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Integrability and continuity of functions represented by trigonometric series: coefficients criteria

by

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Abstract. We study weighted L_p-integrability ($1 \le p < \infty$) of trigonometric series. It is shown how the integrability of a function with weight $x^{-\alpha}$ depends on some regularity conditions on Fourier coefficients. Criteria for the uniform convergence of trigonometric series in terms of their coefficients are also studied.

1. Introduction. Throughout this paper we shall let $f(x)$ and $g(x)$ denote the sums of the series

$$\sum_{n=1}^{\infty} a_n \cos nx \tag{1.1}$$

and

$$\sum_{n=1}^{\infty} b_n \sin nx, \tag{1.2}$$

respectively, whenever they exist. The paper concerns two main questions.

PROBLEM 1. If $f, g \in L_1$, what hypotheses on $\{a_n\}, \{b_n\}$ are equivalent to (imply, follow from)

$$|f(x)|^p x^{-\alpha} \in L \quad \text{or} \quad |g(x)|^p x^{-\alpha} \in L\ ? \tag{1.3}$$

This question is very well studied (see [Bo], [St], [Zy]) and many classical results on Fourier coefficients such as the Hardy–Littlewood–Paley theorem, Pitt's inequality, and the Young–Heywood–Boas results show that in many cases the appropriate condition on the coefficients is

$$\sum_{n=1}^{\infty} n^{\alpha+p-2}|a_n|^p < \infty \quad \text{or} \quad \sum_{n=1}^{\infty} n^{\alpha+p-2}|b_n|^p < \infty. \tag{1.4}$$

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For the case $p = 1$ many authors used the following convenient fact on the multipliers $x^{-\alpha}$ and $n^{\alpha-1}$: when $0 < \alpha < 1$, the function $x^{-\alpha}$ has a Fourier series whose coefficients behave at infinity like $n^{\alpha-1}$, and conversely. Using this, Heywood [He$_2$] proved that (1.4) implies (1.3). It is known that the converse holds under monotonicity of either $\{a_n\}, \{b_n\}$, or f, g. In Section 2 we show that, generally, without the monotonicity condition, the converse does not hold even in a much weaker form.

In Section 3 we consider the case of $1 < p < \infty$ and study one of the most important results in the theory of weighted Fourier inequalities, Pitt's theorem: (1.4) implies (1.3) for $\max(0, 2-p) \le \alpha < 1$. Because of the great importance of this theorem, we reprove it using the multipliers $x^{-\alpha}, n^{\alpha-1}$. In a very simple proof we apply only Hardy's inequalities and do not invoke the interpolation technique (see [St]).

In Section 4 we study trigonometric series with some regularity conditions on the coefficients. Our idea is to consider quantitative characteristics of the condition of bounded variation ($\sum |\Delta a_k| < \infty$), i.e.,

$$\sum_{k=n}^{\infty} |\Delta a_k| < C\beta_n \tag{1.5}$$

and

$$\sum_{k=n}^{2n-1} |\Delta a_k| < C\beta_n. \tag{1.6}$$

Here $\beta = \{\beta_n\}$ is a majorant, that is, a positive sequence; and C is a positive number independent of n.

We denote by $\overline{\mathrm{GM}}(\beta)$ and $\mathrm{GM}(\beta)$ the collections of all null-sequences (generally speaking, complex) satisfying (1.5) and (1.6), respectively. Such sequences are said to be *generally monotone sequences with majorant* β. Note that a class similar to $\overline{\mathrm{GM}}(\beta)$ was introduced by Leindler (see [Le$_2$] ($\beta_n = a_n$) and [Le$_3$]). The class $\mathrm{GM}(\beta)$ was introduced in [Ti$_1$] (see also [LZ], [Ti$_2$] for certain β_n, and the history of the topic in [Ti$_3$]).

In Section 4 we prove that for $a, b \in \overline{\mathrm{GM}}(\beta)$ with appropriate β, conditions (1.3) and (1.4) are equivalent.

PROBLEM 2. What hypotheses on $\{a_n\}, \{b_n\}$ are equivalent to (imply, follow from) the uniform convergence of series (1.1) and (1.2)?

For the cosine series with non-negative coefficients, the answer is clearly $\sum a_n < \infty$. For the sine series, the classical result of Chaundy and Joliffe (see [CJ], [Zy, Vol. 1, p. 183]) states that if $\{b_n\}$ is monotonic, then (1.2) converges uniformly if and only if $\lim_{n\to\infty} nb_n = 0$. In Section 5 we prove a similar result for a fairly wider class of sequences $\{b_n\}$, precisely for generally monotone sequences from the class

$$\left\{b_n : \sum_{k=n}^{2n-1} |\Delta b_k| \le \frac{C}{n} \max_{k\ge[n/c]} \sum_{s=k}^{2k} b_s \text{ for some } c > 1\right\}.$$

This extends many known results (see, e.g., [DT$_2$], [Ti$_3$], [YZZ]). In Section 6, we apply this criterion to the following approximation theory problem: find interrelations between the moduli of smoothness $\omega_k(\psi, 1/n)$ and the best approximation $E_n(\psi)$ in C. It will be shown that the Jackson inequality

$$E_n(\psi) \le C\omega_k(\psi, 1/n)$$

and the weak-type inequality

$$\omega_k(\psi, 1/n) \le \frac{C}{n^k} \sum_{\nu=0}^{n-1} (\nu+1)^{k-1} E_\nu(\psi)$$

(see, e.g., [DL, Ch. 7]) can be sharpened as follows: for even/odd functions represented by trigonometric series with generally monotone coefficients and for even/odd k,

$$\omega_k(\psi, 1/n) \asymp n^{-k} \max_{\nu\in[0,n]} (\nu+1)^k E_\nu(\psi)$$

and

$$\omega_k(\psi, 1/n) \asymp n^{-k} \sum_{\nu=0}^{n-1} (\nu+1)^{k-1} E_\nu(\psi).$$

We conclude with Section 6, where we provide a few remarks. In particular, we prove that our findings concerning the general monotonicity concept (in Sections 4 and 5) do indeed generalize all known results.

Throughout this paper, we denote by C, C_i, c positive constants that may be different on different occasions. In addition, $F \asymp G$ means that $F \le CG$ and $G \le CF$.

2. Weighted L_1-integrability. We start with well-known results on interrelation between the L_1-integrability of $f(x)x^{-\alpha}$ and $g(x)x^{-\alpha}$ and the summability properties of the coefficients $\{a_n\}$ and $\{b_n\}$. The best reference for Theorems 2.1–2.3 below is the monograph by Boas [Bo].

THEOREM 2.1 ([He$_2$]). *Let $\alpha \in (0, 1]$. Let also $f(x)x^{-\alpha} \in L(0, \pi)$. Then for $a_n = a_n(f)$ the series $\sum_{n=1}^\infty a_n n^{\alpha-1}$ converges. If $\alpha \in (-1, 1]$, then a similar result holds true for the sine series as well.*

To prove the converse, we have to assume some conditions on the function or on the sequence of the Fourier coefficients.

