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Strictly singular inclusions of rearrangement invariant spaces and Rademacher spaces

by

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Abstract. If G is the closure of L_∞ in $\exp L_2$, it is proved that the inclusion between rearrangement invariant spaces $E \subset F$ is strictly singular if and only if it is disjointly strictly singular and $E \not\supset G$. For any Marcinkiewicz space $M(\varphi) \subset G$ such that $M(\varphi)$ is not an interpolation space between L_∞ and G it is proved that there exists another Marcinkiewicz space $M(\psi) \subsetneq M(\varphi)$ with the property that the $M(\psi)$ and $M(\varphi)$ norms are equivalent on the Rademacher subspace. Applications are given and a question of Milman answered.

1. Introduction. A linear operator between two Banach spaces E and F is called *strictly singular* (SS for short), or *Kato*, if it fails to be an isomorphism on any infinite-dimensional subspace (cf. [LT1, 2.c.2]). The class of all strictly singular operators is a well-known closed operator ideal with important applications. A weaker notion for Banach lattices, introduced in [HR], is the following: a bounded operator A from a Banach lattice E to a Banach space F is said to be *disjointly strictly singular* (DSS for short) if there is no disjoint sequence of non-null vectors $\{x_n\}_{n=1}^\infty$ in E such that the restriction of A to the subspace $[x_n]$ spanned by the vectors $\{x_n\}$ is an isomorphism. This is a useful tool in comparing structures of rearrangement invariant spaces (cf. [HR], [GHSS]).

This paper deals with the strict singularity of inclusions $E \subset F$ between rearrangement invariant (r.i.) function spaces E and F on the interval $[0,1]$. That means that the norms of E and F are non-equivalent on any (closed) infinite-dimensional subspace of E.

The canonical inclusion $L_\infty \subset E$ is always strictly singular for any r.i. space $E \neq L_\infty$ ([N]), and the case of L_p-spaces is Grothendieck's classical result. Furthermore, this property characterizes the space L_∞ among all r.i.

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spaces ([GHSS]). Concerning the right extreme inclusion $E \subset L_1$, its strict singularity has been characterized in [HNS] by the condition that the r.i. space E does not contain the space G, the closure of L_∞ in the exponential Orlicz space $\exp L_2$. Recall that Rodin and Semenov [RS] (see also [LT2]) proved that the condition $E \supset G$ determines precisely the r.i. spaces E for which the Rademacher function system $\{r_k\}$ is equivalent to the canonical basis of ℓ_2.

One of the aims of this article is to give a complete characterization of the strict singularity of inclusions between arbitrary r.i. spaces in terms of disjoint strict singularity. More precisely, it is proved in Section 3 (Theorem 2) that the inclusion $E \subset F$ is strictly singular if and only if it is disjointly strictly singular and the norms of these spaces are not equivalent on the Rademacher subspace $[r_n]$. This extends some previous results given in [HNS].

In [RS] the following result was proved for the class of r.i. spaces contained in G: Under some additional assumptions, the equivalence of the norms in two r.i. spaces E and F of this class on the Rademacher subspace, i.e.,

$$\Big\| \sum c_k r_k \Big\|_E \asymp \Big\| \sum c_k r_k \Big\|_F,$$

implies the coincidence of E and F up to equivalence of norms, i.e., $E = F$. More recently in [A] this result was obtained under a weaker assumption: the r.i. spaces E and F have to be interpolation spaces between L_∞ and G. It turns out that this interpolation assumption is actually a necessary condition for the above statement to hold. Theorem 9 in Section 4 shows that for any Marcinkiewicz space $M(\varphi) \subset G$ such that $M(\varphi)$ is not an interpolation space between L_∞ and G, there exists another Marcinkiewicz space $M(\psi) \subsetneq M(\varphi)$ with the property that the $M(\psi)$-norm and the $M(\varphi)$-norm are equivalent on the Rademacher subspace $[r_n]$. Also a criterion for the strict singularity of inclusions between Lorentz spaces $\Lambda(\varphi)$ and Marcinkiewicz spaces $M(\psi)$ is given (Theorem 11). In particular, for the class of all proper Lorentz spaces $\Lambda(\varphi)$ which do not contain G, the norms in $\Lambda(\varphi)$ and in the associated Marcinkiewicz space $M(\varphi)$ on the Rademacher subspace are *never* equivalent.

The last part of the paper contains some applications. In particular, we answer in the negative a question of V. Milman [Mi], showing that the r.i. spaces $E = L \log^\lambda L$ and $F = L_1$ satisfy the following conditions: the inclusion $E \subset F$ is not strictly singular and any infinite-dimensional subspace of E on which the norms of E and F are equivalent is an *uncomplemented* subspace of E (Theorem 16). We also prove that any disjointly strictly singular inclusion between r.i. spaces is weakly compact.

Some results of this paper have been announced in [SH].

2. Notation and definitions. Recall that a Banach function space E of measurable functions on $[0,1]$ is called *rearrangement invariant* (r.i. for short) or symmetric (cf. [LT2, 2.a.1], [KPS, 2.4.1]) if

- $|x(t)| \leq |y(t)|$ for all $t \in [0,1]$ and $y \in E$ imply $x \in E$ and $\|x\|_E \leq \|y\|_E$,
- if x and y are equimeasurable and $y \in E$, then $x \in E$ and $\|x\|_E = \|y\|_E$.

As usual we assume that every r.i. space E is separable or isomorphic to the conjugate space of some separable space. If E is an r.i. space then $L_\infty \subset E \subset L_1$ and $\|x\|_{L_1} \leq \|x\|_E \leq \|x\|_{L_\infty}$ for each $x \in L_\infty$, assuming $\|\chi_{(0,1)}\|_E = 1$.

