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Powers of m-isometries

by

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Abstract. A bounded linear operator T on a Banach space X is called an (m,p)-*isometry* for a positive integer m and a real number $p \geq 1$ if, for any vector $x \in X$,

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} \|T^k x\|^p = 0.$$

We prove that any power of an (m,p)-isometry is also an (m,p)-isometry. In general the converse is not true. However, we prove that if T^r and T^{r+1} are (m,p)-isometries for a positive integer r, then T is an (m,p)-isometry. More precisely, if T^r is an (m,p)-isometry and T^s is an (l,p)-isometry, then T^t is an (h,p)-isometry, where $t = \gcd(r,s)$ and $h = \min(m,l)$.

1. Introduction. The m-isometric operators on a Hilbert space H have been introduced in [A]. Given a positive integer m, a (bounded linear) operator T on H is called an m-*isometry* if

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} T^{*k} T^k = 0, \tag{1.1}$$

where T^* denotes the adjoint operator of T. A detailed study of m-isometries was developed by J. Agler and M. Stankus [AS1]–[AS3]. Those operators have been considered by many authors, for example in [At], [BMM], [BJ], [FH1], [FH2], [GR], [H], [PL], [PR] and [R].

A simple manipulation proves that (1.1) is equivalent to

$$\sum_{k=0}^{m}(-1)^k \binom{m}{k} \|T^k x\|^2 = 0 \quad \text{for all } x \in H. \tag{1.2}$$

It is clear that the notions of 1-isometry and isometry coincide. Moreover, any m-isometry is also an $(m+1)$-isometry.

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Sid Ahmed [S] has used (1.2) as the definition of an m-isometric operator T on a Banach space X. However, Bayart [B] has observed that the exponent 2 that appears in the norm does not play a particular role and considered (m,p)-isometries with $p \geq 1$ a real number, which are defined in the following way (X always denotes a Banach space and $L(X)$ the algebra of all bounded linear operators on X).

Definition 1.1. Let $T \in L(X)$, m a positive integer and $p \geq 1$ a real number. We say that T is an (m,p)-*isometry* if, for any $x \in X$,

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} \|T^k x\|^p = 0. \tag{1.3}$$

T is called an m-*isometry* if it is an (m,p)-isometry for some $p \geq 1$.

Recently, Hoffmann et al. [HMS] took off the restriction $p \geq 1$ and defined (m,p)-isometries for all $p > 0$. They studied when an (m,p)-isometry is an (μ,q)-isometry for some pair (μ,q). In particular, for any positive real number p they gave an example of an operator T that is a $(2,p)$-isometry, but is not a $(2,q)$-isometry for any q different from p [HMS, Example 1.2]. In general, for a fixed positive integer m, there is no relation between being an (m,p)-isometry and an $(m,2)$-isometry with $p \neq 2$.

Patel [P, Theorem 2.1] proved that any power of a $(2,2)$-isometry is again a (2, 2)-isometry. In this paper we improve the above property. Indeed, we answer the following natural problems.

(1) Is the class of m-isometries stable under powers?
(2) Find sufficient conditions that guarantee that operators which have some powers in the class of m-isometries are in the same class.

It is a simple observation that if T^r and T^{r+1} are isometries for some positive integer r, then T is an isometry: indeed, $\|x\| = \|T^{r+1}x\| = \|T^r T x\| = \|Tx\|$ for every $x \in X$.

In the study of m-isometries we will use recursive equations. This technique was used by V. Müller in the study of m-contractions [M]. This is our main tool in proving these two theorems:

(1) If $T \in L(X)$ is an (m,p)-isometry, then any power T^r is also an (m,p)-isometry (Theorem 3.1).
(2) If T^r is an (m,p)-isometry and T^s is an (l,p)-isometry, then T^t is an (h,p)-isometry, where t is the greatest common divisor of r and s, and h is the minimum of m and l (Theorem 3.6). In particular, if T^r and T^{r+1} are (m,p)-isometries, then so is T (Corollary 3.7).

