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2-summing multiplication operators

by

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Abstract. Let $1 \leq p < \infty$, $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces and $l_p(\mathcal{X})$ the coresponding vector valued sequence space. Let $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$, $\mathcal{Y} = (Y_n)_{n\in\mathbb{N}}$ be two sequences of Banach spaces, $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$, $V_n : X_n \to Y_n$, a sequence of bounded linear operators and $1 \leq p, q < \infty$. We define the multiplication operator $M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_q(\mathcal{Y})$ by $M_{\mathcal{V}}((x_n)_{n\in\mathbb{N}}) := (V_n(x_n))_{n\in\mathbb{N}}$. We give necessary and sufficient conditions for $M_{\mathcal{V}}$ to be 2-summing when (p, q) is one of the couples $(1, 2)$, $(2, 1)$, $(2, 2)$, $(1, 1)$, $(p, 1)$, $(p, 2)$, $(2, p)$, $(1, p)$, (p, q); in the last case $1 < p < 2$, $1 < q < \infty$.

Introduction and notation. The concept of absolutely summing operator is fundamental in operator theory as the reader can see in the books [2, 3, 9, 11, 15, 16]. The main purpose of this paper is to give necessary and sufficient conditions for the multiplication operator between vector valued sequence spaces to be 2-summing.

Let us fix some notations and terminology. Let $1 \leq p < \infty$ and $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. We write $l_p(\mathcal{X})$ to denote the Banach space of all sequences $(x_n)_{n\in\mathbb{N}}$ with $x_n \in X_n$ for all $n\in\mathbb{N}$, $\sum_{n=1}^{\infty} \|x_n\|_{X_n}^p < \infty$, endowed with the norm $\|(x_n)_{n\in\mathbb{N}}\|_{l_p(\mathcal{X})} := (\sum_{n=1}^{\infty} \|x_n\|_{X_n}^p)^{1/p}$ (see [16]). We consider the canonical mappings $\sigma_k : X_k \to l_p(\mathcal{X})$ and $p_k : l_p(\mathcal{X}) \to X_k$ defined by

$$\sigma_k(x) = (0, \ldots, 0, \underbrace{x}_{k\text{th}}, 0, \ldots), \qquad p_k((x_n)_{n\in\mathbb{N}}) = x_k,$$

where k is a natural number.

Let $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$, $\mathcal{Y} = (Y_n)_{n\in\mathbb{N}}$ be two sequences of Banach spaces, $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$, $V_n : X_n \to Y_n$, a sequence of bounded linear operators, and let $1 \leq p, q < \infty$. We define the multiplication operator $M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_q(\mathcal{Y})$ by

$$M_{\mathcal{V}}((x_n)_{n\in\mathbb{N}}) := (V_n(x_n))_{n\in\mathbb{N}}.$$

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As in the scalar case (see [1, p. 218]), we can prove that $M_{\mathcal{V}}$ is well defined if and only if it is bounded linear if and only if $(\|V_n\|)_{n\in\mathbb{N}} \in l_\infty$, for $p \le q$, respectively $(\|V_n\|)_{n\in\mathbb{N}} \in l_s$ for $q < p$, where $1/q = 1/p + 1/s$.

Given the importance of this issue, a lot of work has been done in order to give necessary and sufficient conditions for some natural operators to be absolutely summing.

For example, L. Schwartz [14] gave necessary and sufficient conditions for some multiplication operators from l_s to l_t to be p-summing, and D. J. H. Garling [4, Theorem 9] gave an almost complete description of the summing properties for multiplication operators from l_s to l_t. Also, E. D. Gluskin, S. V. Kislyakov and O. I. Reinov [5, Lemma I] studied the same problem in a more general context.

In this paper we give necessary and sufficient conditions for $M_{\mathcal{V}}$ to be 2-summing when (p,q) is one of the couples $(1,2)$, $(2,1)$, $(2,2)$, $(1,1)$, $(p,1)$, $(p,2)$, $(2,p)$, $(1,p)$, (p,q); in the last case $1 < p < 2$, $1 < q < \infty$.

As it turns out (see Theorems 3–8, 10–17), we have a full extension of the scalar case shown by D. J. H. Garling [4, Theorem 9]. However, there is one notable exception: in case $(p,1)$, $p > 2$, we need natural cotype 2 assumptions to obtain necessary and sufficient conditions for $M_{\mathcal{V}}$ to be s-summing, $1 \le s \le 2$ resp. $1 \le s < \infty$ (Theorem 9).

Let $1 \le p < \infty$, X a normed space and $(x_k)_{k=1}^n \subset X$. We write

$$w_p((x_k)_{k=1}^n) := \sup_{\|x^*\|\le 1} \Big(\sum_{k=1}^n |x^*(x_k)|^p\Big)^{1/p}.$$

Let $1 \le p < \infty$ and X, Y be Banach spaces. A bounded linear operator $T : X \to Y$ is *p-summing* if there is a constant $C \ge 0$ such that for every $(x_k)_{1\le k\le n} \subset X$,

$$\Big(\sum_{k=1}^n \|T(x_k)\|^p\Big)^{1/p} \le C w_p((x_k)_{k=1}^n).$$

The *p-summing norm* of T is $\pi_p(T) = \inf\{C \mid C \text{ as above}\}$ (see [2, 3, 9, 11, 15, 16]). We denote by $\Pi_p(X,Y)$ the class of all p-summing operators $T : X \to Y$.

One of the main ingredients in the proofs is the famous Grothendieck composition theorem which asserts that the composition of two 2-summing operators is nuclear (see [3, Theorem 5.31], [9, Theorem 17.6.4]).

For sequences $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$, $\mathcal{Y} = (Y_n)_{n\in\mathbb{N}}$, $\mathcal{Z} = (Z_n)_{n\in\mathbb{N}}$ of Banach spaces, and sequences $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$, $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$, $V_n : X_n \to Y_n$, $U_n : Y_n \to Z_n$, of bounded linear operators, we define $\mathcal{U} \circ \mathcal{V} := (U_n \circ V_n)_{n\in\mathbb{N}}$.

For sequences $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$, $\mathcal{Y} = (Y_n)_{n\in\mathbb{N}}$ of Banach spaces, a sequence $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$, $V_n : X_n \to Y_n$, of bounded linear operators and a sequence $a = (a_n)_{n\in\mathbb{N}}$ of scalars, we define $a\mathcal{V} := (a_n V_n)_{n\in\mathbb{N}}$.

If $\mathcal{X} = (X_n)_{n\in\mathbb{N}}$ is a sequence of Banach spaces we define $\mathcal{J} := (I_{X_n})_{n\in\mathbb{N}}$, where $I_{X_n} : X_n \to X_n$ is the identity operator on X_n.

Note that if $1 \le p, q < \infty$ and a is a sequence of scalars such that $M_a : l_p \to l_q$ is well defined, then $M_{a\mathcal{J}} : l_p(\mathcal{X}) \to l_q(\mathcal{X})$ is also well defined and $\|M_{a\mathcal{J}} : l_p(\mathcal{X}) \to l_q(\mathcal{X})\| = \|M_a : l_p \to l_q\|$.

Also if $1 \le p, q, r < \infty$ are such that $M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_q(\mathcal{Y})$ and $M_a : l_q \to l_r$ (resp. $M_a : l_r \to l_p$) are well defined, then $M_{a\mathcal{V}} : l_p(\mathcal{X}) \to l_r(\mathcal{Y})$ (resp. $M_{a\mathcal{V}} : l_r(\mathcal{X}) \to l_q(\mathcal{Y})$) is well defined and $M_{a\mathcal{J}} \circ M_{\mathcal{V}} = M_{a\mathcal{V}}$ (resp. $M_{\mathcal{V}} \circ M_{a\mathcal{J}} = M_{a\mathcal{V}}$).

If $V : X \to Y$ is 2-summing, we define $\operatorname{sgn} V : X \to Y$ by

$$(\operatorname{sgn} V)(x) := \frac{V(x)}{\pi_2(V)} \quad \text{for } V \neq 0,\ x \in X; \quad \operatorname{sgn} 0 := 0.$$

Note that $\operatorname{sgn} V$ is 2-summing, $\pi_2(\operatorname{sgn} V) \le 1$ and $\pi_2(V) \operatorname{sgn} V = V$.

If $\mathcal{V} = (V_n)_{n\in\mathbb{N}}$ and all $V_n : X_n \to Y_n$ are 2-summing, we define $\operatorname{sgn} \mathcal{V} = (\operatorname{sgn} V_n)_{n\in\mathbb{N}}$.

If $1 < p < \infty$ we denote by p^* the conjugate of p, i.e. $1/p + 1/p^* = 1$.

The notations and terminology used along the paper are standard in Banach space theory, as for instance in [2, 3, 9, 11, 15, 16].

The results

The cases $(1, 2), (2, 1), (2, 2), (1, 1)$**.** Our first result, the case $(1, 2)$ (see Theorem 3) was shown by E. D. Gluskin, S. V. Kislyakov and O. I. Reinov [5, Lemma I 3 p. 87, proof on p. 98]. For completeness we include its proof. Recall (see [2]) that if X and Y are Banach spaces, then $X \otimes_\varepsilon Y$ denotes their injective tensor product, i.e. the algebraic tensor product $X \otimes Y$ endowed with the injective cross-norm

$$\varepsilon(u) = \sup_{\|x^*\|\le 1,\ \|y^*\|\le 1} |\langle u, x^* \otimes y^* \rangle|, \quad u \in X \otimes Y.$$

By $X \mathbin{\widehat{\otimes}_\varepsilon} Y$ we denote the completion of $X \otimes_\varepsilon Y$.

