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Semiconjugacy to a map of a constant slope

by

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Abstract. It is well known that any continuous piecewise monotone interval map f with positive topological entropy $h_{\mathrm{top}}(f)$ is semiconjugate to some piecewise affine map with constant slope $e^{h_{\mathrm{top}}(f)}$. We prove this result for a class of Markov *countably* piecewise monotone continuous interval maps.

1. Introduction. Let us consider continuous maps $f: X \to X$ and $g: Y \to Y$, where X, Y are compact Hausdorff spaces and $\varphi: X \to Y$ is continuous such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{f} & X \\ \varphi\downarrow & & \downarrow\varphi \\ Y & \xrightarrow{g} & Y \end{array}$$

commutes, i.e., $\varphi \circ f = g \circ \varphi$. When φ is surjective, we say that f is *semiconjugate* to g via the map φ and in that case the topological entropy $h_{\mathrm{top}}(\cdot)$ satisfies $h_{\mathrm{top}}(f) \geq h_{\mathrm{top}}(g)$ [1].

Let $X = Y = [0,1]$. A continuous map $f: [0,1] \to [0,1]$ is said to be *piecewise monotone* if there are $k \in \mathbb{N}$ and points $0 = c_0 < c_1 < \cdots < c_k = 1$ such that f is monotone on each $[c_i, c_{i+1}]$, $i = 0, \ldots, k-1$. We shall say that a piecewise monotone map g has a *constant slope* s if on each of its pieces of monotonicity it is affine with slope of absolute value s.

In one-dimensional dynamical systems the following interesting result has been proved.

THEOREM 1.1 ([6], [9]). *If f is piecewise monotone and $h_{\mathrm{top}}(f) > 0$ then f is semiconjugate via a nondecreasing map to some map g of constant slope $e^{h_{\mathrm{top}}(f)}$.*

It is known that if g has constant slope s then $h_{\mathrm{top}}(g) = \max(0, \log s)$ [8]. Thus, the slope of g from Theorem 1.1 is maximal possible, i.e., when a

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nondecreasing semiconjugacy φ collapses intervals to points we do not lose any information measurable by the entropy. In this paper we focus on the class of Markov *countably* piecewise monotone continuous interval maps and find a large subclass of it in which the conclusion of Theorem 1.1 remains true.

Some of the notions used are recalled in the Appendix.

2. General observations. An *admissible set* P is a finite or countably infinite closed subset of $[0,1]$ containing the points $0,1$. An interval $[a,b] \subset [0,1]$ is *P-basic* if $a,b \in P$ and $(a,b) \cap P = \emptyset$. The set of all P-basic intervals will be denoted by $B(P)$.

A continuous map $f\colon [0,1] \to [0,1]$ is in the *class $\mathcal{CPM}$* if there is an admissible set P such that $f(P) \subset P$ and f is monotone (perhaps constant) on each P-basic interval. A map $f \in \mathcal{CPM}$ which is not piecewise monotone will be called *countably piecewise monotone*.

For P admissible, we denote by $\mathcal{M}_P$ the set of all (possibly generalized, multi-infinite) matrices indexed by P-basic intervals and with entries from $[0,\infty]$. Also we denote by ℓ^1_P the Banach space of all real absolutely convergent (again possibly multi-infinite) sequences indexed by P-basic intervals, i.e.,

$$\ell^1_P = \Big\{u = (u_I)_{I\in B(P)}\colon \sum_{I\in B(P)} |u_I| < \infty\Big\}. \tag{2.1}$$

The cone of all nonnegative sequences from ℓ^1_P is denoted by $\mathcal{K}^+_P$.

Remark 2.1. For an admissible set P, a matrix $M \in \mathcal{M}_P$ can be modeled as a table $(P \times [0,1]) \cup ([0,1] \times (1-P))$; an entry of M is a number from $[0,\infty]$ in one window $I \times J$, where $I \in B(1-P)$ and $J \in B(P)$. Let us denote by P' the set of all limit points of P. In accordance with the above model, a matrix $M \in \mathcal{M}_P$ will be infinite in the usual sense if $P' = \{1\}$. We call it multi-infinite when $\operatorname{card} P' > 1$. For example, for the choice $P = \{0\} \cup \{\frac{1}{2^m} + \frac{1}{2^n}\}_{m,n\ge 1}$ we get $\operatorname{card} P' = \infty$.

Proposition 2.2. *Let $M = (m_{IJ}) \in \mathcal{M}_P$. Then*

(i) *M represents a bounded linear operator $\mathbb{M}$ on ℓ^1_P defined as*

$$(\mathbb{M}u)_I := \sum_{J\in B(P)} m_{IJ}u_J, \qquad u \in \ell^1_P, \tag{2.2}$$

if and only if $(\|\mathbb{M}\| =)\ \sup_{J\in B(P)} \sum_{I\in B(P)} |m_{IJ}| < \infty$. *In that case the operator $\mathbb{M}$ is $\mathcal{K}^+_P$-positive.*

(ii) *The operator* $\mathbb{M}$ *is compact if and only if its representing matrix* M *satisfies*

$$\forall \varepsilon > 0\ \exists \delta\ \forall J \in B(P): \quad \sum_{|I|<\delta} |m_{IJ}| < \varepsilon.$$

Proof. Recall that the entries of M are from $[0, \infty]$. The result is well known if the set of limit points of the admissible set P equals $\{1\}$ [10, Sec. 4.51, 5.5]. In the general case the arguments are completely analogous. ∎

Definition 2.3. For $f \in \mathcal{CPM}$ we define its matrix $M(f) \in \mathcal{M}_P$: the m_{IJ} entry of $M(f)$ is 1 if $f(I) \supset J$, and 0 otherwise.

