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Quasiaffine transforms of operators

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Abstract. We obtain a new sufficient condition (which may be useful elsewhere) that a compact perturbation of a normal operator be the quasiaffine transform of some normal operator. We also give some applications of this result.

Let $\mathcal{H}$ be a separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. As usual, we will write $\mathbf{K}$ for the ideal of compact operators in $\mathcal{L}(\mathcal{H})$. Recall from [8] that an $X \in \mathcal{L}(\mathcal{H})$ is called a *quasiaffinity* if $\ker X = \ker X^* = (0)$, and that if $S, T \in \mathcal{L}(\mathcal{H})$ and there exists a quasiaffinity $X \in \mathcal{L}(\mathcal{H})$ such that $XS = TX$, then we say that S is a *quasiaffine transform* of T and we write $S \prec T$. If both $S \prec T$ and $T \prec S$ then we say that S and T are *quasisimilar*, and we write $S \sim T$. It is well known that quasisimilarity is an equivalence relation on $\mathcal{L}(\mathcal{H})$ that preserves the existence of nontrivial hyperinvariant subspaces (cf. [5], [8]). One also knows that if S and T are normal and $S \prec T$, then $S \sim T$ and, in fact, S and T are unitarily equivalent. Below we also write $\{T\}'$ for the commutant of an operator T in $\mathcal{L}(\mathcal{H})$ and $\sigma_{\rm p}(T)$ for the point spectrum of T.

The theory of quasiaffine transforms of operators is well developed and plays an important role in the study of operators on Hilbert space (cf., e.g., [2] and [8]). In particular, the following little-known but somewhat interesting result was obtained in [1].

Theorem 1 ([1, Th. 4.3]). *Let T be an arbitrary operator in $\mathcal{L}(\mathcal{H})$ and ε an arbitrary positive number. Then there exist a normal operator N and a compact operator K in $\mathcal{L}(\mathcal{H})$ such that $T \prec N + K$ and $\|K\| < \varepsilon$.*

Thus it is of interest to obtain sufficient conditions in order that an operator $N + K$ as in Theorem 1 be a quasiaffine transform of a normal

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operator M (thus giving $T \prec N+K \prec M$), and we obtain one such condition below (Theorem 5).

Our result depends on an old construction that has been used by many authors (cf., e.g., [4], [6]). We first obtain a new Hilbert space $\mathcal{K}_{\mathcal{H}}$ from $\mathcal{H}$ as follows.

DEFINITION 2. Let $\mathcal{K}_1$ be the linear space of all (bounded) sequences $\{x_n\}_{n\in\mathbb{N}} \subset \mathcal{H}$ such that $x_n \to 0$ weakly, and let LIM be a fixed Banach generalized limit on (the Banach space) l^∞, with all of the properties of such limits (cf., e.g., [3, Ex. 14E]), which we use below without further explicit mention. Define a semi-inner product (and seminorm) on $\mathcal{K}_1$ by

$$\langle\{x_n\},\{y_n\}\rangle_{\mathcal{K}_1} = \mathrm{LIM}\langle x_n, y_n\rangle, \qquad \|\{x_n\}\|^2 = \langle\{x_n\},\{x_n\}\rangle_{\mathcal{K}_1},$$

let $\mathcal{K}_0$ be the linear manifold in $\mathcal{K}_1$ consisting of all $\{x_n\}_{n\in\mathbb{N}}$ in $\mathcal{K}_1$ such that

$$\langle\{x_n\},\{x_n\}\rangle_{\mathcal{K}_1} = 0,$$

and let $\mathcal{K} = \mathcal{K}_{\mathcal{H}}$ be the (Hilbert space) completion of the quotient space $\mathcal{K}_1/\mathcal{K}_0$. We will denote some elements of $\mathcal{K}$ by $[\{x_n\}]$, meaning the equivalence class of $\mathcal{K}$ containing the sequence $\{x_n\}_{n\in\mathbb{N}}$ from $\mathcal{K}_1$. It is easy to see that $\mathcal{K}$ is nonseparable. (Note that if $\{e_n\}_{n\in\mathbb{N}}$ is an orthonormal sequence in $\mathcal{H}$, then $[\{e_n\}]$ is a unit vector in $\mathcal{K}$, and if $\pi : \mathbb{N} \to \mathbb{N}$ is any injective map with no fixed points (or only finitely many), then $[\{e_{\pi(n)}\}]$ is a unit vector in $\mathcal{K}$ orthogonal to $[\{e_n\}]$.) Furthermore, we define a mapping $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ by setting, for every $S \in \mathcal{L}(\mathcal{H})$,

$$\Phi(S)[\{x_n\}] = [\{Sx_n\}], \qquad [\{x_n\}] \in \mathcal{K}. \tag{1}$$

A little cogitation, together with knowledge of the basic properties of generalized Banach limits and compact operators (cf., e.g., [7, Ch. 4]), convinces one of the truth of the following.

LEMMA 3. *The map* $\Phi : \mathcal{L}(\mathcal{H}) \to \mathcal{L}(\mathcal{K})$ *defined by* (1) *is a unital* C^*-*algebra homomorphism with* $\ker\Phi \supset \mathbf{K}$.

Next, let us consider the collection $\mathcal{C}$ of all (bounded) sequences $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}} \subset \mathcal{L}(\mathcal{H})$ such that $A_n \to 0$ in the weak operator topology (WOT) and $A_n^*A_n \to A_0^2 \neq 0$ (WOT), where $A_0 \geq 0$ (which implies, in particular, that for $y \in \mathcal{H}$, $\|A_n y\| \to \|A_0 y\|$). We can now state the following easy lemma, which follows, for instance, from [6, Lemma 1].