THEOREM 2.2. *Let $\alpha \in (0, 1)$. Assume that one of the following two conditions holds true*:

(1) *the function* $f \in L$ *is non-negative on* $(0, \delta)$ *for some* $\delta > 0$;
(2) *the sequence* $\{a_n(f)\}$ *is non-increasing.*

Then the convergence of the series $\sum_{n=1}^{\infty} a_n(f) n^{\alpha-1}$ *implies that* $f(x)x^{-\alpha} \in L(0, \pi)$. *A similar result with* $\alpha \in [0, 1)$ *holds true for the sine series as well.*

Analogous results can be given for absolute convergence, that is,

$$\sum_{n=1}^{\infty} |a_n(f)| n^{\alpha-1} < \infty. \tag{2.1}$$

THEOREM 2.3. *Let* $\alpha \in (0, 1)$. *Assume that the function* $f \in L$ *is non-increasing on* $(0, \delta)$ *and* $f(x) \geq C$ *on* (δ, π). *Then the condition* $f(x)x^{-\alpha} \in L(0, \pi)$ *is equivalent to* (2.1). *A similar result with* $\alpha \in [0, 1)$ *holds true for the sine series as well.*

Below we give an example which shows that both statements in Theorems 2.2 and 2.3 can fail rather dramatically when we do not assume the given conditions on f/g or $\{a_n\}/\{b_n\}$. This answers some questions by Boas [Bo].

THEOREM 2.4. *There exists a function* $\psi(x) \in L(0, 2\pi)$ *such that* $\psi(x)$ *is uniformly bounded on any interval* $[\varepsilon, 2\pi]$, $\varepsilon > 0$, *and* $x^{-\beta}\psi(x) \notin L(0, 2\pi)$ *for* $\beta > 0$ *but*

$$\sum_{n=1}^{\infty} n^{-\alpha}(|a_n(\psi)| + |b_n(\psi)|) < \infty \qquad \text{for any } \alpha > 0.$$

Proof. Let us consider the function sequence

$$\psi_k(x) = \frac{k}{\ln^2 k} \cos(2^k x) \chi_{(1/(k+1), 1/k]}(x), \qquad k \geq 3,$$

and define for $x \in (0, 2\pi]$ a 2π-periodic function $\psi(x)$ by

$$\psi(x) = \sum_{k=3}^{\infty} \psi_k(x).$$

Then $\psi(x) \in L(0, 2\pi)$ and $\psi(x)$ is uniformly bounded on any interval $[\varepsilon, 2\pi]$, where $\varepsilon > 0$. Moreover, $x^{-\beta}\psi(x) \notin L(0, 2\pi)$ for any positive β.

Now we estimate the sum

$$J(\alpha) \equiv \sum_{n=1}^{\infty} n^{-\alpha} |a_n(\psi)|,$$

where α is a positive number. Note that

$$J(\alpha) \leq \sum_{n=1}^{\infty} n^{-\alpha} \sum_{k=3}^{\infty} |a_n(\psi_k)| = \sum_{k=3}^{\infty} \sum_{n=1}^{\infty} n^{-\alpha} |a_n(\psi_k)| \equiv \sum_{k=3}^{\infty} J_k(\alpha).$$

Let us estimate $|a_n(\psi_k)|$. For $k \ge 3$ and $|n-2^k| \ge k$, we get

$$(2.2)\quad |a_n(\psi_k)| \le \frac{Ck}{\ln^2 k}\Big(\Big|\int\limits_{1/(k+1)}^{1/k} \cos{(2^k-n)x}\,dx\Big| + \Big|\int\limits_{1/(k+1)}^{1/k} \cos{(2^k+n)x}\,dx\Big|\Big) \le \frac{Ck}{|2^k-n|}.$$

In the case of $k \ge 3$ and $|n-2^k| < k$, we obtain

$$(2.3)\quad |a_n(\psi_k)| \le \frac{Ck}{\ln^2 k}\Big(\frac{1}{k} - \frac{1}{k+1}\Big) \le \frac{C}{k}.$$

Therefore, for any $\alpha > 0$ and $k \ge 3$, by (2.2) and (2.3), we estimate

$$(2.4)\quad J_k(\alpha) \le C\Big(k\sum_{n=1}^{2^k-k} \frac{n^{-\alpha}}{2^k-n} + \sum_{n=2^k-k+1}^{2^k+k-1} \frac{n^{-\alpha}}{k} + k\sum_{n=2^k+k}^{\infty} \frac{n^{-\alpha}}{n-2^k}\Big) \equiv C(J_{k,1}+J_{k,2}+J_{k,3}).$$

First,

$$(2.5)\quad J_{k,1} \le 2^{-k+1}k\sum_{n=1}^{2^{k-1}} n^{-\alpha} + 2^{-\alpha(k-1)}k\sum_{n=2^{k-1}+1}^{2^k-k} \frac{1}{2^k-n} \le C(\alpha)k^2 2^{-\alpha k}.$$

Second,

$$(2.6)\quad J_{k,2} \le C(\alpha)k2^{-\alpha(k-1)}\frac{1}{k} = C(\alpha)2^{-\alpha k}.$$

Finally,

$$(2.7)\quad J_{k,3} \le 2^{-\alpha k}k\sum_{n=2^k+k}^{2^{k+1}} \frac{1}{n-2^k} + 2k\sum_{n=2^{k+1}+1}^{\infty} n^{-\alpha-1} \le C(\alpha)k^2 2^{-\alpha k}.$$

Then inequalities (2.4)–(2.7) imply

$$J_k(\alpha) \le C(\alpha)k^2 2^{-\alpha k},$$

and hence $J(\alpha) < \infty$. Similarly we estimate the series with the sine coefficients $\sum_{n=1}^{\infty} n^{-\alpha}|b_n(\psi)|$. The proof is now complete.

3. Weighted L_p-integrability: Pitt's theorem. In this section we give a simple proof of Pitt's theorem [Pi]. Note that we make no use of interpolation technique (see, e.g., [St]).

Theorem 3.1. *Let* $1 < p < \infty$ *and* $\max(0, 2-p) \le \alpha < 1$. *Suppose* $\psi(x) \in L(0,\pi)$ *and*

$$\sum_{n=1}^{\infty} n^{\alpha+p-2}(|a_n(\psi)|^p + |b_n(\psi)|^p) < \infty. \tag{3.1}$$

Then $|\psi(x)|^p x^{-\alpha} \in L(0, 2\pi)$.

Proof. For simplicity let $b_n = 0$ and $\psi = f$. We assume first that $p \in [2,\infty)$. Then $\alpha > 0$ and we define $\varphi(x) = x^{-\alpha/p}$. By the classical Paley theorem [Zy, Ch. XII, §3],

$$\sum_n |a_n(f)|^p n^{p-2} < \infty \;\Rightarrow\; f(x) \in L_p.$$

Noting that $x^{-\frac{\alpha}{p}\frac{p}{p-1}} \in L(0,\pi)$ for $0 < \alpha < p-1$, we get $h(x) = f(x)x^{-\alpha/p} \in L(0,\pi)$. The cosine coefficients of $h(x)$ are computed from

$$\begin{aligned} a_n(h) = \frac{a_n(\varphi)a_0(f)}{2} &+ \sum_{r=1}^{n} \frac{a_{n-r}(\varphi) + a_{n+r}(\varphi)}{2}\, a_r(f) \\ &+ \sum_{r=n+1}^{\infty} \frac{a_{r-n}(\varphi) + a_{r+n}(\varphi)}{2}\, a_r(f). \end{aligned} \tag{3.2}$$

