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On the randomized complexity of Banach space valued integration

by

STEFAN HEINRICH (Kaiserslautern) and AICKE HINRICHS (Linz)

Dedicated to Albrecht Pietsch
on the occasion of his 80th birthday

Abstract. We study the complexity of Banach space valued integration in the randomized setting. We are concerned with r times continuously differentiable functions on the d-dimensional unit cube Q, with values in a Banach space X, and investigate the relation of the optimal convergence rate to the geometry of X. It turns out that the nth minimal errors are bounded by $cn^{-r/d-1+1/p}$ if and only if X is of equal norm type p.

1. Introduction. Integration of scalar valued functions is an intensively studied topic in the theory of information-based complexity (see [12], [10], [11]). Motivated by applications to parametric integration, recently the complexity of Banach space valued integration was considered in [2]. It was shown that the behaviour of the nth minimal errors e_n^{ran} of randomized integration in $C^r(Q,X)$ is related to the geometry of the Banach space X in the following way: The infimum of the exponents of the rate is determined by the supremum of p such that X is of type p. In the present paper we further investigate this relation. We establish a connection between nth minimal errors and equal norm type p constants for n vectors. It follows that e_n^{ran} is bounded by $cn^{-r/d-1+1/p}$ if and only if X is of equal norm type p.

2. Preliminaries. Let $\mathbb{N} = \{1, 2, \dots\}$ and $\mathbb{N}_0 = \{0, 1, 2, \dots\}$. We introduce some notation and concepts from Banach space theory needed in what follows. For Banach spaces X and Y let B_X be the closed unit ball of X and

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$\mathscr{L}(X,Y)$ the space of bounded linear operators from X to Y, endowed with the usual norm. If $X=Y$, we write $\mathscr{L}(X)$. The norm of X is denoted by $\|\cdot\|$, while other norms are distinguished by subscripts. We assume that all the Banach spaces considered are defined over the same scalar field $\mathbb{K}=\mathbb{R}$ or $\mathbb{K}=\mathbb{C}$.

Let $Q=[0,1]^d$ and let $C^r(Q,X)$ be the space of all r times continuously differentiable functions $f:Q\to X$ equipped with the norm

$$\|f\|_{C^r(Q,X)}=\max_{0\le|\alpha|\le r,\, t\in Q}\|D^\alpha f(t)\|,$$

where $\alpha=(\alpha_1,\dots,\alpha_d)$, $|\alpha|=|\alpha_1|+\dots+|\alpha_d|$ and D^α denotes the respective partial derivative. For $r=0$ we write $C^0(Q,X)=C(Q,X)$, which is the space of continuous X-valued functions on Q. If $X=\mathbb{K}$, we write $C^r(Q)$ and $C(Q)$.

Let $1\le p\le 2$. A Banach space X is said to be of (Rademacher) *type p* if there is a constant $c>0$ such that for all $n\in\mathbb{N}$ and $x_1,\dots,x_n\in X$,

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^n\varepsilon_i x_i\Big\|^p\Big)^{1/p}\le c\Big(\sum_{k=1}^n\|x_i\|^p\Big)^{1/p}, \tag{1}$$

where $(\varepsilon_i)_{i=1}^n$ is a sequence of independent Bernoulli random variables with $\mathbb{P}\{\varepsilon_i=-1\}=\mathbb{P}\{\varepsilon_i=+1\}=1/2$ on some probability space $(\Omega,\Sigma,\mathbb{P})$ (we refer to [9, 7] for this notion and related facts). The smallest constant satisfying (1) is called the *type p constant* of X and is denoted by $\tau_p(X)$. If there is no such $c>0$, we put $\tau_p(X)=\infty$. The space $L_{p_1}(\mathcal{N},\nu)$ with $(\mathcal{N},\nu)$ an arbitrary measure space and $p_1<\infty$ is of type p with $p=\min(p_1,2)$.

