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The Lebesgue constant for the periodic Franklin system

by

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Abstract. We identify the torus with the unit interval $[0,1)$ and let $n,\nu \in \mathbb{N}$ with $0 \le \nu \le n-1$ and $N := n+\nu$. Then we define the (partially equally spaced) knots

$$t_j = \begin{cases} j/(2n) & \text{for } j = 0,\dots,2\nu, \\ (j-\nu)/n & \text{for } j = 2\nu+1,\dots,N-1. \end{cases}$$

Furthermore, given n,ν we let $V_{n,\nu}$ be the space of piecewise linear continuous functions on the torus with knots $\{t_j : 0 \le j \le N-1\}$. Finally, let $P_{n,\nu}$ be the orthogonal projection operator from $L^2([0,1))$ onto $V_{n,\nu}$. The main result is

$$\lim_{n\to\infty,\,\nu=1} \|P_{n,\nu} : L^\infty \to L^\infty\| = \sup_{n\in\mathbb{N},\,0\le\nu\le n} \|P_{n,\nu} : L^\infty \to L^\infty\| = 2 + \frac{33-18\sqrt{3}}{13}.$$

This shows in particular that the Lebesgue constant of the classical Franklin orthonormal system on the torus is $2 + \frac{33-18\sqrt{3}}{13}$.

1. Introduction. Let $(N_k)_{k\ge 0}$ be an orthonormal basis in $L^2[0,1]$. The Fourier partial sums with respect to this basis are given by

$$P_N(f) = \sum_{k=0}^{N} \langle f, N_k\rangle N_k. \tag{1.1}$$

Clearly, every P_N is a projection onto its (finite-dimensional) range and its norm as an operator from $L^\infty[0,1]$ to $L^\infty[0,1]$ (or as an operator from $L^1[0,1]$ to $L^1[0,1]$) is given by

$$L_N = \operatorname*{ess\,sup}_{s\in[0,1]} \int_0^1 |K_N(s,t)|\,dt,$$

where K_N is the *Dirichlet kernel*

$$K_N(s,t) = \sum_{k=0}^{N} N_k(s)N_k(t).$$

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The *Lebesgue constant* of the basis $(N_k)_{k\geq 0}$ is now defined as

$$L := \sup_{N\geq 0} L_N.$$

As a particular instance of an orthonormal basis in $L^2[0,1]$, we consider the *general Franklin system* $(N_k)_{k\geq 0}$ on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$, that is, we choose a sequence $\mathcal{T} = (t_k)_{k\geq 0}$ of points in $[0,1)$ (we identify this interval with the torus) which is dense in $[0,1)$ and with $t_0 = 0$. The space of piecewise linear and continuous functions on $\mathbb{T}$ with knots $\{t_0,\dots,t_N\}$ is denoted by $V_N(\mathcal{T})$. Then we define $f_0 \equiv 1$ on $\mathbb{T}$ and inductively, for $k \geq 1$, the kth *Franklin function* corresponding to the sequence $\mathcal{T}$ is uniquely determined by the conditions

$$f_k \in V_k(\mathcal{T}), \quad f_k \perp V_{k-1}(\mathcal{T}), \quad \|f_k\|_2 = 1, \quad f_k(t_k) > 0.$$

The Franklin functions f_k are splines of degree $d = 1$. We now make a few comments about the history of calculating or estimating the Lebesgue constant of splines of degree d.

For $d = 0$ (piecewise constant functions), the projection is easily calculated and the Lebesgue constant is 1.

For $d = 1$ (piecewise linear functions), Z. Ciesielski ([2]) proved that for any partition π of $[0,1]$, the L^∞-norm of the projection onto piecewise linear functions with knots π is ≤ 3. He showed this for the non-periodic case, but exactly the same argument gives the upper bound 3 in the periodic case. Moreover, P. Oswald ([15]) and K. Oskolkov ([14]) proved independently that in the non-periodic case, the constant 3 is optimal if one considers arbitrary partitions π. Moreover, Ciesielski ([5]) showed that in the case of uniform partitions the exact upper bound is 2. Some numerical experiments suggested that for the (classical, corresponding to dyadic knots) non-periodic Franklin system, the exact upper bound is $2 + (2-\sqrt{3})^2$ ([7]). Several years later, P. Bechler ([1]) proved that for the piecewise linear Strömberg wavelet, the Lebesgue constant is indeed $2 + (2-\sqrt{3})^2$. Then Z. Ciesielski and A. Kamont ([6]) showed that for the classical non-periodic Franklin system, the Lebesgue constant is $2 + (2-\sqrt{3})^2$, verifying the conjecture in [7].

For splines of higher degree ($d \geq 2$), a problem was the mere existence of a bound C_d for the L^∞-norms of orthogonal projections onto splines of degree d with arbitrary knots, where C_d depends only on d and not on the partition. This was a long-standing conjecture by C. de Boor solved by A. Yu. Shadrin in [16] (in the non-periodic case). Predating Shadrin's result, there were several results for specific degrees (for instance [8] for $d = 2$ in the non-periodic case) or specific sequences of points (for instance [9] and [10], considering the sequence of dyadic partitions both in the non-periodic and periodic case respectively for arbitrary degree d). In the periodic case, there is a further partial result in [13] showing the existence of a bound C_2 for the L^∞-norm of orthogonal projections for $d = 2$ not depending on the knots. The exact values of the Lebesgue constants in the cases $d \geq 2$ are not known.

In the present paper, we determine the Lebesgue constant for the periodic (classical) Franklin system (corresponding to $d = 1$) to be $2 + \frac{33-18\sqrt{3}}{13}$. Our analysis was constantly guided by extensive computer simulations (both numerical and symbolic) involving the Gram matrix and its inverse (see Section 3.1).

2. Formulation of the Main Theorem. Our main result concerns partially equally spaced knots on the torus $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. We choose the special points

$$t_j = \begin{cases} j/(2n) & \text{for } j = 0, \ldots, 2\nu, \\ (j-\nu)/n & \text{for } j = 2\nu+1, \ldots, N-1, \end{cases} \tag{2.1}$$

for arbitrary $n, \nu \in \mathbb{N}$ with $0 \leq \nu \leq n-1$ and $N := n + \nu$. We remark that for $\nu = 0$ or $\nu = n$ we arrive at equally spaced knots. Let $V_{n,\nu}$ be the linear subspace generated by the piecewise linear, continuous functions with knots (2.1) and $P_{n,\nu}$ be the orthogonal projection onto $V_{n,\nu}$. The B-spline basis for $V_{n,\nu}$ with a special choice of parameters n, ν is pictured in Figure 1.

The main theorem now reads as follows:

Main Theorem 2.1. *For all $n \in \mathbb{N}$ and $0 \leq \nu \leq n$, we have the following bound for the norm of the projection operator $P_{n,\nu}$ onto $V_{n,\nu}$:*

$$\|P_{n,\nu}\|_\infty := \|P_{n,\nu} : L^\infty(\mathbb{T}) \to L^\infty(\mathbb{T})\| < 2 + \frac{33-18\sqrt{3}}{13} =: \gamma.$$

Furthermore, for $n \to \infty$ and $\nu = 1$,

$$\lim_{n\to\infty} \|P_{n,1}\|_\infty = \gamma.$$

3. Preliminaries

3.1. Orthogonal projections. Let V be an N-dimensional subspace of $L^2[0,1]$ and $\{N_0, \ldots, N_{N-1}\}$ a basis of V. We first look at the changes in formula (1.1) if the basis functions are no longer orthogonal. In this case, the orthogonal projection P onto V is given by

$$Pf(s) = \sum_{j,k=0}^{N-1} a_{jk} \langle N_k, f\rangle N_j(s),$$

or as an integral operator with kernel $k(s,t) = \sum_{j,k=0}^{N-1} a_{jk} N_j(s) N_k(t)$:

$$Pf(s) = \int_0^1 k(s,t) f(t)\, dt,$$

where (a_{jk}) is the inverse of the Gram matrix (b_{jk}) with $b_{jk} = \langle N_j, N_k \rangle$. The norm of P as a mapping from $L^\infty[0,1]$ to $L^\infty[0,1]$ is

$$\|P\|_\infty = \operatorname*{ess\,sup}_{s\in[0,1]} \int_0^1 |k(s,t)|\, dt. \tag{3.1}$$

Since P is self-adjoint, the norm of P as an operator from $L^1[0,1]$ to $L^1[0,1]$ is the same.

We now consider periodic B-splines of degree one on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. For this let $0 = t_0 < t_1 < \cdots < t_{N-1} < 1$ with an arbitrary natural number $N \geq 2$. Further set $t_{-1} := t_{N-1} - 1$, $t_N := 1$ and $\delta_j := t_{j+1} - t_j$ for $-1 \leq j \leq N-1$. Then we let N_j for $0 \leq j \leq N-1$ be the unique continuous function on $\mathbb{T}$ which is linear on every interval (t_{k-1}, t_k) and has values $N_j(t_k) = \delta_{j,k}$ for $0 \leq k \leq N-1$. Formally we define the functions $N_j : \mathbb{T} \to [0,1]$ for $0 \leq j \leq N-1$ as

$$N_j([t]) := \begin{cases} (s - t_{j-1})/\delta_{j-1} & \text{if } [t] = [s] \text{ for } t_{j-1} < s \leq t_j, \\ (t_{j+1} - s)/\delta_j & \text{if } [t] = [s] \text{ for } t_j < s \leq t_{j+1}, \\ 0 & \text{otherwise,} \end{cases} \tag{3.2}$$

where $[t]$ for $t \in \mathbb{R}$ is its equivalence class in $\mathbb{T}$. From now on we identify $[0,1)$ with $\mathbb{T}$ and furthermore, by a slight abuse of notation, we consider N_j to be defined on $[0,1)$.

Figure 1 shows periodic B-splines of degree one defined in (3.2) for the points in (2.1) with a special choice of parameters n, ν.

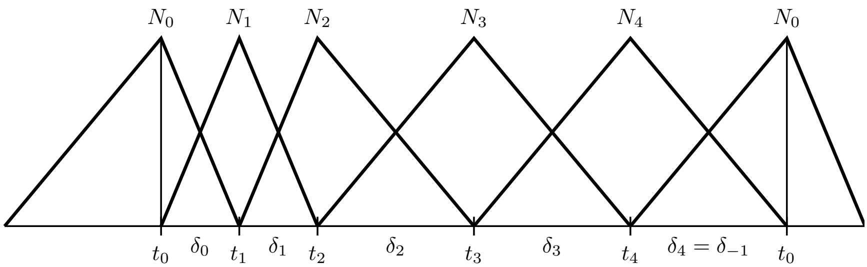

Fig. 1. Situation for $N = 5$, $\nu = 1$, $n = N - \nu = 4$

Let (as above) V be the (finite-dimensional) subspace generated by $\{N_0, \ldots, N_{N-1}\}$ and P be the orthogonal projection from $L^2[0,1)$ onto V.