It is known [Zy, Vol. 1, V, 2] that $a_r(\varphi) \sim r^{\alpha/p-1}$ as $r \to \infty$. Hence (see (3.2)),

$$\begin{aligned} |a_n(h)| &\le C\Big(\sum_{r=0}^{n}(n-r+1)^{\alpha/p-1}|a_r(f)| + \sum_{r=n+1}^{\infty}(r-n)^{\alpha/p-1}|a_r(f)|\Big) \\ &\le C_1\Big(n^{\alpha/p-1}\sum_{r=0}^{[n/2]}|a_r(f)| + \sum_{r=[n/2]+1}^{n}(n-r+1)^{\alpha/p-1}|a_r(f)| \\ &\quad + \sum_{r=n+1}^{2n}(r-n)^{\alpha/p-1}|a_r(f)| + \sum_{r=2n+1}^{\infty} r^{\alpha/p-1}|a_r(f)|\Big). \end{aligned} \tag{3.3}$$

Further, we apply the following Hardy-type inequality (see e.g. [Le$_1$], [Po]): for $\{a_n \ge 0\}$ and $\{\lambda_n > 0\}$,

$$\sum_{n=1}^{\infty}\lambda_n\Big(\sum_{\nu=1}^{n}a_\nu\Big)^p \le p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\Big(\sum_{\nu=n}^{\infty}\lambda_\nu\Big)^p \qquad (p \ge 1). \tag{3.4}$$

We obtain

$$\sum_{n=1}^{\infty} n^{p-2}\Big(n^{\frac{\alpha}{p}-1}\sum_{r=0}^{[n/2]}|a_r(f)|\Big)^p \tag{3.5}$$
$$= \sum_{n=1}^{\infty} n^{\alpha-2}\Big(\sum_{r=0}^{[n/2]}|a_r(f)|\Big)^p \le C(p)\sum_{n=1}^{\infty} n^{(\alpha-2)(1-p)}|a_n(f)|^p\Big(\sum_{r=n}^{\infty} r^{\alpha-2}\Big)^p$$
$$\le C(p,\alpha)\sum_{n=1}^{\infty} n^{\alpha+p-2}|a_n(f)|^p.$$

Let us now take any $\gamma \in (1-\alpha, 1)$, for instance, $\gamma = (2-\alpha)/2$. Then, by Hölder's inequality, we get

$$\sum_{n=1}^{\infty} n^{p-2}\Big(\sum_{r=[n/2]+1}^{n} (n-r+1)^{\alpha/p-1}|a_r(f)|\Big)^p \tag{3.6}$$
$$= \sum_{n=1}^{\infty} n^{p-2}\Big(\sum_{r=[n/2]+1}^{n} (n-r+1)^{(\alpha+\gamma-p)/p}(n-r+1)^{-\gamma/p}|a_r(f)|\Big)^p$$
$$\le \sum_{n=1}^{\infty} n^{p-2}\sum_{r=[n/2]+1}^{n} (n-r+1)^{-\gamma}|a_r(f)|^p\Big(\sum_{r=[n/2]+1}^{n} (n-r+1)^{\frac{\alpha+\gamma-p}{p-1}}\Big)^{p-1}$$
$$\le C(p,\alpha)\sum_{n=1}^{\infty} n^{p-3+\alpha+\gamma}\sum_{r=[n/2]+1}^{n} (n-r+1)^{-\gamma}|a_r(f)|^p$$
$$= C(p,\alpha)\sum_{r=1}^{\infty}|a_r(f)|^p\sum_{n=r}^{2r}(n-r+1)^{-\gamma}n^{p-3+\alpha+\gamma}$$
$$\le C(p,\alpha)\sum_{r=1}^{\infty}|a_r(f)|^p r^{\alpha+p-2}.$$

Similarly,

$$\sum_{n=1}^{\infty} n^{p-2}\Big(\sum_{r=n+1}^{2n} (r-n)^{\alpha/p-1}|a_r(f)|\Big)^p \tag{3.7}$$
$$\le \sum_{n=1}^{\infty} n^{p-2}\sum_{r=n+1}^{2n}(r-n)^{-\gamma}|a_r(f)|^p\Big(\sum_{r=n+1}^{2n}(r-n)^{\frac{\alpha+\gamma-p}{p-1}}\Big)^{p-1}$$
$$\le C(p,\alpha)\sum_{r=1}^{\infty}|a_r(f)|^p r^{\alpha+p-2}.$$

To estimate the last term in (3.3), we apply Hardy's inequality dual to (3.4): for $\{a_n \ge 0\}$ and $\{\lambda_n > 0\}$,

$$\sum_{n=1}^{\infty}\lambda_n\Big(\sum_{\nu=n}^{\infty}a_\nu\Big)^p \le p^p\sum_{n=1}^{\infty}\lambda_n^{1-p}a_n^p\Big(\sum_{\nu=1}^{n}\lambda_\nu\Big)^p \quad (p\ge 1). \tag{3.8}$$

Then

$$\begin{aligned}\sum_{n=1}^{\infty}n^{p-2}\Big(\sum_{r=2n+1}^{\infty}r^{\alpha/p-1}|a_r(f)|\Big)^p &\\ &\le C(p)\sum_{n=1}^{\infty}n^{(p-2)(1-p)}n^{\alpha-p}|a_n(f)|^p\Big(\sum_{r=1}^{n}r^{p-2}\Big)^p\\ &\le C_1(p)\sum_{n=1}^{\infty}n^{\alpha+p-2}|a_n(f)|^p.\end{aligned}\tag{3.9}$$

Collecting (3.3), (3.5)–(3.7), and (3.9), we get $\sum_{n=1}^{\infty}n^{p-2}|a_n(h)|^p<\infty$, and the statement of Theorem 4.1 follows from Paley's theorem.

Let now $p\in(1,2)$. We are going to show that condition (3.1) implies

$$\int|h(x)|^p x^{p-2}\,dx<\infty,\quad h(x)=f(x)\varphi(x),\quad \varphi(x)=x^{-\beta/p},\quad \beta=\alpha+p-2,$$

where $0<\beta<p-1$.

Note that $\varphi(x)\in L_{p'}$ and

$$\sum_n|a_n(f)|^2\le\Big(\sum_n|a_n(f)|^p n^{\alpha+p-2}\Big)^{2/p},$$

i.e., $f\in L_2$ and therefore $f\in L_p$. Then Hölder's inequality gives $h\in L_1$. Therefore, by classical Paley's theorem, it is enough to show that

$$\sum_n|a_n(h)|^p<\infty. \tag{3.10}$$

The rest of the proof is similar to the case $p>2$. We use relations (3.2)–(3.3) with β in place of α. To estimate the first and the last terms, we apply Hardy's inequalities (3.4) and (3.8). To estimate the middle terms, we make use of Hölder's inequality with parameter $\gamma=(2-\beta)/\beta$ (note that $0<\beta<p-1<1$). Finally, we get

$$\sum_n|a_n(h)|^p\le C\sum_n n^\beta|a_n(f)|^p,\qquad 0<\beta<p-1,$$

as desired. The proof is now complete.

Remark 3.2. For $\alpha<\max(0,2-p)$ the statement of the previous theorem does not hold.