Recall some important classes of r.i. spaces. If M is a positive convex function on $[0,\infty)$ with $M(0) = 0$, then the *Orlicz space* L_M consists of all measurable functions on $[0,1]$ for which

$$\|x\|_{L_M} = \inf\Big\{s > 0 : \int_0^1 M(|x(t)|/s)\,dt \leq 1\Big\}.$$

A remarkable example is the Orlicz space L_N generated by the function $N(u) = e^{u^2} - 1$. The space L_N is non-separable and we will denote by G the closure of L_∞ in L_N. Another two special Orlicz spaces that will be considered here are generated by the functions $M(u) = e^{u^\lambda} - 1$ and $M(u) = u\log^\lambda(1+u)$, for $\lambda > 0$, and denoted by $\exp L_\lambda$ and $L\log^\lambda L$.

Let Ω be the set of all increasing concave functions on $[0,1]$ with $\varphi(0) = 0$ and $\varphi(1) = 1$. Each $\varphi \in \Omega$ generates the *Lorentz space* $\Lambda(\varphi)$ endowed with the norm

$$\|x\|_{\Lambda(\varphi)} = \int_0^1 x^*(t)\,d\varphi(t),$$

and the *Marcinkiewicz space* $M(\varphi)$ with

$$\|x\|_{M(\varphi)} = \sup_{0<\tau\leq 1} \frac{\varphi(\tau)}{\tau}\int_0^\tau x^*(t)\,dt,$$

where $x^*(t)$ is the decreasing rearrangement of $|x(t)|$. For any $\varphi \in \Omega$ we have $\Lambda(\varphi) \subset M(\varphi)$ and $\|x\|_{M(\varphi)} \leq \|x\|_{\Lambda(\varphi)}$ for every $x \in \Lambda(\varphi)$. The spaces $\Lambda(\varphi)$ and $M(\varphi)$ coincide up to equivalence of norms if and only if $\varphi(+0) > 0$ or $\lim_{t\to 0}\varphi(t)/t < \infty$.

Recall that the *fundamental function* φ_E of a r.i. space E is defined by $\varphi_E(t) = \|\chi_{[0,t]}\|_E$ for $0 \leq t \leq 1$. The function φ_E is quasi-concave, i.e., $\varphi_E(t)$ and $t/\varphi_E(t)$ increase on $(0,1]$. Up to equivalence of norms, φ_E is a concave function. In that case $\Lambda(\varphi_E) \subset E \subset M(\varphi_E)$ and $\|x\|_{M(\varphi_E)} \leq \|x\|_E \leq \|x\|_{\Lambda(\varphi)}$ for every $x \in \Lambda(\varphi_E)$. The fundamental function $\varphi_E(t)$ is continuous for $t \in (0,1]$. The condition $\varphi_E(+0) > 0$ is necessary and sufficient for the coincidence of the spaces E and L_∞ up to equivalence of norms. If two r.i. spaces E and F coincide as sets then (by the closed graph theorem) the norms $\|\cdot\|_E$ and $\|\cdot\|_F$ are equivalent, and we write $E = F$.

Let $r_k(t) = \operatorname{sign}(\sin 2^k\pi t)$, $k \in \mathbb{N}$, be the Rademacher functions on $[0,1]$. It was proved in [RS] (see also [LT2, Thm. 2.b.4]) that for an r.i. space E the Khinchin inequality

$$\Big\|\sum_{k=1}^{\infty} c_k r_k\Big\|_E \le M\|\{c_k\}\|_{\ell_2}$$

is valid, for some constant $M > 0$, if and only if $E \supset G$. It follows immediately that for r.i. spaces E and F with $E \subset F$ the inclusion $E \subset F$ is not SS provided that $E \supset G$.

The proofs of some statements of this article will make use of interpolation methods. Therefore we recall some concepts and results in the r.i. setting.

Let (E,F) be a pair of r.i. spaces and $x \in E + F$. The Peetre's K-*functional* is defined as

$$K(t,x,E,F) = \inf\{\|u\|_E + t\|v\|_F : x = u + v\}$$

for every $t > 0$. Every Banach lattice Φ on $[0,\infty)$ such that $\min(1,t) \in \Phi$ generates the space $(E,F)^K_\Phi$ of the real interpolation method endowed with the norm

$$\|x\|_{(E,F)^K_\Phi} := \|K(\cdot,x,E,F)\|_\Phi.$$

The space $(E,F)^K_\Phi$ has the interpolation property with respect to the pair (E,F), i.e., every linear operator A bounded in E and F is also bounded in $(E,F)^K_\Phi$ and $\|A\|_{(E,F)^K_\Phi} \le \max(\|A\|_E, \|A\|_F)$. In the classical case of Φ being the lattice on $[0,\infty)$ with the norm

$$\|z\|_\Phi = \bigg(\int_0^\infty (t^{-\theta}|z(t)|)^p \,\frac{dt}{t}\bigg)^{1/p},$$

where $\theta \in (0,1)$ and $p \in [1,\infty]$ (with the usual modification for $p = \infty$), the interpolation spaces $(E,F)^K_\Phi$ are denoted by $(E,F)_{\theta,p}$.

We will denote by $I(E,F)$ the set of all interpolation spaces with respect to the pair (E,F). If, for any $x,y \in E+F$ with $K(t,x,E,F) \le K(t,y,E,F)$ for every $t > 0$, there exists a linear operator A bounded in E and F and such that $x = Ay$, then the set $I(E,F)$ is *described by the real interpolation method*, in the sense that for each space $Q \in I(E,F)$ there exists a Banach lattice Φ such that $\|x\|_Q = \|K(\cdot,x,E,F)\|_\Phi$ (cf. [BK]).

If $f(x)$ and $g(x)$ are positive functions on some set A, we shall write $f \asymp g$ if there exists $C > 0$ such that $C^{-1}f(x) \le g(x) \le Cf(x)$ for every $x \in A$.

We refer to the monographs [LT2] and [KPS] for the above results on r.i. spaces and to [BK] and [BL] for those on interpolation spaces.

3. Strict singularity via disjoint strict singularity. Given an r.i. space E on $[0,1]$, we denote by E_0 the closure of L_∞ in E. The space E_0 is always separable, except for $E = L_\infty$.