Then formula (3.1) for the norm of P simplifies to

$$\|P\|_\infty = \max_{j=0,\dots,N-1} \int_0^1 |k(t_j,t)|\,dt,$$

where $k(s,t) = \sum_{j,k=0}^{N-1} a_{j,k} N_j(s) N_k(t)$ and $(a_{j,k})$ is the inverse of the Gram matrix $(b_{j,k}) = \langle N_j, N_k\rangle$. If we let $\kappa(j) := \int_0^1 |k(t_j,t)|\,dt$, it can be shown by an elementary calculation that

$$\kappa(j) = \sum_{k=0}^{N-1} \frac{\delta_k}{2} \begin{cases} |a_{j,k}| + |a_{j,k+1}| & \text{if } \operatorname{sgn} a_{j,k} = \operatorname{sgn} a_{j,k+1}, \\ \dfrac{a_{j,k}^2 + a_{j,k+1}^2}{|a_{j,k}| + |a_{j,k+1}|} & \text{otherwise,} \end{cases} \tag{3.3}$$

where every subscript is taken modulo N. Observe that $\kappa(j)$ depends on N too. With the rational function $\phi(t) := (1+t^2)/(1+t)^2$, equation (3.3) can be rewritten as

$$\kappa(j) = \sum_{k=0}^{N-1} \frac{\delta_k}{2} (|a_{j,k}| + |a_{j,k+1}|) \cdot \begin{cases} 1 & \text{if } \operatorname{sgn} a_{j,k} = \operatorname{sgn} a_{j,k+1}, \\ \phi(|a_{j,k+1}|/|a_{j,k}|) & \text{otherwise.} \end{cases} \tag{3.4}$$

We now collect a few simple facts about the function ϕ:

LEMMA 3.1. *Let $\phi : (0,\infty) \to [1/2, 1)$ be defined by*

$$t \mapsto \phi(t) = \frac{1+t^2}{(1+t)^2}.$$

Then

$$\phi(t) = \phi(t^{-1}), \qquad \phi'(t) = \frac{2(t-1)}{(1+t)^3}, \qquad \phi''(t) = \frac{4(2-t)}{(1+t)^4}$$

for all $t > 0$. So in particular ϕ is decreasing for $t < 1$ and increasing for $t > 1$ and ϕ' is increasing for $t < 2$ and decreasing for $t > 2$. Furthermore,

$$\phi(\lambda) = \frac{2}{3}, \qquad \phi(4) = \frac{17}{25}, \qquad \phi(6) = \frac{37}{49}, \qquad \phi'(\lambda) = \frac{\lambda^{-1}}{3\sqrt{3}},$$

where $\lambda = 2 + \sqrt{3}$.

By (3.4), exact formulae for the entries of the inverse (a_{jk}) of the Gram matrix are absolutely necessary to determine the exact value of the Lebesgue constant. We will provide this information in Proposition 4.1 for the periodic case. In the non-periodic dyadic case, such exact formulae were given in [3] and they were used in the calculation of the corresponding Lebesgue constant in [6]. For the general Franklin system, there are important estimates for both the non-periodic and periodic cases (see [11] and [12] respectively). To calculate the exact value of the Lebesgue constant, we supplement these estimates with exact formulae.

3.2. Solutions of $f_{k-1} - 4f_k + f_{k+1} = 0$ and their properties. In this section we examine the properties of the solutions of the recurrence $f_{k-1} - 4f_k + f_{k+1} = 0$, which we will use extensively. For an arbitrary real number x, let $A_x := \cosh(\alpha x)$ and $\sqrt{3}B_x := \sinh(\alpha x)$ with $\alpha > 0$ defined by $\cosh\alpha = 2$. For $k \in \mathbb{N}_0$, A_k and B_k can also be defined by the recurrence relations

$$A_{k+1} = 2A_k + 3B_k \quad \text{with } A_0 = 1, \tag{3.5}$$

$$B_{k+1} = A_k + 2B_k \quad \text{with } B_0 = 0. \tag{3.6}$$

This follows from the basic identities

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y, \tag{3.7}$$

$$\sinh(x+y) = \sinh x \cosh y + \cosh x \sinh y. \tag{3.8}$$

We note that it is easy to see (or a special case of Lemma 3.3 below) that

$$A_{k+1} \le 4A_k \quad \text{for } k \in \mathbb{N}_0, \tag{3.9}$$

$$B_{k+1} \le 4B_k \quad \text{for } k \in \mathbb{N}. \tag{3.10}$$

Observe also that

$$A_k = 2A_{k+1} - 3B_{k+1}, \tag{3.11}$$

$$B_k = 2B_{k+1} - A_{k+1}, \tag{3.12}$$

for $k \in \mathbb{N}_0$. Moreover, we have the formulae

$$A_x = \frac{1}{2}(\lambda^x + \lambda^{-x}), \qquad B_x = \frac{1}{2\sqrt{3}}(\lambda^x - \lambda^{-x}), \qquad x \in \mathbb{R}, \tag{3.13}$$

with

$$\lambda = 2 + \sqrt{3}, \qquad \lambda^{-1} = 2 - \sqrt{3}.$$

We remark that $\alpha = \log\lambda$. For reference, we list the first few values of A_n and B_n:

$$(A_0, \dots, A_4) = (1, 2, 7, 26, 97), \qquad (B_0, \dots, B_4) = (0, 1, 4, 15, 56).$$

The crucial fact about A_k and B_k is that they are independent solutions of the linear recursion $f_{k-1} - 4f_k + f_{k+1} = 0$, since λ and λ^{-1} are the two solutions of its characteristic equation $t^2 - 4t + 1 = 0$ and A_k and B_k have the representation (3.13). The recursion $f_{k-1} - 4f_k + f_{k+1} = 0$ in turn takes into account the special form of the Gram matrix for the points (2.1) (see (4.1) and (4.4)). This is important, since we need exact formulae for the inverse of the Gram matrix and these consist of terms depending on A_k and B_k.

LEMMA 3.2. *For $K \in \mathbb{N}_0$,*

$$\sum_{k=0}^{K}(B_k + B_{k+1}) = A_{K+1} - 1, \quad 2\sum_{k=0}^{K} A_k = 3B_{K+1} - A_{K+1} + 1,$$
$$\sum_{k=0}^{K}(A_k + A_{k+1}) = 3B_{K+1}, \qquad 2\sum_{k=0}^{K} B_k = A_{K+1} - B_{K+1} - 1.$$

Proof. The proof uses induction and the recurrences (3.5), (3.6), (3.11) and (3.12) for A_n and B_n. ∎

LEMMA 3.3. *Let $k \in \mathbb{N}_0$. Then*

$$-1 \le -\lambda^{-k} = \lambda B_k - B_{k+1} \le 0, \tag{3.14}$$
$$0 \le \lambda A_k - A_{k+1} = \sqrt{3}\lambda^{-k} \le \sqrt{3}, \tag{3.15}$$
$$-1 \le \lambda^{-k} = \sqrt{3}B_k - A_k \le 0. \tag{3.16}$$

Proof. This follows from (3.13). ∎

LEMMA 3.4. *For all $n \in \mathbb{N}$ and $0 \le k \le n$,*

$$B_k A_{n-k} + A_k B_{n-k} = B_n, \qquad B_n A_{n-k} - B_{n-k} A_n = B_k,$$
$$A_k A_{n-k} + 3B_{n-k} B_k = A_n, \qquad A_n A_{n-k} - 3B_n B_{n-k} = A_k.$$

Proof. This follows directly from (3.7) and (3.8). ∎

4. Proof of the Main Theorem. We begin with a short overview of the main steps of the proof. In Section 4.1 we treat the special case of equally spaced knots, since this is the simplest case and we get an even better Lebesgue constant than the one stated in Theorem 2.1. This serves as some kind of preliminary result, where all important proof steps of more general cases are included:

(i) Compute the inverse of the Gram matrix.
(ii) Estimate the L^∞-norms of the projection operators using (i). For this, it is important to distinguish the cases where the number of points in the knot sequence is even or odd. This comes from the fact that the inverse of the Gram matrix has a different structure in the two cases.
(iii) Determine the asymptotics of these projection operator norms.

In Section 4.2 we calculate the inverse of the Gram matrix for non-equally spaced knots.

Section 4.3 concentrates on estimating $\|P_{n,\nu}\|_\infty$ for $\nu = 1$, where (as we will see) we get the largest values for the projection operator norms. We furthermore determine the asymptotics in this case, which gives us the asserted value $2 + \frac{33-18\sqrt{3}}{13}$ of the Lebesgue constant.

In Section 4.4 we estimate the remaining cases for other choices of ν by employing easy-to-use, but sufficiently sharp estimates on quotients of neighbouring entries of the inverse of the Gram matrix.

4.1. Equally spaced knots. As a preliminary case we consider the points (2.1) for $\nu = 0$ and $N = n$ and show that $\|P_{n,0}\|_\infty < 2$ and $\lim_{n\to\infty} \|P_{n,0}\|_\infty = 2$. For this case of equally spaced knots, the Gram matrix $(b_{jk})_{0\le j,k\le N-1}$ is

$$(b_{jk}) = \frac{1}{6n}\begin{pmatrix} 4 & 1 & & & 1 \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ 1 & & & 1 & 4 \end{pmatrix}, \tag{4.1}$$

where the empty entries are zero. Since all rows in (b_{jk}) are equal up to shifts, the same must be true for the inverse (a_{jk}). For the first row of (a_{jk}), make the ansatz $a_{0,k} = (-1)^k(c_1A_k + c_2B_k)$ with constants c_1, c_2 to be determined. Thus

$$a_{0,k} + 4a_{0,k+1} + a_{0,k+2} = 0 \quad \text{for } k \ge 0.$$

Insert this ansatz into the boundary conditions

$$4a_{0,0} + a_{0,1} + a_{0,N-1} = 1, \quad a_{0,N-2} + 4a_{0,N-1} + a_{0,0} = 0$$

to determine c_1, c_2 and simplify to get

$$a_{0,k} = \frac{6n(-1)^k}{D(N)} g_k$$

with

$$g_k = B_{N-k} + (-1)^N B_k \quad \text{and} \quad D(N) = 2((-1)^{N-1} + A_N). \tag{4.2}$$

Since all rows in (a_{jk}) are equal up to shifts, the quantity (3.4) does not depend on j in this case. So while we consider equally spaced knots, we write κ instead of $\kappa(j)$ for arbitrary $0 \le j \le N-1$. We consider separately the cases of N even and N odd. The difference in the analysis of these two cases comes from the fact that g_k is always positive for N even, whereas for N odd the sign of g_k changes once.