2. Preliminaires. We first recall certain facts about recursive equations.

For $T \in L(X)$ and $x \in X$, denote $a_k := \|T^k x\|^p$ for $k = 0, 1, \ldots$. The equation (1.3) then reads

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_k = 0, \tag{2.1}$$

or equivalently

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_{k+n} = 0 \tag{2.2}$$

for all $n \geq 0$.

Notice that (2.2) is a recursive equation. Let us introduce some classical results to solve this type of equations. We are interested in the sequences $(y_n)_{n\geq 0}$ which satisfy the *recursive equation*

$$y_{n+m} + \gamma_{m-1} y_{n+m-1} + \gamma_{m-2} y_{n+m-2} + \cdots + \gamma_1 y_{n+1} + \gamma_0 y_n = 0 \tag{2.3}$$

for certain $m \geq 1$ and any $n \geq 0$, γ_i being complex numbers $(0 \leq i \leq k-1)$. The *characteristic polynomial* of (2.3) is given by

$$q(z) = z^m + \gamma_{m-1} z^{m-1} + \gamma_{m-2} z^{m-2} + \cdots + \gamma_1 z + \gamma_0,$$

which can be written in the form

$$q(z) = (z - z_1)^{m_1} \cdots (z - z_r)^{m_r}, \tag{2.4}$$

where $m_1 + \cdots + m_r = m$ and $z_i \neq z_j$ for $i \neq j$. It is well known (see for example [KP, Theorem 3.7] and [Ag, p. 104]) that the set of all complex sequences which satisfy (2.3) is a vector subspace of the space $\mathbb{C}^{\mathbb{N}}$ of all complex sequences, it has dimension m and a basis formed by the sequences

$$\begin{array}{l}
(z_1^n)_{n\geq 0}, (n z_1^n)_{n\geq 0}, (n^2 z_1^n)_{n\geq 0}, \ldots, (n^{m_1-1} z_1^n)_{n\geq 0}, \\
(z_2^n)_{n\geq 0}, (n z_2^n)_{n\geq 0}, (n^2 z_2^n)_{n\geq 0}, \ldots, (n^{m_2-1} z_2^n)_{n\geq 0}, \\
\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\
(z_r^n)_{n\geq 0}, (n z_r^n)_{n\geq 0}, (n^2 z_r^n)_{n\geq 0}, \ldots, (n^{m_r-1} z_r^n)_{n\geq 0}.
\end{array} \tag{2.5}$$

There exists an identification between the recursive equation (2.3), the characteristic polynomial (2.4), the subspace of sequences which satisfy the recursive equation and its basis (2.5).

3. Main results. It is clear that if T is an isometry, then so is T^r. Patel [P, Theorem 2.1] proved that any power of a $(2, 2)$-isometry is again a (2,2)-isometry. The next result shows that any power of an (m, p)-isometry is an (m, p)-isometry.

THEOREM 3.1. *Let X be a Banach space, $T \in L(X)$, m be a positive integer and $p \geq 1$ a real number. If T is an (m, p)-isometry, then so is any power T^r.*

Proof. Fix $x \in X$ and denote $a_n := \|T^n x\|^p$ for $n = 0, 1, \ldots$. Then the sequence $(a_n)_{n\ge 0}$ satisfies

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_{k+n} = 0 \tag{3.1}$$

for every $n \ge 0$. The characteristic polynomial associated with (3.1) is $q_1(z) = (z-1)^m$, hence $(a_n)_{n\ge 0}$ is a linear combination of the sequences $(1)_{n\ge 0}, (n)_{n\ge 0}, (n^2)_{n\ge 0}, \ldots, (n^{m-1})_{n\ge 0}$. Consequently, given a positive integer r, $(a_n)_{n\ge 0}$ is also a linear combination of the sequences