If $1 \le p < \infty$, n is a natural number, $l_p^n := \mathbb{K}^n$ ($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$), endowed with the norm $\|\xi\|_p = (\sum_{k=1}^n |\xi_k|^p)^{1/p}$ for $\xi = (\xi_1, \ldots, \xi_n)$, and for every $k = 1, \ldots, n$, $p_k : l_p^n \to \mathbb{K}$ denotes the canonical projection, $p_k((\xi_1, \ldots, \xi_n)) = \xi_k$.

Similarly, if $(X_k)_{k=1}^n$ are Banach spaces, we define $l_p^n((X_k)_{k=1}^n)$ as the cartesian product $\prod_{k=1}^n X_k$ endowed with the norm $\|x\| = (\sum_{k=1}^n \|x_k\|^p)^{1/p}$ for $x = (x_1, \ldots, x_n)$. When $X_1 = \cdots = X_n = X$ we write $l_p^n(X)$.

Let n be a natural number, $J_n : l_1^n \hookrightarrow l_2^n$ the canonical inclusion, i.e. $J_n(\xi) = \xi$, and let $T : Z \to W$ be a bounded linear operator. We define $J_n \otimes T : l_1^n \otimes_\varepsilon Z \to l_2^n(W)$ in the usual way, i.e.

$$(J_n \otimes T)(\xi \otimes z) := \big(p_1(J_n(\xi))T(z), \ldots, p_n(J_n(\xi))T(z)\big) \quad \text{for } \xi \in l_1^n,\ z \in Z$$

and

$$(J_n \otimes T)(u) := \sum_{j=1}^{k} (J_n \otimes T)(\xi_j \otimes z_j) \quad \text{for } u = \sum_{j=1}^{k} \xi_j \otimes z_j \in l_1^n \otimes_\varepsilon Z.$$

PROPOSITION 1. *Let $T : Z \to W$ be a bounded linear operator. Then $J_n \otimes T : l_1^n \otimes_\varepsilon Z \to l_2^n(W)$ is a bounded linear operator, $\|J_n \otimes T\| \le \sqrt{n}\,\|T\|$ and hence, by density, $J_n \otimes T : l_1^n \widehat{\otimes}_\varepsilon Z \to l_2^n(W)$ is also a bounded linear operator.*

Proof. Let $u = \sum_{j=1}^{k} \xi_j \otimes z_j \in l_1^n \otimes_\varepsilon Z$. We have

$$\begin{aligned}(J_n \otimes T)(u) &= \sum_{j=1}^{k} (J_n \otimes T)(\xi_j \otimes z_j) \\ &= \sum_{j=1}^{k} \big(p_1(J_n(\xi_j))T(z_j), \ldots, p_n(J_n(\xi_j))T(z_j)\big) \\ &= \Big(T\Big(\sum_{j=1}^{k} p_1(J_n(\xi_j))z_j\Big), \ldots, T\Big(\sum_{j=1}^{k} p_n(J_n(\xi_j))z_j\Big)\Big)\end{aligned}$$

and thus

$$\|(J_n \otimes T)(u)\|_{l_2^n(W)}^2 = \sum_{i=1}^{n} \Big\|T\Big(\sum_{j=1}^{k} p_i(J_n(\xi_j))z_j\Big)\Big\|^2. \tag{1}$$

For every $i = 1, \ldots, n$ and $w^* \in W^*$ we have

$$\begin{aligned}\Big|\Big\langle T\Big(\sum_{j=1}^{k} p_i(J_n(\xi_j))z_j\Big), w^*\Big\rangle\Big| &= \Big|\sum_{j=1}^{k} (p_i \circ J_n)(\xi_j)\langle z_j, T^*w^*\rangle\Big| \\ &\le \|T^*w^*\| \sup_{\|y^*\|\le 1,\, \|z^*\|\le 1} \Big|\sum_{j=1}^{k} \langle \xi_j, y^*\rangle\langle z_j, z^*\rangle\Big| = \|T^*w^*\|\varepsilon(u),\end{aligned}$$

so

$$\Big\|T\Big(\sum_{j=1}^{k} p_i(J_n(\xi_j))z_j\Big)\Big\| \le \|T^*\|\varepsilon(u) = \|T\|\varepsilon(u).$$

Together with (1) we get

$$\|(J_n \otimes T)(u)\|_{l_2^n(W)} \le \sqrt{n}\,\|T\|\varepsilon(u). \tag{2}$$

Hence $J_n \otimes T$ is well defined. Since it is linear, from (2) we deduce that it is a bounded linear operator and $\|J_n \otimes T\| \le \sqrt{n}\|T\|$. Since $l_1^n \widehat{\otimes}_\varepsilon Z$ is the completion of $l_1^n \otimes_\varepsilon Z$, $J_n \otimes T$ can be extended by continuity to $l_1^n \widehat{\otimes}_\varepsilon Z$. ∎

Lemma 1. *Let X and Y be Banach spaces and $1 \le p < \infty$. Then*

$$w_p((u_i)_{i=1}^n; X \otimes_\varepsilon Y) = \sup_{\|x^*\|\le 1,\, \|y^*\|\le 1} \Big(\sum_{i=1}^n |\langle u_i, x^* \otimes y^*\rangle|^p\Big)^{1/p}$$

for $(u_i)_{i=1}^n \subset X \otimes_\varepsilon Y$.

Proof. We have (see [10, Lemma 1.14, p. 40]),

$$\begin{aligned} w_p((u_i)_{i=1}^n; X \otimes_\varepsilon Y) &= \sup_{\|\lambda\|_{p^*\le 1}} \varepsilon\Big(\sum_{i=1}^n \lambda_i u_i\Big) \\ &= \sup_{\|\lambda\|_{p^*\le 1}} \sup_{\|x^*\|\le 1,\, \|y^*\|\le 1} \Big|\Big\langle \sum_{i=1}^n \lambda_i u_i, x^* \otimes y^*\Big\rangle\Big| \\ &= \sup_{\|x^*\|\le 1,\, \|y^*\|\le 1} \sup_{\|\lambda\|_{p^*\le 1}} \Big|\sum_{i=1}^n \lambda_i \langle u_i, x^* \otimes y^*\rangle\Big| \\ &= \sup_{\|x^*\|\le 1,\, \|y^*\|\le 1} \Big(\sum_{i=1}^n |\langle u_i, x^* \otimes y^*\rangle|^p\Big)^{1/p}. \ \blacksquare \end{aligned}$$

Theorem 1. *If $T : Z \to W$ is 2-summing, then so is $J_n \otimes T : l_1^n \widehat{\otimes}_\varepsilon Z \to l_2^n(W)$ and $\pi_2(J_n \otimes T) \le \pi_2(T)$.*

Proof. Our proof is modelled on the well known proof that the injective tensor product of two p-summing operators is p-summing (see [6, Theorem 3.2], [10, Theorem 1.3.11, p. 51]). Let $u = \sum_{j=1}^k \xi_j \otimes z_j \in l_1^n \otimes_\varepsilon Z$. We have shown in Proposition 1 that

$$\|(J_n \otimes T)(u)\|_{l_2^n(W)}^2 = \sum_{i=1}^n \Big\|T\Big(\sum_{j=1}^k p_i(J_n(\xi_j))z_j\Big)\Big\|^2.$$

Since T is 2-summing, by Pietsch's domination theorem (see [2, 3, 9, 11, 15, 16]), $\|T(z)\|^2 \le [\pi_2(T)]^2 \int_\Omega |\langle z, z^*\rangle|^2 \, d\mu(z^*)$ for $z \in Z$ and some Borel probability measure μ on $\Omega = B_{Z^*}$. We have

$$\begin{aligned} \Big\|T\Big(\sum_{j=1}^k p_i(J_n(\xi_j))z_j\Big)\Big\|^2 &\le [\pi_2(T)]^2 \int_\Omega \Big|\Big\langle \sum_{j=1}^k p_i(J_n(\xi_j))z_j, z^*\Big\rangle\Big|^2 d\mu(z^*) \\ &= [\pi_2(T)]^2 \int_\Omega \Big|\sum_{j=1}^k p_i(J_n(\xi_j))\langle z_j, z^*\rangle\Big|^2 d\mu(z^*). \end{aligned}$$

Since

$$\sum_{i=1}^{n}\Big|\sum_{j=1}^{k}p_i(J_n(\xi_j))\langle z_j,z^*\rangle\Big|^2=\sum_{i=1}^{n}\Big|p_i\Big(\sum_{j=1}^{k}\langle z_j,z^*\rangle J_n(\xi_j)\Big)\Big|^2$$
$$=\Big\|\sum_{j=1}^{k}\langle z_j,z^*\rangle J_n(\xi_j)\Big\|_{l_2^n}^2=\Big\|J_n\Big(\sum_{j=1}^{k}\langle z_j,z^*\rangle\xi_j\Big)\Big\|_{l_2^n}^2$$

we have

$$\|(J_n\otimes T)(u)\|_{l_2^n(W)}^2\le[\pi_2(T)]^2\int_{\Omega}\Big\|J_n\Big(\sum_{j=1}^{k}\langle z_j,z^*\rangle\xi_j\Big)\Big\|_{l_2^n}^2 d\mu(z^*).$$