N **even.** If we assume that N is even, we obtain from (3.4)

$$\kappa = 3D(N)^{-1}\sum_{k=0}^{N-1}(g_k + g_{k+1})\phi\Big(\frac{g_{k+1}}{g_k}\Big).$$

Using the definition of g_k and Lemma 3.3 we see that $\lambda^{-1} < g_{k+1}/g_k < \lambda$, so by Lemma 3.1, $\phi(g_{k+1}/g_k) < \phi(\lambda)$ and thus

$$\kappa < 6\phi(\lambda)D(N)^{-1}\sum_{k=0}^{N-1}(B_k + B_{k+1}).$$

Lemma 3.2 and the fact that $\phi(\lambda) = \frac{2}{3}$ then give us

$$\kappa < 4\frac{A_N - 1}{2(A_N - 1)} = 2.$$

N **odd.** For N odd, we see that (3.4) becomes

$$\kappa = 6D(N)^{-1}\Bigg[B_{(N+1)/2} - B_{(N-1)/2} + \sum_{j=0}^{(N-3)/2}(B_{N-j} + B_{N-j-1} - B_j - B_{j+1})\phi\bigg(\frac{B_{N-j} - B_j}{B_{N-j-1} - B_{j+1}}\bigg)\Bigg].$$

The mean value theorem implies

$$\phi(q_j) \le \phi(\lambda) + (q_j - \lambda)\phi'(\lambda), \quad \text{where} \quad q_j := \frac{B_{N-j} - B_j}{B_{N-j-1} - B_{j+1}},$$

since $\phi'(t)$ is decreasing for $t \ge \lambda \ge 2$ and $q_j \ge \lambda$ by Lemma 3.3. For $q_j - \lambda$, we have again, due to Lemma 3.3 and $0 \le j \le (N-3)/2$,

$$\begin{aligned} q_j - \lambda &= \frac{B_{N-j} - \lambda B_{N-j-1} + \lambda B_{j+1} - B_j}{B_{N-j-1} - B_{j+1}} \le \frac{1 + \lambda B_{j+1}}{B_{N-j-1} - B_{j+1}} \\ &\le \frac{1 + \lambda B_{j+1}}{B_{N-j-1}(1 - \lambda^{-N+2j+2})} \le 2\frac{1 + \lambda B_{j+1}}{B_{N-j-1}}. \end{aligned}$$

Using these facts and the estimates $B_{(N-1)/2} \ge \lambda^{-1}B_{(N+1)/2} - \lambda^{-1}$ (Lemma 3.3) and $-B_j \le 0$, we obtain

$$\begin{aligned} \kappa \le 6D(N)^{-1}\Bigg[&(1 - \lambda^{-1})B_{(N+1)/2} + \lambda^{-1} \\ &+ \phi(\lambda)\sum_{j=0}^{(N-3)/2}(B_{N-j} + B_{N-j-1} - B_{j+1}) \\ &+ 2\phi'(\lambda)\sum_{j=0}^{(N-3)/2}(B_{N-j} + B_{N-j-1} - B_{j+1})\frac{1 + \lambda B_{j+1}}{B_{N-j-1}}\Bigg]. \end{aligned} \tag{4.3}$$

We split the analysis of this expression into a few subcases and introduce the notation $p = (N+1)/2$ to shorten indices.

The sum $\mathrm{I} := \sum_{j=0}^{p-2}(B_{N-j} + B_{N-j-1} - B_{j+1})$. We apply Lemma 3.2 to get

$$\mathrm{I} = \tfrac{1}{2}(2A_N - 3A_p + B_p + 1) \le \tfrac{1}{2}(2A_N - (3\sqrt{3}-1)B_p + 1),$$

by Lemma 3.3.

The sum $\mathrm{II} := \sum_{j=0}^{p-2}(B_{N-j} + B_{N-j-1} - B_{j+1})\frac{1+\lambda B_{j+1}}{B_{N-j-1}}$. Since by Lemma 3.3, $B_{N-j} = \lambda B_{N-j-1} + \lambda^{-N+j+1}$ and $\lambda^{-N+j+1} \le \lambda^{-N+1}B_{j+1}$, we get

$$\mathrm{II} \le (1+\lambda)\sum_{j=0}^{p-2}(1+\lambda B_{j+1}) - (1-\lambda^{-N+1})\sum_{j=0}^{p-2}\frac{B_{j+1}(1+\lambda B_{j+1})}{B_{N-j-1}}.$$

But now, by estimating the second sum by its summand with index $p-2$,

$$\sum_{j=0}^{p-2}\frac{B_{j+1}(1+\lambda B_{j+1})}{B_{N-j-1}} \ge \frac{B_{p-1}(1+\lambda B_{p-1})}{B_p} \ge \lambda\frac{B_{p-1}^2}{B_p},$$

and by Lemmas 3.2 and 3.3,

$$\sum_{j=0}^{p-2} 1 + \lambda B_{j+1} = \frac{N-1}{2} + \frac{\lambda}{2}(A_p - B_p - 1) \le \frac{N-1}{2} + \frac{\lambda(\sqrt{3}-1)}{2}B_p.$$

We thus finally obtain

$$II \le (1+\lambda)\Big(\frac{N-1}{2} + \frac{\lambda(\sqrt{3}-1)}{2}B_p\Big) - (1-\lambda^{-N+1})\lambda\frac{B_{p-1}^2}{B_p}.$$

These estimates together with (4.3) imply, on noting $D(N) \ge 2A_N$ and $\phi(\lambda) = 2/3$, that

$$\kappa \le 2 + \frac{3}{A_N}\Big[\theta B_p + \lambda^{-1} + \frac{1}{3} + 2\phi'(\lambda)\Big((1+\lambda)\frac{N-1}{2} - (1-\lambda^{-N+1})\lambda\frac{B_{p-1}^2}{B_p}\Big)\Big],$$

where

$$\theta = (1-\lambda^{-1}) - (\sqrt{3} - 1/3) + (1+\lambda)\lambda\phi'(\lambda)(\sqrt{3}-1) = 0.$$

Since B_{p-1}^2/B_p dominates $(N-1)/2$ for large N, we finally see that for N sufficiently large ($N \ge 8$),

$$\kappa < 2.$$

In fact, Table 1 on page 274 gives this inequality for all $N \ge 2$. An analogous argument to Section 4.3.3 finally yields $\lim_{N\to\infty}\kappa = 2$, and this completes what we wanted to show in this section.

4.2. Non-equally spaced knots. We now consider the points (2.1) in case $1 \le \nu \le n-1$ (i.e. the knots are not equally spaced anymore). The first step is to calculate the inverse of the Gram matrix in this setting. As before, we take every index concerning the Gram matrix (b_{jk}) or its inverse

(a_{jk}) modulo N. The Gram matrix $(b_{jk}) = (\langle N_j, N_k\rangle)_{0\le j,k\le N-1}$ admits the representation

$$(4.4)\qquad (b_{jk}) = \frac{1}{12n}\begin{pmatrix} 6 & 1 & & & & & & & & 2 \\ 1 & 4 & 1 & & & & & & & \\ & \ddots & \ddots & \ddots & & & & & & \\ & & 1 & 4 & 1 & & & & & \\ & & & 1 & 6 & 2 & & & & \\ & & & & 2 & 8 & 2 & & & \\ & & & & & \ddots & \ddots & \ddots & & \\ & & & & & & 2 & 8 & 2 \\ 2 & & & & & & & 2 & 8 \end{pmatrix},$$

where the row with the pattern $1, 6, 2$ has index 2ν. This leads to the following equations:

$$(4.5)\qquad 6a_{0,k} + a_{1,k} + 2a_{N-1,k} = 12n\delta_{0,k},$$

$$(4.6)\qquad a_{j-1,k} + 4a_{j,k} + a_{j+1,k} = 12n\delta_{j,k} \quad \text{for } j = 1, \dots, 2\nu-1,$$

$$(4.7)\qquad a_{2\nu-1,k} + 6a_{2\nu,k} + 2a_{2\nu+1,k} = 12n\delta_{2\nu,k},$$

$$(4.8)\qquad a_{j-1,k} + 4a_{j,k} + a_{j+1,k} = 6n\delta_{j,k} \quad \text{for } j = 2\nu+1, \dots, N-1,$$

where $\delta_{j,k}$ is the Kronecker delta and $0 \le k \le N-1$. Let

$$(4.9)\qquad D(N,\nu) := 2A_N + \frac{3}{2}B_{2\nu}B_{N-2\nu} - 2(-1)^N.$$

Then we define

$$g(N,\nu,j,k) := \frac{D(N,\nu)a_{j,k}(-1)^{k+j}}{6n}.$$

Observe that $a_{j,k}$ depends on N and ν too. But in the current context, the indices N, ν and also j are fixed, so we write g_k instead of $g(N,\nu,j,k)$. Inserting the definition of g_k into (3.4), we obtain

$$(4.10)\quad \kappa(j) = D(N,\nu)^{-1}\left[\frac{3}{2}\sum_{k=0}^{2\nu-1}(|g_k|+|g_{k+1}|)\xi_{j,k} + 3\sum_{k=2\nu}^{N-1}(|g_k|+|g_{k+1}|)\xi_{j,k}\right]$$

with

$$\xi_{j,k} = \begin{cases} 1 & \text{if } \operatorname{sgn} a_{j,k} = \operatorname{sgn} a_{j,k+1}, \\ \phi(|g_{k+1}|/|g_k|) & \text{else.} \end{cases}$$

In order to determine $(a_{j,k})$, we identify the values of g_k:

Proposition 4.1. *If* $0 \le j \le 2\nu - 1$, *then* g_k *equals*

$$\begin{cases} 2(-1)^N B_{j-k} + B_{N-j+k} + B_{2\nu-j}A_{N-2\nu+k} + B_k(A_{N-j} + 3B_{2\nu-j}B_{N-2\nu}) \\ \hfill \text{if } k \le j, \\ 2(-1)^N B_{k-j} + B_{N-k+j} + B_{2\nu-k}A_{N-2\nu+j} + B_j(A_{N-k} + 3B_{2\nu-k}B_{N-2\nu}) \\ \hfill \text{if } j \le k \le 2\nu, \\ (-1)^N(B_{k-j} + A_{k-2\nu}B_{2\nu-j}) + B_{N-k+j} + B_jA_{N-k} \quad \text{if } 2\nu \le k \le N-1. \end{cases}$$

If $2\nu \le j \le N - 1$, *then* g_k *equals*

$$\begin{cases} (-1)^N(B_{j-k} + A_{j-2\nu}B_{2\nu-k}) + B_{N-j+k} + A_{N-j}B_k & \text{if } k \le 2\nu \le j, \\ (-1)^N B_{j-k} + A_{k-2\nu}B_{N-j+2\nu} + A_{N-j}B_k + \frac{3}{2}B_{k-2\nu}B_{2\nu}B_{N-j} & \\ & \text{if } 2\nu \le k \le j, \\ (-1)^N B_{k-j} + A_{N-k}B_j + A_{j-2\nu}B_{N-k+2\nu} + \frac{3}{2}B_{2\nu}B_{N-k}B_{j-2\nu} & \\ & \text{if } j \le k \le N-1. \end{cases}$$

Proof. If we insert these formulae for g_k into (4.5) and (4.6) for $0 \le j \le 2\nu - 1$ and into (4.7) and (4.8) for $2\nu \le j \le N - 1$, we deduce the assertion by a case-by-case analysis, using the fact that A_n and B_n are solutions of the recurrence $f_{k-1} - 4f_k + f_{k+1} = 0$. Observe that to evaluate (4.5)–(4.8) the recursions (3.5), (3.6), (3.11), (3.12) for A_k and B_k and the identities from Lemma 3.4 are useful. ∎

Remark 4.2. From Proposition 4.1 we can see that for N even, $g_k \ge 0$ for all $0 \le k \le N - 1$, while for N odd, $g_k \ge 0$ for $|k - j| \le (N - 1)/2$ and $g_k \le 0$ for $|k - j| \ge (N + 1)/2$.

