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Majorization of sequences, sharp vector Khinchin inequalities, and bisubharmonic functions

by

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Abstract. Let $X = \sum_{i=1}^{k} a_i U_i$, $Y = \sum_{i=1}^{k} b_i U_i$, where the U_i are independent random vectors, each uniformly distributed on the unit sphere in $\mathbb{R}^n$, and a_i, b_i are real constants. We prove that if $\{b_i^2\}$ is majorized by $\{a_i^2\}$ in the sense of Hardy–Littlewood–Pólya, and if $\Phi : \mathbb{R}^n \to \mathbb{R}$ is continuous and bisubharmonic, then $E\Phi(X) \le E\Phi(Y)$. Consequences include most of the known sharp L^2-L^p Khinchin inequalities for sums of the form X. For radial Φ, bisubharmonicity is necessary as well as sufficient for the majorization inequality to always hold. Counterparts to the majorization inequality exist when the U_i are uniformly distributed on the unit ball of $\mathbb{R}^n$ instead of on the unit sphere.

1. Main results. In this first section, we set up notation and state our main results, about sums of independent random vectors uniformly distributed on spheres in $\mathbb{R}^n$. History and background are provided in §2, Theorems 1, 2, 3 are proved in §3, 4, 5, and §6 contains a formula which permits transfer of the results in §1 to sums of independent random vectors uniformly distributed on balls in $\mathbb{R}^n$.

Throughout the paper, $a = (a_1, \ldots, a_k)$ and $b = (b_1, \ldots, b_k)$ will denote k-tuples of real numbers. Following the terminology of [MO 79], we say that a is *majorized* by b, and write $a \prec b$, or $\{a_i\} \prec \{b_i\}$, if

$$\sum_{i=1}^{k} a_i = \sum_{i=1}^{k} b_i, \quad \text{and} \quad \sum_{i=1}^{j} a_i^* \le \sum_{i=1}^{j} b_i^*, \quad j = 1, \ldots, k,$$

where $\{a_i^*\}$ denotes the sequence $\{a_i\}$ rearranged in decreasing order.

Let Δ denote the Laplace operator in $\mathbb{R}^n$. A continuous function Φ : $\mathbb{R}^n \to \mathbb{R}$ is said to be *bisubharmonic* in $\mathbb{R}^n$ if the distribution $\Delta\Delta\Phi$ equals

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a positive Radon measure on $\mathbb{R}^n$. Equivalently, Φ is bisubharmonic if the distribution $\Delta\Phi$ equals a subharmonic function on $\mathbb{R}^n$, in the sense of distributions. If $\Phi \in C^4(\mathbb{R}^n, \mathbb{R})$, then Φ is bisubharmonic if and only if $\Delta\Delta\Phi \geq 0$ at every point in $\mathbb{R}^n$. A function is said to be *bisuperharmonic* if its negative is bisubharmonic.

For $n \geq 1$, let $U_1, \ldots, U_k$ be independent $\mathbb{R}^n$-valued random variables defined on some probability space, each of which is uniformly distributed on the unit sphere $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\} \subset \mathbb{R}^n$. We use $|x|$ to denote the Euclidean norm of x. When $n = 1$, the U_i take on each of ± 1 with probability $1/2$ and are called *symmetric Bernoulli random variables.* Set, for general n,

$$X = \sum_{i=1}^{k} a_i U_i, \quad Y = \sum_{i=1}^{k} b_i U_i.$$

THEOREM 1. *Let Φ be bisubharmonic and continuous on $\mathbb{R}^n$. If $\{b_i^2\} \prec \{a_i^2\}$, then*

$$E\Phi(X) \leq E\Phi(Y). \tag{1.1}$$

If Φ is bisuperharmonic and continuous on $\mathbb{R}^n$, then Theorem 1 implies that the inequality in (1.1) reverses. The message of the theorem is that spreading out the coefficients in the sum defining X, while keeping their sum of squares constant, increases the Φ moment, when Φ is bisubharmonic. A counterpart of Theorem 1 for balls will be discussed in §6.

In case Φ is radial, that is, $\Phi(x)$ depends only on $|x|$, Theorem 1 admits a converse:

THEOREM 2. *Suppose that $\Phi : \mathbb{R}^n \to \mathbb{R}$ is continuous and radial. If* (1.1) *holds for every pair of k-tuples a, b with $\{b_i^2\} \prec \{a_i^2\}$ and every $k \geq 2$, then Φ is bisubharmonic in $\mathbb{R}^n$.*

For any $\Phi \in C(\mathbb{R}^n, \mathbb{R})$, let Φ_1 be the radial function on $\mathbb{R}^n$ obtained by setting $\Phi_1(x)$ equal to the mean value of Φ on the sphere $|x|S^{n-1}$. We will call Φ_1 the *radialization* of Φ. Since the distribution of X is invariant under rotations, we have $E\Phi(X) = E\Phi_1(X)$. Moreover, if Φ is continuous and bisubharmonic, then so is Φ_1. Using these facts, one sees that Theorems 1 and 2 are equivalent to the following "if and only if" statement:

$$E\Phi(X) \leq E\Phi(Y) \text{ for every } a,\, b \text{ with } \{b_i^2\} \prec \{a_i^2\} \quad \Leftrightarrow \quad \Phi_1 \text{ is bisubharmonic.}$$

The statement above characterizes continuous functions Φ for which a majorization inequality holds for Φ moments. The next statement, a close neighbor to Theorems 1 and 2, gives a characterization of continuous functions which are bisubharmonic.

THEOREM 3. *Let $\Phi : \mathbb{R}^n \to \mathbb{R}$ be continuous. Then Φ is bisubharmonic in $\mathbb{R}^n$ if and only if the function*

$$t \mapsto \int_{S^{n-1}} \Phi(x_0 + t^{1/2}x)\, dx$$

is convex on $(0, \infty)$ for every $x_0 \in \mathbb{R}^n$.

CONVENTION. In integrals over S^{n-1}, dx will denote the uniform probability measure on S^{n-1}. In integrals over the unit ball $\mathbb{B}^n \subset \mathbb{R}^n$, dx will denote the uniform probability measure on $\mathbb{B}^n$.