Since $J_n : l_1^n\hookrightarrow l_2^n$ is 2-summing with $\pi_2(J_n)=1$, again by Pietsch's domination theorem,

$$\|J_n(\xi)\|_{l_2^n}^2\le\int_{\Psi}|\langle\xi,y^*\rangle|^2d\nu(y^*)\qquad\text{for }\xi\in l_1^n$$

and some Borel probability measure ν on $\Psi=B_{(l_1^n)^*}$. Then

$$\Big\|J_n\Big(\sum_{j=1}^{k}\langle z_j,z^*\rangle\xi_j\Big)\Big\|_{l_2^n}^2\le\int_{\Psi}\Big|\Big\langle\sum_{j=1}^{k}\langle z_j,z^*\rangle\xi_j,y^*\Big\rangle\Big|^2 d\nu(y^*)$$
$$=\int_{\Psi}\Big|\sum_{j=1}^{k}\langle\xi_j,y^*\rangle\langle z_j,z^*\rangle\Big|^2 d\nu(y^*)=\int_{\Psi}|\langle u,y^*\otimes z^*\rangle|^2\,d\nu(y^*)$$

and hence

$$\|(J_n\otimes T)(u)\|_{l_2^n(W)}^2\le[\pi_2(T)]^2\int_{\Omega}\int_{\Psi}|\langle u,y^*\otimes z^*\rangle|^2d\nu(y^*)d\mu(z^*).\tag{1}$$

Let $(u_k)_{1\le k\le m}\subset l_1^n\otimes_\varepsilon Z$. From (1) and Lemma 1 we deduce

$$\sum_{k=1}^{m}\|(J_n\otimes T)(u_k)\|_{l_2^n(W)}^2\le[\pi_2(T)]^2\sup_{\|x^*\|\le1,\|y^*\|\le1}\Big(\sum_{k=1}^{m}|\langle u_k,x^*\otimes y^*\rangle|^2\Big)\tag{2}$$
$$=[\pi_2(T)]^2[w_2((u_k)_{k=1}^n;l_1^n\otimes_\varepsilon Z)]^2.$$

By the density of $l_1^n\otimes_\varepsilon Z$ in $l_1^n\widehat{\otimes}_\varepsilon Z$, from (2) we deduce that for every $(u_k)_{1\le k\le m}\subset l_1^n\,\widehat{\otimes}_\varepsilon Z$, we have

$$\Big(\sum_{k=1}^{m}\|(J_n\otimes T)(u_k)\|_{l_2^n(W)}^2\Big)^{1/2}\le\pi_2(T)w_2((u_k)_{k=1}^n;l_1^n\,\widehat{\otimes}_\varepsilon Z),$$

i.e. $J_n\otimes T$ is 2-summing and $\pi_2(J_n\otimes T)\le\pi_2(T)$. ■

Recall (see [3, p. 45], or [9, p. 234]) that if X is a Banach space, then $i_X : X\to C(B_{X^*})$ is the operator defined by $i_X(x)(x^*)=x^*(x)$.

Theorem 2 (Gluskin–Kislyakov–Reinov). *Let $T_k : X_k \to Y_k$ be 2-summing operators with $\pi_2(T_k) \le 1$ for every $k = 1, \ldots, n$ and let $M_T : l_1^n((X_k)_{k=1}^n) \to l_2^n((Y_k)_{k=1}^n)$ be the multiplication operator*

$$M_T(x_1, \ldots, x_n) = (T_1(x_1), \ldots, T_n(x_n)).$$

Then M_T is 2-summing and $\pi_2(M_T) \le 1$.

Proof. Let

$$T_k : X_k \xrightarrow{i_{X_k}} C(B_{X_k^*}) \overset{J_k}{\hookrightarrow} L_2(\mu_k) \xrightarrow{A_k} Y_k$$

be a Pietsch factorization of T_k with $\|A_k\| = \pi_2(T_k)$ (see [3, Corollary 2.16, p. 48], [9, Proposition 17.3.7, p. 234]). Let $\Omega = \prod_{k=1}^n B_{X_k^*}$ and $\mu = \prod_{k=1}^n \mu_k$. We define $S : l_1^n((X_k)_{k=1}^n) \to C(\Omega)$ by

$$[S(x_1, \ldots, x_n)](x_1^*, \ldots, x_n^*) = \sum_{k=1}^n x_k^*(x_k)$$

and denote, as usual, by $J : C(\Omega) \hookrightarrow L_2(\mu)$ the canonical inclusion, $J(f) = \widehat{\widehat{f}}$. Let us define also $V_k : L_2(\mu) \to Y_k$ by

$$V_k(\widehat{\widehat{f}}) = A_k(\widehat{f \circ \sigma_k}),$$

where $\sigma_k : B_{X_k^*} \to \prod_{k=1}^n B_{X_k^*}$ is the canonical injection. Here we denote by $\widehat{\widehat{f}}$ (resp. $\widehat{f \circ \sigma_k}$) the equivalence class of f (resp. $f \circ \sigma_k$) in $L_2(\mu)$ (resp. $L_2(\mu_k)$). We will prove that $T := J \circ S : l_1^n((X_k)_{k=1}^n) \to L_2(\mu)$ is 2-summing, $\pi_2(T) \le 1$ and every T_k has the factorization

$$T_k : X_k \xrightarrow{\sigma_k} l_1^n((X_k)_{k=1}^n) \xrightarrow{S} C(\Omega) \overset{J}{\hookrightarrow} L_2(\mu) \xrightarrow{V_k} Y_k.$$

First, let us show that S is bounded linear. Let $x = (x_1, \ldots, x_n) \in l_1^n((X_k)_{k=1}^n)$. Then, obviously, $S(x)$ is a continuous function on $\Omega = \prod_{k=1}^n B_{X_k^*}$, i.e. $S(x) \in C(\Omega)$. For every $(x_1^*, \ldots, x_n^*) \in \Omega$ we have

$$|S(x)(x_1^*, \ldots, x_n^*)| \le \sum_{k=1}^n |x_k^*(x_k)| \le \sum_{k=1}^n \|x_k\| = \|x\|_{l_1^n((X_k)_{k=1}^n)},$$

i.e.

$$\|S(x)\|_{C(\Omega)} = \sup_{(x_1^*, \ldots, x_n^*) \in \Omega} |S(x)(x_1^*, \ldots, x_n^*)| \le \|x\|_{l_1^n((X_k)_{k=1}^n)}.$$

Since S is linear, it is bounded linear with $\|S\| \le 1$. By measure-theoretical considerations (we omit the details), V_k are well defined, bounded linear with $\|V_k\| \le \|A_k\|$ for $k = 1, \ldots, n$. Since the canonical inclusion $J : C(\Omega) \hookrightarrow L_2(\mu)$ is 2-summing with $\pi_2(J) = 1$, it follows that $T := J \circ S : l_1^n((X_k)_{k=1}^n) \to L_2(\mu)$ is 2-summing with $\pi_2(T) \le 1$.

Let us verify the factorization of every T_k.

Let $x_k \in X_k$ and denote $(S \circ \sigma_k)(x_k) =: f_k \in C(\Omega)$. Then

$$f_k(x_1^*, \ldots, x_n^*) = [S(0, \ldots, 0, x_k, 0, \ldots, 0)](x_1^*, \ldots, x_n^*) = x_k^*(x_k).$$

Also $f_k \circ \sigma_k : B_{X_k^*} \to \mathbb{K}$ is defined by

$$(f_k \circ \sigma_k)(x_k^*) = f_k(0, \ldots, 0, x_k^*, 0, \ldots, 0) = x_k^*(x_k) = i_{X_k}(x_k)(x_k^*),$$

i.e. $f_k \circ \sigma_k = i_{X_k}(x_k) \in C(B_{X_k^*})$. We have

$$\begin{aligned}(V_k \circ J \circ S \circ \sigma_k)(x_k) &= V_k(J(f_k)) = V_k(\widehat{\widehat{f_k}}) = A_k(\widehat{f_k \circ \sigma_k}) \\ &= A_k(J_k(f_k \circ \sigma_k)) \quad \text{ since } f_k \circ \sigma_k \in C(B_{X_k^*}) \\ &= (A_k \circ J_k)(i_{X_k}(x_k)) = (A_k \circ J_k \circ i_{X_k})(x_k) \\ &= T_k(x_k),\end{aligned}$$

the last equality holding by the Pietsch factorization of T_k.

