

Relative Fatou theorem for harmonic functions of rotation invariant stable processes in smooth domains

by

KRZYSZTOF BOGDAN and BARTŁOMIEJ DYDA (Wrocław)

Abstract. For $C^{1,1}$ domains we give exact asymptotics near the domain's boundary for the Green function and Martin kernel of the rotation invariant α -stable Lévy process. We also obtain a relative Fatou theorem for harmonic functions of the stable process.

1. Introduction. The composition of a Markov process with a nonnegative function harmonic for the process (for definitions see Preliminaries) is a nonnegative supermartingale. By Doob's theory of martingales the supermartingale has limits at the lifetime of the process. When the lifetime coincides with the exit time of a domain, an important problem arises how to interpret this probabilistic convergence in analytical or topological terms involving only the function and the domain but not the process.

Such Fatou theorems [F] were first studied for classical harmonic functions, the Brownian motion W_t being the underlying Markov process; see, e.g., [S], [Ca] for purely analytical exposition, [D2] for the probabilistic counterpart, and [BM], [Ba] for both approaches.

The continuous paths of W_t leave the domain by hitting the boundary and, typically, do not creep along the boundary before the exit time [BaB]. This explains why for harmonic functions the notion of nontangential convergence at the domain's boundary ([S], [Wu1]) is an appropriate counterpart of the martingale convergence for the Brownian motion and other diffusion processes. A natural problem in a more general framework is to formulate relevant analytical counterparts of the martingale convergence theorems for discontinuous (jump) processes. Such processes typically leave a sufficiently regular domain without approaching its boundary by jumping to the interior of its complement [Mi], [Sz]; see also [Wu2] in this connection. In that case Doob's theory does not admit a natural and interesting topological interpre-

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tation even though some quantitative analytical results are still available. In this connection we refer the reader to the paper [BaC], devoted to the study of the boundary behavior of the symmetric stable Lévy processes X_t in the unit ball of \mathbb{R}^d .

The subject of the present paper is the same stable process X_t and its boundary behavior on $C^{1,1}$ domains in \mathbb{R}^d (see below for the definitions). However, to obtain a nontrivial result on the boundary convergence of harmonic functions of X_t , we apply Doob's h -transform to the process or, what is the same, we normalize the harmonic functions considered by dividing them by a suitable harmonic function h . We prove the existence of nontangential limits of such quotients at the domain's boundary.

Such *relative* Fatou theorems ([D1], [Wu1]) depend on the approximate factorization of the Martin kernel of the domain. This factorization is related ([B3], [J]) to the boundary Harnack principle for nonnegative harmonic functions [Da], [Wu1], [A], [JK]; see also [B1] for the case of harmonic functions of the stable process X_t .

Doob's h -process converges to the set of poles of the harmonic function h . Thus the discontinuous stable process X_t may be conditioned to leave the domain by hitting its boundary (see, e.g., [CZ], [BB2]). For $C^{1,1}$ (more generally, Lipschitz) domains the appropriate class of h functions is spanned by the Martin kernel of the domain with poles at the boundary introduced in [B2], [CS3]. Such functions are called *singular* α -harmonic (see below for a definition), and they are harmonic for the stable process killed on leaving the domain rather than for the process stopped at the exit time [CS3]; see [MS] for the general representation theorem for α -harmonic functions and [B2] for the special case of Lipschitz domains.

There is a growing literature on fine properties of α -harmonic functions and other potential properties of the stable process X_t (see, e.g., [BB3], [BKN], [M], [BBC]). In this paper we depend on the estimates of the Martin and Green functions of $C^{1,1}$ domains obtained in [K], [CS1], [CS2]. The estimates as well as other recent developments owe much to the explicit calculation of the Green function and the Martin kernel of the ball by M. Riesz [R1], [R2] (see also [BGR]).

In the present paper we do not use probabilistic techniques in an essential way, keeping them only for interpretation as above. Our focus is on expressing explicitly the asymptotic behavior of the relevant harmonic functions in $C^{1,1}$ domains in terms of the distance to the boundary. In Section 2 we introduce the notation used in this paper. In Section 3 we give the exact asymptotics of the Green function of $C^{1,1}$ domains. We note that the main result of this section, Theorem 3.2, is stated in [CS1] but no detailed proof is given there. Our methods are different from those used in [CS1]. They are also simpler and in Section 4 we can pursue our asymptotic investigation fur-

ther to obtain Theorem 4.3, which is our second main technical result. The relative Fatou theorem for α -harmonic functions is given in Theorem 4.2. The theorem answers a question of Jang-Mei Wu about $C^{1,1}$ domains and confirms a hypothesis of Renming Song. Still open remains the case of Lipschitz domains, which is perhaps the most natural setting for this problem.