For each k-tuple a with $\sum_{i=1}^k a_i^2 = 1$ we have $(k^{-1}, \ldots, k^{-1}) \prec \{a_i^2\} \prec (1, 0, \ldots, 0)$, where the first and third sequences are k-tuples. Moreover, the $(k+1)$-tuple $((k+1)^{-1}, \ldots, (k+1)^{-1})$ is majorized by the $(k+1)$-tuple $(k^{-1}, \ldots, k^{-1}, 0)$. For $n \geq 1$, $k \geq 1$, define

$$Z_{n,k} = k^{-1/2} \sum_{i=1}^k U_i,$$

and define Z_n to be the $\mathbb{R}^n$-valued random variable whose components are independent mean-zero real normal random variables, each with variance $1/n$. Let Φ be continuous and bisubharmonic. Then Theorem 1 implies that $E\Phi(Z_{n,k})$ increases as k increases when n is fixed. In §3, we will show that the radialization Φ_1 of Φ satisfies

$$E((\Phi_1(Z_n))^-) < \infty, \tag{1.2}$$

so that $E\Phi_1(Z_n)$ is well defined, and that

$$\lim_{k\to\infty} E\Phi(Z_{n,k}) = E\Phi_1(Z_n). \tag{1.3}$$

Note that for $X = \sum_{i=1}^k a_i U_i$, we have $E|X|^2 = \sum_{i=1}^k a_i^2$. In particular, $E|Z_{n,k}|^2 = E|U_1|^2 = 1$. For fixed n and appropriate Φ, the corollaries to follow assert two things: (a) Among all X with fixed L^2 norm and at most k summands, the extremal Φ moments are achieved by multiples of U_1 and $Z_{n,k}$. (b) Among all X with fixed L^2 norm, the supremum and infimum of the Φ moments are furnished by multiples of U_1 and Z_n.

COROLLARY 1. *If Φ is continuous and bisubharmonic on $\mathbb{R}^n$ and $E|X|^2 = 1$, then*

$$E\Phi(U_1) \leq E\Phi(X) \leq E\Phi(Z_{n,k}) \leq E\Phi_1(Z_n). \tag{1.4}$$

The quantities on the right and left ends are the best possible constants which work simultaneously for all k and fixed n.

The inequalities follow from Theorem 1 and (1.3). The left-hand constant is obviously optimal; optimality of the right-hand constant follows from (1.3). The right-hand constant is sometimes ∞. But if it is finite, then $E|\Phi(Z_n)|$ is also finite, and $E\Phi_1(Z_n) = E\Phi(Z_n)$.

Of course, one could replace the hypothesis $E|X|^2 = 1$ with $E|X|^2 =$ some fixed number M^2, provided a factor M is inserted at appropriate places in the inequalities.

Let us consider the pth moment case: $\Phi(x) = |x|^p$, $p > 0$. For radial Φ on $\mathbb{R}^n$, we have $\Delta\Phi = \Phi_{rr} + ((n-1)/r)\Phi_r$, where the subscript r denotes radial differentiation. Thus, for $x \in \mathbb{R}^n \setminus \{0\}$,

$$\begin{aligned} \Delta|x|^p &= p(p+n-2)|x|^{p-2}, \\ \Delta\Delta|x|^p &= p(p+n-2)(p-2)(p+n-4)|x|^{p-4}. \end{aligned} \tag{1.5}$$

The task of deciding which functions $|x|^p$ are bisub- or bisuperharmonic on all of $\mathbb{R}^n$ is complicated by the singularity at the origin. One way to overcome the difficulty is to consider the functions $u_\varepsilon(x) = (|x|^2 + \varepsilon)^{p/2}$. Then $u_\varepsilon \in C^\infty(\mathbb{R}^n)$ for $\varepsilon > 0$, $p > 0$, and for $x \in \mathbb{R}^n$,

$$\Delta u_\varepsilon(x) = p(p-2)(r^2+\varepsilon)^{p/2-4}(Ar^4 + Br^2 + C),$$

where $r = |x|$ and

$$A = (p+n-2)(p+n-4), \qquad B = 2\varepsilon(n+2)(p+n-4), \qquad C = \varepsilon^2 n(n+2).$$

We deduce that, when $n = 1$, u_ε is bisubharmonic for $p \geq 3$; when $n \geq 2$, u_ε is bisubharmonic for $p \geq 2$; when $n = 3$, u_ε is bisuperharmonic for $1 \leq p \leq 2$; and when $n \geq 4$, u_ε is bisuperharmonic for $0 < p \leq 2$. Since $u_\varepsilon \to |x|^p$ uniformly on compact subsets of $\mathbb{R}^n$ as $\varepsilon \to 0$, it follows that $|x|^p$ is bisub- or bisuperharmonic for the same range of p and n. From Corollary 1 we obtain

COROLLARY 2. *Let X be as in Corollary* 1. *If $p \geq 3$ when $n = 1$, or if $p \geq 2$ when $n \geq 2$, then*

$$1 \leq E|X|^p \leq E|Z_{n,k}|^p \leq E|Z_n|^p = \frac{\Gamma((p+n)/2)}{\Gamma(n/2)}\, 2^{p/2} n^{-p/2}. \tag{1.6}$$

Moreover, when $n = 3$ and $1 \leq p \leq 2$ all the inequalities reverse, and when $n \geq 4$ and $0 < p \leq 2$ all the inequalities reverse. In each of the stated cases for n and p, the quantities on the right and left ends are the best possible constants which work simultaneously for all k and fixed n.

For p and n not covered by Corollary 2, $|x|^p$ is neither bisub- nor bisuperharmonic on $\mathbb{R}^n$. Consider, for example, $n = 2$ and $0 < p < 2$. One can show that the distributional Laplacian of $|x|^p$ on $\mathbb{R}^2$ is the locally integrable function $p^2|x|^{p-2}$. This function is subharmonic in $\mathbb{R}^2 \setminus \{0\}$, but is not a.e. equal to a subharmonic function in $\mathbb{R}^2$, since its limit as $x \to 0$ is ∞. Thus, $|x|^p$ is neither bisub- nor bisuperharmonic on $\mathbb{R}^2$.

Corollary 1 also produces sharp logarithmic Khinchin inequalities, provided the dimension n is sufficiently large. For $x \in \mathbb{R}^n \setminus \{0\}$ and $r = |x|$, the reader is invited to verify that

$$\Delta \log(1+|x|) = \frac{n-1+(n-2)r}{r(1+r)^2}$$

and that

$$r^3(1+r)^4 \Delta\Delta \log(1+|x|) = -\sum_{j=0}^{3} c_j r^j, \tag{1.7}$$

where

$$c_0 = (n-1)(n-3), \qquad c_1 = 4(n-1)(n-3),$$
$$c_2 = (n-1)(5n-19), \quad c_3 = 2(n-2)(n-4).$$

If $n \geq 4$ then all the c_i are nonnegative. Thus, $\log(1+|x|)$ is bisuperharmonic in $\mathbb{R}^n \setminus \{0\}$. By an approximation argument like the one for pth moments, or by other means, one can show that $\log(1+|x|)$ is bisuperharmonic on all of $\mathbb{R}^n$. The same is true for $\log(A+|x|)$ for each nonnegative constant A. Corollary 1 implies

Corollary 3. *Let X be as in Corollary 1, and $n \geq 4$. Then for every nonnegative constant A,*

$$\begin{aligned} E\log(A+|Z_n|) &\leq E\log(A+|Z_{n,k}|) \\ &\leq E\log(A+|X|) \leq \log(A+1). \end{aligned} \tag{1.8}$$

The quantities on the right and left ends are the best possible constants which work simultaneously for all k and fixed $n \geq 4$.