Now since $T : l_1^n((X_k)_{k=1}^n) =: Z \to W := L_2(\mu)$ is 2-summing, Theorem 1 implies that $J_n \otimes T : l_1^n \widehat{\otimes}_\varepsilon l_1^n((X_k)_{k=1}^n) \to l_2^n(L_2(\mu))$ is 2-summing and $\pi_2(J_n \otimes T) \le \pi_2(T) \le 1$. We show that

$$M_T : l_1^n((X_k)_{k=1}^n) \xrightarrow{U} l_1^n \widehat{\otimes}_\varepsilon l_1^n((X_k)_{k=1}^n) \xrightarrow{J_n \otimes T} l_2^n(L_2(\mu)) \xrightarrow{M_V} l_2^n((Y_k)_{k=1}^n)$$

is a factorization of M_T, where

$$U(x) = \sum_{k=1}^n e_k \otimes \sigma_k(x_k) \quad \text{ for } x = (x_1, \ldots, x_n).$$

Indeed, for $x = (x_1, \ldots, x_n) \in l_1^n((X_k)_{k=1}^n)$ we have

$$\begin{aligned}((J_n \otimes T) \circ U)(x_1, \ldots, x_n) &= (J_n \otimes T)\Big(\sum_{k=1}^n e_k \otimes \sigma_k(x_k)\Big) \\ &= \sum_{k=1}^n (J_n \otimes T)(e_k \otimes \sigma_k(x_k)) = (T(\sigma_1(x_1)), \ldots, T(\sigma_n(x_n)))\end{aligned}$$

since for $k = 1, \ldots, n$,

$$\begin{aligned}&(J_n \otimes T)(e_k \otimes \sigma_k(x_k)) \\ &\quad = (p_1(e_k)T(\sigma_k(x_k)), \ldots, p_k(e_k)T(\sigma_k(x_k)), \ldots, p_n(e_k)T(\sigma_k(x_k))) \\ &\quad = (0, \ldots, 0, T(\sigma_k(x_k)), 0, \ldots, 0).\end{aligned}$$

Then, using the equality $V_k \circ T \circ \sigma_k = T_k$ for $k = 1, \ldots, n$, it follows that

$$\begin{aligned}(M_V \circ (J_n \otimes T) \circ U)(x_1, \ldots, x_n) &= M_V(T(\sigma_1(x_1)), \ldots, T(\sigma_n(x_n))) \\ &= (V_1(T(\sigma_1(x_1))), \ldots, V_n(T(\sigma_n(x_n)))) \\ &= (T_1(x_1), \ldots, T_n(x_n)) = M_T(x_1, \ldots, x_n).\end{aligned}$$

Moreover

$$\|M_V\| \leq \sup_{1\leq k\leq n} \|V_k\| \leq \sup_{1\leq k\leq n} \|A_k\| = \sup_{1\leq k\leq n} \pi_2(T_k) \leq 1.$$

Let us show that U is bounded linear. Let $x = (x_1, \ldots, x_n) \in l_1^n((X_k)_{k=1}^n)$. Let $x^* = (\lambda_1, \ldots, \lambda_n) \in (l_1^n)^* = l_\infty^n$ and $\psi \in (l_1^n((X_k)_{k=1}^n))^*$. Then

$$\begin{aligned}|\langle U(x), x^* \otimes \psi\rangle| &= \Big|\sum_{k=1}^n x^*(e_k)\psi(\sigma_k(x_k))\Big| \\ &\leq \sum_{k=1}^n |\lambda_k|\,|\psi(\sigma_k(x_k))| \leq \|x^*\| \sum_{k=1}^n \|\psi\|\,\|\sigma_k(x_k)\| \\ &\leq \|x^*\|\,\|\psi\| \sum_{k=1}^n \|x_k\| = \|x^*\|\,\|\psi\|\,\|x\|.\end{aligned}$$

We deduce $\varepsilon(U(x)) \leq \|x\|$ and since U is linear, it is bounded linear with $\|U\| \leq 1$. By the ideal property, M_T is 2-summing and $\pi_2(M_T) \leq \|M_V\|\pi_2(J_n \otimes T)\|U\| \leq \pi_2(T) \leq 1$. ■

Theorem 3. *Let $M_{\mathcal{V}} : l_1(\mathcal{X}) \to l_2(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_\infty$.*

Moreover, $\pi_2(M_{\mathcal{V}}) = \sup_{n\in\mathbb{N}} \pi_2(V_n)$.

Proof. (i)⇒(ii). Note that $V_n : X_n \xrightarrow{\sigma_n} l_1(\mathcal{X}) \xrightarrow{M_{\mathcal{V}}} l_2(\mathcal{Y}) \xrightarrow{p_n} Y_n$ is a factorization of V_n. From the ideal property, all V_n are 2-summing and $\pi_2(V_n) \leq \pi_2(M_{\mathcal{V}})$ for $n \in \mathbb{N}$, which yields (ii).

(ii)⇒(i). Write $L = \sup_{n\in\mathbb{N}} \pi_2(V_n)$. If $L = 0$, then all V_n are 0, and $M_{\mathcal{V}} = 0$, $\pi_2(M_{\mathcal{V}}) = 0$. Suppose $L > 0$. Let $n \in \mathbb{N}$ and for $k = 1, \ldots, n$ consider $T_k : X_k \to Y_k$ defined by $T_k = V_k/L$. Note that T_k is 2-summing with $\pi_2(T_k) \leq 1$ for every $k = 1, \ldots, n$. Then, by Theorem 2, the multiplication operator

$$M_{(T_1,\ldots,T_n)} : l_1^n((X_k)_{k=1}^n) \to l_2^n((Y_k)_{k=1}^n),$$
$$(x_1, \ldots, x_n) \mapsto (T_1(x_1), \ldots, T_n(x_n)),$$

is 2-summing and $\pi_2(M_{(T_1,\ldots,T_n)}) \leq 1$.

Consider the diagram

$$l_1(\mathcal{X}) \xrightarrow{A_n} l_1^n((X_k)_{k=1}^n) \xrightarrow{M_{(T_1,\ldots,T_n)}} l_2^n((Y_k)_{k=1}^n) \xrightarrow{B_n} l_2(\mathcal{Y}),$$

where $A_n(x_1, \ldots, x_n, \ldots) = (x_1, \ldots, x_n)$, $B_n(y_1, \ldots, y_n) = (y_1, \ldots, y_n, 0, \ldots)$. Then $S_n := B_n \circ M_{(T_1,\ldots,T_n)} \circ A_n : l_1(\mathcal{X}) \to l_2(\mathcal{Y})$ with $S_n(x_1, \ldots, x_n, \ldots) =$

$(T_1(x_1),\dots,T_n(x_n),0,\dots)$ is 2-summing and

$$\pi_2(S_n)\le \pi_2(M_{(T_1,\dots,T_n)})\|B_n\|\,\|A_n\|\le 1.$$

Since $S_n(x)\to \frac{1}{L}M_{\mathcal{V}}(x)$ for $x\in l_1(\mathcal{X})$, from [2, Proposition 17.21, p. 220], $\frac{1}{L}M_{\mathcal{V}}$ is 2-summing and $\pi_2\big(\frac{1}{L}M_{\mathcal{V}}\big)\le 1$, i.e. $M_{\mathcal{V}}$ is 2-summing and $\pi_2(M_{\mathcal{V}})\le L$. ∎

The next result is a completion of Proposition 2.4 in [8] and will be another important ingredient in the proofs.

LEMMA 2. *Let $V: X\to Y$ be a bounded linear operator. The following assertions are equivalent:*

(i) *V is 2-summing.*
(ii) *For each Banach space Z and each 2-summing operator $U: Z\to X$, $V\circ U$ is integral.*
(iii) *For each Banach space Z and each 2-summing operator $U: Z\to X$, $V\circ U$ is nuclear.*

Moreover,

$$\sup_{\pi_2(U)\le 1}\|V\circ U\|_{\mathrm{int}}=\sup_{\pi_2(U)\le 1}\|V\circ U\|_{\mathrm{nuc}}=\pi_2(V).$$

Proof. (i)⇒(iii). From Grothendieck's theorem, $V\circ U$ is nuclear and $\|V\circ U\|_{\mathrm{nuc}}\le \pi_2(V)\pi_2(U)$. Then

$$\sup_{\pi_2(U)\le 1}\|V\circ U\|_{\mathrm{nuc}}\le \pi_2(V). \tag{1}$$

(iii)⇒(ii). This follows from the well known result that each nuclear operator is integral and $\|\cdot\|_{\mathrm{int}}\le\|\cdot\|_{\mathrm{nuc}}$ (see [9, Proposition 6.7.3, p. 101]). Hence

$$\sup_{\pi_2(U)\le 1}\|V\circ U\|_{\mathrm{int}}\le \sup_{\pi_2(U)\le 1}\|V\circ U\|_{\mathrm{nuc}}. \tag{2}$$

(ii)⇔(i). This was shown in [8, Proposition 2.4] together with the equality

$$\sup_{\pi_2(U)\le 1}\|V\circ U\|_{\mathrm{int}}=\pi_2(V). \tag{3}$$

The equality from the statement follows from (1)–(3). ∎

THEOREM 4. *Let $M_{\mathcal{V}}: l_2(\mathcal{X})\to l_1(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}}\in l_1$.*

Moreover, $\pi_2(M_{\mathcal{V}})=\sum_{n=1}^{\infty}\pi_2(V_n)$.

Proof. (i)⇒(ii). Let $\mathcal{Z} = (Z_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces, let $U_n : Z_n \to X_n$ be 2-summing with $\pi_2(U_n) \le 1$ for all $n \in \mathbb{N}$, and denote $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$. From the nontrivial part of Theorem 3, $M_{\mathcal{U}} : l_1(\mathcal{Z}) \to l_2(\mathcal{X})$ is 2-summing and $\pi_2(M_{\mathcal{U}}) \le 1$. By Grothendieck's composition theorem, $M_{\mathcal{V}\circ\mathcal{U}} = M_{\mathcal{V}} \circ M_{\mathcal{U}} : l_1(\mathcal{Z}) \to l_1(\mathcal{Y})$ is nuclear and $\|M_{\mathcal{V}\circ\mathcal{U}}\|_{\mathrm{nuc}} \le \pi_2(M_{\mathcal{V}})\pi_2(M_{\mathcal{U}}) \le \pi_2(M_{\mathcal{V}})$.