Furthermore, given $n\in\mathbb{N}$, let $\sigma_{p,n}(X)$ be the smallest $c>0$ for which (1) holds for any $x_1,\dots,x_n\in X$ with $\|x_1\|=\dots=\|x_n\|$. The contraction principle for Rademacher series (see [7, Th. 4.4]) implies that $\sigma_{p,n}(X)$ is the smallest constant $c>0$ such that for $x_1,\dots,x_n\in X$,

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^n\varepsilon_i x_i\Big\|^p\Big)^{1/p}\le cn^{1/p}\max_{1\le i\le n}\|x_i\|. \tag{2}$$

We say that X is of *equal norm type p* if there is a constant $c>0$ such that $\sigma_{p,n}(X)\le c$ for all $n\in\mathbb{N}$. Clearly, $\sigma_{p,n}(X)\le\tau_p(X)$ and type p implies equal norm type p.

Let us comment a little more on the relation of the different notions of type which are used here and in the literature. The concept of equal norm type p was first introduced and used by R. C. James in the case $p=2$ in [6]. There it is shown that X is of equal norm type 2 if and only if X is of type 2. This result is attributed to G. Pisier. Later, it even turned out in [1] that the sequence $\sigma_{2,n}(X)$ and the corresponding sequence $\tau_{2,n}(X)$ of type 2 constants computed with n vectors are uniformly equivalent. In contrast, for $1 < p < 2$, L. Tzafriri [13] constructed Tsirelson spaces without type p but with equal norm type p. Finally, V. Mascioni introduced and studied the notion of weak type p for $1 < p < 2$ in [8] and showed that, again in contrast to the situation for $p = 2$, a Banach space X is of weak type p if and only if it is of equal norm type p.

Throughout the paper $c, c_1, c_2, \ldots$ are constants, which depend only on the problem parameters r, d, but depend neither on the algorithm parameters n, l etc. nor on the input f. The same symbol may denote different constants, even in a sequence of relations.

For $r, k \in \mathbb{N}$ we let $P_k^{r,X} \in \mathscr{L}(C(Q,X))$ be X-valued composite tensor product Lagrange interpolation of degree r with respect to the partition of $[0,1]^d$ into k^d subcubes of sidelength k^{-1} with disjoint interiors (see [2]). Given $r \in \mathbb{N}_0$ and $d \in \mathbb{N}$, there are constants $c_1, c_2 > 0$ such that for all Banach spaces X and all $k \in \mathbb{N}$,

$$\sup_{f \in B_{C^r(Q,X)}} \|f - P_k^{r,X} f\|_{C(Q,X)} \le c_2 k^{-r} \tag{3}$$

(see [2]).

3. Banach space valued integration. Let X be a Banach space, $r \in \mathbb{N}_0$, and let the integration operator $S^X : C(Q,X) \to X$ be given by

$$S^X f = \int_Q f(t)\,dt.$$

We will work in the setting of information-based complexity theory (see [12, 10, 11]). Below, $e_n^{\mathrm{det}}(S^X, B_{C^r(Q,X)})$ and $e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)})$ denote the nth minimal error of S^X on $B_{C^r(Q,X)}$ in the deterministic, respectively randomized, setting, that is, the minimal possible error among all deterministic, respectively randomized, algorithms approximating S^X on $B_{C^r(Q,X)}$ that use at most n values of the input function f. The precise notions are recalled in the appendix. The following was shown in [2].