Indeed, for $p\ge 2$, defining the series

$$f(x)=\sum_{n=1}^{\infty}(-1)^n n^{1/p-1}\cos nx,$$

we note that $\alpha < 0$ and

$$\sum_{n=1}^{\infty} n^{\alpha+p-2}|a_n|^p = \sum_{n=1}^{\infty} n^{-|\alpha|-1} < \infty.$$

On the other hand, since $f(x) \sim (\pi - x)^{-1/p}$, we get $|f(x)|^p x^{-\alpha} \notin L(0,\pi)$.

Let now $1 < p < 2$ and put $\beta := \alpha + p - 2 < 0$. In this case we consider

$$\sum_{k=2}^{\infty} \frac{1}{\ln^2 k}\, e^{ikx}.$$

It is known that this series converges in $L(0,\pi)$ and diverges in $L_p(0,1)$ ([Zy, Ch. V; Ch. XII]). Thus for any integer l one can find integers $N_l < M_l$ such that

$$S_l(x) = \sum_{k=N_l}^{M_l} \frac{1}{\ln^2 k}\, e^{ikx}$$

and $\|S_l\|_1 < 1/l^2$ and $\|S_l\|_p > l^2$. Let us define

$$c_l := \sum_{k=N_l}^{M_l} \frac{1}{\ln^{2p} k}.$$

Now we choose integers R_l such that

$$R_l + N_l > R_{l-1} + M_{l-1} \quad \text{and} \quad R_l^{\beta} c_l < \frac{1}{l^2} \quad \text{for } l = 2, 3, \ldots.$$

Then the series

$$\sum_{l=1}^{\infty} e^{iR_l x} S_l(\pi - x) \equiv \sum_{l=1}^{\infty} e^{iR_l x} \sum_{k=N_l}^{M_l} \frac{e^{ik(\pi-x)}}{\ln^2 k} \tag{3.11}$$

converges to a function $f(x)$ in $L(0,\pi)$ and

$$\sum_{n=1}^{\infty} n^{\beta}|a_n(f)|^p \le C \sum_{l=1}^{\infty} R_l^{\beta} c_l < \infty.$$

On the other hand, $f(x) \notin L_p(\pi - 1, \pi)$ (since the sequence of partial sums of (3.11) diverges in $L_p(\pi-1,\pi)$), and thus $|f(x)|^p x^{-\alpha} \notin L(0,\pi)$.

4. Weighted L_p-integrability ($1 \le p < \infty$) for the series with monotone type coefficients. In this section we are going to study the L_p-integrability problems with power weights $x^{-\alpha}$ for trigonometric series (1.1) and (1.2) with regularity conditions on the coefficients. Theorem 2.4 and Remark 3.2 show that in order to obtain any positive result of the type

$$\sum_{n=1}^{\infty} n^{\alpha+p-2}|a_n(f)|^p < \infty \ \Rightarrow\ |f(x)|^p x^{-\alpha} \in L(0,\pi)$$

for $p = 1$ or to extend the range $\max(0, 2-p) \le \alpha < 1$ for $1 < p < \infty$, one has to impose some additional conditions on $\{a_n\}$. Note here that since we want to consider a constant function as an example of $f(x)$, we have to assume $\alpha < 1$. For odd functions, generally, it is possible to consider $\alpha < \max(0, 2-p)$ as well as $\alpha \ge 1$.

Typically many authors considered series with monotone (or quasi-monotone) decreasing coefficients. We would like to extend the following result.

THEOREM 4.1. *Let $\{a_n, b_n \ge 0\} \in \mathfrak{M}$, $1 < p < \infty$ and $1 - p < \alpha < 1$. Then*

$$\frac{|\psi(x)|^p}{x^\alpha} \in L(0, \pi) \;\Leftrightarrow\; \sum_{n=1}^{\infty} n^{\alpha+p-2} \lambda_n^p < \infty,$$

where $\psi(x)$ is either $f(x)$ or $g(x)$ and λ_n is either a_n or b_n, respectively.

In the case when $\mathfrak{M}$ denotes the class M of all decreasing sequences, this theorem was proved in [Bo], [He$_1$]; for $\mathfrak{M} = QM$, the class of quasi-monotone sequences, in [AW]; for $\mathfrak{M} = \mathrm{GM}(\overline{\beta})$ in [Ti$_1$]; and for $\mathfrak{M} = \mathrm{GM}(\beta^*)$ in [YZZ], where

$$\mathrm{GM}(\beta) := \Big\{ \{a_k\} : \sum_{k=n}^{2n} |\Delta a_k| \le C \beta_n \Big\}$$

and

$$\overline{\beta}_n = |a_n|, \qquad \beta_n^* = \sum_{k=[n/c]}^{[nc]} \frac{|a_k|}{k} \quad \text{for some } c > 1.$$

Note that ([Ti$_1$], [Ti$_3$])

$$M \subsetneq QM \subsetneq \mathrm{GM}(\overline{\beta}) \subsetneq \mathrm{GM}(\beta^*). \tag{4.1}$$

We would like to consider β-generally monotone coefficients. Let $\theta \in (0, 1]$. By definition, $\overline{\mathrm{GM}}_\theta$ is the class $\overline{\mathrm{GM}}(\beta)$ with

$$\beta_n = n^{\theta-1} \sum_{k=[n/c]}^{\infty} \frac{|a_k|}{k^\theta} \quad \text{for some } c > 1.$$

In other words, $\overline{\mathrm{GM}}_\theta$ is the collection of all sequences such that

$$\sum_{k=n}^{\infty} |\Delta a_k| < C n^{\theta-1} \sum_{k=[n/c]}^{\infty} \frac{|a_k|}{k^\theta} < \infty \quad \text{for some } c > 1. \tag{4.2}$$

For $\theta = 1$ we define $\overline{\mathrm{GM}} \equiv \overline{\mathrm{GM}}_1$.

We have (cf. (4.1))

$$\mathrm{GM}(\beta^*) \subsetneq \overline{\mathrm{GM}} \equiv \overline{\mathrm{GM}}_1 \subseteq \overline{\mathrm{GM}}_{\theta_2} \subseteq \overline{\mathrm{GM}}_{\theta_1}, \qquad 0 < \theta_1 \le \theta_2 \le 1.$$

The fact that $\mathrm{GM}(\beta^*) \subseteq \overline{\mathrm{GM}}$ is clear and an example of a sequence such that $\overline{\mathrm{GM}} \setminus \mathrm{GM}(\beta^*) \ne \emptyset$ will be constructed in the last section.