PROPOSITION 1. *Let E and F be r.i. spaces with $E \subset F$. Then the inclusion $E \subset F$ is disjointly strictly singular if and only if the inclusion $E_0 \subset F$ is disjointly strictly singular.*

Proof. The "only if" part is evident. Assume that the inclusion $E \subset F$ is not DSS. Thus there exist a disjoint sequence $\{x_k\}_{k=1}^\infty$ in E and $M > 0$ such that

$$\Big\| \sum_{k=1}^n c_k x_k \Big\|_E \le M \Big\| \sum_{k=1}^n c_k x_k \Big\|_F$$

for every $n \in \mathbb{N}$ and $c_k \in \mathbb{R}$. We consider separately the cases (i) $E \subset F_0$ and (ii) $E \not\subset F_0$.

In case (i) we have $x_k \in F_0$ for $k \in \mathbb{N}$. Clearly we can assume $\|x_k\|_F = 1$ for $k \in \mathbb{N}$. It is well known that $\lim_{\operatorname{meas} A \to 0} \|x\chi_A\|_F = 0$ for any $x \in F_0$ (cf. [KPS, 2.4.5]). Hence there exists a sequence $\{A_k\}_{k=1}^\infty$ of subsets of $[0,1]$ with $A_k \subset \operatorname{supp} x_k$ such that $y_k = x_k\chi_{A_k} \in L_\infty \subset E_0$ for $k \in \mathbb{N}$ and

$$\sum_{k=1}^\infty \|x_k - y_k\|_F < \frac{1}{2}.$$

Now, by a stability result [LT1, Prop. 1.a.9], the sequences $\{x_k\}$ and $\{y_k\}$ are equivalent in F and

$$\Big\| \sum_{k=1}^n c_k x_k \Big\|_F \le 2 \Big\| \sum_{k=1}^n c_k y_k \Big\|_F$$

for every $n \in \mathbb{N}$ and $c_k \in \mathbb{R}$. Since $|\sum_{k=1}^n c_k y_k| \le |\sum_{k=1}^n c_k x_k|$, we have

$$\Big\| \sum_{k=1}^n c_k y_k \Big\|_E \le \Big\| \sum_{k=1}^n c_k x_k \Big\|_E \le M \Big\| \sum_{k=1}^n c_k x_k \Big\|_F \le 2M \Big\| \sum_{k=1}^n c_k y_k \Big\|_F$$

for any $n \in \mathbb{N}$ and $c_k \in \mathbb{R}$. Therefore the norms $\|\cdot\|_E$ and $\|\cdot\|_F$ are equivalent on the span of $\{y_k\}$ in E_0 and the inclusion $E_0 \subset F$ is not DSS.

(ii) Consider now the case $E \not\subset F_0$. Since $E_0 \subset F_0$ we have $E \setminus E_0 \not\subset F_0$ and $(E \setminus E_0) \cap (F \setminus F_0) \neq \emptyset$. Choose $z = z^* \in (E \setminus E_0) \cap (F \setminus F_0)$. Then

$$d_E(z, L_\infty) = a > 0, \qquad d_F(z, L_\infty) = b > 0$$

where $d_E(z, L_\infty) = \inf\{\|z - u\|_E : u \in L_\infty\}$. Since we have $\|z\chi_{(0,\tau)}\|_E =$

$\lim_{\varepsilon\to 0}\|z\chi_{(\varepsilon,\tau)}\|_E$ and $\|z\chi_{(0,\tau)}\|_E \ge a$ for $0<\tau\le 1$, it follows that

$$\lim_{\varepsilon\to 0}\|z\chi_{(\varepsilon,\tau)}\|_E \ge a.$$

Similarly we have $\lim_{\varepsilon\to 0}\|z\chi_{(\varepsilon,\tau)}\|_F \ge b$ for $0<\tau\le 1$. Hence we can construct a sequence $\tau_k\downarrow 0$ such that

$$\|z\chi_{(\tau_{k+1},\tau_k)}\|_E \ge a/2 \quad \text{and} \quad \|z\chi_{(\tau_{k+1},\tau_k)}\|_F \ge b/2$$

for every natural k. Let $z_k := z\chi_{(\tau_{k+1},\tau_k)}$ for $k\in\mathbb{N}$. Clearly, $z_k\in L_\infty\subset E_0\subset F_0$ and

$$\frac{a}{2}\max_{k\in\mathbb{N}}|c_k| \le \Big\|\sum_{k=1}^{\infty} c_k z_k\Big\|_E \le \|z\|_E \max_{k\in\mathbb{N}}|c_k|$$

and

$$\frac{b}{2}\max_{k\in\mathbb{N}}|c_k| \le \Big\|\sum_{k=1}^{\infty} c_k z_k\Big\|_F \le \|z\|_F \max_{k\in\mathbb{N}}|c_k|,$$

for any sequence $\{c_k\}\in c_0$. Hence the sequence $\{z_k\}_{k=1}^\infty$ is equivalent in E_0 and in F_0 to the canonical basis of c_0. Consequently, the inclusion $E_0\subset F$ is not DSS. ∎

Recall that G is the closure of L_∞ in $\exp L_2$. We can now prove the main result of this section.

THEOREM 2. *Let E and F be r.i. spaces with $E\subset F$. The inclusion $E\subset F$ is strictly singular if and only if it is disjointly strictly singular and $E\not\supset G$.*

Proof. The case of E separable has been proved in [HNS, Theorem 5], so we assume that E is non-separable. Suppose that the inclusion $E\subset F$ is not SS and $E\not\supset G$. We have to prove that the inclusion $E\subset F$ is not DSS. Let Q denote the (closed) infinite-dimensional subspace of E on which the norms $\|\cdot\|_E$ and $\|\cdot\|_F$ are equivalent. Now, if the norms of E and L_1 were equivalent on Q, we would have $E\supset G$, by Theorem 1 of [HNS]. Therefore, we can assume that the norms of E and L_1 are not equivalent on Q.

We first deal with the case of F separable. Consider the real interpolation space $E_1 := (E,F)_{\theta,p}$ for some $0<\theta<1$ and $1<p<\infty$. The separability of F implies $\lim_{\operatorname{meas} A\to 0}\|x\chi_A\|_F = 0$ for any $x\in F$, so also for $x\in E$. Hence $K(t,x,E,F)=K(t,x,E_0,F)$ for $x\in F$, which implies that $E_1=(E_0,F)_{\theta,p}$ (cf. [BL, Thm. 3.4.2]). Therefore E_1 is also separable and $E\subset E_1\subset F$ with $E_1\ne F$.