4.3. The main case $\nu = j = 1$. The first special case to analyze is $\nu = j = 1$. As we will see, this is the main case in the sense that for $N \to \infty$ and $\nu = j = 1$, $\kappa := \kappa(1)$ converges to the Lebesgue constant $2 + \frac{33-18\sqrt{3}}{13}$. In this section, we set $K = N - 1$ for notational convenience. We then find, as a special instance of Proposition 4.1, that $g_k = g(N, 1, 1, k)$ equals

$$\begin{cases} 2[(-1)^N + A_K - B_K] & \text{if } k = 0, \\ 8B_K & \text{if } k = 1, \\ 2[A_{N-k} + B_{N-k} + (-1)^N(A_{k-2} + B_{k-2})] & \text{if } 2 \le k \le N-1. \end{cases}$$

Note that $g_2 = g_0$. Additionally,

$$D(N, 1) = 18B_K - 2A_K - 2(-1)^N.$$

Furthermore the use of the recurrences (3.5), (3.6), (3.11) and (3.12) for A_k and B_k yields

(4.11) $|g_1| + |g_2| = 2(-1)^N + 6B_K + 2A_K,$

(4.12) $|g_k| + |g_{k+1}| = 4|A_{N-k} + (-1)^N A_{k-1}|$ for $k \ge 2, k \ne (N+1)/2,$

(4.13) $|g_{(N+1)/2}| + |g_{(N+3)/2}| = 8A_{K/2}$ for N even,

(4.14) $|g_{(N+1)/2}| + |g_{(N+3)/2}| = 8B_{K/2}$ for N odd.

We recall that all indices are taken modulo N. The quotient of consecutive values of g_k has the following special form:

LEMMA 4.3. *For* $2 \le k \le N-1$,

$$\frac{|g_{k+1}|}{|g_k|} = \frac{A_{|N/2-k|}}{A_{|N/2-k+1|}} \quad \textit{if } N \textit{ is even,} \tag{4.15}$$

$$\frac{|g_{k+1}|}{|g_k|} = \frac{B_{|N/2-k|}}{B_{|N/2-k+1|}} \quad \textit{if } N \textit{ is odd.} \tag{4.16}$$

Proof. Let $k \le N/2$. Then by (3.7), (3.8) and the definitions of A_n and B_n we have

$$\begin{aligned} A_{N-k-1} &= A_{N/2-k}A_{N/2-1} + 3B_{N/2-k}B_{N/2-1}, \\ B_{N-k-1} &= A_{N/2-k}B_{N/2-1} + B_{N/2-k}A_{N/2-1}, \\ A_{k-1} &= A_{N/2-1}A_{N/2-k} - 3B_{N/2-k}B_{N/2-1}, \\ B_{k-1} &= B_{N/2-1}A_{N/2-k} - A_{N/2-1}B_{N/2-k}. \end{aligned}$$

For N even, summing these four equations yields $g_{k+1}/2$ on the left hand side and $A_{N/2-k}$ times a term independent of k on the right hand side. On the other hand, for N odd, summing the first two equations and subtracting the second two gives $|g_{k+1}|/2$ on the left hand side and $B_{N/2-k}$ times a term independent of k on the right hand side. An analogous argument for $k \ge N/2$ completes the proof of the lemma. ∎

4.3.1. *Estimates for N even.* For N even, from (4.10) and the fact $g_0 = g_2$ we get

$$\kappa := \kappa(1) = 3D(N,1)^{-1} \sum_{k=1}^{K} (g_k + g_{k+1}) \phi\left(\frac{g_{k+1}}{g_k}\right).$$

Inserting (4.11)–(4.14) into this expression and recalling $K = N-1$ shows that κ equals

$$3D(N,1)^{-1}\left[(2 + 6B_K + 2A_K)\phi\left(\frac{1 + A_K - B_K}{4B_K}\right) + 8\sum_{k=2}^{K} A_{k-1}\phi\left(\frac{A_{|N/2-k|}}{A_{|N/2-k+1|}}\right)\right]. \tag{4.17}$$

Now observe that Lemmas 4.3 and 3.3 imply $\lambda^{-1} < g_{k+1}/g_k = A_{|N/2-k|}/A_{|N/2-k+1|} < \lambda$ for $k \geq 2$, so using Lemma 3.1, the above expression for κ is strictly less than

$$6D(N,1)^{-1}\Big[(1+3B_K+A_K)\phi\Big(\frac{1+A_K-B_K}{4B_K}\Big)+4\phi(\lambda)\sum_{k=1}^{K-1}A_k\Big].$$

If we use Lemma 3.2 to evaluate the sum and remark that $A_K = \sqrt{3}B_K + \lambda^{-K}$ by Lemma 3.3, we deduce, by setting $\frac{1+A_K-B_K}{4B_K} = \eta + h$ with

$$\eta = \frac{\sqrt{3}-1}{4} \qquad \text{and} \qquad h = h(N) = \frac{1+\lambda^{-K}}{4B_K},$$

that

$$\kappa \leq 6D(N,1)^{-1}[(1+3B_K+A_K)\phi(\eta+h)+2\phi(\lambda)(3B_K-A_K-1)]. \tag{4.18}$$

Since $\phi'(t)$ is increasing for $t \leq 2$ (Lemma 3.1) and $h \leq 1/2$ for $N \geq 2$, the mean value theorem yields

$$\phi(\eta+h) \leq \phi(\eta)+\phi'(\eta+1/2)h. \tag{4.19}$$

Thus, using (4.19) in (4.18) we see that in order to prove $\kappa < \gamma$, it suffices to show that

$$\begin{aligned}6D(N,1)^{-1}[(1+3B_K+A_K)(\phi(\eta)+\phi'(\eta+1/2)h)&\\ +2\phi(\lambda)(3B_K-A_K-1)] &< \gamma.\end{aligned} \tag{4.20}$$

If we multiply this inequality by $D(N,1)$, collect the factors for B_K and A_K and observe that

$$\theta := 6\phi(\eta)+2\gamma-12\phi(\lambda) = \frac{1}{\sqrt{3}}(18\gamma-18\phi(\eta)-36\phi(\lambda)),$$

we see that (4.20) is equivalent to

$$\theta(\sqrt{3}B_K-A_K-1)+6h(N)(1+3B_K+A_K)|\phi'(\eta+1/2)| > 0. \tag{4.21}$$

Now we use again $A_K = \sqrt{3}B_K + \lambda^{-K}$ and insert the definition of $h(N)$ to express the left hand side of (4.21) as

$$(1+\lambda^{-K})\Big[\frac{3}{2B_K}(1+(\sqrt{3}+3)B_K+\lambda^{-K})|\phi'(\eta+1/2)|-\theta\Big].$$

Clearly, this is greater than

$$(1+\lambda^{-K})\Big[\frac{3(\sqrt{3}+3)}{2}|\phi'(\eta+1/2)|-\theta\Big],$$

which is easily seen to be greater than zero. Thus we have shown that $\kappa < \gamma$ for N even and $\nu = j = 1$.

4.3.2. *Estimates for N odd.* For N odd, (4.10) and Remark 4.2 imply that κ equals

$$3D(N,1)^{-1}\Big[\sum_{\substack{k=1\\ k\neq (N+1)/2}}^{K}(|g_k|+|g_{k+1}|)\phi\Big(\frac{|g_k|}{|g_{k+1}|}\Big)+|g_{(N+1)/2}|+|g_{(N+3)/2}|\Big].$$

We now use Lemma 4.3 and the identities (4.11)–(4.14) and recall that $K=N-1$ to obtain, after a little calculation,

$$\begin{aligned}\kappa = 6D(N,1)^{-1}\Big[&(3B_K+A_K-1)\phi\Big(\frac{A_K-B_K-1}{4B_K}\Big)\\ &+4\sum_{k=2}^{K/2}(A_{N-k}-A_{k-1})\phi\Big(\frac{B_{N/2-k}}{B_{N/2-k+1}}\Big)+4B_{K/2}\Big].\end{aligned}\tag{4.22}$$

We first estimate two summands of κ separately.

The term $\mathrm{I} := (3B_K+A_K-1)\phi(\frac{A_K-B_K-1}{4B_K})$. We have $3B_K+A_K-1\le(3+\sqrt{3})B_K$ by Lemma 3.3 and $\frac{A_K-B_K-1}{4B_K}=\eta-h$ with

$$\eta=\frac{\sqrt{3}-1}{4}\quad\text{and}\quad h=\frac{1-\lambda^{-K}}{4B_K},$$

so the mean value theorem implies

$$\begin{aligned}\mathrm{I}&\le(3+\sqrt{3})B_K\phi(\eta-h)\le(3+\sqrt{3})B_K(\phi(\eta)-\phi'(0)h)\\ &=(3+\sqrt{3})B_K(\phi(\eta)+2h),\end{aligned}$$

since ϕ' is increasing for $t\le 2$ and $\phi'(0)=-2$.