THEOREM 1. *Let $r \in \mathbb{N}_0$ and $1 \le p \le 2$. Then there are constants $c_{1-4} > 0$ such that for all Banach spaces X and $n \in \mathbb{N}$, the deterministic nth minimal error satisfies*

$$c_1 n^{-r/d} \le e_n^{\mathrm{det}}(S^X, B_{C^r(Q,X)}) \le c_2 n^{-r/d}.$$

Moreover, if X is of type p and p_X is the supremum of all p_1 such that X is of type p_1, then the randomized nth minimal error fulfills

$$c_3 n^{-r/d-1+1/p_X} \le e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \le c_4 \tau_p(X) n^{-r/d-1+1/p}.$$

As a consequence, we obtain

Corollary 1. *Let $r \in \mathbb{N}_0$ and $1 \le p \le 2$. Then the following are equivalent:*

(i) *X is of type p_1 for all $p_1 < p$.*
(ii) *For each $p_1 < p$ there is a constant $c > 0$ such that for all $n \in \mathbb{N}$,*
$$e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \le cn^{-r/d-1+1/p_1}.$$

The main result of the present paper is the following

Theorem 2. *Let $1 \le p \le 2$ and $r \in \mathbb{N}_0$. Then there are constants $c_1, c_2 > 0$ such that for all Banach spaces X and all $n \in \mathbb{N}$,*

$$\begin{aligned} &c_1 n^{r/d+1-1/p} e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \\ &\qquad \le \sigma_{p,n}(X) \le c_2 \max_{1\le k\le n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}). \end{aligned} \tag{4}$$

This allows us to sharpen Corollary 1:

Corollary 2. *Let $r \in \mathbb{N}_0$ and $1 \le p \le 2$. Then the following are equivalent:*

(i) *X is of equal norm type p.*
(ii) *There is a constant $c > 0$ such that for all $n \in \mathbb{N}$,*
$$e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \le cn^{-r/d-1+1/p}.$$

Recall from the preliminaries that the conditions in the corollary are also equivalent to

(iii) *X is of type 2 if $p = 2$ and of weak type p if $1 < p < 2$.*

For the proof of Theorem 2 we need a number of auxiliary results. The following lemma is a slight modification of Prop. 9.11 of [7], with essentially the same proof, which we include for the sake of completeness.

Lemma 1. *Let $1 \le p \le 2$. Then there is a constant $c > 0$ such that for each Banach space X, each $n \in \mathbb{N}$ and each sequence of independent, essentially bounded, mean zero X-valued random variables $(\eta_i)_{i=1}^n$ on some probability space $(\Omega, \Sigma, \mathbb{P})$,*
$$\Big(\mathbb{E}\Big\|\sum_{i=1}^n \eta_i\Big\|^p\Big)^{1/p} \le c\sigma_{p,n}(X) n^{1/p} \max_{1\le i\le n} \|\eta_i\|_{L_\infty(\Omega,\mathbb{P},X)}.$$

Proof. Let $(\varepsilon_i)_{i=1}^n$ be independent, symmetric Bernoulli random variables on some probability space $(\Omega', \Sigma', \mathbb{P}')$ different from $(\Omega, \Sigma, \mathbb{P})$. Considering $(\eta_i)_{i=1}^n$ and $(\varepsilon_i)_{i=1}^n$ as random variables on the product probability space, we denote the expectation with respect to $\mathbb{P}'$ by $\mathbb{E}'$ (and the expectation with respect to $\mathbb{P}$, as before, by $\mathbb{E}$). Using Lemma 6.3 of [7] and (2), we get

$$\Big(\mathbb{E}\Big\|\sum_{i=1}^n \eta_i\Big\|^p\Big)^{1/p} \le 2\Big(\mathbb{E}\,\mathbb{E}'\Big\|\sum_{i=1}^n \varepsilon_i\eta_i\Big\|^p\Big)^{1/p}$$
$$\le 2\sigma_{p,n}(X)n^{1/p}(\mathbb{E}\max_{1\le i\le n}\|\eta_i\|^p)^{1/p}$$
$$\le 2\sigma_{p,n}(X)n^{1/p}\max_{1\le i\le n}\|\eta_i\|_{L_\infty(\Omega,\mathbb{P},X)}. \ \blacksquare$$