2. Preliminaries. We denote by $|\cdot|$ the Euclidean length of a vector. For $A \subset \mathbb{R}^d$ we put $A^c = \{x \in \mathbb{R}^d : x \notin A\}$. For $x \in \mathbb{R}^d$, $r > 0$ and a set $A \subset \mathbb{R}^d$ we define $B(x, r) = \{y \in \mathbb{R}^d : |x - y| < r\}$ and $\delta_A(x) = \inf\{|x - y| : y \in A^c\}$.

We write D for a nonempty open bounded set in \mathbb{R}^d . We assume that D is a $C^{1,1}$ domain, i.e., there exist $r_0 > 0$ and $\lambda > 0$ such that for each $Q \in \partial D$ there are a $C^{1,1}$ function $\varphi : \mathbb{R}^{d-1} \rightarrow \mathbb{R}$ and an orthonormal coordinate system $y = (y_1, \dots, y_d)$ satisfying

$$D \cap B(Q, r_0) = \{y : y_d > \varphi(y_1, \dots, y_{d-1})\} \cap B(Q, r_0),$$

and, furthermore, $\nabla\varphi$ is Lipschitz with Lipschitz constant not greater than λ .

For the rest of the paper, unless stated otherwise, α is a number in $(0, 2)$ and $d = 1, 2, \dots$. We denote by (X_t, \mathbf{P}_x) the standard [BG] rotation invariant (“symmetric”) α -stable, \mathbb{R}^d -valued Lévy process (i.e. homogeneous, with independent increments) with stability index α and with characteristic function

$$\mathbf{E}_x e^{i\xi(X_t - x)} = e^{-t|\xi|^\alpha}, \quad x \in \mathbb{R}^d, \xi \in \mathbb{R}^d, t \geq 0.$$

For a Borel set $U \subset \mathbb{R}^d$, we put $\tau_U = \inf\{t \geq 0 : X_t \notin U\}$, the first exit time of U . Given $x \in \mathbb{R}^d$, the \mathbf{P}_x distribution of X_{τ_U} is a subprobability measure on U^c (probability measure if U is bounded) called the α -harmonic measure.

When $r > 0$, $|x| < r$ and $B = B(0, r) \subset \mathbb{R}^d$, the corresponding α -harmonic measure has the density function $P_r(x, \cdot)$ (the Poisson kernel) given by

$$(1) \quad P_r(x, y) = C_\alpha^d \left[\frac{r^2 - |x|^2}{|y|^2 - r^2} \right]^{\alpha/2} |y - x|^{-d} \quad \text{if } |y| > r,$$

with $C_\alpha^d = \Gamma(d/2)\pi^{-d/2-1} \sin(\pi\alpha/2)$, and 0 otherwise. The formula for the Poisson kernel for the exterior of the ball, $\{y \in \mathbb{R}^d : |y - x| > r\}$, is similar. Namely, for $|x| > r$ we have

$$(2) \quad \tilde{P}_r(x, y) = C_\alpha^d \left[\frac{|x|^2 - r^2}{r^2 - |y|^2} \right]^{\alpha/2} |y - x|^{-d} \quad \text{if } |y| < r,$$

and $\tilde{P}_r(x, y) = 0$ if $|y| \geq r$. Both (1) and (2) can be found in [BGR].

We say that a function f defined on \mathbb{R}^d is α -harmonic in an open set $D \subset \mathbb{R}^d$ if it has the “mean value property”

$$(3) \quad f(x) = \mathbf{E}_x f(X_{\tau_U}), \quad x \in U,$$

for every bounded open set U with closure contained in D . It is called *regular α -harmonic* in D if (3) holds for $U = D$ and *singular α -harmonic* if $f = 0$ on D^c .

By the strong Markov property a regular α -harmonic function is α -harmonic. The converse is not generally true as shown in [B2], [CS3]. An alternative description of α -harmonic functions as those annihilated by the fractional Laplacian is given in [BB3]. It follows from (1) and (3) that an α -harmonic function f in D satisfies

$$(4) \quad f(x) = \int_{|y-\theta|>r} P_r(x-\theta, y-\theta) f(y) dy, \quad x \in B(\theta, r),$$

for every ball $B(\theta, r)$ with closure contained in D . Note that if f is nonzero in D , then it is positive in D , regardless of the connectedness of D . This is a consequence of the Harnack inequality (see, e.g., [BB1]).