But it is well known (see [9, Proposition 5.5.1, pp. 236–237] for the scalar case, or [12, Theorem 2] for the vector case) that if $T : Z \to l_1(\mathcal{Y})$, $T(x) = (T_n(x))_{n\in\mathbb{N}}$, and T is nuclear, then all T_n are nuclear and $\sum_{n=1}^{\infty} \|T_n\|_{\mathrm{nuc}} = \|T\|_{\mathrm{nuc}}$. In our situation, $\sum_{n=1}^{\infty} \|V_n \circ U_n\|_{\mathrm{nuc}} = \|M_{\mathcal{V}\circ\mathcal{U}}\|_{\mathrm{nuc}}$, and therefore $\sum_{n=1}^{\infty} \|V_n \circ U_n\|_{\mathrm{nuc}} \le \pi_2(M_{\mathcal{V}})$, i.e.

$$\sum_{i=1}^{n} \|V_i \circ U_i\|_{\mathrm{nuc}} \le \pi_2(M_{\mathcal{V}}) \quad \text{for } n \in \mathbb{N}. \tag{1}$$

By Lemma 2, all V_n are 2-summing and $\sup_{\pi_2(U_n)\le 1} \|V_n \circ U_n\|_{\mathrm{nuc}} = \pi_2(V_n)$ for every $n \in \mathbb{N}$. Taking in (1) the supremum, first over $\pi_2(U_1) \le 1$, then over $\pi_2(U_2) \le 1, \ldots, \pi_2(U_n) \le 1$, we get $\sum_{i=1}^{n} \pi_2(V_i) \le \pi_2(M_{\mathcal{V}})$ for $n \in \mathbb{N}$, i.e. $\sum_{n=1}^{\infty} \pi_2(V_n) \le \pi_2(M_{\mathcal{V}})$.

(ii)⇒(i) follows from a general result of [9, Theorem 6.2.3, p. 91].

Another proof of (ii)⇒(i) is the following. Write $a_n = \sqrt{\pi_2(V_n)}$ and note that, by hypothesis, $a = (a_n)_{n\in\mathbb{N}} \in l_2$, thus $M_{a\mathcal{J}} : l_2(\mathcal{Y}) \to l_1(\mathcal{Y})$ is bounded linear. Define

$$T : l_2(\mathcal{X}) \to l_2(\mathcal{Y}), \qquad T(x) = \left(\frac{1}{a_n} V_n(x)\right)_{n\in\mathbb{N}},$$

(we use $0/0 = 0$). By simple calculations, $M_{\mathcal{V}}$ has the factorization $M_{\mathcal{V}} : l_2(\mathcal{X}) \xrightarrow{T} l_2(\mathcal{Y}) \xrightarrow{M_{a\mathcal{J}}} l_1(\mathcal{Y})$. Since all V_n are 2-summing and by hypothesis

$$\sum_{n=1}^{\infty} \left[\pi_2\left(\frac{1}{a_n} V_n\right)\right]^2 = \sum_{n=1}^{\infty} \pi_2(V_n) < \infty,$$

from Nahoum's theorem (see [7, Lemme, p. 5], [16, Lemma 23, p. 274]), T is 2-summing and $\pi_2(T) \le \sqrt{\sum_{n=1}^{\infty} \pi_2(V_n)}$. By the ideal property of the class of 2-summing operators, $M_{\mathcal{V}} = M_{a\mathcal{J}} \circ T$ is 2-summing. ∎

The next result was shown in [13, Corollary 4]. We give a different proof.

THEOREM 5. *Let $M_{\mathcal{V}} : l_2(\mathcal{X}) \to l_2(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_2$.*

Moreover, $[\pi_2(M_{\mathcal{V}})]^2 = \sum_{n=1}^{\infty} [\pi_2(V_n)]^2$.

Proof. (i)⇒(ii). Let $\mathcal{Z} = (Z_n)_{n\in\mathbb{N}}$ be a sequence of Banach spaces. Let $U_n : Z_n \to X_n$ be 2-summing with $\pi_2(U_n) \le 1$ for all $n \in \mathbb{N}$ and set $\mathcal{U} = (U_n)_{n\in\mathbb{N}}$. From the nontrivial part of Theorem 3, $M_{\mathcal{U}} : l_1(\mathcal{Z}) \to l_2(\mathcal{X})$ is 2-summing and $\pi_2(M_{\mathcal{U}}) \le 1$. Let $a = (a_n)_{n\in\mathbb{N}} \in l_2$. Then $M_{a\mathcal{J}} : l_2(\mathcal{Y}) \to l_1(\mathcal{Y})$ is bounded linear and $\|M_{a\mathcal{J}}\| = \|a\|_2$. By Grothendieck's composition theorem,

$$M_{a\mathcal{V}\circ\mathcal{U}} = M_{a\mathcal{J}} \circ M_{\mathcal{V}} \circ M_{\mathcal{U}} : l_1(\mathcal{Z}) \to l_1(\mathcal{Y})$$

is nuclear and

$$\|M_{a\mathcal{V}\circ\mathcal{U}}\|_{\mathrm{nuc}} \le \pi_2(M_{\mathcal{V}})\pi_2(M_{\mathcal{U}})\|M_{a\mathcal{J}}\| \le \pi_2(M_{\mathcal{V}})\|a\|_2.$$

Again, from [9, Proposition 5.5.1, pp. 236–237], or [12, Theorem 2],

$$\sum_{n=1}^{\infty} \|a_n V_n \circ U_n\|_{\mathrm{nuc}} = \|M_{a\mathcal{V}\circ\mathcal{U}}\|_{\mathrm{nuc}},$$

thus

$$\sum_{n=1}^{\infty} \|a_n V_n \circ U_n\|_{\mathrm{nuc}} \le \pi_2(M_{\mathcal{V}})\|a\|_2,$$

i.e.

$$\sum_{i=1}^{n} \|a_i V_i \circ U_i\|_{\mathrm{nuc}} \le \pi_2(M_{\mathcal{V}})\|a\|_2 \quad \text{for } n \in \mathbb{N}. \tag{1}$$

By Lemma 2, each V_n is 2-summing and $\sup_{\pi_2(U_n)\le 1} \|V_n \circ U_n\|_{\mathrm{nuc}} = \pi_2(V_n)$ for all $n \in \mathbb{N}$. Taking in (1) the supremum, first over $\pi_2(U_1) \le 1$, then over $\pi_2(U_2) \le 1, \ldots, \pi_2(U_n) \le 1$, we get $\sum_{i=1}^{n} |a_i|\pi_2(V_i) \le \pi_2(M_{\mathcal{V}})\|a\|_2$ for $n \in \mathbb{N}$, i.e.

$$\sum_{n=1}^{\infty} |a_n|\pi_2(V_n) \le \pi_2(M_{\mathcal{V}})\|a\|_2.$$

We deduce that $\sum_{n=1}^{\infty} [\pi_2(V_n)]^2 < \infty$ and

$$\Big(\sum_{n=1}^{\infty} [\pi_2(V_n)]^2\Big)^{1/2} = \sup_{\|a\|_2 \le 1} \sum_{n=1}^{\infty} |a_n|\pi_2(V_n) \le \pi_2(M_{\mathcal{V}}).$$

(ii)⇒(i). This follows from Nahoum's theorem. ∎

THEOREM 6. *Let $M_{\mathcal{V}} : l_1(\mathcal{X}) \to l_1(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_2$.*

Moreover, $[\pi_2(M_{\mathcal{V}})]^2 = \sum_{n=1}^{\infty} [\pi_2(V_n)]^2$.

Proof. (i)⇒(ii). Let $a = (a_n)_{n\in\mathbb{N}} \in l_2$. Then by (i), $M_{a\mathcal{V}} = M_{\mathcal{V}} \circ M_{a\mathcal{J}} : l_2(\mathcal{X}) \to l_1(\mathcal{Y})$ is 2-summing and $\pi_2(M_{a\mathcal{V}}) \le \pi_2(M_{\mathcal{V}})\|a\|_2$. From the nontrivial part of Theorem 4, all a_nV_n are 2-summing and $\sum_{n=1}^{\infty} \pi_2(a_nV_n) = \pi_2(M_{a\mathcal{V}})$. Then

$$\sum_{n=1}^{\infty} \pi_2(a_nV_n) \le \pi_2(M_{\mathcal{V}})\|a\|_2.$$

Thus all V_n are 2-summing (take $a = e_n$, $n \in \mathbb{N}$) and for each $a = (a_n)_{n\in\mathbb{N}} \in l_2$,

$$\sum_{n=1}^{\infty} |a_n|\pi_2(V_n) \le \pi_2(M_{\mathcal{V}})\|a\|_2.$$

Then, as is well known, $\sum_{n=1}^{\infty}[\pi_2(V_n)]^2 < \infty$ and

$$\Big(\sum_{n=1}^{\infty}[\pi_2(V_n)]^2\Big)^{1/2} = \sup_{\|a\|_2\le 1} \sum_{n=1}^{\infty} |a_n|\pi_2(V_n) \le \pi_2(M_{\mathcal{V}}).$$

(ii)⇒(i). Let us define $a = (\pi_2(V_n))_{n\in\mathbb{N}} \in l_2$ and observe that $M_{\mathcal{V}}$ has the factorization

$$M_{\mathcal{V}} : l_1(\mathcal{X}) \xrightarrow{M_{\operatorname{sgn}\mathcal{V}}} l_2(\mathcal{Y}) \xrightarrow{M_{a\mathcal{J}}} l_1(\mathcal{Y}).$$

Note that by (ii) and the nontrivial part of Theorem 3, $M_{\operatorname{sgn}\mathcal{V}}$ is 2-summing and $\pi_2(M_{\operatorname{sgn}\mathcal{V}}) \le 1$. Also by (ii), $M_{a\mathcal{J}}$ is a bounded linear operator with $\|M_{a\mathcal{J}}\| = \|a\|_2$. Thus $M_{\mathcal{V}}$ is 2-summing and $\pi_2(M_{\mathcal{V}}) \le \|a\|_2$, proving (i). ■

The case $(p, 1)$

THEOREM 7. *Let $1 < p < 2$ and $M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_1(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*

(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_s$, where $1/s = 1/p^* + 1/2$.*

Moreover, $\pi_2(M_{\mathcal{V}}) = (\sum_{n=1}^{\infty}[\pi_2(V_n)]^s)^{1/s}$.