It can be easily seen with the help of Proposition 2.2 that for some f from $\mathcal{CPM}$ (in fact for many of them) the matrix $M(f)$ does not represent a (bounded) operator on ℓ^1_P.

Using the formal equality $0 \cdot \infty = 0$ we can consider the product of nonnegative matrices from $\mathcal{M}_P$: given such $L, M \in \mathcal{M}_P$ we put

$$LM = N \in \mathcal{M}_P, \quad n_{IK} = \sum_{J \in B(P)} l_{IJ} m_{JK}. \tag{2.3}$$

Proposition 2.4. *Let* $M(f) = (m_{IJ}) \in \mathcal{M}_P$ *be the matrix of* $f \in \mathcal{CPM}$. *Then*

(i) *for each* $k \in \mathbb{N}$ *and* $I, J \in B(P)$, *the entry* m^k_{IJ} *of* $M^k(f)$ *is finite,*
(ii) *the entry* m^k_{IJ} *of* $M^k(f)$ *equals* m *if and only if there are closed subintervals* $J_1, \dots, J_m$ *of* I *with pairwise disjoint interiors such that* $f^k(J_i) \supset J$.

Proof. (i) From the continuity of f it follows that $\sum_{I \in B(P)} m_{IJ}$ is finite for each $J \in B(P)$, which directly implies (i). We prove (ii) by induction. For $k = 1$ this is Definition 2.3 of $M(f)$. The induction step follows immediately from the definition of the product of the nonnegative matrices $M(f)$ and $M(f)^{k-1}$ from $\mathcal{M}_P$. ∎

Despite the fact that not every matrix $M(f)$ with $f \in \mathcal{CPM}$ represents a (bounded) operator on ℓ^1_P, we can prove the following general result that can justify further research.

Denote the class of all maps from $\mathcal{CPM}$ of constant slope λ by $\mathcal{CPM}_\lambda$, i.e., $f \in \mathcal{CPM}_\lambda$ if $|f'(x)| = \lambda$ for all $x \in [0, 1]$, possibly except at the points of P. For a continuous nondecreasing map $\psi\colon [0, 1] \to [0, 1]$ its *support* $\mathrm{supp}(\psi)$ is defined as

$$\mathrm{supp}(\psi) = \{x \in [0, 1] \colon \psi(x - \varepsilon, x + \varepsilon) \text{ is nondegenerate for each } \varepsilon > 0\}.$$

Theorem 2.5. *Let* $f \in \mathcal{CPM}$ *with* $M(f) = (m_{IJ}) \in \mathcal{M}_P$. *Then* f *is semiconjugate via a continuous nondecreasing map* ψ *to some map*

$g \in \mathcal{CPM}_\lambda$, $\lambda > 1$, if and only if there is a nonzero vector $v = (v_I)_{I \in B(P)}$ from $\mathcal{K}_P^+$ such that

$$\forall I \in B(P): \quad \sum_{J \in B(P)} m_{IJ} v_J = \lambda v_I. \tag{2.4}$$

Proof. To simplify the technicalities as much as possible we will assume that f is countably piecewise monotone.

In order to show that the condition (2.4) is necessary, assume that for some nondecreasing map ψ and $g \in \mathcal{CPM}_\lambda$,

$$\psi \circ f = g \circ \psi. \tag{2.5}$$

Using the notation $\psi[a, b] := \psi(b) - \psi(a)$ define $v = (v_I)_{I \in B(P)} \in \mathcal{K}_P^+$ by $v_I = \psi[I]$. Assuming $\psi[I] \neq 0$ for $I \in B(P)$, with the help of (2.5) and the definition of $M(f)$ we can write

$$\sum_{J \in B(P)} m_{IJ} v_J = \sum_{J \subset f(I)} \psi[J] = \psi[f(I)] = \lambda \psi[I] = \lambda v_I, \tag{2.6}$$

since if I is a P-basic interval for f then $\psi(I)$ is a $\psi(P)$-basic interval for g. Clearly the equalities (2.6) hold true also when $\psi[I] = v_I = 0$.

Let us prove that the condition (2.4) is sufficient. Our proof is a modified version of the one for piecewise monotone maps in [1, Lemma 4.6.5].

We may assume that f is not constant on any P-basic interval. If f is constant on some P-basic intervals, we can replace f by a map h constructed as follows.

Let f correspond to an admissible set P and call a P-basic interval I *f-vanishing* if $v_I = 0$ (for example, this is true when f is constant on I). More generally, for $u, v \in P$, $u < v$, a block

$$[u, v]_{B(P)} = \{I \in B(P) \colon I \subset [u, v]\}$$

is f-vanishing if it consists of f-vanishing P-basic intervals. Let H denote the union of the interiors of all f-vanishing P-basic intervals. The characteristic function $\chi_{[0,1]\setminus H}$ is defined as usual by

$$\chi_{[0,1]\setminus H}(x) = \begin{cases} 1, & x \notin H, \\ 0, & x \in H. \end{cases}$$