Our main results in this section are the following:

THEOREM 4.2. (Sine) *Let* $\{b_n \geq 0\} \in \overline{\mathrm{GM}}_\theta$, $\theta \in (0,1]$, $1 \leq p < \infty$. *If*

$$1 - \theta p < \alpha < 1 + p,$$

then

$$\frac{|g(x)|^p}{x^\alpha} \in L\,(0,\pi) \;\Leftrightarrow\; \sum_{n=1}^{\infty} n^{\alpha+p-2} b_n^p < \infty.$$

THEOREM 4.3. (Cosine) *Let* $\{a_n \geq 0\} \in \overline{\mathrm{GM}}_\theta$, $\theta \in (0,1]$, $1 \leq p < \infty$. *If*

$$1 - \theta p < \alpha < 1,$$

then

$$\frac{|f(x)|^p}{x^\alpha} \in L(0,\pi) \;\Leftrightarrow\; \sum_{n=1}^{\infty} n^{\alpha+p-2} a_n^p < \infty.$$

REMARK 4.4. We note that the condition

$$\sum_{n=1}^{\infty} n^{\alpha+p-2} \lambda_n^p < \infty, \qquad 1 \leq p < \infty, \qquad \lambda_k \geq 0,$$

always implies (by Hölder's inequality) the condition

$$\sum_{k=1}^{\infty} \frac{\lambda_k}{k^\theta} < \infty \qquad \text{for } \theta \in (0,1],\ \alpha > 1 - \theta p.$$

Proof of Theorem 4.2. Let $x \in (\pi/(n+1), \pi/n]$. Then, by Abel's transform,

$$\begin{aligned} |g(x)| &\leq x \sum_{k=1}^{n} k b_k + \Big| \sum_{k=n+1}^{\infty} b_k \sin kx \Big| \\ &\leq x \sum_{k=1}^{n} k b_k + \sum_{k=n}^{\infty} |(b_k - b_{k+1})(\widetilde{D}_k(x) - \widetilde{D}_n(x))|, \end{aligned}$$

where $\widetilde{D}_k(x) := \sum_{n=1}^{k} \sin nx$ for $k \in \mathbb{N}$. Since $|\widetilde{D}_k(x)| = O(1/x)$ and $\{a_n\} \in \overline{\mathrm{GM}}_\theta$, we get

$$|g(x)| \leq C\Big(x \sum_{k=1}^{n} k b_k + n \sum_{k=n}^{\infty} |b_k - b_{k+1}|\Big) \leq C\bigg(x \sum_{k=1}^{n} k b_k + n^\theta \sum_{k=[n/c]}^{\infty} \frac{|b_k|}{k^\theta}\bigg).$$

Hence,

$$|g(x)| \leq C\bigg(x \sum_{k=1}^{n} k |b_k| + n^\theta \sum_{k=n}^{\infty} \frac{|b_k|}{k^\theta}\bigg).$$

Further (define $\gamma(x)=x^{-\alpha}$ and $\gamma_n=n^{\alpha}$), we have

$$\int_0^\pi \gamma(x)|g(x)|^p\,dx=\sum_{n=1}^\infty \int_{\pi/(n+1)}^{\pi/n}\gamma(x)|g(x)|^p\,dx$$
$$\le C(p,\alpha)\sum_{n=1}^\infty \frac{\gamma_n}{n^{2+p}}\Big(\sum_{k=1}^n k|b_k|\Big)^p+C(p,\alpha)\sum_{n=1}^\infty \frac{n^{\theta p}\gamma_n}{n^2}\Big(\sum_{k=n}^\infty \frac{|b_k|}{k^\theta}\Big)^p.$$

By Hardy's inequalities (3.4) and (3.8), we obtain

$$\int_0^\pi \gamma(x)|g(x)|^p\,dx\le C(p,\alpha)\sum_{n=1}^\infty \gamma_n n^{p-2}|b_n|^p$$

for $1-\theta p<\alpha<1+p$.

Let us now prove the "⇒" part. Note that if $1-p<\alpha$, then $g(x)\in L(0,\pi)$. Integrating $g(x)$, we have

$$F(x):=\int_0^x g(t)\,dt=\sum_{n=1}^\infty \frac{b_n}{n}(1-\cos nx)=2\sum_{n=1}^\infty \frac{b_n}{n}\sin^2\frac{nx}{2}.$$

Since $\{b_n\ge 0\}$, this gives

$$F\Big(\frac{\pi}{k}\Big)\ge C\sum_{n=[k/2]}^{k}\frac{b_n}{n}.$$

As the sequence $\{b_s\}$ is generally monotone, we estimate

$$b_s\le\sum_{l=s}^\infty|\Delta b_l|\le Cs^{\theta-1}\sum_{l=[s/c]}^\infty \frac{b_l}{l^\theta}\le Cs^{\theta-1}\sum_{l=[s/c]}^\infty\frac{1}{l^\theta}\sum_{n=[l/2]}^{l}\frac{b_n}{n}$$
$$\le Cs^{\theta-1}\sum_{l=[s/c]}^\infty \frac{1}{l^\theta}F\Big(\frac{\pi}{l}\Big).$$

Applying this, we get

$$J:=\sum_{k=1}^\infty \gamma_k k^{p-2}b_k^p\le C(p)\sum_{k=1}^\infty \gamma_k k^{p-2}k^{(\theta-1)p}\Big(\sum_{\nu=[k/c]}^\infty \frac{1}{\nu^\theta}F\Big(\frac{\pi}{\nu}\Big)\Big)^p.$$

Further, because $\gamma_{[cn]}\asymp\gamma_n$ for $c>0$, we use inequality (3.8):

$$J\le C(p)\sum_{k=1}^\infty \gamma_k k^{p-2}k^{(\theta-1)p}\Big(\sum_{\nu=k}^\infty\frac{1}{\nu^\theta}F\Big(\frac{\pi}{\nu}\Big)\Big)^p$$
$$\le C(p)\sum_{\nu=1}^\infty \frac{F^p(\pi/\nu)}{\nu^{\theta p}}\,(\gamma_\nu \nu^{\theta p-2})^{1-p}\Big\{\sum_{k=1}^{\nu}\gamma_k k^{\theta p-2}\Big\}^p.$$

Since $\alpha > 1-\theta p$, we obtain

$$J \le C(p)\sum_{\nu=1}^{\infty}\frac{F^p(\pi/\nu)}{\nu^{\theta p}}\,(\gamma_\nu\nu^{\theta p-2})^{1-p}\Big\{\sum_{k=1}^{\nu}\gamma_k k^{\theta p-2}\Big\}^p$$

$$\le C(p)\sum_{\nu=1}^{\infty}F^p\Big(\frac{\pi}{\nu}\Big)(\gamma_\nu\nu^{p-2}).$$

Defining $d_\nu := \int_{\pi/(\nu+1)}^{\pi/\nu}|g(x)|\,dx$ ($\nu\in\mathbb{N}$), we immediately get

$$F\Big(\frac{\pi}{k}\Big) \le \sum_{\nu=k}^{\infty} d_\nu.$$

Hence

$$J \le C(p)\sum_{\nu=1}^{\infty}F^p\Big(\frac{\pi}{\nu}\Big)(\gamma_\nu\nu^{p-2}) \le C(p)\sum_{\nu=1}^{\infty}(\gamma_\nu\nu^{p-2})\Big(\sum_{s=\nu}^{\infty}d_s\Big)^p.$$

Using (3.8) for $\alpha > 1-\theta p \ge 1-p$, we obtain

$$J \le C(p)\sum_{\nu=1}^{\infty}\gamma_\nu\nu^{2p-2}d_\nu^p.$$

By Hölder's inequality ($p\in(1,\infty)$, $p' = p/(p-1)$),

$$d_\nu^p = \Big(\int_{\pi/(\nu+1)}^{\pi/\nu}|g(x)|\,dx\Big)^p \le C(p)\int_{\pi/(\nu+1)}^{\pi/\nu}|g(x)|^p\,dx\,\Big(\frac{1}{\nu^2}\Big)^{p/p'} \tag{4.3}$$

$$= C(p)\nu^{2(1-p)}\int_{\pi/(\nu+1)}^{\pi/\nu}|g(x)|^p\,dx.$$

Finally,

$$\sum_{k=1}^{\infty}\gamma_k k^{p-2}b_k^p \le C(p,\alpha)\int_0^{\pi}|g(x)|^p\gamma(x)\,dx,$$

which completes the proof.