2. A brief history of sharp Khinchin inequalities. First, we review work in dimension $n = 1$, so that $X = \sum_{i=1}^k a_i U_i$ is a real linear combination of symmetric Bernoulli random variables. We remind the reader that Z_1 denotes a real-valued standard normal random variable, and that

$$Z_{1,k} = k^{-1/2} \sum_{i=1}^{k} U_i.$$

In this whole section, we shall always assume the L^2-normalization

$$E|X|^2 = \sum_{i=1}^{k} a_i^2 = 1.$$

Khinchin [Kh 23] proved existence of constants A_p and B_p, $0 < p < \infty$, such that $A_p \leq E|X|^p \leq B_p$. We shall denote the best such constants also by A_p, B_p, and call them the *best L^2-L^p constants.* Hölder's inequality implies that $B_p = 1$ for $0 < p \le 2$ and $A_p = 1$ for $2 \le p < \infty$. The first nontrivial best constants were found by Whittle [W 60], who asserted that $B_p = E|Z_1|^p$ for $2 < p < \infty$. However, as pointed out in [F 97, p. 998], Whittle's proof is valid only for $3 \le p < \infty$. The "Gaussian bound" $B_p = E|Z_1|^p$, $p \ge 3$, was proved again by Young [Y 76]. Earlier, Stechkin [Ste 61] had proved $B_{2m} = E|Z_1|^{2m}$ for positive integers m. Presumably, Young and Stechkin were unaware of Whittle's paper. Szarek [Sz 76] proved that $A_p = E|Z_1|^p$ for $p = 1$ or p slightly larger than 1.

To find the remaining best constants turned out to be much more difficult. This feat was accomplished by Haagerup [Ha 82], who proved that $B_p = E|Z_1|^p$ for all $p > 2$, while for $0 < p < 2$, A_p is the smaller of $E|Z_1|^p$ and $E|Z_{1,2}|^p$. Letting $p_1 \approx 1.8$ be the unique solution $p \in (0,2)$ to the equation $\Gamma\big(\frac{1}{2}(p+1)\big) = \frac{1}{2}\sqrt{\pi}$, the smaller is $E|Z_1|^p$ when $p_1 < p < 2$, and is $E|Z_{1,2}|^p$ when $0 < p < p_1$.

Haagerup's proof requires lots of integrals and estimates. An ingenious alternative proof for $0 < p < 2$, shorter but still complicated, can be found in [NP 96].

Turning now to majorization inequalities, still in dimension $n = 1$, set $Y = \sum_{i=1}^{k} b_i U_i$. Eaton [E 70] proved that $\{b_i^2\} \prec \{a_i^2\} \Rightarrow E\Phi(X) \le E\Phi(Y)$ for functions $\Phi : \mathbb{R} \to \mathbb{R}$ which satisfy a certain condition. Komorowski [Kom 88] showed that Eaton's condition is satisfied by $K(x) = |x|^p$ when $p \ge 3$, and thereby obtained the $n = 1$ case of our Corollary 2.

It turns out that Eaton's condition on Φ is in fact equivalent to convexity of Φ''. Thus, after tending to a few technicalities, one sees that Eaton's Theorem coincides with our Theorem 1 for $n = 1$.

When $n = 2$ the U_i are uniformly distributed on the unit circle S^1, and are sometimes called *Steinhaus random variables*. Suppressing their dependence on the dimension, we continue to denote the corresponding best L^2-L^p Khinchin constants by A_p and B_p. The trivial bounds $A_p = 1$ for $p \ge 2$, $B_p = 1$ for $p \le 2$ are still true. Haagerup (see [Pe 85, p. 151]) conjectured that $B_p = E|Z_2|^p$ for $p > 2$, and $A_p = \min(E|Z_2|^p, E|Z_{2,2}|^p)$ for $0 < p < 2$. Here, for $0 < p < 2$, the Gaussian is smaller if and only if $p_2 < p < 2$, where $p_2 \approx 0.48$. Haagerup's conjecture was verified by Sawa [S 85] for $p = 1$ or p close to 1. Sawa stated that he could prove $A_p = E|Z_2|^p$ for every $p \in [p_2, 2)$ and $B_p = E|Z_2|^p$ for every $p > 2$, but the proofs were never published.

Peškir [Peš 95] proved that when $n = 2$ one has the majorization inequality $\{b_i^2\} \prec \{a_i^2\} \Rightarrow E|X|^{2m} \le E|Y|^{2m}$ for positive integers m. A consequence is that $B_{2m} = E|Z_2|^{2m}$ when $n = 2$. König [Kö 98] and Culverhouse [C 98] independently confirmed that $B_p = E|Z_2|^p$ for every $p > 2$. In fact, they proved that the n-dimensional best L^2-L^p Khinchin constant B_p is $B_p = E|Z_n|^p$ for every $p > 2$ and $n \ge 2$. In addition, König proved that the best n-dimensional constant A_p satisfies $A_p = E|Z_n|^p$ for $1 \le p < 2$

when $n \geq 2$, and for $0 < p < 2$ when $n \geq 3$. All of these results follow from Theorem 1 and are included in Corollary 2 of the present paper, except for $1 \leq p < 2$ when $n = 2$, and $0 < p < 1$ when $n = 3$.

For each $n \geq 1$ and $p \geq 3$, the bound $B_p = E|Z_n|^p$ can also be obtained by clever application of Theorem 1 in [F 97]. We thank S. Kwapień for pointing this out to us.

Let $V_1, \ldots, V_k$ be independent random variables, each of which is uniformly distributed on the unit interval $[-1, 1]$. Latała and Oleszkiewicz [LO 95] proved that

$$\{b_i^2\} \prec \{a_i\}^2 \;\Rightarrow\; E\Big|\sum_{i=1}^{k} a_i V_i\Big|^p \leq E\Big|\sum_{i=1}^{k} b_i V_i\Big|^p \tag{2.1}$$

for $2 \leq p < \infty$, while for $1 \leq p \leq 2$ the inequality reverses. As with spheres, the majorization inequalities permit identification of the corresponding best Khinchin L^2-L^p constants for random variables of the form $\sum_{i=1}^{k} a_i V_i$. Culverhouse [C 98], building on ideas in [LO 95], obtained the analogues of (2.1) and its reverse inequality for V_i uniformly distributed on balls of any dimension when $1 \leq p < \infty$. [C 98] also contains majorization inequalities for $\Phi(\sum_{i=1}^{k} a_i V_i)$ and $\Phi(\sum_{i=1}^{k} b_i V_i)$ for some more general Φ, and the S^{n-1}-analogue of the majorization inequality (2.1) for $n \geq 2$ and $p \geq 2$.