The map $\psi_1 \colon [0, 1] \to [0, 1]$ given by

$$\psi_1(x) = \int_0^x \chi_{[0,1]\setminus H}(t)\, dt$$

is continuous, nondecreasing and $\operatorname{supp}(\psi_1) = [0, 1] \setminus H$. In particular, ψ_1 is increasing if and only if $H = \emptyset$. We have assumed that $v \in \mathcal{K}_P^+$, is nonzero, which implies that $\psi_1([0, 1])$ is not degenerate. Notice that by (2.4) if $v_I = 0$

and $m_{IJ} = 1$ then $v_J = 0$, i.e., if $I \in B(P)$ is f-vanishing and $f(I)$ contains a P-basic interval J then J is also f-vanishing. More generally, if a block $[u, v]_{B(P)}$ is f-vanishing then so is the block $f([u, v])_{B(P)}$. This property together with the continuity of ψ_1 and the countability of P implies that if $\psi_1|C$ is constant, so is $\psi_1|f(C)$. Thus, the map $h\colon [0, \psi_1(1)] \to [0, \psi_1(1)]$ satisfying

$$\psi_1 \circ f = h \circ \psi_1 \quad \text{ on } [0, 1]$$

is (uniquely) well defined; it is also continuous by Proposition 4.4 (see Appendix). We can assume that $h\colon [0, 1] \to [0, 1]$ by rescaling (by an affine conjugacy). Obviously $h \in \mathcal{CPM}$ and its matrix $M(h) \in \mathcal{M}_{\psi_1(P)}$ arises from $M(f)$ by omitting the rows and columns corresponding to all f-vanishing P-basic intervals. If we denote by $u = (u_I)_{I\in B(\psi_1(P))}$ the vector from $\mathcal{K}^+_{\psi_1(P)} \subset \ell^1_{\psi_1(P)}$ obtained from $v = (v_I)_{I\in B(P)}$ by omitting the coordinates corresponding to the f-vanishing P-basic intervals, we clearly obtain

$$\forall I \in B(\psi_1(P)) : \qquad \sum_{J\in B(\psi_1(P))} m_{IJ} u_J = \lambda u_I.$$

It can be easily seen that h is not constant on any $\psi_1(P)$-basic interval (we do not claim that h is strictly monotone on basic intervals).

We will need a genealogical tree $(P_n)_{n=0}^{\infty}$ of P with respect to f (see [1, p. 64]). It is defined inductively as follows.

We set $P_0 = P$. By the above, f is not constant on any P_0-basic interval.

Suppose that P_n is already defined and f is not constant on any P_n-basic intervals. Since f is countably piecewise monotone, $f^{-1}(P_n)\cap[0, 1]$ is a union of (at most) countably many closed intervals (perhaps degenerate). Since f was not constant on any P_n-basic interval, no component of $f^{-1}(P_n) \cap [0, 1]$ contains more than one element of P_n. From each of these components we choose one point, if possible an element of P_n, and we define P_{n+1} to be the set of those chosen points. Thus $P_n \subset P_{n+1}$ and P_{n+1} is invariant since $f(P_{n+1}) \subset P_n$. By construction, P_{n+1} is a countable set and f is not constant on any P_{n+1}-basic interval.

Denote by $\mathcal{J}_n$ the set of all P_n-basic intervals. In particular, $\mathcal{J}_0 = B(P)$. Let $v = (v_I)_{I\in B(P)} \in \mathcal{K}^+_P$ be a normalized vector satisfying (2.4).

In order to define the map $\psi\colon Q = \bigcup_{n=0}^{\infty} P_n \to [0, 1]$ we put $\psi(0) = 0$ and for $x \in P_n \cap (0, 1]$,

$$\psi(x) = \lambda^{-n} \sum_{J\in\mathcal{J}_n,\, J\le x} v_{f^n(J)}. \tag{2.7}$$

Since for any fixed $K \in \mathcal{J}_n$, $f^n|K$ is monotone and $f^n(K) \in \mathcal{B}(P)$, using (2.4) one gets

$$(2.8)\qquad \lambda^{-n-1}\sum_{J\in\mathcal{J}_{n+1},\, J\subset K} v_{f^{n+1}(J)} = \lambda^{-n-1}\sum_{J\in\mathcal{J}_1,\, J\subset f^n(K)} v_{f(J)}$$
$$= \lambda^{-n-1}\sum_{J\in B(P)} m_{f^n(K)J}v_J = \lambda^{-n-1}\lambda v_{f^n(K)} = \lambda^{-n}v_{f^n(K)},$$

hence also

$$\sum_{J\in\mathcal{J}_n} v_{f^n(J)} = \lambda\sum_{J\in\mathcal{J}_{n-1}} v_{f^{n-1}(J)} = \lambda^2\sum_{J\in\mathcal{J}_{n-2}} v_{f^{n-2}(J)}$$
$$= \cdots = \lambda^n\sum_{J\in\mathcal{J}_0} v_J = \lambda^n.$$

This shows that the map ψ is well defined, nondecreasing and $\psi(1) = 1$. From the fact that v is normalized and (2.7) we get

$$(2.9)\qquad \sup_{[x,y]\in\mathcal{J}_n} |\psi(x)-\psi(y)| \le \lambda^{-n}.$$

Moreover, if $x\in P_n$ is a left limit point of P_n then

$$\lim_{\varepsilon\to 0_+}\lambda^{-n}\sum_{J\in\mathcal{J}_n\cap(x-\varepsilon,x)} v_{f^n(J)} = 0$$

and similarly for x being a right limit point of P_n. Let $x\in(0,1)\setminus Q$. Then for each n there is an interval $J = J(n)$ such that $x\in J\in\mathcal{J}_n$. From the above properties of ψ we obtain

$$\lim_{x\to 0_+,\, x\in Q}\psi(x) = 0, \qquad \lim_{x\to 1_-,\, x\in Q}\psi(x) = 1,$$

and for each $x\in(0,1)$,

$$\sup_{y<x,\, y\in Q}\psi(y) = \inf_{y>x,\, y\in Q}\psi(y).$$

Thus ψ can be continuously extended from Q to the whole interval $[0,1]$, constant on the components of $[0,1]\setminus Q$.