We write $G_D(\cdot, \cdot)$ for the *Green function* of D , i.e.

$$\mathbf{E}_x \int_0^{\tau_D} f(X_t) dt = \int G_D(x, y) f(y) dy$$

for all nonnegative Borel measurable functions f (see [K], [L]). Recall from [BGR] that

$$G_{B(0,r)}(x, y) = \mathcal{B}_{d,\alpha} |x-y|^{\alpha-d} \int_0^{w(x/r, y/r)} \frac{s^{\alpha/2-1}}{(s+1)^{d/2}} ds, \quad x, y \in B(0, r), \quad x \neq y,$$

and

$$G_{B(0,r)^c}(x, y) = \mathcal{B}_{d,\alpha} |x-y|^{\alpha-d} \int_0^{w(x/r, y/r)} \frac{s^{\alpha/2-1}}{(s+1)^{d/2}} ds, \quad x, y \in B(0, r)^c, \quad x \neq y,$$

where $w(x, y) = (1-|x|^2)(1-|y|^2)/|x-y|^2$, $\mathcal{B}_{d,\alpha} = 2^{-\alpha} \pi^{-d/2} \Gamma(\alpha/2)^{-2} \Gamma(d/2)$.

We denote by $M^{x_0}(\cdot, \cdot)$ the *Martin kernel* of D based at $x_0 \in D$:

$$M^{x_0}(x, Q) = \lim_{D \ni y \rightarrow Q} \frac{G_D(x, y)}{G_D(x_0, y)}, \quad x \in \mathbb{R}^d, \quad Q \in \partial D.$$

The existence of the above limit follows from the boundary Harnack principle (see [B1]). There are constants c and C , depending only on D , α and x_0 , such that

$$(5) \quad c \frac{\delta_D(x)^{\alpha/2}}{|x-Q|^d} \leq M^{x_0}(x, Q) \leq C \frac{\delta_D(x)^{\alpha/2}}{|x-Q|^d}$$

for all $x \in D$ and $Q \in \partial D$ (see [CS3]).

Every nonnegative function singular α -harmonic on D has a unique representation

$$(6) \quad f(x) = \int_{\partial D} M^{x_0}(x, Q) \mu(dQ), \quad x \in \mathbb{R}^d,$$

where μ is a finite nonnegative measure on ∂D . Conversely, for every signed measure μ on ∂D the formula (6) defines a (singular) α -harmonic function on D (see [B2], [CS3]).

3. The Green function. We define

$$f_r(x, y) = \frac{2}{\alpha} \mathcal{B}_{d,\alpha} r^{-\alpha} |x - y|^{-d} \left| |x|^2 - r^2 \right|^{\alpha/2} (|y| + r)^{\alpha/2}$$

for all $x, y \in \mathbb{R}^d$, $x \neq y$.

LEMMA 3.1. For $x, y \in B(0, r)$, $x \neq y$, we have

$$\frac{f_r(x, y)}{(w(x/r, y/r) + 1)^{d/2}} \leq \frac{G_{B(0,r)}(x, y)}{\delta_{B(0,r)}(y)^{\alpha/2}} \leq f_r(x, y),$$

and for $x, y \in \text{int } B(0, r)^c$, $x \neq y$, we have

$$\frac{f_r(x, y)}{(w(x/r, y/r) + 1)^{d/2}} \leq \frac{G_{B(0,r)^c}(x, y)}{\delta_{B(0,r)^c}(y)^{\alpha/2}} \leq f_r(x, y).$$

Proof. For $x, y \in B(0, r)$, $x \neq y$, we have

$$\begin{aligned} \frac{G_{B(0,r)}(x, y)}{\delta_{B(0,r)}(y)^{\alpha/2}} &= \mathcal{B}_{d,\alpha} |x - y|^{\alpha-d} \left| |y| - r \right|^{-\alpha/2} \int_0^{w(x/r, y/r)} \frac{s^{\alpha/2-1}}{(s+1)^{d/2}} ds \\ &\leq \mathcal{B}_{d,\alpha} |x - y|^{\alpha-d} \left| |y| - r \right|^{-\alpha/2} \int_0^{w(x/r, y/r)} s^{\alpha/2-1} ds = f_r(x, y). \end{aligned}$$

We obtain the lower bound similarly. The same proof works for the second part of the lemma. ■

Note that

$$\lim_{D \ni y \rightarrow Q} \frac{G_{B(0,r)}(x, y)}{\delta_{B(0,r)}(y)^{\alpha/2}} = f_r(x, Q)$$

and similarly for $G_{B(0,r)^c}$.

Now we will introduce some notation. Our assumptions on D imply existence of an $R > 0$ with the following property: for every $z \in \partial D$ there are two balls $B(z) \subset D$ and $E(z) \subset D^c$ with radius R , such that $z \in \overline{B(z)}$ and $z \in \overline{E(z)}$. The balls $B(z)$ and $E(z)$ are uniquely determined; their centers will be denoted by $b(z)$ and $e(z)$, respectively.