Next we introduce an algorithm for the aproximation of $S^X f$. Let $n\in\mathbb{N}$ and let $\xi_i:\Omega\to Q$ $(i=1,\dots,n)$ be independent random variables on some probability space $(\Omega,\Sigma,\mathbb{P})$, uniformly distributed on Q. For $f\in C(Q,X)$ define

$$A^{0,X}_{n,\omega}f=\frac{1}{n}\sum_{i=1}^n f(\xi_i(\omega)) \tag{5}$$

and, if $r\ge 1$, put $k=\lceil n^{1/d}\rceil$ and

$$A^{r,X}_{n,\omega}f=S^X(P^{r,X}_k f)+A^{0,X}_{n,\omega}(f-P^{r,X}_k f). \tag{6}$$

These are the Banach space valued versions of the standard Monte Carlo method $(r=0)$ and the Monte Carlo method with separation of the main part $(r\ge 1)$. The following extends the second part of Proposition 1 of [2].

PROPOSITION 1. *Let $r\in\mathbb{N}_0$ and $1\le p\le 2$. Then there is a constant $c>0$ such that for all Banach spaces X, $n\in\mathbb{N}$, and $f\in C^r(Q,X)$,*

$$(\mathbb{E}\,\|S^X f-A^{r,X}_{n,\omega}f\|^p)^{1/p}\le c\sigma_{p,n}(X)n^{-r/d-1+1/p}\|f\|_{C^r(Q,X)}. \tag{7}$$

Proof. Let us first consider the case $r=0$. Let $f\in C(Q,X)$ and put

$$\eta_i(\omega)=\int_Q f(t)\,dt-f(\xi_i(\omega)).$$

Clearly, $\mathbb{E}\,\eta_i(\omega)=0$,

$$S^X f-A^{0,X}_{n,\omega}f=\frac{1}{n}\sum_{i=1}^n\eta_i(\omega)\quad\text{and}\quad\|\eta_i(\omega)\|\le 2\|f\|_{C(Q,X)}.$$

An application of Lemma 1 gives (7). If $r\ge 1$, we have

$$S^X f-A^{r,X}_{n,\omega}f=S^X(f-P^{r,X}_k f)-A^{0,X}_{n,\omega}(f-P^{r,X}_k f)$$

and the result follows from (3) and the case $r=0$. $\blacksquare$

LEMMA 2. *Let $1\le p\le 2$. Then there are constants $c>0$ and $0<\gamma<1$ such that for each Banach space X, each $n\in\mathbb{N}$, and $(x_i)_{i=1}^n\subset X$ there is a subset $I\subseteq\{1,\dots,n\}$ with $|I|\ge\gamma n$ and*

$$\mathbb{E}\Big\|\sum_{i\in I}\varepsilon_i x_i\Big\|\le cn^{1/p}\|(x_i)\|_{\ell^n_\infty(X)}\max_{1\le k\le n}k^{r/d+1-1/p}e^{\mathrm{ran}}_k(S^X,B_{C^r(Q,X)}).$$

Proof. Since for all $n \in \mathbb{N}$,

$$\max_{1\le k\le n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \ge e_1^{\mathrm{ran}}(S^{\mathbb{K}}, B_{C^r(Q,\mathbb{K})}) > 0,$$

the statement is trivial for $n < 8^d$. Therefore we can assume $n \ge 8^d$. Clearly, we can also assume $\|(x_i)\|_{\ell_\infty^n(X)} > 0$. Let $m \in \mathbb{N}$ be such that

$$m^d \le n < (m+1)^d, \tag{8}$$

hence

$$m \ge 8. \tag{9}$$

Let ψ be an infinitely differentiable function on $\mathbb{R}^d$ such that $\psi(t) > 0$ for $t \in (0,1)^d$ and $\operatorname{supp}\psi \subset [0,1]^d$. Let $(Q_i)_{i=1}^{m^d}$ be the partition of Q into closed cubes of side length m^{-1} with disjoint interiors, let t_i be the point in Q_i with minimal coordinates and define $\psi_i \in C(Q)$ by

$$\psi_i(t) = \psi(m(t-t_i)) \quad (i = 1,\dots,m^d).$$

It is easily verified that there is a constant $c_0 > 0$ such that for all $(\alpha_i)_{i=1}^{m^d}$ in $[-1,1]^{m^d}$,

$$\Big\|\sum_{i=1}^{m^d} \alpha_i x_i \psi_i\Big\|_{C^r(Q,X)} \le c_0 m^r \|(x_i)\|_{\ell_\infty^n(X)}.$$