Claim 2.6. *For any $x,y\in[0,1]$ such that f is monotone on $[x,y]$,*

$$(2.10)\qquad |\psi(f(y))-\psi(f(x))| = \lambda|\psi(y)-\psi(x)|.$$

Proof. Let $[x,y]\in\mathcal{J}_n$. Then as in (2.8),

$$|\psi(f(y))-\psi(f(x))| = \sum_{J\in\mathcal{J}_1,\, J\subset f^n([x,y])} v_{f(J)}$$
$$= \lambda v_{f^n([x,y])} = \lambda|\psi(y)-\psi(x)|.$$

By taking sums and limits, we obtain (2.10) for every $x,y\in\overline{Q}$ such that f is monotone on $[x,y]$. If $z<w$ and $Q\cap(z,w)=\emptyset$ then f is monotone on $[z,w]$ and by the construction of the genealogical tree of P and Q also $Q\cap f((z,w))=\emptyset$. The map ψ is constant on each component of $[0,1]\setminus Q$.

Hence $\psi(z) = \psi(w)$ and $\psi(f(z)) = \psi(f(w))$. This proves that (2.10) holds for all $x, y \in [0, 1]$ with f monotone on $[x, y]$. ∎

Let us define g. If $z \in [0, 1]$ then $\psi^{-1}(z)$ is a closed interval. It contains countably many subintervals J_k satisfying

- f is monotone on each J_k,
- $f(\psi^{-1}(z)) \setminus \bigcup_k f(J_k)$ is countable.

By (2.10), the map ψ is constant on each image $f(J_k)$, so by the second property, ψ is constant on the whole $f(\psi^{-1}(z))$. This means that $f(\psi^{-1}(z)) \subset \psi^{-1}(w)$ for some $w \in [0, 1]$. We then set $g(z) = w$. For every $v \in \psi^{-1}(z)$ we have $\psi(f(v)) = g(\psi(v)) = w$. This shows that with our definition of g we get $\psi \circ f = g \circ \psi$ on $[0, 1]$, hence Proposition 4.4 and (2.10) imply $g \in \mathcal{CPM}_\lambda$. ∎

Remark 2.7. In Theorem 2.5, if f is transitive then ψ is increasing, i.e., f is *conjugate* to g (see [1, Proposition 4.6.9]).

Example 2.8. In order to illustrate Theorem 2.5 put $P = \{1\} \cup \{x_n = 1 - 1/n\}_{n \geq 1}$ with P-basic interval $I(n) = [x_n, x_{n+1}]$ and consider a map f from $\mathcal{CPM}$ such that $f(x_2) = x_1 = 0$ and

$$f(x_n) = \begin{cases} 1, & n \geq 1 \text{ odd}, \\ x_{n-2}, & n \geq 4 \text{ even}. \end{cases}$$

Then

$$M(f) = \begin{pmatrix}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
0&0&0&0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
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\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot\\
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\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{pmatrix}.$$

Put $v = (v_{I(n)})_{I(n) \in B(P)}$ with

$$v_{I(2k+1)} = v_{I(2k+2)} = \frac{k+1}{\lambda} \left(\frac{\lambda - 1}{2\lambda} \right)^k, \qquad k \geq 0,$$

and $\lambda = 3+\sqrt{8}$. The reader can directly verify that $v \in \mathcal{K}_P^+$, v is normalized and the condition (2.4) is fulfilled. Thus, by Theorem 2.5 our map f is semiconjugate via a nondecreasing map ψ to some map $g \in \mathcal{CPM}_{3+\sqrt{8}}$ (in fact one can show that $h_{\mathrm{top}}(f) = \log(3+\sqrt{8})$).

From Proposition 2.2 it follows that the matrix $M(f)$ does not represent a bounded linear operator on ℓ_P^1.

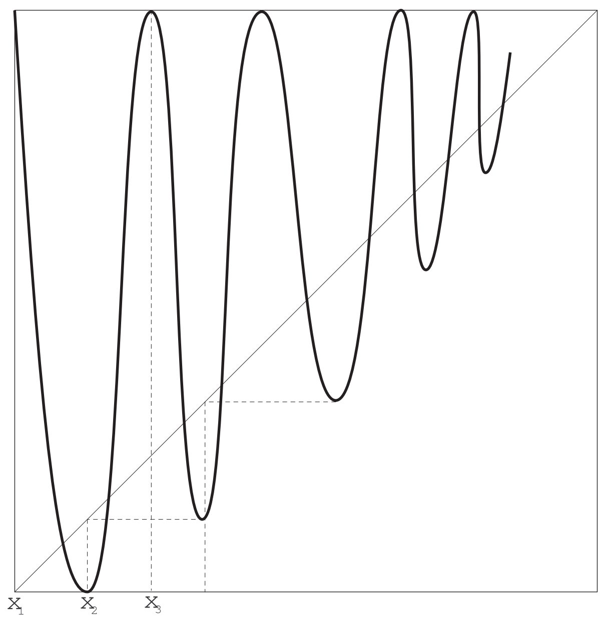

Fig. 1. A transitive map $f \in \mathcal{CPM}$ from Example 2.8

3. Case of bounded operators. In order to use Theorem 2.5 effectively we restrict our attention to a (still sufficiently rich) subclass of maps from $\mathcal{CPM}$.