We will write $l(z)$ for the open line segment with end points z and $b(z)$. Every point $y \in D$ such that $\delta_D(y) < R$ belongs to $l(z)$ for some $z \in \partial D$. Indeed, we may take $z \in \partial D$ with $|y - z| = \delta_D(y)$.

If $y \in l(z)$ for some $z \in \partial D$, then

$$\delta_D(y) = \delta_{B(z)}(y) = \delta_{E(z)^c}(y) = |y - z|.$$

We fix $P = (p_1, \dots, p_d) \in \partial D$ and put $p = (p_1, \dots, p_{d-1})$. To simplify the notation we assume as we may that there is a $C^{1,1}$ function φ defined on $\{x \in \mathbb{R}^{d-1} : |p - x| < r_0\}$ such that $D \cap B(P, r_0) = \{(x_1, \dots, x_{d-1}, x_d) : x_d > \varphi(x_1, \dots, x_{d-1})\} \cap B(P, r_0)$ and, last but not least, $\nabla\varphi(p) = 0$.

We take $Q \in \partial D \cap B(P, r_0)$. Again, we put $q = (q_1, \dots, q_{d-1})$, where $Q = (q_1, \dots, q_{d-1}, q_d)$, so that $Q = (q, \varphi(q))$ and $P = (p, \varphi(p))$. Since $(-\nabla\varphi(q), 1) = (-\frac{\partial\varphi}{\partial x_1}(q), \dots, -\frac{\partial\varphi}{\partial x_{d-1}}(q), 1)$ is a normal vector at Q ,

$$b(Q) = Q + R \frac{(-\nabla\varphi(q), 1)}{\sqrt{|\nabla\varphi(q)|^2 + 1}},$$

which gives

$$(7) \quad |b(Q) - b(P)| \leq (1 + 2R\lambda)|Q - P|,$$

where λ is the Lipschitz constant of $\nabla\varphi$. This shows that $\partial D \ni Q \mapsto b(Q) \in D$ is a Lipschitz function. The same argument works for the function $\partial D \ni Q \mapsto e(Q) \in D^c$.

We denote by $\langle \cdot, \cdot \rangle$ the usual inner product in \mathbb{R}^d or \mathbb{R}^{d-1} . We have

$$\begin{aligned} | \langle b(Q) - Q, Q - P \rangle | &= \left| \left\langle R \frac{(-\nabla\varphi(q), 1)}{\sqrt{|\nabla\varphi(q)|^2 + 1}}, (q - p, \varphi(q) - \varphi(p)) \right\rangle \right| \\ &\leq \frac{R}{\sqrt{|\nabla\varphi(q)|^2 + 1}} (|\nabla\varphi(q)| |q - p| + |\varphi(q) - \varphi(p)|) \\ &\leq \frac{R}{\sqrt{|\nabla\varphi(q)|^2 + 1}} 2\lambda |q - p|^2 \leq 2R\lambda |Q - P|^2, \end{aligned}$$

and so

$$(8) \quad \left| |b(Q) - P|^2 - R^2 \right| = |2\langle b(Q) - Q, Q - P \rangle + \langle Q - P, Q - P \rangle| \leq (4R\lambda + 1)|Q - P|^2.$$

One of the main results of this paper is the following theorem.

THEOREM 3.2. *Let $x_0 \in D$. For every $Q \in \partial D$ the limit*

$$g(Q) = \lim_{D \ni y \rightarrow Q} \frac{G_D(x_0, y)}{\delta_D(y)^{\alpha/2}}$$

exists and is a positive number. The function g defined above is continuous on ∂D .

Proof. Fix $Q \in \partial D$. We first prove that

$$(9) \quad g_0(Q) = \lim_{l(Q) \ni y \rightarrow Q} \frac{G_D(x_0, y)}{\delta_D(y)^{\alpha/2}}$$

exists and is a positive number. We consider $0 < \eta \leq 1$ and let $x = x(Q) =$

$\eta b(Q) + (1 - \eta)Q$. Let $l(Q) \ni y \rightarrow Q$. We have

$$(10) \quad \begin{aligned} \frac{G_D(x_0, y)}{\delta_D(y)^{\alpha/2}} &= \frac{G_D(x_0, y)}{G_D(x, y)} \frac{G_D(x, y)}{\delta_{E(Q)^c}(y)^{\alpha/2}} \leq \frac{G_D(x_0, y)}{G_D(x, y)} \frac{G_{E(Q)^c}(x, y)}{\delta_{E(Q)^c}(y)^{\alpha/2}} \\ &\leq \frac{G_D(x_0, y)}{G_D(x, y)} f_R(x - e(Q), y - e(Q)) \\ &\rightarrow M^{x_0}(x, Q)^{-1} f_R(x - e(Q), Q - e(Q)) < \infty, \end{aligned}$$

and

$$\begin{aligned} \frac{G_D(x_0, y)}{\delta_D(y)^{\alpha/2}} &\geq \frac{G_D(x_0, y)}{G_D(x, y)} \frac{f_R(x - b(Q), y - b(Q))}{(w(\frac{x-b(Q)}{R}, \frac{y-b(Q)}{R}) + 1)^{d/2}} \\ &\rightarrow M^{x_0}(x, Q)^{-1} f_R(x - b(Q), Q - b(Q)) > 0. \end{aligned}$$