Set

$$f_i = c_0^{-1} m^{-r} \|(x_i)\|_{\ell_\infty^n(X)}^{-1} x_i \psi_i;$$

it follows that

$$\sum_{i=1}^{m^d} \alpha_i f_i \in B_{C^r(Q,X)} \quad \text{for all } (\alpha_i)_{i=1}^{m^d} \in [-1,1]^{m^d}.$$

Moreover, with $\sigma = \int_Q \psi(t)\,dt$ we have

$$\begin{aligned}\Big\|\sum_{i=1}^{m^d} \alpha_i S^X f_i\Big\| &= c_0^{-1} m^{-r} \|(x_i)\|_{\ell_\infty^n(X)}^{-1} \Big\|\sum_{i=1}^{m^d} \alpha_i x_i \int_Q \psi_i(t)\,dt\Big\| \\ &= c_0^{-1} \sigma m^{-r-d} \|(x_i)\|_{\ell_\infty^n(X)}^{-1} \Big\|\sum_{i=1}^{m^d} \alpha_i x_i\Big\|.\end{aligned}$$

Next we use Lemmas 5 and 6 of [3] with $K = X$ (although stated for $K = \mathbb{R}$, Lemma 6 is easily seen to hold for $K = X$ as well) to obtain, for all $l \in \mathbb{N}$ with $l < m^d/4$,

$$\begin{aligned} e_l^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) &\ge \frac{1}{4} \min_{I \subseteq \{1,\dots,m^d\},\, |I| \ge m^d - 4l} \mathbb{E} \Big\|\sum_{i\in I} \varepsilon_i S^X f_i\Big\| \\ &\ge c m^{-r-d} \|(x_i)\|_{\ell_\infty^n(X)}^{-1} \mathbb{E} \Big\|\sum_{i\in I} \varepsilon_i x_i\Big\|.\end{aligned}$$

We put $l = \lfloor m^d/8 \rfloor$. Then

$$m^d/16 < l \le m^d/8. \tag{10}$$

Indeed, by (9) the left-hand inequality clearly holds for $m^d < 16$, while for $m^d \ge 16$ we get $\lfloor m^d/8 \rfloor > m^d/8 - 1 \ge m^d/16$. We conclude that there is an $I \subseteq \{1, \dots, m^d\}$ with $|I| \ge m^d - 4l \ge m^d/2$ and

$$\begin{aligned}\mathbb{E}\Big\|\sum_{i\in I}\varepsilon_i x_i\Big\| &\le cm^{r+d}\|(x_i)\|_{\ell_\infty^n(X)} e_l^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \\ &\le cm^{r+d} l^{-r/d+1/p-1}\|(x_i)\|_{\ell_\infty^n(X)} \max_{1\le k\le n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \\ &\le cn^{1/p}\|(x_i)\|_{\ell_\infty^n(X)} \max_{1\le k\le n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}),\end{aligned}$$

where we used (8) and (10). Finally, (8) and (9) give

$$|I| \ge m^d/2 \ge \frac{m^d}{2(m+1)^d} n \ge \frac{8^d}{2\cdot 9^d} n. \ \blacksquare$$

Proof of Theorem 2. The left-hand inequality of (4) follows directly from Proposition 1, since the number of function values involved in $A_{n,\omega}^{r,X}$ is bounded by $ck^d + n \le cn$; see also (16).