To this end denote by $\mathcal{P}$ the set of all pairs (P,φ) such that

(A1) P is admissible,
(A2) $\varphi\colon P \to P$ is continuous,
(A3) the continuous 'connect-the-dots' map $\varphi_P\colon [0,1] \to [0,1]$ defined by $\varphi_P|_P = \varphi$, $\varphi_P|_J$ affine for any interval $J \subset \mathrm{conv}(P)$ such that $J \cap P = \emptyset$, satisfies

$$\exists L = L(P,\varphi) > 0 \ \forall I \in B(P) \ \forall y \in I^\circ : \quad \mathrm{card}\,\varphi_P^{-1}(y) < L.$$

In this section we will deal with *restrictively countably piecewise monotone continuous maps* from the class $\mathcal{RCPM}$, where $f \in \mathcal{RCPM}$ if and only if it corresponds to some pair $(P,\varphi)\in\mathcal{P}$, i.e., $f|P=\varphi$ and f is monotone on each P-basic interval.

Below, for each $f\in\mathcal{RCPM}$ we consider the matrix $M(f)$ of Definition 2.3.

PROPOSITION 3.1. *Let $f\in\mathcal{RCPM}$. Then $M(f)$ represents a bounded $\mathcal{K}_P^+$-positive linear operator on ℓ_P^1.*

Proof. Let $M(f)=(m_{IJ})$, and let $L=L(P,\varphi)$ be as in (A3). Then each column $(m_{IJ})_{I\in B(P)}$ contains at most L units, hence by Proposition 2.2(i), $M(f)$ represents an operator $\mathbb{M}$ satisfying $\|\mathbb{M}\|\le L$. ∎

PROPOSITION 3.2. *Let $f\in\mathcal{RCPM}$. Then for each $k\in\mathbb{N}$ and $I,J\in B(P)$ the entry m_{IJ}^k of $M^k(f)$ is less than or equal to L^{k-1}, where $L=L(P,\varphi)$ is the constant given by* (A3).

Proof. We leave the proof to the reader. ∎

Using Theorem 4.6 below for the operator $\mathbb{M}$ represented by a matrix $M(f)$ associated to $f\in\mathcal{RCPM}$, we find that $r(\mathbb{M})\in\sigma(\mathbb{M})$. Up to now we have no information on the relationship of the entropy of $f\in\mathcal{RCPM}$ and its spectral radius $r(\mathbb{M})$. This is provided by the following theorem.

THEOREM 3.3. *Let $f\in\mathcal{RCPM}$, denote by $\mathbb{M}$ the operator on ℓ_P^1 represented by $M(f)$, and assume that $h_{\mathrm{top}}(f)>0$. Then $r(\mathbb{M})\ge e^{h_{\mathrm{top}}(f)}$.*

Proof. This is a consequence of Theorem 4.3. In what follows we repeatedly use the following: if some continuous map g satisfies $g([a,b])\supset[c,d]$ with $[a,b],[c,d]$ compact intervals, then $[a,b]$ is g-optimal for $[c,d]$ if $g((a,b))=(c,d)$. Then either $g(a)=c$ and $g(b)=d$ (increasing type) or $g(a)=d$ and $g(b)=c$ (decreasing type).

So let closed intervals $J_1\le\cdots\le J_{s_n}$ with pairwise disjoint interiors create an s_n-horseshoe of f^{k_n}, i.e.,

$$f^{k_n}(J_i)\supset\bigcup_{i=1}^{s_n}J_i\quad\text{ for each } i\in\{1,\dots,s_n\}.\tag{3.1}$$

We can assume that each J_i is f^{k_n}-optimal for the interval $[\min J_1,\max J_{s_n}]$. Since each J_i has its monotonicity type and the map f^{k_n} has its local extrema in f^{k_n}-preimages of P, each J_i can be enlarged (if necessary) to satisfy

- (3.1),
- $\min\bigcup J_i,\max\bigcup J_i\in P$,
- $J_1\le\cdots\le J_{s_n}$,
- J_i is optimal for $[\min\bigcup J_i,\max\bigcup J_i]$.

Clearly this means that there is a P-basic interval K such that $f^{k_n}(J_i) \supset K$ for each i and it enables us to consider for each i a P_{k_n}-basic interval $L_i \subset J_i$ such that L_i is f^{k_n}-optimal for K. Notice that

- each L_i lies in some P-basic interval,
- two neighbors L_i, L_{i+1} can/need not lie in the same P-basic interval,
- $L_1 \le \cdots \le L_{s_n}$,

In any case, using Proposition 2.4(ii) we obtain

$$\sum_{I \in B(P)} m_{IK}^{k_n} \ge s_n. \tag{3.2}$$

From (3.2) and Proposition 2.2(i) we get

$$\|\mathbb{M}^{k_n}\| \ge s_n,$$

hence Gelfand's formula gives

$$r(\mathbb{M}) = \lim_{n\to\infty} \|\mathbb{M}^{k_n}\|^{1/k_n} \ge \lim_{n\to\infty} s_n^{1/k_n} = e^{h_{\mathrm{top}}(f)}. \ \blacksquare$$

In Appendix we introduce the Radon–Nikol'skiĭ operators. For that class of operators we can say even more.