Proof. (i)⇒(ii). Let $a = (a_n)_{n\in\mathbb{N}} \in l_{s^*}$. From $1/p = 1/2 + 1/s^*$, $M_{a\mathcal{J}} : l_2(\mathcal{X}) \to l_p(\mathcal{X})$ is bounded linear and so, by (i), $M_{a\mathcal{V}} = M_{\mathcal{V}} \circ M_{a\mathcal{J}} : l_2(\mathcal{X}) \to l_1(\mathcal{Y})$ is 2-summing and $\pi_2(M_{a\mathcal{V}}) \le \pi_2(M_{\mathcal{V}})\|a\|_{s^*}$. From the nontrivial part of Theorem 4, all a_nV_n are 2-summing and $\sum_{n=1}^{\infty} \pi_2(a_nV_n) = \pi_2(M_{a\mathcal{V}})$. Thus all V_n are 2-summing and for each $a = (a_n)_{n\in\mathbb{N}} \in l_{s^*}$,

$$\sum_{n=1}^{\infty} |a_n|\pi_2(V_n) \le \pi_2(M_{\mathcal{V}})\|a\|_{s^*}.$$

As is well known, it follows that $\sum_{n=1}^{\infty}[\pi_2(V_n)]^s < \infty$ and

$$\Big(\sum_{n=1}^{\infty}[\pi_2(V_n)]^s\Big)^{1/s} = \sup_{\|a\|_{s^*}\le 1} \sum_{n=1}^{\infty} |a_n|\pi_2(V_n) \le \pi_2(M_{\mathcal{V}}).$$

(ii)⇒(i). Define $a = (\pi_2(V_n))_{n\in\mathbb{N}} \in l_s$. From the equality $1/s = 1/p^* + 1/2$, we get the factorization

$$M_{\mathcal{V}} : l_p(\mathcal{X}) \xrightarrow{M_{c\mathcal{J}}} l_1(\mathcal{X}) \xrightarrow{M_{\operatorname{sgn}\mathcal{V}}} l_2(\mathcal{Y}) \xrightarrow{M_{b\mathcal{J}}} l_1(\mathcal{Y})$$

where $c_n = [\pi_2(V_n)]^{s/p^*}$, $b_n = [\pi_2(V_n)]^{s/2}$; note that $bc(\operatorname{sgn}\mathcal{V}) = \mathcal{V}$. By (ii) and the nontrivial part of Theorem 3, $M_{\operatorname{sgn}\mathcal{V}}$ is 2-summing, $\pi_2(M_{\operatorname{sgn}\mathcal{V}}) \le 1$, and by (ii), $M_{b\mathcal{J}}$ is bounded linear with $\|M_{b\mathcal{J}}\| = \|b\|_2$, and $M_{c\mathcal{J}}$ is bounded linear with $\|M_{c\mathcal{J}}\| = \|c\|_{p^*}$. Thus $M_{\mathcal{V}}$ is 2-summing, $\pi_2(M_{\mathcal{V}}) \le \|b\|_2\|c\|_{p^*} = \|a\|_s$, proving (i). ∎

THEOREM 8. *Let* $2 < p < \infty$ *and* $M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_1(\mathcal{Y})$. *The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_1$.*

Moreover, $\pi_2(M_{\mathcal{V}}) = \sum_{n=1}^{\infty} \pi_2(V_n)$.

Proof. (i)⇒(ii). From $p > 2$, $J : l_2(\mathcal{X}) \hookrightarrow l_p(\mathcal{X})$ and since $M_{\mathcal{V}}$ is 2-summing, $M_{\mathcal{V}} \circ J : l_2(\mathcal{X}) \to l_1(\mathcal{Y})$ is 2-summing with $\pi_2(M_{\mathcal{V}} \circ J) \le \pi_2(M_{\mathcal{V}})$. By the nontrivial part of Theorem 4 we get (ii) and $\sum_{n=1}^{\infty} \pi_2(V_n) = \pi_2(M_{\mathcal{V}} \circ J)$.

(ii)⇒(i). By (ii) and [9, Theorem 6.2.3, p. 91], $M_{\mathcal{V}}$ is 2-summing and $\pi_2(M_{\mathcal{V}}) \le \sum_{n=1}^{\infty} \pi_2(V_n)$. ∎

The next result requires some natural cotype 2 assumptions.

THEOREM 9. *Let* $2 < p < \infty$ *and* $M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_1(\mathcal{Y})$.

(a) *Suppose that all X_n have cotype 2 with $C_2(\mathcal{X}) := \sup_{n\in\mathbb{N}} C_2(X_n) < \infty$ and let $1 \le s \le 2$. The following assertions are equivalent:*

  (i) *$M_{\mathcal{V}}$ is s-summing.*
  (ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_1$.*
  (iii) *$M_{\mathcal{V}}$ is 1-summing.*

(b) *Suppose that all X_n, Y_n have cotype 2 with $C_2(\mathcal{X}) := \sup_{n\in\mathbb{N}} C_2(X_n) < \infty$, $C_2(\mathcal{Y}) := \sup_{n\in\mathbb{N}} C_2(Y_n) < \infty$, and let $1 \le s < \infty$. The following assertions are equivalent:*

  (i) *$M_{\mathcal{V}}$ is s-summing.*
  (ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_1$.*
  (iii) *$M_{\mathcal{V}}$ is 1-summing.*

Proof. (a) (i)⇒(ii). Since $1 \le s \le 2$, by (i), $M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_1(\mathcal{Y})$ is 2-summing. From Theorem 8 we get (ii).

(ii)⇒(iii). Since X_n has cotype 2, $\Pi_1(X_n, \cdot) = \Pi_2(X_n, \cdot)$ and there exists a universal constant $c > 0$ such that

$$\pi_2(\cdot) \le \pi_1(\cdot) \le cC_2(X_n)\sqrt{1 + \log C_2(X_n)}\,\pi_2(\cdot)$$

(see [15, Corollary 10.18(i), p. 71]). Then, by hypothesis,

$$\pi_2(\cdot) \le \pi_1(\cdot) \le cC_2(\mathcal{X})\sqrt{1+\log C_2(\mathcal{X})}\,\pi_2(\cdot).$$

Since by (ii), $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_1$, we deduce $(\pi_1(V_n))_{n\in\mathbb{N}} \in l_1$ and by [9, Theorem 6.2.3, p. 91], $M_{\mathcal{V}}$ is 1-summing.

(iii)⇒(i) is well known.

(b) (i)⇒(ii). From $p > 2$, $J : l_2(\mathcal{X}) \hookrightarrow l_p(\mathcal{X})$, and because $M_{\mathcal{V}}$ is s-summing, $M_{\mathcal{V}} \circ J : l_2(\mathcal{X}) \to l_1(\mathcal{Y})$ is s-summing. Since, by hypothesis, $\sup_{n\in\mathbb{N}} C_2(X_n) < \infty$ and $\sup_{n\in\mathbb{N}} C_2(Y_n) < \infty$, it follows that $l_2(\mathcal{X})$ and $l_1(\mathcal{Y})$ both have cotype 2 (see [16, Exercise 18, p. 109]). Then, by the coincidence theorem (see [3, Corollary 11.16, p. 224]), $M_{\mathcal{V}} \circ J : l_2(\mathcal{X}) \to l_1(\mathcal{Y})$ is 2-summing and thus by the nontrivial part of Theorem 4 we get (ii).

The implication (ii)⇒(iii) was shown in (a), and (iii)⇒(i) is well known. ∎

The case $(p, 2)$

THEOREM 10. *Let $1 < p < 2$ and $M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_2(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_{p^*}$.*

Moreover, $\pi_2(M_{\mathcal{V}}) = (\sum_{n=1}^{\infty}[\pi_2(V_n)]^{p^*})^{1/p^*}$.