Now, since the norms of E, E_1 and F are equivalent on Q, the norms of E_1 and L_1 are not. Hence, applying the Kadec–Pełczyński method ([LT2], see [HNS, Thm. 5]) we can find a normalized sequence $\{x_n\}$ in Q and a sequence of disjoint measurable sets $A_n\subset \operatorname{supp} x_n$, $n\in\mathbb{N}$, such that $y_n := x_n\chi_{A_n}\in L_\infty$ and the sequence $\{x_n\}$ is equivalent to $\{y_n\}$ in E_1 and in F.

Now, using that the fact $|\sum c_n y_n| \leq |\sum c_n x_n|$ and the equivalence of the norms of E and E_1 on $[x_n]$, we have

$$\begin{aligned}\Big\|\sum_n c_n y_n\Big\|_E &\leq \Big\|\sum_n c_n x_n\Big\|_E \\ &\leq M_1\Big\|\sum_n c_n x_n\Big\|_{E_1} \leq M_2\Big\|\sum_n c_n y_n\Big\|_{E_1} \leq M_3\Big\|\sum_n c_n y_n\Big\|_E\end{aligned}$$

for any scalar sequence $\{c_n\}$ and for some constants $M_1, M_2, M_3 > 0$. Therefore, the sequences $\{x_n\}$ and $\{y_n\}$ are also equivalent in E. Thus the norms of E and F are equivalent on $[y_n]$ and the inclusion $E \subset F$ is not DSS.

Finally, assume that E and F are non-separable. We distinguish two cases: $E \subset F_0$ and $E \not\subset F_0$. If $E \subset F_0$, this inclusion cannot be SS and since F_0 is separable, we deduce as earlier that the inclusion $E \subset F$ is not DSS. In the case of $E \not\subset F_0$, we get the same conclusion by proceding as in the second part of the proof of Proposition 1. ∎

Notice that Theorem 2 may be reformulated as follows: the inclusion $E \subset F$ is strictly singular if and only if it is disjointly strictly singular and the norms of E and F are not equivalent on the Rademacher subspace $[r_n]$.

Corollary 3. *Let E and F be r.i. spaces with $E \subset F$ and $E \not\supset G$. If the norms of E and F are equivalent on $[r_n]$ then there exists a disjoint sequence $\{x_n\}$ in E for which the norms of E and F are equivalent on $[x_n]$.*

Corollary 4. *Let E and F be r.i. spaces with $E \subset F$. The inclusion $E \subset F$ is strictly singular if and only if the inclusion $E_0 \subset F$ is strictly singular.*

Proof. The "only if" part is trivial. Suppose that the inclusion $E \subset F$ is not SS. It follows from Theorem 2 that either $E \supset G$, or the inclusion $E \subset F$ is not DSS. If $E \supset G$ then $E_0 \supset G$ since G is separable. And if the inclusion $E \subset F$ is not DSS then, by Proposition 1, neither is the inclusion $E_0 \subset F$; all the more, it is not SS. ∎

4. Strict singularity and Rademacher spaces. In this section we study couples of r.i. spaces E and F "close" to L_∞ with equivalence of norms on the Rademacher subspace. For that we will make use of some interpolation results.

Given a r.i. space E, consider the sequence space $R(E)$ of Rademacher coefficients $\{a_k\}$ endowed with the norm

$$\|\{a_k\}\|_{R(E)} = \Big\|\sum_{k=1}^{\infty} a_k r_k\Big\|_E.$$

It is easy to check that $R(E)$ is an interpolation space between ℓ_1 and ℓ_2, i.e., $R(E) \in I(\ell_1, \ell_2)$. Moreover, it is known that the set $I(\ell_1, \ell_2)$ is described by the real interpolation method (cf. [LS]). Therefore there exists a Banach lattice F of measurable functions on $[0, \infty)$ with respect to the measure dt/t such that $\min(1, t) \in F$ and

$$R(E) = (\ell_1, \ell_2)_F^K = \{a \in \ell_2 : K(t, a, \ell_1, \ell_2) \in F\} \tag{1}$$

(cf. [BK, Thms. 4.4.5 and 4.4.38]).

We can consider the r.i. space $\widetilde{E}$ associated to E defined by

$$\widetilde{E} := (L_\infty, G)_F^K$$

with its canonical norm $\|x\|_{\widetilde{E}} = \|K(\cdot, x, L_\infty, G)\|_F$, where G is the closure of L_∞ in $L_N \equiv \exp L_2$. It is known ([A, Thm. 1.4]) that $R(E) = R(\widetilde{E})$, i.e.,

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\widetilde{E}} \asymp \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_E. \tag{2}$$

Moreover, $E = \widetilde{E}$ if and only if $E \in I(L_\infty, G)$ ([A, Thm. 1.5]).

Given $x \in L_N$, denote by Sx the function

$$Sx(t) := \log^{1/2} \frac{e}{t} \sup_{0<u\le t} x^*(u) \log^{-1/2} \frac{e}{u} \tag{3}$$

for $0 < t \le 1$. The following statement gives a simple description of the r.i. space $\widetilde{E}$.

PROPOSITION 5. *If E is an r.i. space then $\|x\|_{\widetilde{E}} \asymp \|Sx\|_E$.*

Proof. It follows from [M, Cor. 2.2] that there exist absolute constants $C_1, C_2, \beta > 0$ such that for all $a \in \ell_2$,

$$\Big(\sum_{k=1}^{\infty} a_k r_k\Big)^*(t) \le C_1 K(\log^{1/2}(e/t), a, \ell_1, \ell_2)$$

and

$$\Big(\sum_{k=1}^{\infty} a_k r_k\Big)^*(\beta t) \ge C_2 K(\log^{1/2}(e/t), a, \ell_1, \ell_2)$$

for every $0 < t \le 1$. Hence

$$\|a\|_{R(E)} \asymp \|K(\log^{1/2}(e/t), a, \ell_1, \ell_2)\|_E. \tag{4}$$

The sets of K-functionals corresponding to the Banach pairs (L_∞, G) and (ℓ_1, ℓ_2) coincide up to equivalence ([A]). If $x \in \widetilde{E}$, then $K(t, x, L_\infty, G) \in F$ and there exists $a \in l_2$ such that

$$K(t, x, L_\infty, G) \asymp K(t, a, \ell_1, \ell_2) \tag{5}$$

for $t > 0$. Hence $a \in R(E)$ and, by (4), we have $K(\log^{1/2}(e/t), a, \ell_1, \ell_2) \in E$. Using (5) we get $K(\log^{1/2}(e/t), x, L_\infty, G) \in E$.