Theorem 3.4. *Let $f \in \mathcal{RCPM}$, denote by $\mathbb{M}$ the operator on ℓ_P^1 represented by $M(f)$ and assume that $h_{\mathrm{top}}(f) > 0$. If $\tau(\mathbb{M})$ is a Radon–Nikol'skiĭ operator on ℓ_P^1 for some τ holomorphic in the neighborhood of the spectrum $\sigma(\mathbb{M})$ then $r(\mathbb{M}) = e^{h_{\mathrm{top}}(f)}$.*

Proof. Fix $\varepsilon > 0$. Let $\mathbb{M}_\delta$ be the operator represented by the matrix $M_\delta = (m_{IJ}(\delta)) \in \mathcal{M}_P$ from Theorem 4.9. By that theorem for some sufficiently small $\delta > 0$,

$$r(\mathbb{M}_\delta) > r(\mathbb{M}) - \varepsilon. \tag{3.3}$$

Since by (4.1) the matrix M_δ contains only finitely many nonzero elements, just as for every finite matrix [1, Lemma 4.4.2] we get

$$r(\mathbb{M}_\delta) = \lim_{k\to\infty} \sqrt[k]{\sum_{I\in B(P)} m_{II}^k(\delta)} = \lim_{k\to\infty} \sqrt[k]{\sum_{n=1}^{\ell} m_{I_nI_n}^k(\delta)}, \tag{3.4}$$

where the matrix $M_\delta^k = (m_{IJ}^k(\delta)) \in \mathcal{M}_P$ represents the kth power $\mathbb{M}_\delta^k$ of the operator $\mathbb{M}_\delta$ and the P-basic intervals $I_1, \dots, I_\ell$ do not depend on k. From (3.4), (3.3) for some $j \in \{1, \dots, \ell\}$ and a sufficiently large k we obtain $\ell < (1+\varepsilon)^k$ and

$$\sqrt[k]{\ell \cdot m_{I_jI_j}^k(\delta)} \ge \sqrt[k]{\sum_{n=1}^{\ell} m_{I_nI_n}^k(\delta)} > r(\mathbb{M}) - \varepsilon, \tag{3.5}$$

hence

$$m = m_{I_jI_j}^k(\delta) \geq \left(\frac{1}{1+\varepsilon}\sqrt[k]{\sum_{n=1}^{\ell} m_{I_nI_n}^k(\delta)}\right)^k$$
$$> \left(\frac{r(\mathbb{M})-\varepsilon}{1+\varepsilon}\right)^k.$$

Using Proposition 3.2(ii) one can see that there are closed subintervals $J_1, \ldots, J_m$ of I_j with pairwise disjoint interiors such that $f^k(J_i) \supset I_j$, i.e., the map f^k has an m-horseshoe; hence by Proposition 4.2,

$$e^{h_{\mathrm{top}}(f)} \geq \sqrt[k]{m} > \frac{r(\mathbb{M})-\varepsilon}{1+\varepsilon}.$$

Since ε was arbitrary, we have

$$e^{h_{\mathrm{top}}(f)} \geq r(\mathbb{M}).$$

The opposite inequality follows from Theorem 3.3. ■

The previous results lead to the following theorem.

THEOREM 3.5. *Let $f \in \mathcal{RCPM}$, denote by $\mathbb{M}$ the operator on ℓ_P^1 represented by $M(f)$, and assume that $h_{\mathrm{top}}(f) = \log\beta > 0$. If $\tau(\mathbb{M})$ is a Radon–Nikol'skiĭ operator on ℓ_P^1 for a suitable τ holomorphic in the neighborhood of the spectrum $\sigma(\mathbb{M})$ then f is semiconjugate via a nondecreasing map ψ to some map $g \in \mathcal{RCPM}_\beta$. In particular this is so when $\mathbb{M}$ itself is a Radon–Nikol'skiĭ operator.*

Proof. By our assumptions Theorem 4.8 can be applied to the operator $\mathbb{M}$, the real Banach space ℓ_P^1 and the cone $\mathcal{K}_P^+$. By that theorem $r(\mathbb{M}) \in P_\sigma(\mathbb{M})$ with corresponding eigenvector in $\mathcal{K}_P^+$. Since from Theorem 3.4 we get $r(\mathbb{M}) = e^{h_{\mathrm{top}}(f)} = \beta > 1$, our conclusion follows from Theorem 2.5 with $\lambda = \beta$. ■

DEFINITION 3.6. For an integer $m > 1$, we say that a pair $(P, \varphi) \in \mathcal{P}$ is *m-ruled* if there are P-basic intervals $I_1, \ldots, I_m$ such that

- $\varphi_P\colon [0,1] \to [0,1]$ has an m-horseshoe created by the intervals $I_1, \ldots, I_m$ (see Definition 4.1),
- $\forall I \in B(P)\ \forall y \in I^\circ : \mathrm{card}[\varphi_P^{-1}(y) \cap ([0,1] \setminus \bigcup_{i=1}^m I_i)] < m$.

THEOREM 3.7. *Let $f \in \mathcal{RCPM}$ correspond to an m-ruled pair $(P, \varphi) \in \mathcal{P}$. Then f is semiconjugate via a nondecreasing map ψ to some map $g \in \mathcal{RCPM}_\beta$ with $\beta = e^{h_{\mathrm{top}}(f)}$.*

Proof. Let $M(f) = (m_{IJ}) \in \mathcal{M}_P$ be the matrix of $f \in \mathcal{RCPM}$, and denote by $\mathbb{M}$ the operator on ℓ_P^1 represented by $M(f)$. We show that under our assumptions $\mathbb{M}$ is a Radon–Nikol'skiĭ operator.