The term $\mathrm{II} := \sum_{k=2}^{K/2}(A_{N-k}-A_{k-1})\phi(\frac{B_{N/2-k+1}}{B_{N/2-k}})$. Since $B_{L+1}=\lambda B_L+\lambda^{-L}$, the mean value theorem and the fact that ϕ' is decreasing for $t\ge 2$ yield

$$\phi\Big(\frac{B_{N/2-k+1}}{B_{N/2-k}}\Big)\le\phi(\lambda)+\phi'(\lambda)\frac{\lambda^{k-N/2}}{B_{N/2-k}}.$$

Now, use the identity $2\sum_{k=0}^{L}A_k=3B_{L+1}-A_{L+1}+1$ from Lemma 3.2 and simplify using the recurrences for A_k and B_k to obtain

$$\begin{aligned}\sum_{k=2}^{K/2}(A_{N-k}-A_{k-1})&=\frac{1}{2}(3B_K-A_K-6B_{K/2}+1)\\ &\le\frac{1}{2}((3-\sqrt{3})B_K-6B_{K/2}+1),\end{aligned}$$

by Lemma 3.3. Next, we get

$$S := \sum_{k=2}^{K/2} A_{N-k}\frac{\lambda^{k-N/2}}{B_{N/2-k}} = \sqrt{3}\sum_{k=2}^{K/2}\frac{\lambda^{k-N/2}(\lambda^{N-k}+\lambda^{k-N})}{\lambda^{N/2-k}-\lambda^{k-N/2}}$$
$$= \sqrt{3}\sum_{k=2}^{K/2}\frac{\lambda^{N-k}+\lambda^{k-N}}{\lambda^{N-2k}-1},$$

by (3.13). Since $1 \le \lambda^{N-2k}/2$, we estimate

$$\begin{aligned}
S &\le 2\sqrt{3}\sum_{k=2}^{K/2}\frac{\lambda^{N-k}+\lambda^{k-N}}{\lambda^{N-2k}} = 2\sqrt{3}\sum_{k=2}^{K/2}\lambda^k+\lambda^{3k-2N}\\
&= 2\sqrt{3}\Big[\frac{\lambda^{K/2+1}-\lambda^2}{\lambda-1}+\lambda^{-2N}\frac{\lambda^{3(K/2+1)}-\lambda^6}{\lambda^3-1}\Big]\\
&\le 2\sqrt{3}\Big[\frac{\lambda^{K/2+1}}{\lambda-1}+\lambda^{-2N}\frac{\lambda^{3(K/2+1)}}{\lambda-1}\Big]\\
&= 4\sqrt{3}\frac{A_{K/2}}{1-\lambda^{-1}} \le 4\sqrt{3}\frac{\sqrt{3}B_{K/2}+1}{1-\lambda^{-1}}.
\end{aligned}$$

Altogether, we get

$$\mathrm{II} \le \frac{\phi(\lambda)}{2}((3-\sqrt{3})B_K - 6B_{K/2}+1) + 4\sqrt{3}\phi'(\lambda)\frac{\sqrt{3}B_{K/2}+1}{1-\lambda^{-1}}.$$

Let us now return to (4.22). The estimate $h \le \frac{1}{4B_K}$, combined with the estimates for I and II, gives

$$\kappa \le 6D(N,\nu)^{-1}(\sigma B_K - \tau B_{K/2} + \vartheta)$$

with

$$\sigma = (3+\sqrt{3})\phi(\eta) + 2\phi(\lambda)(3-\sqrt{3}), \qquad \tau = 12\phi(\lambda) - 4 - \frac{48\phi'(\lambda)}{1-\lambda^{-1}} > 0,$$
$$\vartheta = \frac{3+\sqrt{3}}{2} + 2\phi(\lambda) + \frac{16\sqrt{3}\phi'(\lambda)}{1-\lambda^{-1}}.$$

Now recall that $D(N,1) = 18B_K - 2A_K + 2 \ge (18-2\sqrt{3})B_K$ by Lemma 3.3, so in order to prove $\kappa < \gamma$, it suffices to show

$$\frac{\gamma(18-2\sqrt{3})B_K}{6} > \sigma B_K - \tau B_{K/2} + \vartheta.$$

Since $\sigma = \frac{\gamma}{6}(18-2\sqrt{3})$, this is equivalent to

$$\tau B_{K/2} - \vartheta > 0,$$

which is true for $N \ge 7$. For $N < 7$ we get the desired bound for κ from Table 1 on page 274.

4.3.3. *Asymptotic behaviour.* In this section, we calculate the limit of κ as $N \to \infty$ for $\nu = j = 1$. In the following, the symbol $\sim$ will denote asymptotic equality for $N \to \infty$. Since $A_N \sim \sqrt{3}B_N$, $A_{N+1} \sim \lambda A_N$ (by Lemma 3.3), recalling the definition of $D(N,1) = 18B_K - 2A_K - 2$ (where as above, $K = N-1$) we get for N even from (4.17)

$$\begin{aligned}\kappa = 6D(N,1)^{-1}\biggl[&(1+3B_K+A_K)\phi\Bigl(\frac{1+A_K-B_K}{4B_K}\Bigr)\\&\qquad+4\sum_{k=2}^{K}A_{k-1}\phi\Bigl(\frac{A_{|N/2-k|}}{A_{|N/2-k+1|}}\Bigr)\biggr]\\ \sim \frac{6}{(18-2\sqrt{3})B_K}&\biggl[(3+\sqrt{3})B_K\phi\Bigl(\frac{\sqrt{3}-1}{4}\Bigr)+4\phi(\lambda)\sum_{k=3N/4}^{K-1}A_k\biggr].\end{aligned}$$

Using the identity $2\sum_{k=0}^{L}A_k = 3B_{L+1} - A_{L+1} + 1$ from Lemma 3.2, we further get

$$\begin{aligned}\kappa &\sim \frac{6}{(18-2\sqrt{3})B_K}\biggl[(3+\sqrt{3})B_K\phi\Bigl(\frac{\sqrt{3}-1}{4}\Bigr)+2\phi(\lambda)(3-\sqrt{3})B_K\biggr]\\ &\sim \frac{6}{18-2\sqrt{3}}\biggl[(3+\sqrt{3})\phi\Bigl(\frac{\sqrt{3}-1}{4}\Bigr)+2\phi(\lambda)(3-\sqrt{3})\biggr]\\ &= \gamma = 2+\frac{33-18\sqrt{3}}{13}.\end{aligned}$$

If on the other hand N is odd, from (4.22) we obtain

$$\begin{aligned}\kappa \sim \frac{6}{(18-2\sqrt{3})B_K}\biggl[&(3+\sqrt{3})B_K\phi\Bigl(\frac{\sqrt{3}-1}{4}\Bigr)\\&\qquad+4\sum_{k=2}^{K/4}A_{N-k}\phi\Bigl(\frac{B_{N/2-k}}{B_{N/2-k+1}}\Bigr)\biggr].\end{aligned}$$

Again, the identity $2\sum_{k=0}^{L}A_k = 3B_{L+1} - A_{L+1} + 1$ and $B_{N+1} \sim \lambda B_N$ imply $\kappa \sim \gamma$ in the same way as above. The estimates of this section together with the numerical results from Table 1 show that for $\nu = j = 1$, we have $\kappa < \gamma$ and $\lim_{N\to\infty}\kappa = \gamma$. We will see in the next section that this is the critical case, since $\kappa < \gamma$ for all other values of ν and j.

4.4. Estimating $\kappa(j)$. In this section we derive bounds for $\kappa(j)$ for all remaining values of ν, j, which will allow us to deduce that $\|P_{n,\nu}\|_\infty < \gamma$ for all $n, \nu \in \mathbb{N}$ with $0 \le \nu \le n$. To derive these estimates we first need some bounds for the quotients of consecutive values of g:

LEMMA 4.4. *Let N be even. Then*

$$6^{-1} \le \frac{g_{k+1}}{g_k} \le 6 \quad \text{if } k = 0 \text{ or } k = 2\nu - 1, \tag{4.23}$$

$$4^{-1} \le \frac{g_{k+1}}{g_k} \le 4 \quad \text{if } k \neq 0 \text{ and } k \neq 2\nu - 1. \tag{4.24}$$

For $j = k = 0$, we have the better estimate

$$4^{-1} \le \frac{g_{k+1}}{g_k} \le 4.$$

We get analogous estimates for N odd, but we have to add a further restriction to the domain of validity of the inequalities:

LEMMA 4.5. *Let $N \ge 7$ be odd and $|k - j| \le (N - 5)/2$ or $|k - j| \ge (N + 5)/2$. Then*

$$6^{-1} \le \frac{|g_{k+1}|}{|g_k|} \le 6 \quad \text{if } k = 0 \text{ or } k = 2\nu - 1,$$

$$4^{-1} \le \frac{|g_{k+1}|}{|g_k|} \le 4 \quad \text{if } k \neq 0 \text{ and } k \neq 2\nu - 1.$$

Additionally, for $j = k = 0$ we have the better estimate

$$4^{-1} \le \frac{|g_{k+1}|}{|g_k|} \le 4.$$

For the proof of Lemma 4.4 and parts of the proof of Lemma 4.5, see Appendix A.

We note that in the following, we only treat the case of N even. In fact, as we will show later (in Section 4.4.4), the case of N odd will follow from these estimates. Combining (4.10) with Remark 4.2 shows for N even that $\kappa(j)$ equals

$$D(N, \nu)^{-1} \left[\frac{3}{2} \sum_{k=0}^{2\nu-1} (g_k + g_{k+1}) \phi\left(\frac{g_{k+1}}{g_k} \right) + 3 \sum_{k=2\nu}^{N-1} (g_k + g_{k+1}) \phi\left(\frac{g_{k+1}}{g_k} \right) \right]. \tag{4.25}$$

In estimating $\kappa(j)$, we consider the three cases $j = 0$, $1 \le j \le 2\nu - 1$ and $2\nu \le j \le N - 1$ separately.