Proof of Theorem 4.3. First we remark

$$\frac{|g(x)|^p}{x^\alpha}\in L\,(0,\pi) \;\Leftrightarrow\; \int_{[-\pi,\pi]}|g(x)|^p|x|^{-\alpha}\,dx < \infty.$$

It was shown in [Bab] that

$$\int_{[-\pi,\pi]}|\widetilde{g(x)}|^p|x|^{-\alpha}\,dx \le C(p,\alpha)\int_{[-\pi,\pi]}|g(x)|^p|x|^{-\alpha}\,dx$$

for $-1 < -\alpha < p-1$. A more general result was obtained in [HMW] for the A_p-Muckenhoupt weights (note that $|x|^{-\alpha}$ is an A_p-weight if and only if

$-1 < -\alpha < p-1$): for $\omega \in A_p$ we have

$$\int_{[-\pi,\pi]} |\widetilde{g(x)}|^p \omega(x)\,dx \le C(p,\omega) \int_{[-\pi,\pi]} |g(x)|^p \omega(x)\,dx.$$

Using these inequalities twice with $-1 < -\alpha < \theta p - 1$ (note that $\theta p - 1 < p-1$), we get

$$\int_{[-\pi,\pi]} |g(x)|^p |x|^{-\alpha} dx \le C(p,\alpha) \int_{[-\pi,\pi]} |f(x)|^p |x|^{-\alpha}\,dx$$
$$\le C(p,\alpha) \int_{[-\pi,\pi]} |g(x)|^p |x|^{-\alpha}\,dx.$$

Since

$$-1 < -\alpha < \theta p - 1 \ \Leftrightarrow\ 1 - \theta p < \alpha < 1,$$

the proof of the theorem is finished.

5. Uniform convergence. First, let us consider the case of the cosine series. One has the following results. If either

(1) $a_n \ge 0$, or
(2) $\{a_n\} \in \mathrm{GM}(\beta)$ and $n\beta_n = o(1)$ as $n \to \infty$,

then

$$\text{series (1.1) converges uniformly on } [0,2\pi] \ \Leftrightarrow\ \sum_n a_n \text{ converges.}$$

In the case when $a_n \ge 0$ this criterion is clear; for the series with β-generally monotone coefficients it was proved in [DT$_1$].

For the sine series, Chaundy and Joliffe ([CJ], [Zy, Vol. 1, p. 183]) proved the following: a necessary and sufficient condition for series (1.2), where $b_n \ge b_{n+1} \ge \cdots$, to be uniformly convergent on $[0,2\pi]$ is $nb_n \to 0$ as $n \to \infty$.

We remark that this problem has been extensively studied recently. For instance, series with β-generally monotone coefficients were considered for $\overline{\beta}_n = |a_n|$, $\widetilde{\beta}_n = |a_n| + \cdots + |a_{n+N}|$, $\beta_n^* = \sum_{k=[n/c]}^{[cn]} |a_k|/k$ (see the history of the question in [Ti$_3$]). We note that $\mathrm{GM}(\beta^*)$ is the largest known class for which Chaundy–Joliffe's criterion holds true.

Theorem 5.1. *Let $b \in \mathrm{GM}(\beta)$.*

(A) *If*

$$\sum_{k=n}^{\infty} |\Delta b_k| = o(1/n) \tag{5.1}$$

as $n \to \infty$, *then series* (1.2) *converges uniformly on* $[0,2\pi]$ *and*

$$\|g(x) - S_n(g,x)\|_\infty \le C \max_{\nu \ge n} \nu \sum_{k=\nu}^{\infty} |\Delta b_k|.$$

(B) *Let a non-negative sequence* $b = \{b_n\}$ *satisfy*

$$b_n \le \frac{C}{n} \max_{k \ge [n/c]} \sum_{s=k}^{2k} b_s \quad \text{for some } c > 1. \tag{5.2}$$

Then the uniform convergence of series (1.2) *implies*

$$nb_n = o(1) \quad \text{as } n \to \infty. \tag{5.3}$$

Even though the proof of this theorem is relatively straightforward, both statements are sharp.

Remark 5.2. First, in (5.1) we cannot substitute $O(1/n)$ for $o(1/n)$. According to (B), we have the following result. For any non-decreasing positive sequence $\{\varphi(n)\}_{n\in\mathbb{N}}$ satisfying $\varphi(n) \to \infty$ as $n \to \infty$, there exists an odd function $g(x) \in C([0,\pi])$ with uniformly convergent Fourier series such that its sine coefficients are non-negative and satisfy

$$b_n \le C \frac{\varphi(n)}{n} \max_{k \ge [n/c]} \sum_{s=k}^{2k} b_s \quad \text{for some } c > 1$$

for all n, but $nb_n \not\to 0$ as $n \to \infty$.

Let us now present a generalization of Chaundy–Joliffe's criterion as well as its extensions.

Corollary 5.3. *Let* $\{b_n \ge 0\} \in \mathrm{GM}(\overline{\beta})$, *where*

$$\overline{\beta}_n = \frac{1}{n} \max_{k \ge [n/c]} \sum_{s=k}^{2k} b_k \quad \text{for some } c > 1.$$

Then series (1.2) *converges uniformly on* $[0,2\pi]$ *if and only if condition* (5.3) *holds.*

Remark 5.4. We have $\mathrm{GM}(\beta^*) \subsetneq \mathrm{GM}(\overline{\beta})$, where

$$\beta_n^* = \sum_{k=[n/c]}^{[cn]} \frac{|b_k|}{k} \quad \text{and} \quad \overline{\beta}_n = \frac{1}{n} \max_{k \ge [n/c]} \sum_{s=k}^{2k} b_k \quad \text{for some } c > 1.$$

This remark will be proved in the last section.

Proof of Theorem 5.1. Item (A) was actually proved in [DT$_1$, Th. 2.1]. To prove the second part, if series (1.2) with non-negative coefficients converges uniformly, then

$$\frac{1}{n}\sum_{k=1}^{n} kb_k = o(1).$$

This gives

$$\sum_{k=n}^{2n} b_k = o(1).$$

Further, by (5.2), we get

$$nb_n \le C \max_{k\ge[n/c]} \sum_{s=k}^{2k} b_k = o(1) \quad \text{as } n\to\infty,$$

that is, condition (5.3) is satisfied.

Proof of Remark 5.2. First, we consider the sine series

$$\sum_n \frac{\sin nx}{n}.$$

Here the condition $\sum_{k=n}^{\infty} |\Delta b_k| = O(1/n)$ holds but the series does not converge uniformly on $[0, 2\pi]$.

Let us now prove the second part. Without loss of generality, we assume that $\varphi(n) \le \ln(n+1)$, $\varphi(2n) \le 2\varphi(n)$ as $n \ge 1$ and $\varphi(1) \ge 1$. Define $\psi(n) := \sqrt{\varphi([n/2])}$ for $n \ge 2$ and $\psi(1) := \frac{1}{2}\sqrt{\varphi(1)}$.

