence, $c_0 \not\hookrightarrow L^1(m)$ [2, Theorem 14.12].

(ii) Since $L^1(m)$ is weakly sequentially complete (cf. proof of part (i)), it follows from Rosenthal's Theorem [2, p. 247] that either $L^1(m)$ is reflexive or $\ell^1 \hookrightarrow L^1(m)$.

Suppose that $L^1(m)$ is reflexive. It then follows that $I_m : L^1(m) \to X$, being already completely continuous, is compact, and so $L^1(m) = L^1(|m|)$ (see Section 1). Let $\{A(n)\}_{n=1}^{\infty}$ be any sequence of measurable, pairwise disjoint, non-m-null sets. Then each function $\varphi_n := (|m|(A(n)))^{-1}\chi_{A(n)}$ belongs to $L^1(|m|)$ with $\|\varphi_n\|_{L^1(|m|)} = 1$ for $n \in \mathbb{N}$. It is routine to check that the linear map $u \mapsto \sum_{n=1}^{\infty} u_n\varphi_n$ for $u = (u_n)_{n=1}^{\infty} \in \ell^1$ is a bicontinuous linear isomorphism of ℓ^1 onto a closed subspace of $L^1(|m|)$. This contradicts the reflexivity of $L^1(|m|) = L^1(m)$. Hence, $L^1(m)$ is not reflexive, i.e., $\ell^1 \hookrightarrow L^1(m)$. ∎

Remark 4.6. (i) The analogue of Proposition 4.5(ii) with $\mathcal{A}_c$ in place of $\mathcal{A}_{cc}$ is known [10, Claim, p. 3800].

(ii) Let $m : \Sigma \to X$ be a σ-decomposable, Banach-space-valued vector measure. According to Proposition 4.5(ii) we have $I_m \notin \mathcal{A}_{cc}$ whenever $\ell^1 \not\hookrightarrow L^1(m)$ (equivalently, whenever the Banach lattice $L^1(m)^*$ has the RNP; see Section 1). The condition $\ell^1 \not\hookrightarrow L^1(m)$ has some useful consequences. For instance, it implies that the ideal in the dual Banach lattice $L^1(m)^*$ which is generated by the family of continuous linear functionals $\varphi_{x^*,A} : f \mapsto \int_A f\,d\langle m, x^*\rangle$ for $f \in L^1(m)$, for all $x^* \in X^*$ and $A \in \Sigma$ (cf. (1.2)), is *dense* in $L^1(m)^*$. This in turn implies that weak convergence of bounded nets in $L^1(m)$ is characterized by weak convergence (in X) of the integrals over arbitrary sets, i.e., if $\sup_\alpha \|f_\alpha\|_{L^1(m)} < \infty$, then $\lim_\alpha f_\alpha = f$ weakly in $L^1(m)$ iff $\lim_\alpha I_m(f_\alpha\chi_A) = I_m(f\chi_A)$ weakly in X for every $A \in \Sigma$ [9, Theorem 4].

(iii) The converse of Proposition 4.5(ii) is false. Let $X = \ell^2$. Section 6 of [10] exhibits a vector measure $m : \Sigma \to X$ (denoted there by ν) and a bounded basic sequence $\{f_n\}_{n=1}^\infty$ in $L^1(m)$, equivalent to the canonical basis of ℓ^1, such that

$$\lim_{n\to\infty} \varphi_{x^*,A}(f_n) = 0, \qquad \forall x^* \in X^*,\ A \in \Sigma. \tag{4.2}$$

In particular, $\ell^1 \hookrightarrow L^1(m)$ and $\{f_n\}_{n=1}^\infty$ is not weakly convergent to zero in $L^1(m)$. If $I_m \in \mathcal{A}_{cc}$, then $m(\Sigma)$ is relatively compact in X [38, p. 153]. Hence, (4.2) and [31, Proposition 17] imply that $\{f_n\}_{n=1}^\infty$ converges weakly to zero in $L^1(m)$; contradiction! So, $I_m \notin \mathcal{A}_{cc}$.

(iv) Since $L^1(m)$ is a Banach lattice with order continuous norm [38, Theorem 3.7(iii)], it is known that $\ell^1 \not\hookrightarrow L^1(m)$ iff $c_0 \not\hookrightarrow L^1(m)^*$ [2, p. 246, Ex. 13].

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