LEMMA 4. *For every* $\mathcal{A} = \{A_n\}_{n\in\mathbb{N}} \in \mathcal{C}$, *there exists a nonzero bounded operator* $X = X_{\mathcal{A}} : \mathcal{H} \to \mathcal{K}$ *defined by*

$$Xy = [\{A_n y\}], \qquad y \in \mathcal{H}, \tag{2}$$

and $\ker X \supset \{y \in \mathcal{H} : \|A_n y\| \to 0\}$. *Moreover*, *if* $T \in \mathcal{L}(\mathcal{H})$ *and* $\{A_n\}_{n\in\mathbb{N}} \in \mathcal{C} \cap \{T\}'$, *then*

$$XT = \Phi(T)X. \tag{3}$$

Our first theorem, which uses J. Thomson's deep result [9] on the existence of analytic bounded point evaluations, and which we believe to be new, is the following.

THEOREM 5. *Suppose that* $T = N + K \in \mathcal{L}(\mathcal{H})$ *with* N *normal and* $K \in \mathbf{K}$, *and that the* WOT *on the unit ball of* $\{T\}'$ *is strictly weaker than the* SOT *there* (*equivalently*, *there exists a sequence* $\{A_n\} \subset \{T\}'$ *such that* $A_n \to 0$ (WOT) *but* $A_n \nrightarrow 0$ (SOT)). *Then either* T *has a nontrivial invariant subspace or there exists a normal operator* $M \in \mathcal{L}(\mathcal{H})$ *such that* $T \prec M$.

Proof. By dropping down to a subsequence (without changing the notation) we may suppose that $A_n^* A_n \to A_0^2 \neq 0$ (WOT) where $A_0 \geq 0$. Thus $\{A_n\} \in \mathcal{C}$, and by Lemma 4 this sequence generates a bounded nonzero operator $X : \mathcal{H} \to \mathcal{K}$ satisfying

$$Xy = [\{A_n y\}], \quad \|Xy\|^2 = \|A_0 y\|^2, \quad y \in \mathcal{H},$$

and also

$$XT = \Phi(T)X = \Phi(N)X. \tag{4}$$

It now follows immediately from (4) that if $\ker X \neq (0)$ (i.e., A_0 is not a quasiaffinity), then $\ker X$ is a nontrivial invariant subspace for T, so, regarding X as a linear transformation from $\mathcal{H}$ to $\mathcal{R} = (\operatorname{ran} X)^-$, we may suppose that X is a quasiaffinity and that $\mathcal{R}$ is an invariant subspace for the normal operator $\Phi(N)$. Thus (4) readily implies that $T \prec \Phi(N)|_{\mathcal{R}}$, and it now suffices to show that $\Phi(N)|_{\mathcal{R}}$ is normal. Suppose, to the contrary, that $\Phi(N)|_{\mathcal{R}}$ is a nonnormal, subnormal operator. We may also suppose that every $y \neq 0$ in $\mathcal{H}$ is cyclic for T, and consequently $\Phi(N)|_{\mathcal{R}}$ has a cyclic vector too. But then the pure subnormal part S of $\Phi(N)|_{\mathcal{R}}$ has a cyclic vector, and by the deep theorem of J. Thomson [9], $\sigma_{\mathrm{p}}(S^*)$ is nonvoid. Thus also $\sigma_{\mathrm{p}}((\Phi(N)|_{\mathcal{R}})^*) \neq \emptyset$, and taking adjoints in (4), we get $\sigma_{\mathrm{p}}(T^*) \neq \emptyset$, which leads immediately to the existence of a nontrivial invariant subspace for T. ∎

This allows us to recover the following theorem, which, of course, dates from 1980 and thus was originally proved independently of Theorem 5.

THEOREM 6 ([6]). *Let* $T = N + K \in \mathcal{L}(\mathcal{H})$ *with* N *normal and* $K \in \mathbf{K}$, *and suppose that on the unit balls of* $\{T\}'$ *and* $\{T^*\}'$ *the* WOT *is strictly weaker than the* SOT. *Then* T *has a nontrivial invariant subspace.*

Proof. According to Theorem 5, either T has a nontrivial invariant subspace or there exist normal operators M_1 and M_2 such that $T \prec M_1$ and $T^* \prec M_2$. But then, as was noted above, M_1 and M_2^* are unitarily equivalent, and consequently T is quasisimilar to M_1, from which the result follows. ∎

Remark 7. To our knowledge, Lomonosov [6] was the first to realize the utility of the hypothesis that on the unit balls of $\{T\}'$ and $\{T^*\}'$ the WOT and SOT differ.

The following result, a special case of which is known (cf. [8, Chapter II, Prop. 5.3]), is an application of Theorem 5.

Theorem 8. *Suppose $T = N + K \in \mathcal{L}(\mathcal{H})$ with N normal and $K \in \mathbf{K}$, and $\|T\| = r(T) = 1$, where $r(T)$ is the spectral radius of T. If T does not belong to the class C_{00} (defined in [8]), then either T has a nontrivial invariant subspace or there exists a normal operator M satisfying $T \prec M$ or $M \prec T$.*

Proof. As is well-known, if T has a unitary part, then T has a nontrivial hyperinvariant subspace ([8]). Thus we may suppose that T is completely nonunitary. Furthermore, since $T \notin C_{00}$, if neither T nor T^* belongs to the class C_{10} (defined in [8]) then again T has a nontrivial hyperinvariant subspace. Thus, by taking adjoints if necessary, we may suppose that $T \in C_{10}$. Since $T^n \to 0$ (WOT) via the H^∞-functional calculus for completely nonunitary contractions, and $T^n \nrightarrow 0$ (SOT) by definition of the class C_{10}, Theorem 5 is applicable. ∎

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