The proofs in [C 98] required power series computations which were pleasant for $n = 2$ but quite taxing for $n \geq 3$. Accordingly, the computations for $n \geq 3$ were omitted from the thesis [C 98], the intention being to include them in a subsequent paper based on the thesis. Meanwhile, we heard from Professor König about his work showing equivalence of ball and sphere problems, and this inspired us to take a fresh look at all the results in [C 98]. Eventually, we found the tie between Khinchin-type inequalities and bisubharmonic functions embodied in Theorems 1–3. The present Theorem1 and its ball counterpart—stated in §6—contain results more general than those stated in [C 98], and are proved with much less effort.

For $\Phi(x) = |x|^p$ and p and n as in Corollary 2, the L^p majorization inequalities supplied by our Theorem 1 are proved also in [KK 01], via a somewhat different path. See Remark (ii), p. 122 of [KK 01]. A key element of the KK proofs is convexity of the function $t \mapsto E|x_0 + t^{1/2} U|^p$ for appropriate p and n, where U is uniformly distributed on S^{n-1}. It was after seeing this that we were led to formulate our Theorem 3 and Lemma 1. (We had already proved Theorems 1 and 2, by arguments less simple than those used here.) The KK proof of convexity makes use of heavy-duty one- and two-variable calculus, whereas our more general Theorem 3 uses only light-duty n-variable calculus: an application of the divergence theorem to the appropriate integral proves the convexity almost immediately.

We remark also that Theorem 1 can be derived by combining Theorem 3 with some arguments in [KK 01] or [LO 95], but we have opted to give a self-contained proof of Theorem 1, to highlight the salutary role the divergence theorem can play in the study of vector Khinchin inequalities.

To recapitulate: On spheres S^{n-1}, sharp L^2-L^p Khinchin inequalities, and much more, can be painlessly proved when $n = 1$ and $p \geq 3$, when $n = 2$ and $p \geq 2$, when $n = 3$ and $p \geq 1$, and when $n = 4$ and $p > 0$ (Theorem 1, Corollary 2). With vigorous hard analysis, sharp Khinchin inequalities have also been proved when $n = 1$ and $0 < p < 2$ or $2 < p < 3$, when $n = 2$ and $1 \leq p < 2$, and when $n = 3$ and $0 < p < 1$ ([Ha 82], [Kö 98], [KK 01]). To illustrate the difficulty gap between the two sets of results, one may note that KK prove their results which overlap ours within a space of about 12 pages, but require approximately 25 pages full of estimates and integrals of Bessel functions and other quantites to dispose of $n = 2$, $p \in [1, 2)$.

For $n = 2$ and $0 < p < 1$, Haagerup's conjecture for the best A_p remains open, except for p close enough to 1 so that Sawa's proof is valid. The logarithmic counterpart to Haagerup's Conjecture seems especially interesting: When $n = 2$, is it true that

$$E \log |X| \geq E \log |Z_{1,2}|? \tag{2.2}$$

With some absolute constant on the right, (2.2) was proved, independently, by Favorov [Fa 87] and Ullrich [U 88]. Gorin and Favorov [GF 87] (see also [Fa 98]) generalized this result by proving n-dimensional inequalities for negative moments: $E|X|^{-p} \leq C_{p,n}$ when $n \geq 2$ and $0 < p < n - 1$. As far as we know, the best constants $C_{p,n}$ are not known in any dimension. On the other hand, our Corollary 3 shows that if $n \geq 4$ the best constant on the right in (2.2) is $E \log |Z_n|$.

There remain some other open questions about which of the inequalities in Theorem 1 and Corollary 2 continue to hold when n and p do not satisfy the hypotheses of Corollary 2. We will forgo a systematic discussion, but will present two examples which furnish negative results.

Example 1. Take $n = 1$. The function $\psi(p) = 4p - 5 - 3^{p-1}$ is concave and has two zeros on $\mathbb{R}$, at $p = 2$ and $p = p^*$, where $2.34 < p^* < 2.35$. For each $p \in (2, p^*)$, there exists $\varepsilon_p > 0$ such that if $X = \sum_{i=1}^{3} a_i U_i$, with $a_1 = (2/3)^{1/2} \cos\theta$, $a_2 = (2/3)^{1/2} \sin\theta$, $a_3 = 3^{-1/2}$, and $|\theta - \pi/4| < \varepsilon_p$, then $E|X|^p > E|Z_{1,3}|^p$. Thus, the second inequality in (1.6) can fail when $2 < p < p^*$, whereas $E|X|^p \leq E|Z_1|^p$, by Haagerup's Theorem.

Example 2. Take $n = 2$. For $\delta > 0$, set

$$X = U_1 + \delta U_3, \qquad Y = U_1 + \frac{1}{\sqrt{2}}\,\delta U_2 + \frac{1}{\sqrt{2}}\,\delta U_3.$$

For $p > 0$ write $\alpha = p/2$. Then, as $\delta \to 0$,

$$E|X|^p = 1 + \alpha^2\delta^2 + \frac{\alpha^2(\alpha-1)^2}{4}\,\delta^4 + O(\delta^6),$$
$$E|Y|^p = 1 + \alpha^2\delta^2 + \frac{3}{2}\frac{\alpha^2(\alpha-1)^2}{4}\,\delta^4 + O(\delta^6).$$

Thus, $E|X|^p < E|Y|^p$ for sufficiently small δ and all p except $p = 2$. In particular, a conceivable strengthening of Haagerup's Conjecture,

$$\{b_i^2\} \prec \{a_i^2\} \;\Rightarrow\; E|Y|^p \le E|X|^p,\; p \in [p_2, 2),$$

is false for every $p \in [p_2, 2)$.

3. Proof of Theorem 1. For continuous $f \in C(\mathbb{R}^n, \mathbb{R})$, set

$$I(r) = I(r, f) = \int\limits_{S^{n-1}} f(rx)\,dx, \qquad r \ge 0. \tag{3.1}$$

We remind the reader of our convention that in integrals over S^{n-1}, dx denotes normalized uniform measure on S^{n-1}, and that in integrals over unit balls $\mathbb{B}^n = \{x \in \mathbb{R}^n : |x| < 1\}$, dx denotes normalized Lebesgue measure on $\mathbb{B}^n$.

The proof of Theorem 1 involves mostly elementary integral identities, two of which are stated in the following lemmas.