Proof. (i)⇒(ii). Define r by $1/p = 1/2 + 1/r$. For $a = (a_n)_{n\in\mathbb{N}} \in l_2$, $M_{a\mathcal{J}} : l_2(\mathcal{Y}) \to l_1(\mathcal{Y})$ is bounded linear and, by (i), $M_{a\mathcal{V}} = M_{a\mathcal{J}} \circ M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_1(\mathcal{Y})$ is 2-summing and $\pi_2(M_{a\mathcal{V}}) \le \pi_2(M_{\mathcal{V}})\|a\|_2$. From Theorem 7, all a_nV_n are 2-summing, $(\pi_2(a_nV_n))_{n\in\mathbb{N}} \in l_{r^*}$ and

$$\pi_2(M_{a\mathcal{V}}) = \Big(\sum_{n=1}^{\infty}[\pi_2(a_nV_n)]^{r^*}\Big)^{1/r^*}.$$

It follows that all V_n are 2-summing and for each $a = (a_n)_{n\in\mathbb{N}} \in l_2$ we have

$$\Big(\sum_{n=1}^{\infty}|a_n|^{r^*}[\pi_2(V_n)]^{r^*}\Big)^{1/r^*} \le \pi_2(M_{\mathcal{V}})\|a\|_2.$$

Since $1/r^* = 1/2 + 1/p^*$, we deduce $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_{p^*}$ and

$$\Big(\sum_{n=1}^{\infty}[\pi_2(V_n)]^{p^*}\Big)^{1/p^*} = \sup_{\|a\|_2\le 1}\Big(\sum_{n=1}^{\infty}|a_n|^{r^*}[\pi_2(V_n)]^{r^*}\Big)^{1/r^*} \le \pi_2(M_{\mathcal{V}}),$$

proving (i).

(ii)⇒(i). We consider $a = (\pi_2(V_n))_{n\in\mathbb{N}} \in l_{p^*}$ and we note that $M_{\mathcal{V}}$ has the factorization $M_{\mathcal{V}} : l_p(\mathcal{X}) \xrightarrow{M_{a\mathcal{J}}} l_1(\mathcal{X}) \xrightarrow{M_{\operatorname{sgn}\mathcal{V}}} l_2(\mathcal{Y})$. From (ii) and the nontrivial part of Theorem 3 we get (i). ∎

Theorem 11. *Let $2 < p < \infty$ and $M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_2(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_2$.*

Moreover, $\pi_2(M_{\mathcal{V}}) = (\sum_{n=1}^{\infty}[\pi_2(V_n)]^2)^{1/2}$.

Proof. (i)⇒(ii). Since $p > 2$, $l_2(\mathcal{X}) \hookrightarrow l_p(\mathcal{X})$ and by (i), $M_{\mathcal{V}} : l_2(\mathcal{X}) \to l_2(\mathcal{Y})$ is 2-summing. By the nontrivial part of Theorem 5 we get (ii).

(ii)⇒(i) follows from Nahoum's theorem. ∎

The case $(2, p)$

Theorem 12. *Let $1 < p < 2$ and $M_{\mathcal{V}} : l_2(\mathcal{X}) \to l_p(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_p$.*

Moreover, $\pi_2(M_{\mathcal{V}}) = (\sum_{n=1}^{\infty}[\pi_2(V_n)]^p)^{1/p}$.

Proof. (i)⇒(ii). Let $a = (a_n)_{n\in\mathbb{N}} \in l_{p^*}$. Since $M_{\mathcal{V}}$ is 2-summing, $M_{a\mathcal{V}} = M_{a\mathcal{J}} \circ M_{\mathcal{V}} : l_2(\mathcal{X}) \to l_1(\mathcal{Y})$ is 2-summing and $\pi_2(M_{a\mathcal{V}}) \le \pi_2(M_{\mathcal{V}})\|a\|_{p^*}$. From the nontrivial part of Theorem 4, all a_nV_n are 2-summing, and $\sum_{n=1}^{\infty}\pi_2(a_nV_n) = \pi_2(M_{a\mathcal{V}})$. Thus all V_n are 2-summing and for each $a = (a_n)_{n\in\mathbb{N}} \in l_{p^*}$,

$$\sum_{n=1}^{\infty}|a_n|\pi_2(V_n) \le \pi_2(M_{\mathcal{V}})\|a\|_{p^*}.$$

As is well known, $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_p$ and

$$\Big(\sum_{n=1}^{\infty}[\pi_2(V_n)]^p\Big)^{1/p} = \sup_{\|a\|_{p^*}\le 1}\sum_{n=1}^{\infty}|a_n|\pi_2(V_n) \le \pi_2(M_{\mathcal{V}}).$$

(ii)⇒(i). We consider $a = (\pi_2(V_n))_{n\in\mathbb{N}} \in l_p$ and define r by $1/p = 1/2 + 1/r$. Set $b_n = [\pi_2(V_n)]^{p/2}$, $c_n = [\pi_2(V_n)]^{p/r}$. Then $b = (b_n)_{n\in\mathbb{N}} \in l_2$, $c = (c_n)_{n\in\mathbb{N}} \in l_r$ and since $bc(\operatorname{sgn}\mathcal{V}) = \mathcal{V}$ we find that $M_{\mathcal{V}}$ has the factorization

$$M_{\mathcal{V}} : l_2(\mathcal{X}) \xrightarrow{M_{b\,\operatorname{sgn}\mathcal{V}}} l_2(\mathcal{Y}) \xrightarrow{M_{c\mathcal{J}}} l_p(\mathcal{Y}).$$

By (ii) and Nahoum's theorem, $M_{b\,\operatorname{sgn}\mathcal{V}}$ is 2-summing, $\pi_2(M_{b\,\operatorname{sgn}\mathcal{V}}) \le \|b\|_2$, and by (ii), $M_{c\mathcal{J}}$ is bounded linear with $\|M_{c\mathcal{J}}\| = \|c\|_r$. Thus $M_{\mathcal{V}}$ is 2-summing and $\pi_2(M_{\mathcal{V}}) \le \|b\|_2\|c\|_r = \|a\|_p$, proving (i). ∎

Theorem 13. *Let $2 < p < \infty$ and $M_{\mathcal{V}} : l_2(\mathcal{X}) \to l_p(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_2$.*

Moreover, $\pi_2(M_{\mathcal{V}}) = (\sum_{n=1}^{\infty}[\pi_2(V_n)]^2)^{1/2}$.

Proof. (i)⇒(ii). Since $M_{\mathcal{V}}$ is 2-summing, from Theorem 3 in [13], all $M_{\mathcal{V}} \circ \sigma_n$ are 2-summing, and

$$\sum_{n=1}^{\infty}[\pi_2(M_{\mathcal{V}} \circ \sigma_n)]^2 < \infty \quad \text{and} \quad \sum_{n=1}^{\infty}[\pi_2(M_{\mathcal{V}} \circ \sigma_n)]^2 \le [\pi_2(M_{\mathcal{V}})]^2.$$

Since $M_{\mathcal{V}} \circ \sigma_n = \sigma_n \circ V_n$ we deduce that all V_n are 2-summing, $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_2$ and $(\sum_{n=1}^{\infty}[\pi_2(V_n)]^2)^{1/2} \le \pi_2(M_{\mathcal{V}})$.

(ii)⇒(i). From (ii) and Nahoum's theorem, $M_{\mathcal{V}} : l_2(\mathcal{X}) \to l_2(\mathcal{Y})$ is 2-summing with $\pi_2(M_{\mathcal{V}}) \le (\sum_{n=1}^{\infty}[\pi_2(V_n)]^2)^{1/2}$ and thus $M_{\mathcal{V}} : l_2(\mathcal{X}) \to l_p(\mathcal{Y})$ is 2-summing since $l_2(\mathcal{Y}) \hookrightarrow l_p(\mathcal{Y})$, $p \ge 2$. ■

The case $(1,p)$

THEOREM 14. *Let $1 < p < 2$ and $M_{\mathcal{V}} : l_1(\mathcal{X}) \to l_p(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*

(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_r$, where $1/p = 1/2 + 1/r$.*

Moreover, $\pi_2(M_{\mathcal{V}}) = (\sum_{n=1}^{\infty}[\pi_2(V_n)]^r)^{1/r}$.

Proof. (i)⇒(ii). Let $b = (b_n)_{n\in\mathbb{N}} \in l_2$. Then $M_{b\mathcal{J}} : l_2(\mathcal{X}) \to l_1(\mathcal{X})$ is bounded linear, thus by (i), $M_{b\mathcal{V}} = M_{\mathcal{V}} \circ M_{b\mathcal{J}} : l_2(\mathcal{X}) \to l_p(\mathcal{Y})$ is 2-summing and $\pi_2(M_{b\mathcal{V}}) \le \pi_2(M_{\mathcal{V}})\|b\|_2$. From the nontrivial part of Theorem 12, all b_nV_n are 2-summing, $(\pi_2(b_nV_n))_{n\in\mathbb{N}} \in l_p$ and

$$\Big(\sum_{n=1}^{\infty}[\pi_2(b_nV_n)]^p\Big)^{1/p} = \pi_2(M_{b\mathcal{V}}).$$

Then all V_n are 2-summing and

$$\Big(\sum_{n=1}^{\infty}|b_n|^p[\pi_2(V_n)]^p\Big)^{1/p} \le \pi_2(M_{\mathcal{V}})\|b\|_2.$$

As is well known, this implies that $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_r$ and

$$\Big(\sum_{n=1}^{\infty}[\pi_2(V_n)]^r\Big)^{1/r} = \sup_{\|b\|_2\le 1}\Big(\sum_{n=1}^{\infty}|b_n|^p[\pi_2(V_n)]^p\Big)^{1/p} \le \pi_2(M_{\mathcal{V}}).$$

(ii)⇒(i). We consider $a = (\pi_2(V_n))_{n\in\mathbb{N}} \in l_r$ and we note that

$$M_{\mathcal{V}} : l_1(\mathcal{X}) \xrightarrow{M_{\operatorname{sgn}\mathcal{V}}} l_2(\mathcal{Y}) \xrightarrow{M_{a\mathcal{J}}} l_p(\mathcal{Y})$$

is a factorization of $M_{\mathcal{V}}$. From (ii) and the nontrivial part of Theorem 3, $M_{\operatorname{sgn}\mathcal{V}}$ is 2-summing, $\pi_2(M_{\operatorname{sgn}\mathcal{V}}) \le 1$ and thus $M_{\mathcal{V}} : l_1(\mathcal{X}) \to l_p(\mathcal{Y})$ is 2-summing with $\pi_2(M_{\mathcal{V}}) \le \|a\|_r$, i.e. (i) holds. ■

THEOREM 15. *Let $2 \le p < \infty$ and $M_{\mathcal{V}} : l_1(\mathcal{X}) \to l_p(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_\infty$.*

Moreover, $\pi_2(M_{\mathcal{V}}) = \sup_{n\in\mathbb{N}} \pi_2(V_n)$.