Similar arguments show that the converse holds: if $K(\log^{1/2}(e/t), x, L_\infty, G) \in E$, then $x \in \widetilde{E}$. Thus the space $\widetilde{E}$ and the (Banach) space endowed with the norm $\|K(\log^{1/2}(e/t), \cdot, L_\infty, G)\|_E$ coincide as sets, so, by the closed graph theorem, both norms are equivalent, i.e.,

$$\|x\|_{\widetilde{E}} \asymp \|K(\log^{1/2}(e/t), x, L_\infty, G)\|_E. \tag{6}$$

Finally, it is easy to see that

$$K(t, x, L_\infty, G) = K(t, x, L_\infty, L_N) \tag{7}$$

for any $x \in G$ and $t > 0$. And it is well known ([Lo]) that

$$\|x\|_{L_N} \asymp \|x\|_{M(\varphi_0)} \tag{8}$$

for $x \in L_N$ where $\varphi_0(t) = \log^{-1/2}(e/t)$, and clearly, $L_\infty = M(\varphi_1)$ for the function $\varphi_1(t) = 1$. Therefore we can consider the Banach pair (L_∞, L_N) as a pair of Marcinkiewicz spaces and apply a formula for the K-functional from [CN]. Thus we have

$$K(\log^{1/2}(e/t), x, L_\infty, L_N) \asymp \log^{1/2}(e/t) \sup_{0<u\le t} x^*(u) \log^{-1/2}(e/u), \tag{9}$$

and the needed equivalence follows now from (6)–(9). ∎

Note that $Sx(t) \ge x^*(t)$. Hence the above proposition implies that $\widetilde{E} \subset E$ and $\|x\|_E \le C\|x\|_{\widetilde{E}}$ for every $x \in \widetilde{E}$ and some constant $C > 0$. In particular, $\varphi_E(t) \le C\varphi_{\widetilde{E}}(t)$ for every $t \in [0, 1]$.

We can now give a characterization of the Lorentz and Marcinkiewicz spaces which are interpolation spaces between L_∞ and G.

Proposition 6. *Let $\psi \in \Omega$.*

(i) *A Lorentz space $\Lambda(\psi)$ belongs to the set $I(L_\infty, G)$ if and only if $\varphi_{\widetilde{\Lambda(\psi)}}(t) \le C\psi(t)$ for some $C > 0$ and $0 \le t \le 1$.*

(ii) *A Marcinkiewicz space $M(\psi)$ belongs to the set $I(L_\infty, L_N)$ if and only if $\varphi_{\widetilde{M(\psi)}}(t) \le C\psi(t)$ for some $C > 0$ and $0 \le t \le 1$.*

Proof. (i) If $\Lambda(\psi) \in I(L_\infty, G)$ then $\widetilde{\Lambda(\psi)} = \Lambda(\psi)$ ([A]) and hence the functions $\varphi_{\widetilde{\Lambda(\psi)}}$ and ψ are equivalent.

Conversely, if $\varphi_{\widetilde{\Lambda(\psi)}}(t) \le C\psi(t)$, then Proposition 5 implies that the quasi-linear operator S on $\Lambda(\psi)$ defined in (3) is uniformly bounded on the set of characteristic functions. Hence [KPS, Lemma 2.5.2] shows that S is bounded in $\Lambda(\psi)$. Therefore $\|Sx\|_{\Lambda(\psi)} \le C\|x\|_{\Lambda(\psi)}$ for some $C > 0$, and Proposition 5 yields $\widetilde{\Lambda(\psi)} = \Lambda(\psi)$, and hence $\Lambda(\psi) \in I(L_\infty, G)$.

(ii) If $M(\psi) \in I(L_\infty, L_N)$ then $M(\psi) \cap G \in I(L_\infty, G)$. Indeed, since the set $I(L_\infty, L_N)$ can be described by the real interpolation method ([A]) we have $M(\psi) = (L_\infty, L_N)^K_F$ for some Banach lattice F on $[0, \infty)$. And, by (7), we have $M(\psi)\cap G=\widetilde{(L_\infty,G)^K_F}$. This means that $M(\psi)\cap G\in I(L_\infty,G)$ and $M(\psi)\cap G=\widetilde{M(\psi)\cap G}$. Moreover, since $M(\psi)\subset L_N$ the fundamental functions of the spaces $M(\psi)$ and $M(\psi)\cap G$ are equivalent, and

$$\Big\|\sum a_k r_k\Big\|_{M(\psi)}\asymp\Big\|\sum a_k r_k\Big\|_{M(\psi)\cap G},$$

therefore $\widetilde{M(\psi)}=\widetilde{M(\psi)\cap G}$. Hence,

$$\varphi_{\widetilde{M(\psi)}}(t)\asymp\varphi_{\widetilde{M(\psi)\cap G}}(t)\asymp\varphi_{M(\psi)\cap G}(t)\asymp\varphi_{M(\psi)}(t)=\psi(t).$$