Lemma 1. *Suppose that $f \in C^2(\mathbb{R}^n, \mathbb{R})$. Then*

$$\frac{d}{dt} I(t^{1/2}, f) = \frac{1}{2n} \int\limits_{\mathbb{B}^n} \Delta f(t^{1/2}x)\,dx, \qquad 0 \le t < \infty. \tag{3.2}$$

Proof. An argument using the divergence theorem, and the relation $s = nb$ between the unnormalized measures s of S^{n-1} and b of $\mathbb{B}^n$, gives

$$I'(r) = \frac{r}{n} \int\limits_{\mathbb{B}^n} \Delta f(rx)\,dx,$$

from which (3.2) follows. ∎

In the next lemma, e_1 denotes the unit vector $(1, 0, \ldots) \in \mathbb{R}^n$.

Lemma 2. *Suppose that $f \in C(\mathbb{R}^n, \mathbb{R})$. Then, for $\alpha, \beta \in \mathbb{R}$,*

$$\int\limits_{(S^{n-1})^2} f(\alpha x + \beta y)\,dx\,dy = \int\limits_{S^{n-1}} I(|\alpha e_1 + \beta y|, f)\,dy. \tag{3.3}$$

Proof. Fix $n \ge 2$, and write $S = S^{n-1}$. Let G denote the special orthogonal group $\mathrm{SO}(n)$, and dg denote normalized Haar measure on G. Then for continuous functions F on $\mathbb{R}^n$ and $z \in \mathbb{R}^n$, we have $\int_G F(gz)\,dg =$

$\int_S F(|z|x)\,dx$, where gz denotes the action of g on z. Thus

$$\begin{aligned}\int\limits_{S\times S} f(\alpha x+\beta y)\,dx\,dy &= \int\limits_{G\times G} f(\alpha g_1 e_1+\beta g_2 e_1)\,dg_1\,dg_2\\ &= \int\limits_G dg_1 \int\limits_G f(g_1(\alpha e_1+\beta g_1^{-1} g_2 e_1))\,dg_2\\ &= \int\limits_G dg_1 \int\limits_G f(g_1(\alpha e_1+\beta g_2 e_1))\,dg_2\\ &= \int\limits_G dg_2 \int\limits_G f(g_1(\alpha e_1+\beta g_2 e_1))\,dg_1\\ &= \int\limits_G dg_2 \int\limits_S f(|\alpha e_1+\beta g_2 e_1|y)\,dy\\ &= \int\limits_G I(|\alpha e_1+\beta g_2 e_1|, f)\,dg_2\\ &= \int\limits_S I(|\alpha e_1+\beta y|, f)\,dy. \ \blacksquare\end{aligned}$$

Proof of Theorem 1. We may assume that each of the a_i and each of the b_i are nonnegative, and, via the usual approximation arguments, that the bisubharmonic function Φ is in C^4. Let $\mathbb{R}^k_+$ be the set of all points $s=(s_1,\ldots,s_k)\in\mathbb{R}^k$ with each $s_i\ge 0$. Define $Q:\mathbb{R}^k_+\to\mathbb{R}$ by

$$Q(s)=E\Phi\Big(\sum_{j=1}^k s_j^{1/2}U_i\Big)=\int\limits_{(S^{n-1})^k}\Phi\Big(\sum_{j=1}^k s_j^{1/2}x_j\Big)\,dx_1\ldots dx_k. \tag{3.4}$$

To prove Theorem 1, we must show that if a and b are points of $\mathbb{R}^k_+$ with $b=(b_1,\ldots,b_k)\prec(a_1,\ldots,a_k)=a$, then $Q(a)\le Q(b)$. Now

$$\begin{aligned}Q(b)-Q(a) &= \int\limits_0^1 \frac{d}{dt}Q((1-t)a+tb)\,dt\\ &= \int\limits_0^1\sum_{j=1}^k \partial_j Q((1-t)a+tb)(b_j-a_j)\,dt,\end{aligned} \tag{3.5}$$

where $\partial_j Q=\partial Q/\partial s_j$. For $i=1,\ldots,k$, set $\Omega_i=S^{n-1}\times\ldots\times\mathbb{B}^n\times\ldots\times S^{n-1}$, where $\mathbb{B}^n$ is the ith factor. Fix $i\in\{1,\ldots,k\}$, integrate in (3.4) first with respect to x_i, differentiate with respect to s_i, and apply Lemma 1 to the function $x_i\mapsto\Phi(x_i+\sum_{j\ne i}s_j^{1/2}x_j)$. The result is

$$\partial_i Q(s)=\frac{1}{2n}\int\limits_{\Omega_i}\Delta\Phi\Big(\sum_{j=1}^k s_j^{1/2}x_j\Big)\,dx_1\ldots dx_k,\qquad 1\le i\le k. \tag{3.6}$$

Taking $i = 1$ in (3.6), and using the relation

$$\int_{\mathbb{B}^n} F(x)\,dx = \int_0^1 nr^{n-1}\,dr \int_{S^{n-1}} F(rx)\,dx,$$

we obtain

$$\partial_1 Q(s) = \frac{1}{2} \int_{(S^{n-1})^{k-2}} dx_3 \ldots dx_k \ldots \tag{3.7}$$

$$\int_0^1 r^{n-1}\,dr \int_{(S^{n-1})^2} \Delta\Phi\Big(s_1^{1/2} r x_1 + s_2^{1/2} x_2 + \sum_{j=3}^{k} s_j^{1/2} x_j\Big)\,dx_1\,dx_2.$$

Fix the s_j and x_j for $j \geq 3$, and set $f(z) = \Delta\Phi(z + \sum_{j=3}^k s_j^{1/2} x_j)$. By Lemma 2, the $dx_1 dx_2$ integral in (3.7) equals $I(|rs_1^{1/2} + s_2^{1/2}|, f)$. Formula (3.7) holds also for $\partial_2 Q(s)$, provided that $I(|rs_1^{1/2} + s_2^{1/2}|, f)$ is changed to $I(|s_1^{1/2} + rs_2^{1/2}|, f)$. If $s_1 \geq s_2$ and $0 \leq r \leq 1$, then

$$\begin{aligned} |rs_1^{1/2} + s_2^{1/2}|^2 &= r^2 s_1 + 2r s_1^{1/2} s_2^{1/2} + s_2 \\ &\leq s_1 + 2r s_1^{1/2} s_2^{1/2} + r^2 s_2 = |s_1^{1/2} + rs_2^{1/2}|^2. \end{aligned}$$

Now f is subharmonic on $\mathbb{R}^n$, so $I(r, f) \nearrow$ as $r \nearrow$. Thus, we have shown that $s_1 \geq s_2 \Rightarrow \partial_1 Q(s) \leq \partial_2 Q(s)$. The same argument shows that $s_i \geq s_{i+1} \Rightarrow \partial_i Q(s) \leq \partial_{i+1} Q(s)$ for each $i = 1, \ldots, k-1$. Hence:

$$\text{If } s_1 \geq \ldots \geq s_k, \text{ then } \partial_1 Q(s) \leq \ldots \leq \partial_k Q(s). \tag{3.8}$$

Let us return now to (3.5). The function Q is permutation invariant, so we may assume that the components of a and of b decrease as i increases. Then for $0 \leq t \leq 1$ the components of $(1-t)a + tb$ also decrease. By (3.8), the sequence $\partial_j Q((1-t)a + tb)$ increases as j increases. By assumption, $b \prec a$. If $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a length k sequence with increasing terms, it is easy to show, using summation by parts for example, that $b \prec a \Rightarrow \sum_{j=1}^k \lambda_j b_j \geq \sum_{j=1}^k \lambda_j a_j$. Thus, the integrand on the right-hand side of (3.5) is nonnegative, and hence $Q(b) \geq Q(a)$, as required. ∎

Functions Q with the above property, that $b \prec a \Rightarrow Q(a) \leq Q(b)$, are said to be *Schur-concave* on $\mathbb{R}_+^k$.