4.4.1. $j = 0$. Invoking Lemma 4.4, we get the bound

$$D(N, \nu)\kappa(0) \le \tfrac{3}{2}\phi(6) I_1 + \tfrac{3}{2}\phi(4) I_2 + 3\phi(4) I_3 =: J,$$

where

$$I_1 = g_{2\nu-1} + g_{2\nu}, \qquad I_2 = \sum_{k=0}^{2\nu-2} (g_k + g_{k+1}), \qquad I_3 = \sum_{k=2\nu}^{N-1} (g_k + g_{k+1}).$$

PROPOSITION 4.6. *For $j = 0$,*

$$I_1 = 2(B_{2\nu-1} + B_{2\nu}) + A_{N-2\nu+1},$$
$$I_2 = 2A_{2\nu-1} - 2 + A_N - A_{N-2\nu+1} + A_{N-2\nu}(A_{2\nu} - 2),$$
$$I_3 = 2A_N - A_{2\nu} + A_{N-2\nu} - A_{2\nu}A_{N-2\nu} - 1.$$

Proof. Insert the formulae from Proposition 4.1, use the recurrences (3.11), (3.12) for A_k and B_k and Lemmas 3.2 and 3.4. ∎

With this proposition and the identity $A_N = A_{N-2\nu}A_{2\nu} + 3B_{N-2\nu}B_{2\nu}$ from Lemma 3.4 we see that

$$\begin{aligned} J = {} & \tfrac{3}{2}\phi(6)[2(B_{2\nu-1} + B_{2\nu}) + A_{N-2\nu+1}] \\ & + \tfrac{3}{2}\phi(4)[4A_N - 4 + 3B_{2\nu}B_{N-2\nu} - A_{N-2\nu+1} + 2(A_{2\nu-1} - A_{2\nu})]. \end{aligned}$$

Now recall that $D(N,\nu) = 2A_N + \frac{3}{2}B_{2\nu}B_{N-2\nu} - 2$. If we then use the recurrences (3.11), (3.12) for $A_{2\nu-1}$ and $B_{2\nu-1}$ and set $s := \frac{3}{2}(\phi(6) - \phi(4)) = \frac{138}{1225}$ it follows with $\phi(4) = \frac{17}{25}$ that

$$J = \frac{51}{25}D(N,\nu) + s(6B_{2\nu} - 2A_{2\nu} + A_{N-2\nu+1}). \tag{4.26}$$

Plugging in the estimate for $B_{2\nu}$ from Lemma 3.3 and remarking that $2\nu \le N-1$ and $N - 2\nu + 1 \le N - 1$, we get

$$J \le \frac{51}{25}D(N,\nu) + s(2\sqrt{3} - 1)A_{N-1}. \tag{4.27}$$

Using again Lemma 3.3 on A_{N-1}, we obtain

$$J \le \frac{51}{25}D(N,\nu) + (A_N + \sqrt{3})\frac{s}{\lambda}(2\sqrt{3} - 1).$$

Finally, the definition of $D(N,\nu)$ and the fact that the function $\nu \mapsto B_{2\nu} B_{N-2\nu}$ is concave for $1 \le \nu \le (N-1)/2$ and therefore attains its minimum at the border for $2\nu = N - 1$ yield

$$A_N + \sqrt{3} \le \frac{D(N,\nu)}{2} = A_N + \frac{3}{4}B_{2\nu}B_{N-2\nu} - 1 \quad \text{for } N \ge 3.$$

Thus,

$$\kappa(0) \le \frac{51}{25} + \frac{s}{2\lambda}(2\sqrt{3} - 1) \approx 2.07719 \quad \text{for } N \ge 3.$$

For $N < 3$, this follows from the numerical results of Table 1 on page 274.

4.4.2. $1 \le j \le 2\nu - 1$. Just as for $j = 0$, Lemma 4.4 yields the bound, for $1 \le j \le 2\nu - 1$,

$$D(N,\nu)\kappa(j) \le \tfrac{3}{2}\phi(6)I_1 + \tfrac{3}{2}\phi(4)I_2 + 3\phi(4)I_3 =: J,$$

where now

$$I_1 = g_0 + g_1 + g_{2\nu-1} + g_{2\nu}, \qquad I_2 = \sum_{k=1}^{2\nu-2} (g_k + g_{k+1}), \qquad I_3 = \sum_{k=2\nu}^{N-1} (g_k + g_{k+1}).$$

PROPOSITION 4.7. *For* $1 \leq j \leq 2\nu - 1$,

$$\begin{aligned} I_1 &= 2(B_j + B_{j-1} + B_{2\nu-j} + B_{2\nu-j-1}) \\ &\quad + 3B_{N-2\nu+1}(B_j + B_{2\nu-j}) + A_{N-j+1} + A_{N-2\nu+j+1}, \\ I_2 &= 2D(N,\nu) - 3(B_{2\nu-j} + B_j)(B_{N-2\nu+1} + 2B_{N-2\nu}) \\ &\quad + 2(A_{j-1} - A_{N-j} + A_{2\nu-j-1} - A_{N-2\nu+j}) - A_{N-j+1} - A_{N-2\nu+1+j}, \\ I_3 &= A_{N-j} + A_{N-2\nu+j} - A_{2\nu-j} - A_j + 3B_{N-2\nu}(B_j + B_{2\nu-j}). \end{aligned}$$

Proof. As in the case $j = 0$, it suffices to insert the formulae from Proposition 4.1, use Lemmas 3.2 and 3.4 and the recurrences (3.5), (3.6), (3.11) and (3.12). ■

Now recall that we defined $s = \frac{3}{2}(\phi(6) - \phi(4)) = \frac{138}{1225}$ and $\phi(4) = \frac{17}{25}$; thus inserting the formulas of Proposition 4.7 into the definition of J and using the recursions (3.11) and (3.12) for $A_{j-1}, B_{j-1}, A_{2\nu-j-1}$, $B_{2\nu-j-1}$ yields

$$\begin{aligned} J &= \frac{51}{25} D(N,\nu) + 2s(3B_j - A_j + 3B_{2\nu-j} - A_{2\nu-j}) \\ &\quad + s(A_{N-j+1} + A_{N-2\nu+j+1} + 3B_{N-2\nu+1}(B_{2\nu-j} + B_j)) =: J_1 + J_2 + J_3. \end{aligned}$$

From Lemma 3.3 we deduce

$$J_2 \leq 2s(3 - \sqrt{3})(B_j + B_{2\nu-j}).$$

Since the functions $x \mapsto A_x + A_{K-x}$ and $x \mapsto B_x + B_{K-x}$ are convex for $K > 0$ and $0 \leq x \leq K$, we see that the maximum is attained at the border, so J is less than or equal to

$$\frac{51}{25} D(N,\nu) + 2s(3-\sqrt{3})(1+B_{2\nu-1}) + s(A_N + A_{N-2\nu+2} + 3B_{N-2\nu+1}(1+B_{2\nu-1})).$$

We now require $\nu \geq 2$. Since we are in the case $1 \leq j \leq 2\nu - 1$, we see that the only case missing is $\nu = 1$, $j = 1$, which was treated in Section 4.3. Using the estimates

- $B_{2\nu-1} \leq \lambda^{-1} B_{2\nu} \leq \lambda^{-1} B_{2\nu} B_{N-2\nu}$ (Lemma 3.3),
- $A_{N-2\nu+2} \leq A_{N-2} \leq \lambda^{-2} A_N + \frac{\sqrt{3}}{\lambda}(1 + \lambda^{-1})$ (Lemma 3.3),
- $3B_{N-2\nu+1} B_{2\nu-1} \leq A_N/2$ (Lemmas 3.3 and 3.4),
- $3B_{N-2\nu+1} \leq 3B_{N-3} \leq 3\lambda^{-3} B_N \leq \sqrt{3}\lambda^{-3} A_N$ (Lemma 3.3),

we get

$$J - \frac{51}{25} D(N,\nu) \le s(a_1 + a_2 A_N + a_3 B_{2\nu} B_{N-2\nu})$$
$$= s\Big(a_1 + a_2 + \frac{a_2}{2} D(N,\nu) - \Big(\frac{3a_2}{4} - a_3\Big) B_{2\nu} B_{N-2\nu}\Big)$$

with $a_1 = 2(3-\sqrt{3}) + \frac{\sqrt{3}}{\lambda}(1+\lambda^{-1})$, $a_2 = \frac{3}{2} + \lambda^{-2} + \sqrt{3}\lambda^{-3}$ and $a_3 = \frac{2}{\lambda}(3-\sqrt{3})$. Since the function $\nu \mapsto B_{2\nu} B_{N-2\nu}$ is concave and therefore attains its minimum for $2\nu = N-1$ we conclude with the exact value of $3a_2/4 - a_3 \ge 0$ that

$$J \le D(N,\nu)\left[\frac{51}{25} + \frac{sa_2}{2}\right] \quad \text{for } N \ge 4.$$

Thus finally

$$\kappa(j) \le \frac{51}{25} + \frac{sa_2}{2} \le 2.130411 \quad \text{for } N \ge 4. \tag{4.28}$$

Once again, Table 1 on page 274 shows that we have the same bound if $N < 4$.

4.4.3. $2\nu \le j \le N-1$. We invoke Lemma 4.4 again to get

$$D(N,\nu)\kappa(j) \le \tfrac{3}{2}\phi(6) I_1 + \tfrac{3}{2}\phi(4) I_2 + 3\phi(4) I_3 =: J,$$

where

$$I_1 = g_0 + g_1 + g_{2\nu-1} + g_{2\nu}, \qquad I_2 = \sum_{k=1}^{2\nu-2} (g_k + g_{k+1}), \qquad I_3 = \sum_{k=2\nu}^{N-1} (g_k + g_{k+1}).$$

PROPOSITION 4.8. *For* $2\nu \le j \le N-1$,

$$I_1 = (1 + B_{2\nu} + B_{2\nu-1})(A_{j-2\nu} + A_{N-j}) + B_j + B_{j-1} + B_{N-j} + B_{N-j+1}$$
$$+ B_{j-2\nu} + B_{j-2\nu+1} + B_{N-j+2\nu} + B_{N-j+2\nu-1},$$
$$I_2 = A_{j-1} - A_{j-2\nu+1} + (A_{j-2\nu} + A_{N-j})(A_{2\nu-1} - 2) + A_{N-j+2\nu-1} - A_{N-j+1},$$
$$I_3 = D(N,\nu) + (1 - A_{2\nu})(A_{j-2\nu} + A_{N-j}) - \tfrac{3}{2} B_{2\nu}(B_{N-j} + B_{j-2\nu}).$$

Proof. Insert the formulae for g from Proposition 4.1 and use Lemmas 3.2, 3.4 and the recurrences (3.5), (3.6), (3.11) and (3.12). ∎

If we apply the recurrences (3.5), (3.6), (3.11) and (3.12), Lemma 3.4 and Proposition 4.8 to J, we see that it simplifies to (recall that $s = \frac{3}{2}(\phi(6) - \phi(4)) = \frac{138}{1225}$ and $\phi(4) = \frac{17}{25}$)

$$J = \frac{51}{25} D(N,\nu) + s[3B_j - A_j + (A_{j-2\nu} + A_{N-j})(3B_{2\nu} - A_{2\nu})$$
$$+ 3B_{N-j+2\nu} - A_{N-j+2\nu} + A_{N-j+1} + A_{j-2\nu+1}].$$

Remember that $2\nu \le j \le N-1$. Since the functions $j \mapsto A_{N-j+1} + A_{j-2\nu+1}$, $j \mapsto 3B_j - A_j + 3B_{N-j+2\nu} - A_{N-j+2\nu}$, $j \mapsto A_{j-2\nu} + A_{N-j}$ are convex, they attain their maximum at the border, in our case for $j = 2\nu$, so

$$J \le \frac{51}{25} D(N,\nu) + s[6B_{2\nu} - 2A_{2\nu} + 3B_N - A_N + A_{N-2\nu}(3B_{2\nu} - A_{2\nu}) + 2 + A_{N-2\nu+1}].$$