Proof. (i)⇒(ii). Note that $V_n : X_n \xrightarrow{\sigma_n} l_1(\mathcal{X}) \xrightarrow{M_{\mathcal{V}}} l_p(\mathcal{Y}) \xrightarrow{p_n} Y_n$ is a factorization of V_n. From the ideal property, all V_n are 2-summing and $\pi_2(V_n) \le \pi_2(M_{\mathcal{V}})$ for $n \in \mathbb{N}$, i.e. (ii) holds.

(ii)⇒(i). From (ii) and the nontrivial part of Theorem 3, $M_{\mathcal{V}} : l_1(\mathcal{X}) \to l_2(\mathcal{Y})$ is 2-summing with $\sup_{n\in\mathbb{N}} \pi_2(V_n) \le \pi_2(M_{\mathcal{V}})$, and since $J : l_2(\mathcal{Y}) \hookrightarrow l_p(\mathcal{Y})$ $(2 \le p < \infty)$, $M_{\mathcal{V}} : l_1(\mathcal{X}) \to l_p(\mathcal{Y})$ is 2-summing with $\sup_{n\in\mathbb{N}} \pi_2(V_n) \le \pi_2(M_{\mathcal{V}})$. ■

The case (p, q), $1 < p < 2$ and $1 < q < \infty$. The next case is analogous to the case shown by L. Schwartz in [14, Théorème XXVI, 3.5, p. 15]; see also [4, Theorem 1(iii)].

THEOREM 16. *Let $1 < p < 2$, $1 < q < 2$ and $M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_q(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_s$, where $1/s = 1/q - 1/p + 1/2$.*

Moreover, $\pi_2(M_{\mathcal{V}}) = (\sum_{n=1}^{\infty} [\pi_2(V_n)]^s)^{1/s}$.

Proof. (i)⇒(ii). Define r by $1/p = 1/2 + 1/r$ and note that $2 < r < \infty$. Let $a = (a_n)_{n\in\mathbb{N}} \in l_r$. Since $M_{\mathcal{V}}$ is 2-summing, $M_{a\mathcal{V}} = M_{\mathcal{V}} \circ M_{a\mathcal{J}} : l_2(\mathcal{X}) \to l_q(\mathcal{Y})$ is 2-summing and $\pi_2(M_{a\mathcal{V}}) \le \pi_2(M_{\mathcal{V}})\|a\|_r$. From the nontrivial part of Theorem 12, all a_nV_n are 2-summing, and

$$\Big(\sum_{n=1}^{\infty} [\pi_2(a_nV_n)]^q\Big)^{1/q} = \pi_2(M_{a\mathcal{V}}) \le \pi_2(M_{\mathcal{V}})\|a\|_r.$$

Thus V_n are 2-summing and for each $a = (a_n)_{n\in\mathbb{N}} \in l_r$,

$$\Big(\sum_{n=1}^{\infty} |a_n|^q [\pi_2(V_n)]^q\Big)^{1/q} \le \pi_2(M_{\mathcal{V}})\|a\|_r.$$

Since $q < 2 < r$, as is well known, $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_s$ and

$$\Big(\sum_{n=1}^{\infty} [\pi_2(V_n)]^s\Big)^{1/s} = \sup_{\|a\|_r \le 1} \Big(\sum_{n=1}^{\infty} |a_n|^q [\pi_2(V_n)]^q\Big)^{1/q} \le \pi_2(M_{\mathcal{V}}),$$

where $1/s = 1/q - 1/r$, i.e. $1/s = 1/q - 1/p + 1/2$.

(ii)⇒(i). We write $a = (\pi_2(V_n))_{n\in\mathbb{N}} \in l_s$. Since $1/s = 1/q + 1/p^* - 1/2$, define $1/v = 1/q - 1/2$ and note that $1/s = 1/p^* + 1/v$. Now define $b_n = [\pi_2(V_n)]^{s/p^*}$, $c_n = [\pi_2(V_n)]^{s/v}$. Then $b = (b_n)_{n\in\mathbb{N}} \in l_{p^*}$, $c = (c_n)_{n\in\mathbb{N}} \in l_v$,

$1/q = 1/2 + 1/v$ and since $bc(\operatorname{sgn}\mathcal{V}) = \mathcal{V}$, we get the factorization

$$M_{\mathcal{V}} : l_p(\mathcal{X}) \xrightarrow{M_{b\mathcal{J}}} l_1(\mathcal{X}) \xrightarrow{M_{\operatorname{sgn}\mathcal{V}}} l_2(\mathcal{Y}) \xrightarrow{M_{c\mathcal{J}}} l_q(\mathcal{Y}).$$

By (ii) and the nontrivial part of Theorem 3, $M_{\operatorname{sgn}\mathcal{V}}$ is 2-summing with $\pi_2(M_{\operatorname{sgn}\mathcal{V}}) \le 1$, and by (ii), $M_{b\mathcal{J}}$ is bounded linear with $\|M_{b\mathcal{J}}\| = \|b\|_{p^*}$, while $M_{c\mathcal{J}}$ is bounded linear with $\|M_{c\mathcal{J}}\| = \|c\|_v$. Thus $M_{\mathcal{V}}$ is 2-summing and $\pi_2(M_{\mathcal{V}}) \le \|b\|_{p^*}\|c\|_v = \|a\|_u$, i.e. (i) holds. ∎

THEOREM 17. *Let $1 < p < 2 \le q < \infty$ and $M_{\mathcal{V}} : l_p(\mathcal{X}) \to l_q(\mathcal{Y})$. The following assertions are equivalent:*

(i) *$M_{\mathcal{V}}$ is 2-summing.*
(ii) *All V_n are 2-summing and $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_{p^*}$.*

Moreover, $\pi_2(M_{\mathcal{V}}) = (\sum_{n=1}^{\infty}[\pi_2(V_n)]^{p^*})^{1/p^*}$.

Proof. (i)⇒(ii). Define r by $1/p = 1/2 + 1/r$ and note that $2 < r < \infty$. Let $a = (a_n)_{n\in\mathbb{N}} \in l_r$. Since $M_{\mathcal{V}}$ is 2-summing, $M_{a\mathcal{V}} = M_{\mathcal{V}} \circ M_{a\mathcal{J}} : l_2(\mathcal{X}) \to l_q(\mathcal{Y})$ is 2-summing and $\pi_2(M_{a\mathcal{V}}) \le \pi_2(M_{\mathcal{V}})\|a\|_r$. From the nontrivial part of Theorem 13, all a_nV_n are 2-summing and

$$\Big(\sum_{n=1}^{\infty}[\pi_2(a_nV_n)]^2\Big)^{1/2} \le \pi_2(M_{\mathcal{V}})\|a\|_r.$$

Thus V_n are 2-summing and for each $a = (a_n)_{n\in\mathbb{N}} \in l_r$,

$$\Big(\sum_{n=1}^{\infty}|a_n|^2[\pi_2(V_n)]^2\Big)^{1/2} \le \pi_2(M_{\mathcal{V}})\|a\|_r.$$

Then, since $1/2 = 1/r + 1/p^*$, we deduce $(\pi_2(V_n))_{n\in\mathbb{N}} \in l_{p^*}$ and

$$\Big(\sum_{n=1}^{\infty}[\pi_2(V_n)]^{p^*}\Big)^{1/p^*} = \sup_{\|a\|_r\le 1}\Big(\sum_{n=1}^{\infty}|a_n|^2[\pi_2(V_n)]^2\Big)^{1/2} \le \pi_2(M_{\mathcal{V}}).$$

(ii)⇒(i). Write $a = (\pi_2(V_n))_{n\in\mathbb{N}} \in l_{p^*}$. Then $M_{\mathcal{V}}$ has the factorization

$$M_{\mathcal{V}} : l_p(\mathcal{X}) \xrightarrow{M_{a\mathcal{J}}} l_1(\mathcal{X}) \xrightarrow{M_{\operatorname{sgn}\mathcal{V}}} l_2(\mathcal{Y}) \hookrightarrow l_q(\mathcal{Y}).$$

By (ii) and the nontrivial part of Theorem 3, $M_{\operatorname{sgn}\mathcal{V}}$ is 2-summing with $\pi_2(M_{\operatorname{sgn}\mathcal{V}}) \le 1$, and by (ii), $M_{a\mathcal{J}}$ is bounded linear with $\|M_{a\mathcal{J}}\| = \|a\|_{p^*}$. Thus $M_{\mathcal{V}}$ is 2-summing and $\pi_2(M_{\mathcal{V}}) \le \|a\|_{p^*}$, i.e. (i) holds. ∎

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