The Gaussian in Corollary 1. With the situation of Corollary 1, we need to verify that

$$E((\Phi_1(Z_n))^-) < \infty, \tag{3.9}$$

and that

$$\lim_{k\to\infty} E\Phi(Z_{n,k}) = E\Phi_1(Z_n). \tag{3.10}$$

If Φ is continuous and bisubharmonic then so is Φ_1. Moreover, for rotationally-invariant $\mathbb{R}^n$-valued random variables W, $E|\Phi(W)| < \infty$ if and only if $E|\Phi_1(W)| < \infty$, in which case $E\Phi(W) = E\Phi_1(W)$. Thus, to prove (3.9) and (3.10) we may assume that Φ is itself radial. Set $u = \Delta\Phi$. Then u is radial and subharmonic. Write $u(r) = u(re_1)$, and $\Phi(r) = \Phi(re_1)$. Since $u(r)$ is the mean of a subharmonic function on $|x| = r$, it follows that u is finite, continuous and increasing on $(0,\infty)$. Moreover, Φ satisfies the distributional o.d.e.

$$r^{1-n}\,\frac{d}{dr}(r^{n-1}\Phi'(r)) = u(r). \tag{3.11}$$

It is easy to verify that continuous distributional solutions of (3.11) on $(0,\infty)$ are in C^2. Thus, $\Phi \in C^2(0,\infty)$.

Let us assume now that $n \geq 3$. Integrate (3.11) from 1 to r. The result is, for $0 < r < \infty$,

$$\begin{aligned}\Phi(r) = \Phi(1) &+ \frac{1}{n-2}\,\Phi'(1)(1 - r^{2-n}) \\ &+ \frac{1}{n-2}\int_1^r (1 - t^{n-2}r^{2-n})tu(t)\,dt.\end{aligned} \tag{3.12}$$

Since $u(t) \geq u(1)$ for $t \geq 1$, (3.12) implies existence of positive constants a and b such that

$$\Phi(r) \geq -ar^2 - b, \qquad 1 \leq r < \infty. \tag{3.13}$$

Since Φ is continuous on $\mathbb{R}^n$, it is bounded below in $|x| \leq 1$, so (3.13) still holds for $r \in [0,\infty)$, with perhaps a larger b. The statement (3.9) for $n \geq 3$ follows from (3.13).

If u is bounded above on $(0,\infty)$, then from (3.12) it follows that $|\Phi(r)| \leq ar^2 + b$ for some constants a, b and all $r \in (0,\infty)$. Then (3.10), for $n \geq 3$, follows from either of the central limit type Theorems 2.2.11 or 2.2.20 in [St 93].

If u is not bounded above, define $v : (0,\infty) \to \mathbb{R}$ by $u(r) = v(r^{2-n})$. A calculation shows that subharmonicity of u is equivalent to convexity of v. Moreover, our v is decreasing, and tends to infinity as $r \to 0$. Let $\{r_m\}$ be a positive sequence in $(0,1)$ which decreases to zero. Set $v_m = v$ on $[r_m,\infty)$, and define v_m on $[0,r_m]$ to be the linear function with $v_m(r_m) = v(r_m)$ and slope equal to the derivative of v from the right at r_m. Then each v_m is convex on $(0,\infty)$, and $v_m(r) \nearrow v(r)$ as $m \to \infty$ for each $r \in (0,\infty)$. Set $u_m(r) = v_m(r^{2-n})$. Define $\Phi_m = \Phi$ on $[0,1]$, and on $[1,\infty)$ define Φ_m by replacing u by u_m in (3.12). Extend Φ_m to $\mathbb{R}^n$ by defining $\Phi_m(x) = \Phi_m(|x|)$. Then $\{\Phi_m\}$ is a sequence of continuous radial bisubharmonic functions which increases to Φ pointwise on $(0,\infty)$, and each $\Delta\Phi_m = u_m$ is bounded above.

Thus $E\Phi_m(Z_{n,k}) \nearrow E\Phi_m(Z_n)$ for each fixed m. Hence

$$E\Phi_m(Z_{n,k}) \le E\Phi_m(Z_n) \le E\Phi(Z_n) \quad \text{for all } k, m.$$

Letting $m \to \infty$ and applying the Monotone Convergence Theorem, we see that $E\Phi(Z_{n,k}) \le E\Phi(Z_n)$ for each k, so that $\lim_{k\to\infty} E\Phi(Z_{n,k}) \le E\Phi(Z_n)$. The opposite inequality can be obtained by writing $\Phi = \Phi^+ - \Phi^-$, then applying [St 93, 2.2.2] to Φ^+, and (3.13) and [St 93, 2.11] or [St 93, 2.20] to Φ^-. The proof of (3.10) is complete when $n \ge 3$.

Suppose $n = 2$. Then instead of (3.12) we have

$$\Phi(r) = \Phi(1) + \Phi'(1)\log r + \int_1^r \left(\log \frac{r}{t}\right) tu(t)\,dt, \qquad 1 < r < \infty. \tag{3.14}$$

The lower bound (3.13) is still true, and hence so is (3.9). If u is bounded above then u is constant, which implies that $\Phi(r)$ has the form ar^2+b, so that (3.10) is trivial. To handle nonconstant u, define $v : (-\infty,\infty) \to \mathbb{R}$ by $v(x) = u(e^x)$. Then v is convex and increasing. Approximate v by an increasing sequence of convex functions, each of which is linear for sufficiently large x. The corresponding approximants to Φ satisfy bounds of the form $\Phi_m(r) = O(r^2 \log r)$ as $r \to \infty$. Theorem 2.2.20 of [St 93] still applies to each Φ_m. The proof of (3.10) for $n = 2$ can be now accomplished like the one for $n \ge 3$, with a few small changes. A proof of (3.9) and (3.10) for $n = 1$ can be constructed along the same lines as for $n = 2$.