Let us now prove the converse. Assume $\varphi_{\widetilde{M(\psi)}}(t)\le C\psi(t)$. Then, by Proposition 5, there is $C>0$ such that

$$\|S\chi_{(0,\tau)}\|_{M(\psi)}\le C\psi(\tau) \tag{10}$$

for $\tau\in[0,1]$. Now, since

$$S\chi_{(0,\tau)}(t)=\begin{cases}1, & 0<t\le\tau\\ \left(\dfrac{\log(e/t)}{\log(e/\tau)}\right)^{1/2}, & \tau\le t\le 1,\end{cases} \tag{11}$$

and $\int_0^t\log^{1/2}(e/s)\,ds\asymp t\log^{1/2}(e/t)$, for $0<t\le 1$, we get

$$\begin{aligned}\frac{1}{t}\int_0^t S\chi_{(0,\tau)}(s)\,ds&\asymp\chi_{(0,\tau)}(t)+\left(\frac{\tau}{t}+\left(\frac{\log(e/t)}{\log(e/\tau)}\right)^{1/2}\right)\chi_{(\tau,1)}(t)\\&\asymp S\chi_{(0,\tau)}(t).\end{aligned} \tag{12}$$

Hence

$$\|S\chi_{(0,\tau)}\|_{M(\psi)}\asymp\left(\log\frac{e}{\tau}\right)^{-1/2}\sup_{\tau\le t\le 1}\psi(t)\log^{1/2}\frac{e}{t}. \tag{13}$$

Therefore (10) can be rewritten as

$$\frac{\psi(t)}{\psi(\tau)}\le C_1\left(\frac{\log(e/\tau)}{\log(e/t)}\right)^{1/2},\qquad 0<\tau\le t\le 1.$$

Since $\varphi_{L_N}(t)=\log^{-1/2}(e/t)$ the above inequality proves, by [S], that $M(\psi)\in I(L_\infty,L_N)$. ∎

Corollary 7. *Let E be an r.i. space with $E\in I(L_\infty,G)$. Then the Marcinkiewicz space $M(\varphi_E)$ belongs to $I(L_\infty,L_N)$.*

Proof. We have $E\subset M(\varphi_E)$ (cf. [KPS, Thm. 2.5.7]) and, using [A] once more, we get $E=\widetilde{E}\subset\widetilde{M(\varphi_E)}\subset M(\varphi_E)$. Now, since E and $M(\varphi_E)$ have the same fundamental functions, the spaces $\widetilde{M(\varphi_E)}$ and $M(\varphi_E)$ have equivalent fundamental functions. This implies, by Proposition 6 above, that $M(\varphi_E)\in I(L_\infty,L_N)$. ∎

COROLLARY 8. *Given a Marcinkiewicz space $M(\varphi)$, there exists a Marcinkiewicz space $M(\psi)$ with $M(\psi)\in I(L_\infty, L_N)$ such that $\widetilde{M(\varphi)} = M(\psi)\cap G$.*

Proof. Let ψ denote the fundamental function of $\widetilde{M(\varphi)}$. Since $\widetilde{M(\varphi)} \in I(L_\infty, G)$, Corollary 7 yields $M(\psi) \in I(L_\infty, L_N)$. Now, from the inclusions $\widetilde{M(\varphi)} \subset M(\psi) \cap G \subset M(\varphi)$ and (2), we deduce that

$$\Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{\widetilde{M(\varphi)}} \asymp \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{M(\psi)\cap G} \asymp \Big\| \sum_{k=1}^{\infty} a_k r_k \Big\|_{M(\varphi)}$$

for **all** sequences $a \in \ell_2$. Since it was proved in Proposition 6 that $M(\psi)\cap G \in I(L_\infty, G)$, an application of [A, Thm. 1.5] shows that $\widetilde{M(\varphi)} = M(\psi) \cap G$. ∎

We are now in a position to present one of the main results of this section.

THEOREM 9. *If a Marcinkiewicz space $M(\varphi) \subset G$ does not belong to $I(L_\infty, G)$, then there exists another Marcinkiewicz space $M(\psi)$ such that $M(\psi) \subsetneq M(\varphi)$ and $R(M(\varphi)) = R(M(\psi))$.*

Proof. We have $\widetilde{M(\varphi)} \subset M(\varphi) \subset G$. Hence, by Corollary 8, if ψ is the fundamental function of $\widetilde{M(\varphi)}$ then $\widetilde{M(\varphi)} = M(\psi) \cap G = M(\psi)$ and $M(\psi) \in I(L_\infty, G)$. Moreover, the Marcinkiewicz spaces $M(\varphi)$ and $M(\psi)$ do not coincide because $M(\varphi) \notin I(L_\infty, G)$. Finally, by (2), we have $R(M(\varphi)) = R(M(\psi))$. ∎

An analogous result is also valid for Lorentz spaces.

THEOREM 10. *If a Lorentz space $\Lambda(\varphi) \subset G$ does not belong to $I(L_\infty, G)$, then there exists another Lorentz space $\Lambda(\psi)$ such that $\Lambda(\psi) \subsetneq \Lambda(\varphi)$ and $R(\Lambda(\varphi)) = R(\Lambda(\psi))$.*

Proof. Let $X := \widetilde{\Lambda(\varphi)}$. Since $X \in I(L_\infty, G)$ and $\Lambda(\varphi) \notin I(L_\infty, G)$, we have $X \subsetneq \Lambda(\varphi)$. It is easily checked that the Köthe dual X' satisfies $X' \supsetneq M(t/\varphi)$. Therefore, there exists a positive decreasing function $a(\cdot) \in X'$ such that

$$a(t) \ge \varphi'(t) \quad (0 < t \le 1) \quad \text{and} \quad \limsup_{t\to+0} \frac{1}{\varphi(t)} \int_0^t a(s)\, ds = \infty.$$

Define $\psi(t) := \int_0^t a(s)\, ds$ $(0 \le t \le 1)$. Since for every $x \in X$ we have

$$\int_0^1 x^*(t)\, d\psi(t) = \int_0^1 x^*(t) a(t)\, dt < \infty,$$

it follows that $X \subset \Lambda(\psi)$. Moreover, $\Lambda(\psi) \subsetneq \Lambda(\varphi)$, so the equality $R(\Lambda(\varphi)) = R(\Lambda(\psi))$ follows from (2). ∎

In particular, the above inclusions $M(\psi) \subset M(\varphi)$ and $\Lambda(\psi) \subset \Lambda(\varphi)$ are not strictly singular and, by Corollary 3, not disjointly strictly singular either. By contrast, we have the following:

THEOREM 11. *Let $\varphi, \psi \in \Omega$ be such that $\Lambda(\varphi) \subset M(\psi)$. The inclusion $\Lambda(\varphi) \subset M(\psi)$ is strictly singular if and only if $\Lambda(\varphi) \not\supset G$ and $\psi(+0) = 0$.*

Proof. The necessity part is well known. If $\Lambda(\varphi) \supset G$ then $R(\Lambda(\varphi)) = R(M(\psi)) = \ell_2$ ([RS]). In the case when $\psi(+0) > 0$ we have $M(\psi) = L_\infty = \Lambda(\varphi)$.