Therefore the limit (9), if exists, is positive and finite. Since

$$\frac{\overline{\lim} G_D(x_0, y)/\delta_D(y)^{\alpha/2}}{\underline{\lim} G_D(x_0, y)/\delta_D(y)^{\alpha/2}} \leq \frac{f_R(x - e(Q), Q - e(Q))}{f_R(x - b(Q), Q - b(Q))} = \left(\frac{2 + \eta}{2 - \eta}\right)^{\alpha/2},$$

by letting $\eta \rightarrow 0$ we get the existence of the limit (9).

For $Q' \in \partial D$ we have

$$\left(\frac{2 - \eta}{2 + \eta}\right)^{\alpha/2} \frac{M^{x_0}(x(Q), Q)}{M^{x_0}(x(Q'), Q')} \leq \frac{g_0(Q')}{g_0(Q)} \leq \left(\frac{2 + \eta}{2 - \eta}\right)^{\alpha/2} \frac{M^{x_0}(x(Q), Q)}{M^{x_0}(x(Q'), Q')}$$

and since the kernel $M^{x_0}(\cdot, \cdot)$ is continuous in $D \times \partial D$, and so is the function $\partial D \ni Q \mapsto x(Q) \in D$, the continuity of g_0 follows by letting $\eta \rightarrow 0$.

Now, let $y \in D$ be arbitrary and let $y \rightarrow Q$. Let $Q' \in \partial D$ be such that $y \in l(Q')$. As in (10) we have

$$\begin{aligned} \frac{G_D(x_0, y)}{\delta_D(y)^{\alpha/2}} &\leq \frac{G_D(x_0, y)}{G_D(x(Q), y)} \frac{G_D(x(Q), y)}{G_D(x(Q'), y)} f_R(x(Q') - e(Q'), y - e(Q')) \\ &\rightarrow M^{x_0}(x(Q), Q)^{-1} f_R(x(Q) - e(Q), Q - e(Q)). \end{aligned}$$

The fact that $\lim_{y \rightarrow Q} G_D(x(Q), y)/G_D(x(Q'), y) = 1$, used above, follows from α -harmonicity of $G_D(\cdot, y)$ in a ball centered at $x(Q)$ and (4), or from the gradient estimates of [BKN]. We obtain similarly

$$\underline{\lim}_{y \rightarrow Q} \frac{G_D(x_0, y)}{\delta_D(y)^{\alpha/2}} \geq M^{x_0}(x(Q), Q)^{-1} f_R(x(Q) - b(Q), Q - b(Q))$$

and we proceed as before. ■

4. Convergence of α -harmonic functions. We fix a point $x_0 \in D$ and denote by σ the $(d-1)$ -dimensional Hausdorff measure on ∂D . We write δ_x for $\delta_D(x)$.

Recall that the *Lebesgue set* of a function $f \in L^1(\partial D, \sigma)$ is the set of all $P \in \partial D$ satisfying

$$\lim_{r \rightarrow 0^+} \frac{\int_{\partial D \cap B(P, r)} |f(Q) - f(P)| \sigma(dQ)}{\sigma(\partial D \cap B(P, r))} = 0.$$

In particular

$$\sup_{r > 0} \frac{\int_{\partial D \cap B(P, r)} |f(Q)| \sigma(dQ)}{\sigma(\partial D \cap B(P, r))} < \infty$$

for all points P in the Lebesgue set of such an f . It is well known that the set of Lebesgue points is of full measure σ .

We introduce a suitable normalizing function:

$$N(x) = \int_{\partial D} M^{x_0}(x, Q) \sigma(dQ).$$

By a remark after (6), N is singular α -harmonic on D , and this function will play a role of Doob's h function as discussed in the introduction.

There are constants c_1 and c_2 , depending only on D , α and x_0 , such that

$$(11) \quad 0 < c_1 \leq \frac{N(x)}{\delta_x^{\alpha/2-1}} \leq c_2 < \infty$$

for all $x \in D$. The proof of this fact is similar to the proof of Lemma 4.1 but simpler, and is left to the reader.