4. Proof of Theorem 2. Let Φ be a radial function on $\mathbb{R}^n$, with the property that $b \prec a \Rightarrow Q(a) \le Q(b)$ for each pair of nonnegative k-tuples $a, b \in \mathbb{R}^k_+$, where Q is defined as in (3.4). Assume for now that $\Phi \in C^4$. Take $R > 0$, $B \ge 0$, $\varepsilon > 0$, $k = 3$, $a = (R^2, 0, B^2)$, and $b = (R^2 - \varepsilon, \varepsilon, B^2)$. Then $b \prec a$, so $Q(a) - Q(b) \le 0$. Dividing by ε and letting $\varepsilon \to 0$, we obtain $\partial_1 Q(R^2, 0, B^2) - \partial_2 Q(R^2, 0, B^2) \le 0$. Set $u = \Delta\Phi$. If we write $x = x_1$, $y = x_3$, it follows from (3.6) that

$$\int_{\mathbb{B}^n \times S^{n-1}} u(Rx + By)\,dx\,dy \le \int_{S^{n-1}\times S^{n-1}} u(Rx + By)\,dx\,dy. \tag{4.1}$$

As in the previous section, dx and dy denote uniform probability measures on $\mathbb{B}^n$ or S^{n-1}, according to the specified domain of integration. From (4.1), (3.3) and a simple computation, we obtain

$$\int_{\mathbb{B}^n} I(|Rx + z_0|, u)\,dx \le \int_{S^{n-1}} I(|Rx + z_0|, u)\,dx, \tag{4.2}$$

where z_0 is any point of $\mathbb{R}^n$ with $|z_0| = B$. But u is radial, so $I(|Rx+z_0|, u) = u(Rx + z_0)$. Thus, the right-hand integral in (4.2) equals the mean value of u over the sphere $|x - z_0| = R$. Similarly, the left-hand integral equals the mean over the ball $|x-z_0| \le R$. We have thus shown that for every $z_0 \in \mathbb{R}^n$ and every $R > 0$,

$$\int\limits_{\mathbb{B}^n} u(z_0+Rx)\,dx \le \int\limits_{S^{n-1}} u(z_0+Rx)\,dx. \tag{4.3}$$

Now $u \in C^2(\mathbb{R}^n,\mathbb{R})$. If u were not subharmonic, there would exist $z_0 \in \mathbb{R}^n$ such that $\Delta u(z_0) < 0$. Then u would be strictly superharmonic in some ball centered at z_0, the spherical mean on the right in (4.3) would be a strictly decreasing function of R for small R, and the left side would be strictly larger than the right side for small R. This contradiction shows that u must be subharmonic, so that Φ must be bisubharmonic. Theorem 2 is proved for radial $\Phi \in C^4$.

Let now Φ be radial, continuous and satisfy the hypotheses of Theorem 2. Take $x_0 \in \mathbb{R}^n$, and set $\Psi(x) = \Phi(x+x_0)$, $B = |x_0|$. By converting to integrals over products of spheres, or otherwise, one may show that

$$E\Psi\Big(\sum_{i=1}^{k} a_i U_i\Big) = E\Phi\Big(\sum_{i=1}^{k} a_i U_i + BU_{k+1}\Big)$$

for sequences $(a_1,\dots,a_k)$, where U_{k+1} is uniformly distributed on S^{n-1} and is independent of $U_1,\dots,U_k$. If $(b_1^2,\dots,b_k^2) \prec (a_1^2,\dots,a_k^2)$, then $(b_1^2,\dots,b_k^2,B^2) \prec (a_1^2,\dots,a_k^2,B^2)$. Since (1.1) holds for Φ and the augmented sequences, it holds also for Ψ and the original sequences. Consequently, for any nonnegative integrable compactly supported function K on $\mathbb{R}^n$, if $\{b_i^2\} \prec \{a_i^2\}$ then (1.1) holds with Φ replaced by the convolution $K*\Phi$. If we take K to be also C^∞ and radial, then $K*\Phi$ is C^∞ and radial, so the first part of the proof implies that $K*\Phi$ is bisubharmonic. Standard arguments, like those in [Hö 94, p.148], imply that Φ is bisubharmonic. ∎

5. Proof of Theorem 3. The usual smoothing methods work straightforwardly in this situation, so we will only consider the case when $\Phi \in C^4(\mathbb{R}^n,\mathbb{R})$. Fix $x_0 \in \mathbb{R}^n$, and set

$$h(t) = \int\limits_{S^{n-1}} \Phi(x_0+t^{1/2}x)\,dx, \qquad u(x) = \Delta\Phi(x_0+x).$$

Lemma 1, with $f(x) = \Phi(x+x_0)$, implies that $h'(t)$ is $1/(2n)$ times the mean value of u over the ball $|x-x_0| < t^{1/2}$, so that

$$h'(t) = \frac{1}{2}t^{-n/2}\int\limits_0^{t^{1/2}} I(s,u)s^{n-1}\,ds = \frac{1}{2n}\,t^{-n/2}\int\limits_0^{t^{n/2}} I(r^{1/n},u)\,dr. \tag{5.1}$$

If Φ is bisubharmonic, then $I(s,u)\nearrow$ as $s\nearrow$, so $h'(t)\nearrow$ as $t\nearrow$, hence h is convex. Conversely, if the C^4 function Φ were not bisubharmonic, there would exist $x_0 \in \mathbb{R}^n$ such that $\Delta\Delta\Phi(x) < 0$ in some neighborhood of x_0.

The corresponding u would be strictly superharmonic in a ball around the origin, $I(s,u)$ would strictly decrease for all small s, and (5.1) would show that $h'(t)$ strictly decreases for all small t. Thus, h would not be convex. ■

6. Sharp Khinchin inequalities for balls. Let $V_1,\dots,V_k$ be independent $\mathbb{R}^n$-valued random variables, each of which is uniformly distributed on the unit ball $\mathbb{B}^n$ of $\mathbb{R}^n$. We continue to let $a_1,\dots,a_k$ denote a real sequence of k elements. H. König [Kö 98], [KK 01] discovered the following remarkable identity:

KÖNIG'S IDENTITY. *For $n \ge 1$,*

$$E\Big|\sum_{i=1}^{k} a_i V_i\Big|^p = \frac{n}{n+p} E\Big|\sum_{i=1}^{k} a_i U_i\Big|^p, \qquad 0<p<\infty, \tag{6.1}$$

where $U_1,\dots,U_k$ are independent $\mathbb{R}^{n+2}$-valued random variables, each of which is uniformly distributed on S^{n+1}.