Conversely, since $\psi(+0)=0$ we have $M(\psi)\neq L_\infty$, and clearly $\Lambda(\varphi)\neq L_1$. Hence the spaces $\Lambda(\varphi)$ and $M(\psi)$ do not coincide. Thus the statement is known for the left extreme case of $\Lambda(\varphi) = L^\infty$ ([N]) and also for the right extreme case of $M(\psi) = L^1$ since $\Lambda(\varphi) \not\supset G$ ([HNS, Thm. 1]). Now, using the fact that any normalized disjoint sequence in $\Lambda(\varphi)$ (resp. $M(\psi)$) contains a subsequence equivalent to the canonical basis of ℓ_1 [FJT] (resp. of c_0, cf. [Se]) we deduce that the inclusion $\Lambda(\varphi) \subset M(\psi)$ is DSS. Hence, by Theorem 2, it is also SS. ∎

In particular: *the canonical inclusion $\Lambda(\varphi) \subset M(\varphi)$ is strictly singular if and only if $\Lambda(\varphi) \not\supset G$ and $\varphi(+0) = 0$.*

A direct consequence is

COROLLARY 12. *Let $\varphi \in \Omega$. Then $R(\Lambda(\varphi)) = R(M(\varphi))$ if and only if $\Lambda(\varphi) \supset G$ or $\varphi(+0) > 0$.*

5. Applications. In this section we give some applications of the main results.

PROPOSITION 13. *Let E and F be r.i. spaces with $E \not\supset G$. If*

$$\int_0^1 \left(\frac{t}{\varphi_E(t)}\right)' \varphi_F'(t)\, dt < \infty, \tag{14}$$

then $E \subset F$ and this inclusion is strictly singular.

Proof. It was proved in Theorem 3.1 of [GHSS] that condition (14) implies the inclusion $E \subset F$ and that this inclusion is DSS. Hence, using Theorem 2, we get the statement. ∎

COROLLARY 14. *Let E and F be r.i. spaces such that $\varphi_E(t) \geq a \log^{-\alpha}(e/t)$ and $\varphi_F(t) \leq b \log^{-\beta}(e/t)$ for some $0 < \alpha < \min(\beta, 1/2)$ and constants $a, b > 0$. Then the inclusion $E \subset F$ is strictly singular.*

Proof. We may assume that the functions $t/\varphi_E(t)$ and φ_F are concave on $(0,1]$. Then $(t/\varphi_E(t))'$ and φ_F' decrease on $(0,1]$. Now, applying twice the property 2.2.19 in [KPS], we get

$$\int_0^1 \left(\frac{t}{\varphi_E(t)}\right)' \varphi_F'(t)\,dt \le \int_0^1 \left(\frac{t}{a\log^{-\alpha}(e/t)}\right)' \left(b\log^{-\beta}\frac{e}{t}\right)' dt$$

$$\le \frac{b\beta}{a}\int_0^1 \log^{\alpha-\beta-1}\frac{e}{t}\,\frac{dt}{t} = \frac{b\beta}{a(\beta-\alpha)} < \infty.$$

By [KPS, Thm. 2.5.7]), the assumption $\alpha < 1/2$ implies $E \subsetneq G$. Hence the statement follows from the above proposition. ∎

Proposition 15. *Let E and F be r.i. spaces with $E \subset F$. If the inclusion $E \subset F$ is disjointly strictly singular then the inclusion operator is weakly compact.*

Proof. We can assume that $E \subset F_0$. Indeed, otherwise, reasoning as in the proof of Proposition 1 we construct a disjoint sequence $\{z_k\}$ in E_0 which is equivalent in E_0 and in F_0 to the canonical basis of c_0. So the inclusion $E \subset F$ is not DSS.

Now, let $E \subset F$ with F separable, hence order continuous. Assume that $E \subset F$ is not weakly compact. Consider the real interpolation space $(E,F)_{\theta,p}$ for $0 < \theta < 1$, $1 < p < \infty$, which is not reflexive by [B, Thm. 3.1]. Hence the lattice $(E,F)_{\theta,p}$ contains a subspace Q isomorphic to ℓ_1 or to c_0 (cf. [LT2]). Now if Q is isomorphic to ℓ_1, we find, by [B, Prop. 2.3.3], that the inclusion $E \hookrightarrow F$ preserves an ℓ_1-isomorphic copy. In the case of Q isomorphic to c_0, an analogous statement is also true [Ma, Cor. 4.1]. Now, using ([Me, Thms. 3.4.11–3.4.17]), we deduce that $E \subset F$ also preserves a disjoint ℓ_1-sequence or a disjoint c_0-sequence. ∎

V. Milman [Mi] posed the following question: Given two Banach spaces E and F and a non-strictly singular operator A from E into F, does there exist a *complemented* subspace Q in E such that the restriction of the operator A to Q is an isomorphism?

We give a negative answer to this question using the above results. First note that the inclusions $L\log^\lambda L \subset L_1$ are not strictly singular for $\lambda > 0$ because the Rademacher spaces satisfy $R(L\log^\lambda L) = R(L_1) = \ell_2$.

Recall that an operator $A : E \to F$ between two Banach spaces E and F is said to be *strictly cosingular* (or *Pełczyński*) if there is no infinite-dimensional space H and onto operators $h : E \to H$ and $g : F \to H$ such that $h = gA$. Note that this class of operators is somewhat related by duality to strictly singular operators ([P]).