$$\begin{gathered} (1)_{n\ge 0}, (n)_{n\ge 0}, (n^2)_{n\ge 0}, \ldots, (n^{m-1})_{n\ge 0}, \\ (x_2^n)_{n\ge 0}, (n x_2^n)_{n\ge 0}, (n^2 x_2^n)_{n\ge 0}, \ldots, (n^{m-1} x_2^n)_{n\ge 0}, \\ \cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ (x_r^n)_{n\ge 0}, (n x_r^n)_{n\ge 0}, (n^2 x_r^n)_{n\ge 0}, \ldots, \ldots, (n^{m-1} x_r^n)_{n\ge 0}, \end{gathered} \tag{3.2}$$

where $x_1 := 1, x_2, \ldots, x_r$ are the r-roots of unity. Hence the sequence $(a_n)_{n\ge 0}$ satisfies the equation

$$\sum_{k=0}^{m} (-1)^k \binom{m}{k} a_{kr+n} = 0 \tag{3.3}$$

for any $n \ge 0$. Since $x \in X$ is arbitrary, we conclude that T^r is an (m,p)-isometry. ∎

REMARK 3.2. Related to the previous result we have the following problem. Given an operator $T \in L(X)$ that is an m-isometry, determine the class of functions f such that the operator $f(T) \in L(X)$ is also an m-isometry. It is known that the spectrum of an m-isometry is the closed unit disc or a closed subset of its boundary ([AS1, Lemma 1.21] & [B, Proposition 2.3]). So, the eligible functions must leave invariant the unit disc and its boundary. In particular:

(1) If $f(z) = z^r$, then the result is true by Theorem 3.1.
(2) If $f(z) = e^{it} z^r$ for some real t, then the result is true by $\|e^{it} T^k x\| = \|T^k x\|$ and Theorem 3.1.
(3) If $f(z) = (2z-1)/(2-z)$ and T is an isometry, then $f(T)$ is not always an isometry. Indeed, consider

$$f(z) = \frac{2z-1}{2-z} = -2 + \sum_{n=0}^{\infty} \frac{3}{2^{n+1}} z^n$$

and $T(x_1, x_2, \ldots) = (0, x_1, x_2, \ldots)$ defined on $\ell_p(\mathbb{N})$ such that $p \ge 1$ with $p \ne 2$. Then $\|f(T)(e_1 + e_2)\|^p \ne \|e_1 + e_2\|^p$.

In general the converse of Theorem 3.1 is false.

EXAMPLE 3.3. Let S_w be the unilateral weighted forward shift on $\ell_2(\mathbb{N})$ with weight sequence (w_n) given by $w_{2n+1} := \frac{1}{2}$ and $w_{2n} := 2$ for all $n > 0$, that is,

$$S_w(x_1, x_2, \dots) = \big(0, \tfrac{1}{2}x_1, 2x_2, \tfrac{1}{2}x_3, 2x_4, \dots\big).$$

Then $\|S_w^2(x_1, x_2, \dots)\| = \|(x_1, x_2, \dots)\|$, so S_w^2 is an isometry, but S_w is not.

LEMMA 3.4 ([BMN, Remark 3.9]). *Let S_w be the unilateral weighted forward shift operator on $\ell_2(\mathbb{N})$ with weight sequence $w = (w_n)_{n\geq 1} \in \ell_\infty(\mathbb{N})$. Then S_w is a $(2,2)$-isometry if and only if*

$$|w_n|^2 = \frac{n|w_1|^2 - (n-1)}{(n-1)|w_1|^2 - (n-2)} > 0$$

for $n \geq 1$. Observe that

$$\frac{n|w_1|^2 - (n-1)}{(n-1)|w_1|^2 - (n-2)} > 0 \;\Leftrightarrow\; |w_1| \geq 1.$$

EXAMPLE 3.5. The converse of Theorem 3.1 is not true for the class of unilateral weighted shifts. Set $w_2 := w_{2n+1} := 4$ and $w_{2n+2} := \big(\frac{3n+4}{6n+2}\big)^2$ for all $n \geq 1$. By Lemma 3.4, S_w^2 is a $(2,2)$-isometry but S_w is not.