LEMMA 4.1. *If $P \in \partial D$ is a Lebesgue point of $f \in L^1(\partial D, \sigma)$, then*

$$\lim_{D \ni x \rightarrow P} \int_{\partial D \cap \{Q : |Q-P| \geq |x-P|^{2/3}\}} M^{x_0}(x, Q) \delta_x^{1-\alpha/2} f(Q) \sigma(dQ) = 0.$$

Proof. We may assume $f \geq 0$. We fix $P \in \partial D$ from the Lebesgue set of f . Let $D \ni x \rightarrow P$. We put $B_n = B(P, 2^n |x - P|^{2/3})$ and $A_n = B_n \setminus B_{n-1}$ for $n = 1, 2, \dots$. Our assumptions on D imply $\sigma(\partial D \cap B_n) \leq c(2^n |x - P|^{2/3})^{d-1}$ for some constant c depending only on D . By (5) we have

$$\begin{aligned} & \int_{\partial D \cap \{Q : |Q-P| \geq |x-P|^{2/3}\}} M^{x_0}(x, Q) \delta_x^{1-\alpha/2} f(Q) \sigma(dQ) \\ & \leq c' \sum_{n=1}^{\infty} \int_{\partial D \cap A_n} \frac{\delta_x^{\alpha/2}}{|x-Q|^d} \delta_x^{1-\alpha/2} f(Q) \sigma(dQ) \\ & \leq c' \delta_x \sum_{n=1}^{\infty} \left(\frac{1}{4} 2^n |x - P|^{2/3}\right)^{-d} \frac{\int_{\partial D \cap B_n} f(Q) \sigma(dQ)}{\sigma(\partial D \cap B_n)} c(2^n |x - P|^{2/3})^{d-1} \\ & \leq c'' \delta_x |x - P|^{-2/3} \leq c'' |x - P|^{1/3} \rightarrow 0, \end{aligned}$$

and the lemma follows. ■

For $f \in L^1(\partial D, \sigma)$ we put

$$u(x) = \int_{\partial D} M^{x_0}(x, Q) f(Q) \sigma(dQ)$$

(cf. (6)).

THEOREM 4.2. *Let $f \in L^1(\partial D, \sigma)$. If f is continuous at $P \in \partial D$ then*

$$(12) \quad \lim_{x \rightarrow P} \frac{u(x)}{N(x)} = f(P).$$

Generally, (12) holds at every Lebesgue point P of f , but the limit should be taken nontangentially.

Proof. Let P be a Lebesgue point of f . We have

$$\begin{aligned} & \left| \frac{u(x)}{N(x)} - f(P) \right| \\ & \leq \frac{\int_{\partial D \cap \{Q: |Q-P| \geq |x-P|^{2/3}\}} M^{x_0}(x, Q) \delta_x^{1-\alpha/2} |f(Q) - f(P)| \sigma(dQ)}{\delta_x^{1-\alpha/2} N(x)} \\ & \quad + \frac{\int_{\partial D \cap \{Q: |Q-P| < |x-P|^{2/3}\}} M^{x_0}(x, Q) |f(Q) - f(P)| \sigma(dQ)}{N(x)} \\ & = I_1(x) + I_2(x). \end{aligned}$$

From Lemma 4.1 with $|f(\cdot) - f(Q)|$ in place of f and (11) it follows that $\lim_{x \rightarrow P} I_1(x) = 0$.

Put $B_n = B(P, 2^n|x - P|)$ and $A_n = B_{n+1} \setminus B_n$ for $n \geq 1$, $A_0 = B_1$. Denote by n_0 the smallest integer for which $B(P, |x - P|^{2/3}) \subset B_{n_0}$, so $-\frac{1}{3} \log_2 |x - P| \leq n_0 < -\frac{1}{3} \log_2 |x - P| + 1$. Note that $|x - Q| \geq \frac{1}{2} \delta_x 2^n$ for $Q \in A_n$. By (5) we have

$$\begin{aligned} N(x) I_2(x) & \leq c \delta_x^{\alpha/2} \sum_{n=0}^{n_0} \int_{\partial D \cap A_n} \frac{|f(Q) - f(P)|}{|x - Q|^d} \sigma(dQ) \\ & \leq c' \delta_x^{\alpha/2} \sum_{n=0}^{n_0} (\delta_x 2^n)^{-d} \frac{\int_{\partial D \cap B_{n+1}} |f(Q) - f(P)| \sigma(dQ)}{\sigma(\partial D \cap B_{n+1})} (2^n |x - P|)^{d-1} \\ & \leq c' \delta_x^{\alpha/2-d} |x - P|^{d-1} \sup_{0 < r < 2|x-P|^{2/3}} \frac{\int_{\partial D \cap B(P, r)} |f(Q) - f(P)| \sigma(dQ)}{\sigma(\partial D \cap B(P, r))}. \end{aligned}$$

From (11), $N(x) \geq c_1 \delta_x^{\alpha/2-1}$, so

$$I_2(x) \leq c'' \delta_x^{1-d} |x - P|^{d-1} \sup_{0 < r < 2|x-P|^{2/3}} \frac{\int_{\partial D \cap B(P, r)} |f(Q) - f(P)| \sigma(dQ)}{\sigma(\partial D \cap B(P, r))}.$$

If $x \rightarrow P$ nontangentially then $\delta_x \geq \gamma|x - P|$ for some $\gamma > 0$ and consequently $I_2(x) \rightarrow 0$.