Theorem 16. *Let $0 < \lambda < 1/2$. If Q is a subspace of $L\log^\lambda L$ on which the $L\log^\lambda L$-norm and the L_1-norm are equivalent, then Q is not complemented in $L\log^\lambda L$.*

Proof. Suppose the contrary and denote by P a projection from $L\log^\lambda L$ onto Q. There exists a reflexive r.i. space E with $L\log^\lambda L \subset E \subset L_1$ ([FS]). Therefore Q is a reflexive subspace of L_1. It follows from Rosenthal's theorem [R, Thm. 8] that Q embeds isomorphically into L_p for some $p > 1$, i.e., there exists an operator $T : (Q, \|\cdot\|_{L_1}) \to L_p$ which is an isomorphism onto its image. Set $Z = T(Q)$.

Now, consider the inclusion operator $i : L_p \hookrightarrow L\log^\lambda L$ which is not strictly cosingular since there exist onto operators $R = TPi : L_p \to Z$ and $TP : L\log^\lambda L \to Z$ with $TPi = R$. On the other hand, by Corollary 14, the adjoint operator $i^* : \exp L_\mu \hookrightarrow L_{p'}$ is SS because $\mu > 2$ (here $\mu = 1/\lambda$ and $p' = p/(p-1)$). Hence, using [P, Prop. 1] we conclude that the inclusion operator i is strictly cosingular, which gives a contradiction. Thus the subspace Q cannot be complemented in $L\log^\lambda L$ (and hence not in L_1 either). ■

Note that the assumption $0 < \lambda < 1/2$ is essential since the Rademacher subspace $[r_n]$ is complemented in $L\log^\lambda L$ for $\lambda \geq 1/2$ (cf. [LT2, Prop. 2.b.4]).

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References

[A] S. V. Astashkin, *About interpolation of subspaces of rearrangement invariant spaces generated by Rademacher system*, Int. J. Math. Math. Sci. 25 (2001), 451–465.

[B] B. Beauzamy, *Espaces d'interpolation réels : topologie et géométrie*, Lecture Notes in Math. 666, Springer, Berlin, 1978.

[BL] J. Bergh and J. Löfström, *Interpolation Spaces. An Introduction*, Springer, Berlin, 1976.

[BK] Yu. A. Brudnyĭ and N. Ya. Krugljak, *Interpolation Functors and Interpolation Spaces*, North-Holland, Amsterdam, 1991.

[CN] M. Cwikel and P. Nilsson, *Interpolation of Marcinkiewicz spaces*, Math. Scand. 56 (1985), 29–42.

[FJT] T. Figiel, W. B. Johnson and L. Tzafriri, *On Banach lattices and spaces having local unconditional structure*, *with applications to Lorentz function spaces*, J. Approx. Theory 13 (1975), 395–412.

[FS] C. Franchetti and E. M. Semenov, *A Hilbert space characterization among function spaces*, Anal. Math. 21 (1995), 85–93.

[GHSS] A. García del Amo, F. L. Hernández, V. M. Sánchez and E. M. Semenov, *Disjointly strictly-singular inclusions between rearrangement invariant spaces*, J. London Math. Soc. 62 (2000), 239–252.

[HNS] F. L. Hernández, S. Ya. Novikov and E. M. Semenov, *Strictly singular embeddings between rearrangement invariant spaces*, Positivity 7 (2003), 119–124.

[HR] F. L. Hernández and B. Rodríguez-Salinas, *On l^p-complemented copies in Orlicz spaces II*, Israel J. Math. 68 (1989), 27–55.

[KPS] S. G. Krein, Ju. I. Petunin and E. M. Semenov, *Interpolation of Linear Operators*, Amer. Math. Soc., Providence, RI, 1982.

[LT1] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces. I. Sequence Spaces*, Springer, Berlin, 1977.

[LT2] —, —, *Classical Banach Spaces. II. Function Spaces*, Springer, Berlin, 1979.

[Lo] G. G. Lorentz, *Relations between function spaces*, Proc. Amer. Math. Soc. 12 (1961), 127–132.

[LS] G. G. Lorentz and T. Shimogaki, *Interpolation theorems for the pair of spaces* (L^1, L^p) *and* (L^q, L^∞), Trans. Amer. Math. Soc. 159 (1973), 207–221.

[Ma] M. Mastyło, *Interpolation Banach lattices containing no isomorphic copies of* c_0, J. Math. Anal. Appl. 341 (2008), 676–688.

[Me] P. Meyer-Nieberg, *Banach Lattices*, Springer, 1991.

[Mi] V. D. Milman, *Operators of class* C_0 *and* C_0^*, in: Theory of Functions, Functional Analysis and Applications 10, Kharkov, 1970, 15–26 (in Russian).

[M] S. J. Montgomery-Smith, *The distribution of Rademacher sums*, Proc. Amer. Math. Soc. 109 (1990), 517–522.

[N] S. Ya. Novikov, *Boundary spaces for inclusion map between rearrangement invariant spaces*, Collect. Math. 44 (1997), 211–215.

[P] A. Pełczyński, *On strictly singular and strictly cosingular operators. I.* Bull. Acad. Polon. Sci. Sér. Sci. Math. Astronom. Phys. 13 (1965), 31–36.

[RS] V. A. Rodin and E. M. Semenov, *Rademacher series in symmetric spaces*, Anal. Math. 1 (1975), 207–222.

[R] H. P. Rosenthal, *On subspaces of* L^p, Ann. of Math. 97 (1973), 344–373.

[Se] E. M. Semenov, *On the stability of the real interpolation method in the class of rearrangement invariant spaces*, in: Function Spaces, Interpolation Spaces, and Related Topics (Haifa, 1995), Israel Math. Conf. Proc. 13, Bar-Ilan Univ., Ramat Gan, 1999, 172–182.

[SH] E. M. Semenov and F. L. Hernández, *Strictly singular and disjointly strictly singular inclusions*, Dokl. Ross. Akad. Nauk 413 (2007), 461–462 (in Russian).

[S] R. Sharpley, *Characterization of intermediate spaces of* M_φ, in: Linear Operators and Approximation II (Oberwolfach, 1974), P. L. Butzer and B. Sz.-Nagy (eds.), Birkhäuser, 1976, 205–214.

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