However, if we impose that two suitable different powers of T are m-isometries, then T is m-isometry.

THEOREM 3.6. *Let X be a Banach space and $T \in L(X)$. Let r, s, m, l be positive integers and $p \geq 1$ a real number. If T^r is an (m,p)-isometry and T^s is an (l,p)-isometry, then T^t is an (h,p)-isometry, where t is the greatest common divisor of r and s, and h is the minimum of m and l.*

Proof. Fix $x \in X$ and denote $a_n := \|T^n x\|^p$ for $n = 0, 1, \dots$. As T^r is an m-isometry, the sequence $(a_n)_{n\geq 0}$ satisfies the recursive equation

$$\sum_{k=0}^{m} \binom{m}{k} (-1)^k a_{rk+n} = 0 \tag{3.4}$$

for all $n \geq 0$. This equation has characteristic polynomial $q(z) := q_r(z)^m := (z^r - 1)^m$. Let V_r^m be the subspace of $\mathbb{C}^{\mathbb{N}}$ formed by all complex sequences which satisfy (3.4). Then $\dim V_r^m = mr$ and a basis B_r of V_r^m is formed by the sequences $(n^i x_j^n)_{n\geq 0}$, where $0 \leq i \leq m-1$, $1 \leq j \leq r$ and x_j are the roots of $q_r(z) = z^r - 1$.

Analogously, as T^s is an l-isometry, the sequence $(a_n)_{n\geq 0}$ also satisfies

$$\sum_{k=0}^{l} \binom{l}{k} (-1)^k a_{sk+n} = 0 \tag{3.5}$$

for all $n \geq 0$. Now the set of all sequences which satisfy (3.5) is a vector subspace V_s^l of $\mathbb{C}^{\mathbb{N}}$, $\dim V_s^l = sl$ and a basis B_s of V_s^l is formed by the sequences $(n^i y_j^n)_{n\geq 0}$, where $0 \leq i \leq l-1$, $1 \leq j \leq s$ and y_j are the roots of $q_s(z) := z^s - 1$; moreover, $p(z) := q_s(z)^l = (z^s-1)^l$ is the characteristic polynomial of (3.5).

Using the factorization $z^n - 1 = (z-1)q(z)$ for a polynomial $q(z)$, we obtain $q_r(z) = q_t(z)q_1(z)$ and $q_s(z) = q_t(z)q_2(z)$ for some polynomials $q_1(z)$ and $q_2(z)$, where $t = \gcd(r,s)$.

Therefore the sequences which satisfy both (3.4) and (3.5) are those in the subspace $V_r^m \cap V_s^l$, whose characteristic polynomial is $\gcd(q_r(z)^m, q_s(z)^l) = q_t(z)^h$, where $t = \gcd(r,s)$ and $h = \min(m,l)$. As x is arbitrary, T^t is an (h,p)-isometry. ∎

In the following result we have some particular cases of Theorem 3.6.

Corollary 3.7. *Let X be a Banach space and $T \in L(X)$. Let r, s, m positive integers and $p \geq 1$ a real number.*

(1) *If T is an (m,p)-isometry and T^s is an isometry, then T is an isometry.*
(2) *If T^r and T^{r+1} are (m,p)-isometries, then so is T.*
(3) *If T^r is an (m,p)-isometry and T^{r+1} is an (n,p)-isometry with $m < n$, then T is an (m,p)-isometry.*

Corollary 3.8. *Let X be a Banach space and $T \in L(X)$. Let m be a positive integer and $p \geq 1$ a real number. If T is a proper (m,p)-isometry, in the sense that it is not an $(m-1,p)$-isometry, then any power of T is a proper (m,p)-isometry.*

Example 3.9. It is not difficult to prove that $T := \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$ defined in $\mathbb{C}^2$ with the euclidean norm is a proper 3-isometry. Using Theorem 3.1 and Corollary 3.8 we find that $T^k = \left(\begin{smallmatrix} 1 & k \\ 0 & 1 \end{smallmatrix}\right)$ is a proper 3-isometry.

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