For $2\nu = N-1$, we see with an estimate utilizing Lemma 3.3 and the recurrences for A_k and B_k that $\kappa(j) \le \frac{J}{D(N,\nu)} \le \frac{51}{25} + \frac{3}{4}s \approx 2.1245$ for $N \ge 4$. If $2\nu \le N-2$, we use the estimates

- $\sqrt{3}B_{2\nu} \le A_{2\nu}$ (Lemma 3.3),
- $3B_N \le \sqrt{3}A_N$ (Lemma 3.3),
- $A_{N-2\nu+1} \le A_{N-1}$,
- $A_{N-2\nu} \le \sqrt{3}B_{N-2\nu} + 1$ (Lemma 3.3),
- $3B_{N-2\nu}B_{2\nu} \le A_N/2$ (Lemmas 3.3 and 3.4),
- $A_{N-1} \le \lambda^{-1}(A_N + \sqrt{3})$ (Lemma 3.3),
- $B_{2\nu} \le B_{2\nu}B_{N-2\nu}/4$ ($2\nu \le N-2$)

and obtain further

$$\begin{aligned} J - \frac{51}{25}D(N,\nu) &\le s[a_1 + a_2 A_N + a_3 B_{2\nu}B_{N-2\nu}] \\ &= s\left(a_1 + a_2 + \frac{a_2}{2}D(N,\nu) - \left(\frac{3a_2}{4} - a_3\right)B_{2\nu}B_{N-2\nu}\right) \end{aligned}$$

with $a_1 = 2 + \sqrt{3}\lambda^{-1}$, $a_2 = \frac{3}{2}(\sqrt{3}-1) + \lambda^{-1}$, $a_3 = \frac{3}{4}(3-\sqrt{3})$. Since $\frac{3}{4}a_2 - a_3 > 0$, we conclude that

$$\kappa(j) = \frac{J}{D(N,\nu)} \le \frac{51}{25} + \frac{sa_2}{2} \approx 2.117 \qquad \text{for } N \ge 5.$$

For $N < 5$, see Table 1 on page 274.

Summary. Up to now we have shown that in particular for $N \ge 5$ even, for all $1 \le \nu \le (N-1)/2$ and all $0 \le j \le N-1$ (*except* the case $\nu = 1, j = 1$)

$$\kappa(j) \le 2.130411 \qquad \text{(see (4.28))}. \tag{4.29}$$

4.4.4. $\kappa(j)$ *for* N *odd.* Now let N be odd. We recall the formula (4.10) for $\kappa(j)$,

$$\kappa(j) = D(N,\nu)^{-1}\left[\frac{3}{2}\sum_{k=0}^{2\nu-1}(|g_k| + |g_{k+1}|)\cdot\xi_{j,k} + 3\sum_{k=2\nu}^{N-1}(|g_k| + |g_{k+1}|)\cdot\xi_{j,k}\right],$$

where

$$\xi_{j,k} = \begin{cases} 1 & \text{if } \operatorname{sgn} a_{j,k} = \operatorname{sgn} a_{j,k+1}, \\ \phi(|g_{k+1}|/|g_k|) & \text{otherwise.} \end{cases}$$

If we write this formula in the form $\kappa(j) = \sum_{k=0}^{N-1} s_k$, then every summand

s_k admits the (trivial) bound

$$s_k \le \frac{3(|g_k| + |g_{k+1}|)}{D(N,\nu)},$$

since $\phi(t) \le 1$ for all $t \ge 0$. We now denote by $D^e(N,\nu)$ and g_k^e the expressions for $D(N,\nu)$ and g_k respectively, but for N even. That is, write 1 instead of $(-1)^N$ in (4.9) and in the expressions for g_k in Proposition 4.1, no matter if N is even or odd. Then we get

$$s_k \le \frac{3(g_k^e + g_{k+1}^e)}{D^e(N,\nu)}. \tag{4.30}$$

Easy estimates for g_k^e and $D^e(N,\nu)$ supply now

$$\frac{3(g_k^e + g_{k+1}^e)}{D^e(N,\nu)} \le 10^{-3}, \tag{4.31}$$

provided $(N-3)/2 \le |k-j| \le (N+3)/2$ and $N \ge 19$. So, let $N \ge 19$. Define the index set $\Lambda = \{(N-3)/2, (N-1)/2, (N+1)/2, (N+3)/2\}$. Then

$$\kappa(j) = \sum_{k=0}^{N-1} s_k = \sum_{k \notin \Lambda} s_k + \sum_{k \in \Lambda} s_k.$$

We find that $\sum_{k\notin\Lambda} s_k$ equals

$$D(N,\nu)^{-1}\Bigg[\frac{3}{2}\sum_{\substack{k=0\\ k\notin\Lambda}}^{2\nu-1} (|g_k| + |g_{k+1}|)\phi(|g_{k+1}|/|g_k|) + 3\sum_{\substack{k=2\nu\\ k\notin\Lambda}}^{N-1} (|g_k| + |g_{k+1}|)\phi(|g_{k+1}|/|g_k|)\Bigg]$$

and by the above considerations this is at most

$$D^e(N,\nu)^{-1}\Bigg[\frac{3}{2}\sum_{\substack{k=0\\ k\notin\Lambda}}^{2\nu-1} (g_k^e + g_{k+1}^e)\phi(|g_{k+1}|/|g_k|) + 3\sum_{\substack{k=2\nu\\ k\notin\Lambda}}^{N-1} (g_k^e + g_{k+1}^e)\phi(|g_{k+1}|/|g_k|)\Bigg].$$

We apply Lemma 4.5 to see that the terms $\phi(|g_{k+1}|/|g_k|)$ admit the same bounds as for N even. Thus, if we first apply the estimate and then omit the restriction $k \notin \Lambda$ for the summation range, we arrive at estimating the same sum as for N even. Since for N even we got the bound (4.29) (except for $\nu = j = 1$), we finally obtain

$$\sum_{k\notin\Lambda} s_k \le 2.130411.$$

Table 1. Values of $\|P_{n,\nu}\|_\infty$ for different values of $\nu, N = n + \nu$ obtained with MATHEMATICA. The numbers are rounded to the last digit. We have with the same precision $\gamma \approx 2.14023734$. The MATHEMATICA notebook used to generate this table may be found attached at arXiv:1103.1950v3.

$\nu \rightarrow$ / $N \downarrow$	0	1	2	3	4	5	6	7	8
2	1.66666667								
3	1.77777778	1.84444444							
4	1.88888889	2.00000000							
5	1.94696970	2.06951872	1.99530864						
6	1.96835017	2.09951691	2.03615841						
7	1.98631436	2.12227384	2.05943912	2.03242817					
8	1.99137719	2.12904795	2.06731688	2.05587710					
9	1.99637151	2.13550178	2.07350359	2.06635304	2.04175181				
10	1.99767916	2.13721808	2.07535577	2.06916162	2.06184314				
11	1.99903054	2.13897416	2.07709926	2.07093598	2.06877403	2.04423294			
12	1.99937738	2.13942680	2.07756643	2.07147842	2.07063242	2.06343762			
13	1.99974043	2.13989929	2.07804184	2.07192928	2.07140616	2.06942343	2.04489705		
14	1.99983312	2.14002005	2.07816469	2.07206051	2.07160716	2.07106530	2.06386325		
15	1.99993046	2.14014679	2.07829271	2.07218617	2.07173511	2.07157865	2.06959951	2.04507495	
16	1.99995528	2.14017911	2.07832546	2.07221958	2.07177393	2.07171306	2.07118008	2.06397716	
17	1.99998137	2.14021308	2.07835981	2.07225375	2.07180634	2.07176873	2.07162518	2.06964688	2.04512262
18	1.99998802	2.14022174	2.07836857	2.07226256	2.07181575	2.07178315	2.07174416	2.07121066	2.06400767
19	1.99999501	2.14023084	2.07837778	2.07227176	2.07182477	2.07179234	2.07178110	2.07163781	2.06965959
20	1.99999679	2.14023316	2.07838012	2.07227411	2.07182717	2.07179513	2.07179076	2.07175241	2.07121883

The remaining sum $\sum_{k\in\Lambda} s_k$ is now estimated using (4.30) and (4.31):

$$\sum_{k\in\Lambda} s_k \le 4\cdot 10^{-3},$$

so altogether we get

$$\kappa(j) \le 2.134411$$

for all $N \ge 19$, ν, j (no matter if N is odd or even) *except* the case $\nu = j = 1$.

Summary. Combining the present section with Sections 4.1 and 4.3 shows that for all $N \ge 19$, $0 \le \nu \le n$ and $0 \le j \le N-1$, we have the bound

$$\kappa(j) < \gamma.$$

The numerical results of Table 1 yield this estimate for $N \le 20$, so we get the first assertion of our main theorem (i.e. $\|P_{n,\nu}\|_\infty < \gamma$ for all $n \in \mathbb{N}, 0 \le \nu \le n$). The asymptotic value γ for $\|P_{n,1}\|_\infty$ (as $n \to \infty$) was already identified in Section 4.3. So, the proof Theorem 2.1 is complete.

Appendix

Proof of Lemma 4.4. In order to prove (4.23) and (4.24) we recall the bounds (3.9) and (3.10):

$$B_{l+1} \le 4B_l \quad \text{for } l \ge 1, \quad A_{l+1} \le 4A_l \quad \text{for } l \ge 0. \tag{A.1}$$

We consider several cases depending on the values of j, k, ν:

I. $0 \le j \le 2\nu - 1$

I.a. $k = 0$, $j \ne 0$. From (A.1) and the formula for g_k in Proposition 4.1, we get $6g_1 - g_0 \ge 0$ immediately. For the reverse inequality we get, since we assumed $j \ge 1$,

$$\begin{aligned} 6g_0 - g_1 &= (12B_j - 2B_{j-1}) + (6B_{N-j} - B_{N-j+1}) \\ &\quad + B_{2\nu-j}(6A_{N-2\nu} - A_{N-2\nu+1}) \\ &\quad - A_{N-j} - 3B_{2\nu-j}B_{N-2\nu} \\ &\ge 10B_j + 2B_{N-j} + 2B_{2\nu-j}A_{N-2\nu} \\ &\quad - A_{N-j} - 3B_{2\nu-j}B_{N-2\nu}, \end{aligned}$$

by (A.1). Since $2B_{N-j} \ge A_{N-j}$ (for $N - j \ge 1$, which is satisfied) and $A_{N-2\nu} \ge \sqrt{3}B_{N-2\nu}$, we see that this is ≥ 0.