For a set $K \subset [0,1]$ define a matrix $M(f,K) = (m_{IJ}(K)) \in \mathcal{M}_P$ by

$$m_{IJ}(K) = \begin{cases} m_{IJ}, & I \subset K, \\ 0, & \text{otherwise}. \end{cases} \tag{3.6}$$

Since the pair $(P,\varphi) \in \mathcal{P}$ is m-ruled, the map φ_P (hence also f) has an m-horseshoe created by P-basic intervals $I_1,\dots,I_m$. Let

$$C = M\Big(f, \bigcup_{i=1}^{m} I_i\Big) \quad \text{and} \quad B = M\Big(f, [0,1] \setminus \bigcup_{i=1}^{m} I_i\Big).$$

Then $M(f) = C + B$ and by Proposition 2.2(i) the matrices C, B represent bounded operators on ℓ_P^1; denote them $\mathbb{C}, \mathbb{B}$, and set $\mathbb{M} = \mathbb{C} + \mathbb{B}$. Clearly by our definition the matrix C has (finitely many) nonzero I_i-rows, hence by Proposition 2.2(ii) the operator $\mathbb{C}$ is compact. Due to Definition 4.7 it is sufficient to verify that $r(\mathbb{M}) > r(\mathbb{B})$. Using Proposition 2.2 and Gelfand's formula we consequently get

$$r(\mathbb{M}) \geq r(\mathbb{C}) \geq m > m - 1 \geq \|\mathbb{B}\| \geq r(\mathbb{B}).$$

This shows that the operator $\mathbb{M}$ is Radon–Nikol'skiĭ and the conclusion follows from Theorem 3.5. ∎

Example 3.8. Let $P = \{a_n : n = 0, 1, \dots, \infty\}$ be an admissible set with the only limit point equal to 1. Assume that $0 = a_0 < a_1 < \cdots < a_\infty = 1$ and define the map $\varphi\colon P \to P$ by

$$\varphi(a_0) = \varphi(a_2) = \varphi(a_4) = a_\infty, \quad \varphi(a_1) = \varphi(a_3) = \varphi(a_5) = a_0, \quad \varphi(a_6) = a_6$$

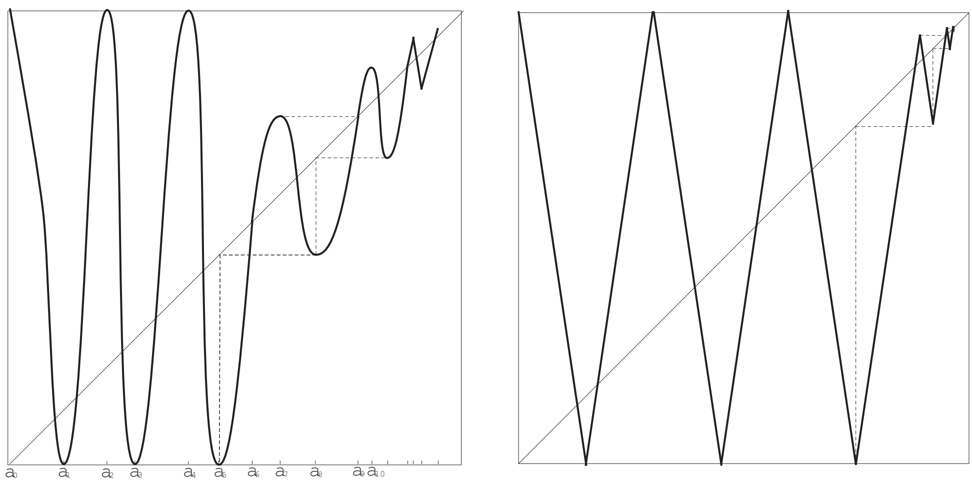

Fig. 2. $f \in \mathcal{RCPM}$ transitive (left), $g \in \mathcal{RCPM}_{r(\mathbb{M})}$, $r(\mathbb{M}) = e^{h_{\mathrm{top}}(f)} \sim 6.5616$ from Example 3.8

and for each $k \geq 0$,

$$\varphi(a_{3k+7}) = a_{3k+9}, \quad \varphi(a_{3k+8}) = a_{3k+5}, \quad \varphi(a_{3k+9}) = a_{3k+9}.$$

Then $(P, \varphi) \in \mathcal{P}$ and it is 6-ruled, where φ_P has a 6-horseshoe created by the P-basic intervals $[a_0, a_1], \dots, [a_5, a_6]$. Let us consider a map $f \in \mathcal{RCPM}$ that corresponds to (P, φ). By Theorem 3.7 the map f is semiconjugate via a nondecreasing map ψ to some map $g \in \mathcal{RCPM}_\beta$ with $\beta = e^{h_{\mathrm{top}}(f)}$. In particular, the maps f, g are conjugate when f is transitive—see Remark 2.7.