This identity permits deduction of sharp L^2-L^p Khinchin inequalities for balls from the corresponding inequalities for spheres. For example, using (1.6), one obtains, for $n \ge 1$, $p \ge 2$ and $\sum_{i=1}^k a_i^2 = 1$,

$$\frac{n}{n+p} \le E\Big|\sum_{i=1}^{k} a_i V_i\Big|^p \le E\Big|k^{-1/2}\sum_{i=1}^{k} V_i\Big|^p \le \frac{n}{n+p} E|Z_{n+2}|^p.$$

The quantities on the right and left ends are the smallest constants which work for all k and fixed n. For $n=1$, these inequalities, and the reverse inequalities for $1 \le p \le 2$, were first proved by Latała and Oleszkiewicz [LO 95].

We shall state a Φ moment generalization of König's identity. For simplicity, we confine attention to radial moments. Thus, let Φ be a continuous radial function on $\mathbb{R}^n$. Then $\Phi(x) = \Phi_1(|x|)$ for a continuous function Φ_1 on $[0,\infty)$. Define $\Psi_1 : (0,\infty) \to \mathbb{R}$ by

$$\Psi_1(r) = nr^{-n}\int_0^r \Phi_1(s)s^{n-1}\,ds, \qquad r \ge 0, \tag{6.2}$$

and define a radial function Ψ on $\mathbb{R}^{n+2}$ by $\Psi(x) = \Psi_1(|x|)$.

THEOREM 4. *Let the U_i and V_i be as in König's identity. Then*

$$E\Phi\Big(\sum_{i=1}^{k} a_i V_i\Big) = E\Psi\Big(\sum_{i=1}^{k} a_i U_i\Big). \tag{6.3}$$

From Theorem 4, one sees that the inequalities (1.1) and (1.2) of Theorem 1 and Corollary 1 hold when $X = \sum_{i=1}^k a_i U_i$, $Y = \sum_{i=1}^k b_i U_i$, and $Z_{n,k}$ are replaced by $\sum_{i=1}^k a_i V_i$, $\sum_{i=1}^k b_i V_i$, and $k^{-1/2}\sum_{i=1}^k V_i$, provided Φ

is radial and Ψ is bisubharmonic on $\mathbb{R}^{n+2}$. The reader may verify that a smooth radial Φ has biharmonic Ψ if and only if Φ_1 satisfies the differential inequality

$$r^2 {\Phi_1}'''(r) + (n^2 - n)(r{\Phi_1}''(r) - {\Phi_1}'(r)) \geq 0, \qquad 0 < r < \infty. \tag{6.4}$$

Proof of Theorem 4. Let f and g denote the density functions of $|\sum_{i=1}^k a_i U_i|$ and $|\sum_{i=1}^k a_i V_i|$, respectively. We claim that

$$f(r) = -\frac{r^n}{n}\,(r^{1-n} g(r))'. \tag{6.5}$$

Assuming (6.5), and using also (6.2), we have

$$\begin{aligned} E\Psi\Big(\sum_{i=1}^k a_i U_i\Big) &= \int_0^\infty \Psi_1(r) f(r)\,dr = -\int_0^\infty \Psi_1(r)\,\frac{r^n}{n}\,(r^{1-n}g(r))'\,dr \\ &= n^{-1}\int_0^\infty (r^n \Psi_1(r))' r^{1-n} g(r)\,dr = \int_0^\infty \Phi_1(r) g(r)\,dr \\ &= E\Phi\Big(\sum_{i=1}^k a_i V_i\Big). \end{aligned}$$

To complete the proof of the theorem, we must verify (6.5). To do this, we use some ideas from a proof of (6.1) in [KK 01, p. 126] which is attributed to Latała. Its point of departure is the observation that if U is uniform on S^{n+1} and V is uniform on $\mathbb{B}^n$, then $U \cdot e_1$ and $V \cdot e_1$ have the same distribution, given by the density $c_n(1-t^2)^{(n-1)/2}$ for $|t| < 1$ and suitable constant c_n.

Let F and G denote the density functions, defined on $\mathbb{R}^{n+2}$ and $\mathbb{R}^n$, respectively, of $\sum_{i=1}^k a_i U_i$ and $\sum_{i=1}^k a_i V_i$. Then F and G are radial, and so are their Fourier transforms $\widehat{F}$ and $\widehat{G}$. With the normalizations of [SW 71], we have

$$\begin{aligned} \widehat{F}(re_1) &= E\Big(\exp\Big(-2\pi i r e_1 \cdot \sum_{i=1}^k a_i U_i\Big)\Big) \\ &= E\Big(\exp\Big(-2\pi i r e_1 \cdot \sum_{i=1}^k a_i V_i\Big)\Big) = \widehat{G}(re_1), \quad r \geq 0. \end{aligned} \tag{6.6}$$

Write $H(r) = \widehat{F}(re_1) = \widehat{G}(re_1)$, and define, for $t \geq 0$, $n \geq 2$,

$$P_n(t) = \int_{S^{n-1}} e^{2\pi i t e_1 \cdot x}\,dx = c\int_0^\pi e^{2\pi i t\cos\theta} \sin^{n-2}\theta\,d\theta. \tag{6.7}$$

Here, and below, c will denote a constant depending on n which can change from identity to identity. When $n = 2$ the term $\sin^{n-2}\theta$ should be replaced by the constant 1. When $n = 1$, define $P_1(t) = \cos 2\pi t$.

The Fourier inversion formula and conversion to polar coordinates [SW 71, pp. 11, 154, 155] give

$$
\begin{aligned}
F(re_1) &= \int_{\mathbb{R}^{n+2}} \widehat{F}(\xi) e^{2\pi i r e_1 \cdot \xi}\, d\xi = c \int_0^\infty P_{n+2}(tr) H(t) t^{n+1}\, dt, \\
G(re_1) &= \int_{\mathbb{R}^{n}} \widehat{G}(\xi) e^{2\pi i r e_1 \cdot \xi}\, d\xi = c \int_0^\infty P_{n}(tr) H(t) t^{n-1}\, dt.
\end{aligned} \tag{6.8}
$$

Differentiation, then integration by parts in the last integral in (6.7), leads to $P_n'(t) = -ctP_{n+2}(t)$. Using this with (6.8), we deduce

$$\frac{\partial G}{\partial r}(re_1) = -crF(re_1). \tag{6.9}$$

The density functions f, F, g, G are related by

$$f(r) = cF(re_1)r^{n+1}, \qquad g(r) = cG(re_1)r^{n-1}. \tag{6.10}$$

From (6.9) and (6.10), it follows that $f(r) = -cr^n(r^{1-n}g(r))'$. Using $\int_0^\infty f(r)\,dr = \int_0^\infty g(r)\,dr = 1$, one can show that the constant c in the last identity equals $1/n$. The proof of (6.5), and of Theorem 4, is complete. ∎

The function P_n is expressible in terms of Bessel functions:

$$P_n(t) = \frac{2\pi}{\omega_{n-1}}\, t^{-(n-2)/2} J_{(n-2)/2}(2\pi t),$$

where ω_{n-1} is the area of S^{n-1}. See, for example, [SW 71, p. 154].

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