I.b. $1 \le k \le j-1$. Again, with (A.1) and the assumption $k \le j-1$ we get $4g_k - g_{k+1} \ge 0$ immediately. The second inequality is only critical for $k = j-1$ and in this case we get (with (A.1))

$$\begin{aligned} 4g_{k+1} - g_k &= -2 + 4B_N - B_{N-1} + (\text{positive term}) \\ &\ge 3B_N - 2 \ge 0 \quad \text{for } N \ge 1. \end{aligned}$$

I.c. $j \le k \le 2\nu - 2$. For $4g_{k+1} - g_k \ge 0$, it suffices to use (A.1), and similarly for $4g_k - g_{k+1} \ge 0$ in the case $k \ne j$. For $k = j$,

$$4g_k - g_{k+1} = -2 + 4B_N - B_{N-1} + (\text{positive term}) \ge 0 \quad \text{for } N \ge 1.$$

I.d. $k = 2\nu - 1$. As in I.c, the distinction between the cases $k = j$ and $k > j$ gives $6g_k - g_{k+1} \ge 0$. On the other hand (recall that $k = 2\nu - 1$, $j \le 2\nu - 1$)

$$\begin{aligned} 6g_{k+1} - g_k &= (12B_{2\nu-j} - 2B_{2\nu-1-j}) + (6B_{N-2\nu+j} - B_{N-2\nu+1+j}) \\ &\quad - A_{N-2\nu+j} + (6B_j A_{N-2\nu} - B_j A_{N-2\nu+1}) - 3B_j B_{N-2\nu} \\ &\ge 0 + 2B_{N-2\nu+j} - A_{N-2\nu+j} + 2B_j A_{N-2\nu} - 3B_j B_{N-2\nu}, \end{aligned}$$

by (A.1). The inequalities $2B_{N-2\nu+j} \ge A_{N-2\nu+j}$ (observe $N - 2\nu + j \ge 1$) and $A_{N-2\nu} \ge \sqrt{3} B_{N-2\nu}$ then yield $6g_{k+1} - g_k \ge 0$.

I.e. $2\nu \le k \le N-1$. Since now $k > j$, an application of (A.1) suffices for $4g_k - g_{k+1} \ge 0$. The same reasoning provides $4g_{k+1} - g_k \ge 0$ in the case $k \ne N - 1 \vee j \ne 0$, and for $k = N - 1$, $j = 0$ we have

$$\begin{aligned} 4g_{k+1} - g_k &= (4B_N - B_{N-1}) + B_{2\nu}(4A_{N-2\nu} - A_{N-1-2\nu}) - 1 \\ &\ge 3B_N - 1 \ge 0 \quad \text{for } N \ge 1. \end{aligned}$$

II. $2\nu \le j \le N - 1$

II.a. $k = 0$. Again, $6g_{k+1} - g_k \ge 0$ is a trivial consequence of (A.1). Furthermore, by (A.1),

$$\begin{aligned} 6g_k - g_{k+1} &= 6B_{N-j} - B_{N-j+1} - A_{N-j} + (\text{positive terms}) \\ &\ge 2B_{N-j} - A_{N-j} \ge 0. \end{aligned}$$

II.b. $1 \le k \le 2\nu - 2$. Here, both $4g_k - g_{k+1} \ge 0$ and $4g_{k+1} - g_k \ge 0$ are consequences of (A.1).

II.c. $k = 2\nu - 1$. The bound $6g_k - g_{k+1} \ge 0$ follows from (A.1). For the converse we get

$$\begin{aligned} 6g_{k+1} - g_k &= (6B_{j-2\nu} - B_{j-2\nu+1}) - A_{j-2\nu} \\ &\quad + (6B_{N-j+2\nu} - B_{N-j+2\nu-1}) + (\text{positive term}). \end{aligned}$$

If $j > 2\nu$, we have $6B_{j-2\nu} - B_{j-2\nu+1} \ge 2B_{j-2\nu}$, which is greater than $A_{j-2\nu}$; if $j = 2\nu$, then $6g_{k+1} - g_k \ge -2 + 5B_N \ge 0$.

II.d. $2\nu \le k \le j - 1$. For $k > 2\nu$, $4g_k - g_{k+1} \ge 0$ is a consequence of (A.1). If $k = 2\nu$, we have

$$4g_k - g_{k+1} = 2B_{N-j+2\nu} - \frac{3}{2} B_{2\nu} B_{N-j} + (\text{positive term}).$$

Since $2B_{N-j+2\nu} \ge A_{N-j+2\nu}$ and $3B_{2\nu} B_{N-j} = A_{N-j+2\nu} - A_{2\nu} A_{N-j} \le A_{N-j+2\nu}$, we get $4g_k - g_{k+1} \ge 0$. The converse estimate $4g_{k+1} - g_k \ge 0$ follows once more from (A.1) provided

$k < j - 1$. If on the other hand $k = j - 1$, we see that

$$4g_{k+1} - g_k = -1 + A_{N-j}(4B_{k+1} - B_k) + (\text{positive term}) \geq 0,$$

since $k = j - 1 \geq 2\nu \geq 2$.

II.e. $j \leq k \leq N-1$. The estimate $4g_k - g_{k+1} \geq 0$ follows from (A.1) if $k > j$, as does $4g_{k+1} - g_k \geq 0$ for $k < N - 1$. For $k = j$ resp. $k = N - 1$, the calculations are similar to those in II.d. ∎

Proof of Lemma 4.5. If N is odd, the proof is similar to that of Lemma 4.4 but with twice as many case distinctions, since one has to consider the cases $|k - j| \leq (N - 5)/2$ and $|k - j| \geq (N + 5)/2$ separately. We pick out one special case and omit all the others since they involve very similar arguments. We will treat values of ν, k, j where $2\nu \leq k \leq j - 1$ and consider the two cases mentioned above:

I. $|j-k| \leq (N - 5)/2$. From Proposition 4.1 and Remark 4.2 we obtain

$$\begin{aligned}|g_k| &= -B_{j-k} + A_{k-2\nu}B_{N-j+2\nu} + A_{N-j}B_k + \tfrac{3}{2}B_{k-2\nu}B_{2\nu}B_{N-j},\\ |g_{k+1}| &= -B_{j-k-1} + A_{k+1-2\nu}B_{N-j+2\nu} + A_{N-j}B_{k+1}\\ &\quad + \tfrac{3}{2}B_{k+1-2\nu}B_{2\nu}B_{N-j}.\end{aligned}$$

The inequality $4|g_k| - |g_{k+1}| \geq 0$ for $k = 2\nu$ is a simple consequence of Lemmas 3.3 and 3.4. Utilizing Lemma 3.3, we get, for $k \geq 2\nu + 1$,

$$4|g_k| - |g_{k+1}| \geq -4B_{j-k} + (4 - \lambda)A_{k-2\nu}B_{N-j+2\nu}. \tag{A.2}$$

Since $N - j + 2\nu \geq 3$, $A_3 = 26$ and $2\nu \leq k$, Lemma 3.4 implies that

$$A_{k-2\nu} \leq \frac{A_{k-2\nu}A_{N-j+2\nu}}{A_3} = \frac{A_{k-2\nu}A_{N-j+2\nu}}{26} \leq \frac{A_{N-j+k}}{26}.$$

This estimate, the definition of the recurrences A_k and B_k and Lemmas 3.3 and 3.4 yield

$$\begin{aligned}A_{k-2\nu}B_{N-j+2\nu} &\geq \frac{1}{\sqrt{3}}(A_{k-2\nu}A_{N-j+2\nu} - A_{k-2\nu})\\ &\geq \frac{1}{2\sqrt{3}}(A_{N-j+k} - 2A_{k-2\nu}) \geq \frac{2\sqrt{3}}{13}A_{N-j+k}.\end{aligned}$$

Thus, this estimate and (A.2) imply

$$\begin{aligned}4|g_k| - |g_{k+1}| &\geq (4 - \lambda)\frac{2\sqrt{3}}{13}A_{N-j+k} - 4B_{j-k}\\ &\geq (4 - \lambda)\frac{6}{13}B_{N-j+k} - 4B_{j-k}\\ &\geq \left(\lambda^5(4 - \lambda)\frac{6}{13} - 4\right)B_{(N-5)/2} \geq 0,\end{aligned}$$

if we use Lemma 3.3 in conjunction with our hypothesis $|j-k| \leq (N-5)/2$. The estimate $4|g_{k+1}| - |g_k| \geq 0$ follows analogously.

II. $|j-k| \geq (N+5)/2$. From Proposition 4.1 and Remark 4.2 we obtain

$$|g_k| = B_{j-k} - A_{k-2\nu}B_{N-j+2\nu} - A_{N-j}B_k - \tfrac{3}{2}B_{k-2\nu}B_{2\nu}B_{N-j},$$
$$|g_{k+1}| = B_{j-k-1} - A_{k+1-2\nu}B_{N-j+2\nu} - A_{N-j}B_{k+1} - \tfrac{3}{2}B_{k+1-2\nu}B_{2\nu}B_{N-j}.$$

If we employ Lemma 3.3 three times, we obtain

$$4|g_k| - |g_{k+1}| \geq 3B_{j-k} - (4-\lambda)[B_{N-j+2\nu}A_{k-2\nu} + B_kA_{N-j} + \tfrac{3}{2}B_{2\nu}B_{N-j}B_{k-2\nu}]$$

Since by Lemma 3.4 every summand in the square brackets is majorized by B_{N-j+k}, we finally get

$$4|g_k| - |g_{k+1}| \geq 3(B_{j-k} - (4-\lambda)B_{N-j+k}) \geq 0,$$

by the hypothesis $|j-k| \geq (N+5)/2$. For the inequality $4|g_{k+1}| - |g_k| \geq 0$, we first omit some positive terms to get

$$4|g_{k+1}| - |g_k| \geq 4B_{j-k-1} - B_{j-k} - 4A_{k+1-2\nu}B_{N-j+2\nu} - 4A_{N-j}B_{k+1} - 6B_{k+1-2\nu}B_{2\nu}B_{N-j}.$$

As above, Lemmas 3.4 and 3.3 yield

$$4|g_{k+1}| - |g_k| \geq 4B_{j-k-1} - B_{j-k} - 10B_{N-j+k+1} \geq (4-\lambda)B_{j-k-1} - 1 - 10B_{N-j+k+1}.$$

Now we employ again Lemma 3.3 and the fact that $|j-k| \geq (N+5)/2$ to get

$$4|g_{k+1}| - |g_k| \geq (\lambda^3(4-\lambda) - 10)B_{(N-3)/2} - 1 \geq 0. \blacksquare$$

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