Moreover, if $f \in C(\partial D)$, then

$$I_2(x) \leq \sup\{|f(Q) - f(P)| : Q \in \partial D, |Q - P| < |x - P|^{2/3}\} \rightarrow 0$$

as $x \rightarrow P$. The proof is complete. ■

The exact asymptotics of $N(x)$ for x near the boundary is described in the following

THEOREM 4.3. *The limit*

$$\nu(P) = \lim_{D \ni x \rightarrow P} \frac{N(x)}{\delta_x^{\alpha/2-1}}$$

exists for all $P \in \partial D$. The function ν defined above is continuous on ∂D .

Proof. The proof is very similar to that of Theorem 3.2, however, it is more complicated. Fix $P \in \partial D$. We start by proving the existence of the limit

$$(13) \quad \nu_0(P) = \lim_{l(P) \ni x \rightarrow P} N(x) \delta_x^{1-\alpha/2}$$

(see also (11)). Let $x \in l(P)$; then $\delta_x = |x - P|$.

Fix $0 < \eta \leq 1$ and let $x_1 = x_1(P) = \eta b(P) + (1 - \eta)P$. We define $A(P, x) = \{Q \in \partial D : |Q - P| < |x - P|^{2/3}\}$. Taking $f \equiv 1$ in Lemma 4.1 we see that

$$\lim_{l(P) \ni x \rightarrow P} N(x) \delta_x^{1-\alpha/2} = \lim_{l(P) \ni x \rightarrow P} \int_{A(P, x)} M^{x_0}(x, Q) \delta_x^{1-\alpha/2} \sigma(dQ)$$

if either limit exists. From now on we assume $Q \in A(P, x)$. We have

$$\begin{aligned} |x - b(Q)|^2 - R^2 &= \langle x - P + P - b(Q), x - P + P - b(Q) \rangle - R^2 \\ &= |x - P|^2 + 2\langle x - P, P - b(P) + b(P) - b(Q) \rangle \\ &\quad + |P - b(Q)|^2 - R^2 \\ &= |x - P|^2 - 2R|x - P| + 2\langle x - P, b(P) - b(Q) \rangle \\ &\quad + |P - b(Q)|^2 - R^2 \end{aligned}$$

and similarly

$$\begin{aligned} |x - e(Q)|^2 - R^2 &= |x - P|^2 + 2R|x - P| + 2\langle x - P, e(P) - e(Q) \rangle \\ &\quad + |P - e(Q)|^2 - R^2. \end{aligned}$$

From (7) and (8) we see that

$$|\langle x - P, b(P) - b(Q) \rangle| \leq (1 + 2R\lambda)|x - P|^{5/3}$$

and

$$||P - b(Q)|^2 - R^2| \leq (4R\lambda + 1)|x - P|^{4/3},$$

hence for $x \in l(P)$ close enough to P we have

$$(14) \quad 2R(1 - \eta)|x - P| \leq R^2 - |x - b(Q)|^2 \leq 2R(1 + \eta)|x - P|,$$

$$(15) \quad 2R(1-\eta)|x-P| \leq |x-e(Q)|^2 - R^2 \leq 2R(1+\eta)|x-P|.$$

In particular, $x, x_1 \in B(Q)$.

Note that $M^{x_0}(x, Q) = M^{x_0}(x_1, Q)M^{x_1}(x, Q)$. By Lemma 3.1 and (15) we have

$$\begin{aligned} M^{x_1}(x, Q) &= \lim_{l(Q) \ni y \rightarrow Q} \frac{G_D(x, y)}{G_D(x_1, y)} \\ &\leq \overline{\lim}_{l(Q) \ni y \rightarrow Q} \frac{G_{E(Q)^c}(x, y)}{\delta_{E(Q)^c}(y)^{\alpha/2}} \frac{\delta_{B(Q)}(y)^{\alpha/2}}{G_{B(Q)}(x_1, y)} \\ &= \frac{||x-e(Q)|^2 - R^2|^{\alpha/2} |x-Q|^{-d}}{||x_1-b(Q)|^2 - R^2|^{\alpha/2} |x_1-Q|^{-d}} \\ &\leq \frac{(2R(1+\eta)|x-P|)^{\alpha/2} |x-Q|^{-d}}{||x_1-b(Q)|^2 - R^2|^{\alpha/2} |x_1-Q|^{-d}}, \end{aligned}$$

and similarly by Lemma 3.1 and (14),

$$M^{x_1}(x, Q) \geq \frac{(2R(1-\eta)|x-P|)^{\alpha/2} |x-Q|^{-d}}{||x_1-e(Q)|^2 - R^2|^{\alpha/2} |x_1-Q|^{-d}}.$$