The matrices $M(f), C, B$ from Example 3.8 are

$$M(f) = \left(\begin{array}{ccccccccccccccccccccccc}
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&\cdot&\cdot\\
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\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot&\cdot
\end{array}\right)$$

$$= \left(\begin{array}{ccccccccccccccccccccccc}
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\end{pmatrix}.
$$

4. Appendix

Definition 4.1. A function $f\colon [a,b] \to [a,b]$ is said to have a *d-horseshoe* if there exist d subintervals $I_1, \dots, I_d$ of $[a,b]$ with disjoint interiors such that $f(I_i) \supset I_j$ for all $1 \le i, j \le d$.

Proposition 4.2 ([7]). *If $f\colon [a,b] \to [a,b]$ is continuous and has a d-*horseshoe *then $h_{\mathrm{top}}(f) \ge \log d$.*

Theorem 4.3 ([7]). *Assume that f has positive entropy. Then there exist sequences $\{k_n\}_{n=1}^{\infty}$ and $\{s_n\}_{n=1}^{\infty}$ of positive integers such that $\lim_{n\to\infty} k_n = \infty$, for each n the map f^{k_n} has an s_n-horseshoe and*

$$\lim_{n\to\infty} \frac{1}{k_n} \log s_n = h_{\mathrm{top}}(f).$$

Proposition 4.4 ([1, Lemma 4.6.1]). *Let $f\colon [0,1] \to [0,1]$ be continuous and $\psi\colon [0,1] \to [0,1]$ a nondecreasing continuous map satisfying $\psi([0,1]) = [0,1]$. Assume that $g\colon [0,1] \to [0,1]$ satisfies $\psi \circ f = g \circ \psi$ on $[0,1]$. Then g is continuous and if f is nondecreasing (respectively nonincreasing) on some interval J then g is nondecreasing (respectively nonincreasing) on $\psi(J)$.*

Let $\mathcal{E}$ be a real Banach space.

Definition 4.5. A closed set $\mathcal{K} \subset \mathcal{E}$ is called a *cone* if it satisfies

(i) $\mathcal{K} + \mathcal{K} \subset \mathcal{K}$,
(ii) $a\mathcal{K} \subset \mathcal{K}$ for $a \in \mathbb{R}_+$, where $\mathbb{R}_+ = [0,\infty)$,
(iii) $\mathcal{K} \cap (-\mathcal{K}) = \{\theta\}$, where θ is the zero element from $\mathcal{E}$.

A cone $\mathcal{K}$ is called

(iv) *reproducing* if $\mathcal{K} - \mathcal{K} = \mathcal{E}$,
(v) *normal* if there exists a constant $b > 0$ such that $\|x\| \leq b\|y\|$ whenever $y - x \in \mathcal{K}$.

An operator T on $\mathcal{E}$ is called $\mathcal{K}$-*positive* if $T\mathcal{K} \subset \mathcal{K}$.

For a bounded linear operator $\mathbb{A}$ on a Banach space $\mathcal{E}$ we will consider its spectrum $\sigma(\mathbb{A}) = P_\sigma(\mathbb{A}) \cup R_\sigma(\mathbb{A}) \cup C_\sigma(\mathbb{A})$ partitioned into the point, residual and continuous part respectively.

THEOREM 4.6 ([2], [3]). *Let $\mathcal{K}$ be a normal reproducing cone in a real Banach space $\mathcal{E}$. Then for every bounded $\mathcal{K}$-positive operator $\mathbb{A}$ the spectral radius $r(\mathbb{A})$ of $\mathbb{A}$ belongs to the spectrum $\sigma(\mathbb{A})$.*

Following [5] we introduce the Radon–Nikol'skiĭ operators.

DEFINITION 4.7. A bounded linear operator $\mathbb{A}$ defined on a (complex) Banach space $\mathcal{F}$ will be called *Radon–Nikol'skiĭ* if $\mathbb{A}$ may be represented as $\mathbb{A} = \mathbb{C} + \mathbb{B}$, where

(i) $\mathbb{C}$ is compact,
(ii) $r(\mathbb{A}) > r(\mathbb{B})$.

THEOREM 4.8 ([5, Theorem 3.2]). *Let τ be a function holomorphic in the neighborhood of the spectrum $\sigma(\mathbb{A})$ of the operator $\mathbb{A}$. Assume that $\mathbb{A}$ is $\mathcal{K}$-positive and $\tau(\mathbb{A})$ is a Radon–Nikol'skiĭ operator on a real Banach space $\mathcal{E}$. Then $r(\mathbb{A}) \in P_\sigma(\mathbb{A})$ with corresponding eigenvector in $\mathcal{K}$.*

Let $\mathcal{P}$, $\mathcal{M}_P$ and ℓ^1_P be as in Section 2.

THEOREM 4.9 ([5]). *Given $P \in \mathcal{P}$ let $M = (m_{IJ}) \in \mathcal{M}_P$ be a matrix representing an operator $\mathbb{M}$ on ℓ^1_P. Assume that for some function τ holomorphic in the neighborhood of $\sigma(\mathbb{M})$ the operator $\tau(\mathbb{M})$ is a Radon–Nikol'skiĭ operator on ℓ^1_P. For $\delta > 0$, denote by $\mathbb{M}_\delta$ the operator on ℓ^1_P represented by the matrix $M_\delta = (m_{IJ}(\delta)) \in \mathcal{M}_P$ defined as*

$$m_{IJ}(\delta) = \begin{cases} m_{IJ}, & \min\{|I|, |J|\} \geq \delta, \\ 0, & \textit{otherwise}. \end{cases} \tag{4.1}$$

Then

$$\lim_{\delta \to 0} r(\mathbb{M}_\delta) = r(\mathbb{M}).$$

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