Let $b_{n,1} := 1/n\psi(n)$ for $n \ge 1$. Then for any n we have

$$\frac{4}{n}\sum_{k=n+1}^{2n} b_{k,1} \ge \frac{2}{n\psi(2n)} \ge b_{n,1}. \tag{5.4}$$

We next define the sequence

$$b_{n,2} := \begin{cases} 2^{-k}\psi(2^k) & \text{for } n = 2^k,\ k = 1, 2, \dots, \\ 0 & \text{otherwise.} \end{cases}$$

For any k, according to (5.4) we have

$$\frac{4\varphi(2^k)}{2^k}\sum_{k=2^k+1}^{2^{k+1}} b_{k,1} \ge \frac{\varphi(2^k)}{2^k\psi(2^k)} \ge b_{2^k,2}. \tag{5.5}$$

Inequalities (5.4) and (5.5) imply that the sequence $b_n = b_{n,1} + b_{n,2}$ satisfies

$$b_n \le 8\,\frac{\varphi(n)}{n}\sum_{k=n+1}^{2n} b_k \le 8\,\frac{\varphi(n)}{n}\max_{k\ge[n/c]}\sum_{s=k}^{2k} b_k.$$

This is the sequence of Fourier coefficients of the function $g(x) = g_1(x) +$

$g_2(x)$, where

$$g_i(x)=\sum_{n=1}^{\infty} b_{n,i}\sin nx \qquad \text{for } i=1,2. \tag{5.6}$$

The function $g_1(x)$ is the sum of a sine series with monotone coefficients $\{b_{n,1}\}$ such that $nb_{n,1}\to 0$ as $n\to\infty$. By the Chaundy–Joliffe theorem, $g_1(x)\in C([0,\pi])$ and series (5.6) converges uniformly. As regards $g_2(x)$, it is the sum of an absolutely convergent series and therefore it is continuous. Then the series $\sum_{n=1}^{\infty} b_n \sin nx$ converges uniformly. Finally, we remark that $nb_{n,2}\nrightarrow 0$ as $n\to\infty$.

Proof of Corollary 5.3. First, condition (5.3) implies $\sum_{k=n}^{2n} b_k = o(1)$. Since $\{b_n\}\in \mathrm{GM}(\overline{\beta})$, we have

$$\begin{aligned}\sum_{k=n}^{\infty}|\Delta b_k| &\le 2\sum_{k=[n/2]}^{\infty}\frac{1}{k}\sum_{s=k}^{2k-1}|\Delta b_s|\\ &\le C\Big(\max_{k\ge[n/c]}\sum_{s=k}^{2k} b_k\Big)\sum_{k=[n/2]}^{\infty}\frac{1}{k^2}\le \frac{C}{n}\max_{k\ge[n/c]}\sum_{s=k}^{2k} b_k.\end{aligned}$$

Combining this with $\sum_{k=n}^{2n} b_k = o(1)$, we arrive at (5.2). Then from Theorem 5.1(A) series (1.2) converges uniformly on $[0,2\pi]$.

Conversely, using

$$b_n\le \sum_{k=n}^{\infty}|\Delta b_k|\le \frac{C}{n}\max_{k\ge[n/c]}\sum_{s=k}^{2k} b_k, \tag{5.7}$$

we apply Theorem 5.1(B). Thus (5.3) follows.

6. Applications to approximation theory. Let $f\in C([0,2\pi])$ and let $E_n(f)$ be the best approximation of f by trigonometric polynomials of order n. Let also $\omega_k(f,t)$ be the modulus of smoothness of f of order $k>0$, i.e.,

$$\omega_k(f,t)=\sup_{|h|\le t}\Big\|\sum_{\nu=0}^{\infty}(-1)^{\nu}\binom{k}{\nu}f(x+(k-\nu)h)\Big\|,$$

where $\binom{k}{\nu}=k(k-1)\cdots(k-\nu+1)/\nu!$ for $\nu\ge 1$, $\binom{k}{\nu}=1$ for $\nu=0$, and $\|f(\cdot)\|=\max_{x\in[0,2\pi]}|f(x)|$.

It is well known (see, e.g., [DL, pp. 205, 208]) that the best approximation and the modulus of smoothness are related as follows:

$$C(k)E_n(f)\le \omega_k\Big(f,\frac{1}{n}\Big)\le C(k)\,\frac{1}{n^k}\sum_{\nu=0}^{n-1}(\nu+1)^{k-1}E_{\nu}(f). \tag{6.1}$$

Our main goal in this section is to study classes of trigonometric series for which one can check the sharpness of the left-hand side inequality (Jackson-type estimate) and the right-hand side inequality (Bernstein–Stechkin type estimate). For the history of this question we recommend [Ti$_3$].

As in Section 5, we will deal with a sequence $\{d_n\} \in \mathrm{GM}(\overline{\beta})$, i.e.,

$$\sum_{k=n}^{2n-1} |\Delta d_k| \le \frac{C}{n} \max_{k \ge [n/c]} \sum_{s=k}^{2k} |d_s| \quad \text{for some } c > 1.$$

THEOREM 6.1. (Cosine, sine) *Let $a = \{a_n\}_{n=1}^{\infty}, b = \{b_n\}_{n=1}^{\infty} \in \mathrm{GM}(\overline{\beta})$ be non-negative sequences. Then*

$$\omega_k\left(f, \frac{1}{n}\right) \asymp n^{-k} \sum_{\nu=0}^{n-1} (\nu+1)^{k-1} E_\nu(f) \quad \text{for } k \ne 2l-1 \ (l \in \mathbb{N}), \tag{6.2}$$

$$\omega_k\left(f, \frac{1}{n}\right) \asymp n^{-k} \max_{\nu \in [0,n]} (\nu+1)^{k} E_\nu(f) \quad \text{for } k = 2l-1 \ (l \in \mathbb{N}), \tag{6.3}$$

$$\omega_k\left(g, \frac{1}{n}\right) \asymp n^{-k} \sum_{\nu=0}^{n-1} (\nu+1)^{k-1} E_\nu(g) \quad \text{for } k \ne 2l \ (l \in \mathbb{N}), \tag{6.4}$$

$$\omega_k\left(g, \frac{1}{n}\right) \asymp n^{-k} \max_{\nu \in [0,n]} (\nu+1)^{k} E_\nu(g) \quad \text{for } k = 2l \ (l \in \mathbb{N}). \tag{6.5}$$

THEOREM 6.2. (Cosine, sine) *Let $a = \{a_n\}_{n=1}^{\infty}, b = \{b_n\}_{n=1}^{\infty} \in \mathrm{GM}(\overline{\beta})$ be non-negative sequences. Then for any $\varepsilon > 0$ we have*

$$\omega_k\left(f, \frac{1}{n}\right) \asymp n^{-k} \sum_{\nu=1}^{n} \nu^{k-1} \omega_{k+\varepsilon}\left(f, \frac{1}{\nu}\right) \quad \text{for } k \ne 2l-1 \ (l \in \mathbb{N}),$$

$$\omega_k\left(f, \frac{1}{n}\right) \asymp n^{-k} \max_{\nu \in [1,n]} \nu^{k} \omega_{k+\varepsilon}\left(f, \frac{1}{\nu}\right) \quad \text{for } k = 2l-1 \ (l \in \mathbb{N}),$$

$$\omega_k\left(g, \frac{1}{n}\right) \asymp n^{-k} \sum_{\nu=1}^{n} \nu^{k-1} \omega_{k+\varepsilon}\left(g, \frac{1}{\nu}\right) \quad \text{for } k \ne 2l \ (l \in \mathbb{N}),$$

$$\omega_k\left(g, \frac{1}{n}\right) \asymp n^{-k} \max_{\nu \in [1,n]} \nu^{k} \omega_{k+\varepsilon}\left(g, \frac{1}{\nu}\right) \quad \text{for } k = 2l \ (l \in \mathbb{N}).$$

These results were first presented in [Be] for series with quasi-monotone coefficients (without proof). A generalization was given in [Ti$_3$]. The proofs of Theorems 6.1 and 6.2 can be obtained as in [Ti$_3$] and are based on the following theorem.