This gives

$$\begin{aligned} &\int_{A(P,x)} \frac{M^{x_0}(x, Q)}{\delta_x^{\alpha/2-1}} \sigma(dQ) \\ &\leq \delta_x \int_{A(P,x)} \frac{(2R(1+\eta))^{\alpha/2} |x-Q|^{-d} M^{x_0}(x_1, Q)}{||x_1-b(Q)|^2 - R^2|^{\alpha/2} |x_1-Q|^{-d}} \sigma(dQ) \\ &\leq \sup_{Q \in A(P,x)} \frac{(2R(1+\eta))^{\alpha/2} |x_1-Q|^d}{||x_1-b(Q)|^2 - R^2|^{\alpha/2}} \\ &\quad \times \sup_{Q \in A(P,x)} M^{x_0}(x_1, Q) \cdot \delta_x \int_{A(P,x)} |x-Q|^{-d} \sigma(dQ) \\ &\rightarrow \left(\frac{2(1+\eta)}{\eta(2-\eta)} \right)^{\alpha/2} (\eta R)^d \cdot M^{x_0}(x_1, P) \cdot \int_{\mathbb{R}^{d-1}} (|q|^2 + 1)^{-d} dq \end{aligned}$$

as $x \rightarrow P$. Similarly we obtain

$$\begin{aligned} \underline{\lim}_{l(P) \ni x \rightarrow P} \int_{A(P,x)} \frac{M^{x_0}(x, Q)}{\delta_x^{\alpha/2-1}} \sigma(dQ) &\geq \left(\frac{2(1-\eta)}{\eta(2+\eta)} \right)^{\alpha/2} (\eta R)^d \cdot M^{x_0}(x_1, P) \\ &\quad \times \int_{\mathbb{R}^{d-1}} (|q|^2 + 1)^{-d} dq. \end{aligned}$$

Thus

$$\frac{\overline{\lim} \int_{A(P,x)} M^{x_0}(x, Q) \delta_x^{1-\alpha/2} \sigma(dQ)}{\underline{\lim} \int_{A(P,x)} M^{x_0}(x, Q) \delta_x^{1-\alpha/2} \sigma(dQ)} \leq \left(\frac{(1+\eta)(2+\eta)}{(1-\eta)(2-\eta)} \right)^{\alpha/2}.$$

Since $\eta > 0$ can be arbitrarily small, the existence of the limit (13) follows.

The rest of the proof is very similar to the corresponding part of the proof of Theorem 3.2 and will be omitted. ■

COROLLARY 4.4. *Let $f \in L^1(\partial D, \sigma)$. If f is continuous at $P \in \partial D$ then*

$$(16) \quad \lim_{x \rightarrow P} \frac{u(x)}{\delta_x^{\alpha/2-1}} = \nu(P)f(P).$$

Generally, (16) holds at every Lebesgue point P of f , but the limit should be taken nontangentially.

EXAMPLE 4.5. Let D be the ball $B(0, r) \subset \mathbb{R}^d$, $r > 0$, and $x_0 = 0$. The corresponding Martin kernel M is

$$(17) \quad M(x, Q) = r^{d-\alpha} \frac{(r^2 - |x|^2)^{\alpha/2}}{|x - Q|^d}, \quad |x| < r,$$

for all $Q \in \partial B(0, r)$. Formula (17) is an exact analogue of the classical Poisson kernel for the ball. It follows easily that multiplication by $r^{2-\alpha}(r^2 - |x|^2)^{\alpha/2-1}$ is an isomorphism from the nonnegative functions harmonic in $B(0, r)$ onto the nonnegative singular α -harmonic functions in $B(0, r)$. Clearly, there is no such isomorphism for general $C^{1,1}$ domains, as can be seen by considering a disconnected domain D .

We note that Theorem 4.2 readily generalizes to other normalizing functions $\tilde{N}(x) = \int_{\partial D} M^{x_0}(x, Q) r(Q) \sigma(dQ)$ for nonnegative $r \in L^1(\partial D, \sigma)$ by using (12) twice. Further generalization in the spirit of [Wu1] requires a separate argument and should perhaps be considered in the more general context of Lipschitz domains.

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Institute of Mathematics
Wrocław University of Technology
50-370 Wrocław, Poland
E-mail: bogdan@im.pwr.wroc.pl
bdyda@im.pwr.wroc.pl

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