THEOREM 6.3. (Cosine, sine) *Let* $a = \{a_n\}_{n=1}^{\infty}$, $b = \{b_n\}_{n=1}^{\infty} \in \mathrm{GM}(\overline{\beta})$ *be non-negative sequences with* $\sum_n a_n < \infty$ *and* $nb_n = o(1)$. *Then*

$$\omega_k\Big(f, \frac{1}{n}\Big) \asymp n^{-k} \sum_{m=1}^{n} m^k a_m + \sum_{m=n}^{\infty} a_m \qquad \text{for } k \neq 2l-1 \ (l \in \mathbb{N}), \tag{6.6}$$

$$\omega_k\Big(f, \frac{1}{n}\Big) \asymp n^{-k} \max_{m\in[1,n]} m^{k+1} a_m + \sum_{m=n}^{\infty} a_m \qquad \text{for } k = 2l-1 \ (l \in \mathbb{N}), \tag{6.7}$$

$$\omega_k\Big(g, \frac{1}{n}\Big) \asymp n^{-k} \sum_{m=1}^{n} m^k b_m + \max_{m\ge n} m b_m \qquad \text{for } k \neq 2l \ (l \in \mathbb{N}), \tag{6.8}$$

$$\omega_k\Big(g, \frac{1}{n}\Big) \asymp n^{-k} \max_{m\in[1,n]} m^{k+1} b_m + \max_{m\ge n} m b_m \qquad \text{for } k = 2l \ (l \in \mathbb{N}). \tag{6.9}$$

Since this theorem actually follows from the results of [Ti$_3$] and [Ti$_2$], we will only give the proof of (6.8), say.

"$\le$". This follows from

$$\omega_k\bigg(g, \frac{1}{n}\bigg) \le C\bigg(n^{-k} \sum_{m=1}^{n} m^k |b_m| + \max_{m\ge n} m(|b_m| + \overline{\beta}_m)\bigg)$$

(see [Ti$_3$, Th. 4.2(A)]) and the definition of $\overline{\beta}_m$.

"$\ge$". In [Ti$_3$, Th. 4.4(A)] we proved

$$C\omega_k\bigg(g, \frac{1}{n}\bigg) \ge n^{-k} \sum_{m=1}^{n} m^k b_m + \max_{m\ge n} \sum_{\nu=[m/c]}^{[cm]} b_\nu, \qquad c > 1.$$

Therefore, since

$$n|b_n| \le C \max_{k\ge[n/c]} \sum_{s=k}^{2k} |b_k|$$

(see (5.7)), we get

$$\begin{aligned} \omega_k\bigg(g, \frac{1}{n}\bigg) &\ge C\bigg(n^{-k} \sum_{m=1}^{n} m^k b_m + \max_{m\ge n} \max_{s\ge[m/c]} \sum_{\nu=s}^{2s-1} b_\nu\bigg) \\ &\ge C\bigg(n^{-k} \sum_{m=1}^{n} m^k b_m + \max_{m\ge n} m b_m\bigg), \qquad c > 1, \end{aligned}$$

and (6.8) follows.

7. Concluding remarks

1. The condition $\overline{\mathrm{GM}}_\theta$ for $0 < \theta < 1$ is equivalent to the condition $\mathrm{GM}(\beta)$, where

$$\beta_n \equiv n^{\theta-1} \sum_{k=[n/c]}^{\infty} \frac{|a_k|}{k^\theta} \quad \text{for some } c>1; \tag{7.1}$$

i.e.,

$$\sum_{k=n}^{2n-1} |\Delta a_k| \le C n^{\theta-1} \sum_{k=[n/c]}^{\infty} \frac{|a_k|}{k^\theta} \quad \text{for some } c>1.$$

2. Let us construct $\{a_n\} \in \overline{\mathrm{GM}} \equiv \overline{\mathrm{GM}}_1$ such that $\{a_n\} \notin \mathrm{GM}(\beta^*)$. Since one clearly has $\overline{\mathrm{GM}} \subseteq \mathrm{GM}(\overline{\beta})$, we immediately prove Remark 5.4 as well.

Set

$$N_1 = 1, \qquad N_{\xi+1} = N_\xi + 2[N_\xi \exp(N_\xi)].$$

Then we define $a = \{a_k\}$ as follows:

$$a_k = \begin{cases} 10^{-4}, & 1 \le k < N_4, \\ (-1)^k 10^{-\xi}, & N_\xi \le k < 2N_\xi,\ \xi \ge 4, \\ 10^{-\xi}, & 2N_\xi \le k < 2N_\xi + [N_\xi \exp(N_\xi)],\ \xi \ge 4, \\ 0, & N_\xi + [N_\xi \exp(N_\xi)] \le k < N_{\xi+1},\ \xi \ge 4. \end{cases}$$

Let us verify that $\{a_n\} \notin \mathrm{GM}(\beta^*)$. For any k we take $s \in \mathbb{N}$ such that $N_s \le k < N_{s+1}$. Then for $k = N_s$ we have

$$\sum_{l=k}^{2k-1} |\Delta a_l| \ge C\, \frac{k}{10^s} \asymp \frac{N_s}{10^s}.$$

But

$$\sum_{l=[k/c]}^{[ck]} \frac{|a_l|}{l} \le \frac{C}{10^s} \sum_{l=[k/c]}^{[ck]} \frac{1}{l} \asymp \frac{1}{10^s},$$

which contradicts

$$\sum_{l=k}^{2k-1} |\Delta a_l| \le C \sum_{l=[k/c]}^{[ck]} \frac{|a_l|}{l}.$$

Now we show that $\{a_n\} \in \overline{\mathrm{GM}}$. Considering $k \in (N_\xi + 1, N_{\xi+1})$, we get

$$\sum_{l=k}^{\infty} |\Delta a_l| \le \sum_{j=\xi}^{\infty} \sum_{l=N_j+1}^{N_{j+1}} |\Delta a_l| \le C \sum_{j=\xi}^{\infty} \frac{N_j}{10^j} \le C \sum_{j=\xi+1}^{\infty} \frac{N_j}{10^j}$$

$$\le C \sum_{j=\xi+1}^{\infty} \frac{1}{10^j} \sum_{l=N_j+1}^{N_j+[N_j \exp(N_j)]} \frac{1}{l} \le C \sum_{j=\xi+1}^{\infty} \sum_{l=N_j+1}^{N_{j+1}} \frac{|a_l|}{l} \le C \sum_{l=k}^{\infty} \frac{|a_l|}{l}.$$

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