To prove the right-hand inequality of (4), let $n \in \mathbb{N}$ and $x_1, \dots, x_n \in X$. We construct by induction a partition of $K = \{1, \dots, n\}$ into a sequence of disjoint subsets $(I_l)_{l=1}^{l^*}$ such that for $1 \le l \le l^*$,

$$|I_l| \ge \gamma\Big|K \setminus \bigcup_{j<l} I_j\Big| \tag{11}$$

and

$$\begin{aligned}\mathbb{E}\Big\|\sum_{i\in I_l}\varepsilon_i x_i\Big\| \\ \le c\Big|K \setminus \bigcup_{j<l} I_j\Big|^{1/p}\|(x_i)\|_{\ell_\infty^n(X)} \max_{1\le k\le n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}),\end{aligned} \tag{12}$$

where c and γ are the constants from Lemma 2. For $l = 1$ the existence of an I_1 satisfying (11)–(12) follows directly from Lemma 2. Now assume that we already have a sequence of disjoint subsets $(I_l)_{l=1}^m$ of K satisfying (11)–(12). If

$$J := K \setminus \bigcup_{j\le m} I_j \ne \emptyset,$$

we apply Lemma 2 to $(x_i)_{i\in J}$ to find $I_{m+1} \subseteq J$ with

$$|I_{m+1}| \ge \gamma|J| \tag{13}$$

and

$$\mathbb{E}\Big\| \sum_{i\in I_{m+1}} \varepsilon_i x_i\Big\| \le c|J|^{1/p}\|(x_i)_{i\in J}\|_{\ell_\infty(J,X)} \max_{1\le k\le |J|} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}). \tag{14}$$

Observe that for $l = m+1$, (13) is just (11) and (14) implies (12). Furthermore, (11) implies

$$\Big|K \setminus \bigcup_{j\le l} I_j\Big| \le (1-\gamma)\Big|K \setminus \bigcup_{j\le l-1} I_j\Big|$$

and therefore

$$\Big|K \setminus \bigcup_{j\le l} I_j\Big| \le (1-\gamma)^l n. \tag{15}$$

It follows that the process stops with $K = \bigcup_{j\le l} I_j$ for a certain $l = l^* \in \mathbb{N}$. This completes the construction.

Using the equivalence of moments (Theorem 4.7 of [7]), we find from (12) and (15) that

$$\begin{aligned}\Big(\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\|^p\Big)^{1/p} &\le c\mathbb{E}\Big\|\sum_{i=1}^n \varepsilon_i x_i\Big\| \le c\sum_{l=1}^{l^*}\mathbb{E}\Big\|\sum_{i\in I_l}\varepsilon_i x_i\Big\| \\ &\le cn^{1/p}\|(x_i)\|_{\ell_\infty^n(X)} \max_{1\le k\le n} k^{r/d+1-1/p} e_k^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) \sum_{l=1}^{l^*}(1-\gamma)^{(l-1)/p}.\end{aligned}$$

This gives the upper bound of (4). ∎

Let us mention that results analogous to Theorem 2 and Corollary 2 above also hold for Banach space valued indefinite integration (see [2] for the definition) and for the solution of initial value problems for Banach space valued ordinary differential equations [5]. Indeed, an inspection of the respective proofs together with Lemma 1 of the present paper shows that Proposition 2 of [2] also holds with $\tau_p(X)$ replaced by $\sigma_{p,n}(X)$, and similarly Proposition 3.4 of [5]. Moreover, in both papers the lower bounds on e_n^{ran} are obtained by reduction to (definite) integration and thus the right-hand side inequality of (4) carries over directly.

4. Appendix. In this appendix we recall some basic notions of information-based complexity—the framework we used above. We refer to [10, 12] for more on this subject and to [3, 4] for the particular notation applied here. First we introduce the class of deterministic adaptive algorithms of varying cardinality, $\mathcal{A}^{\det}(C(Q,X),X)$. It consists of tuples $A=((L_i)_{i=1}^\infty,(\varrho_i)_{i=0}^\infty,(\varphi_i)_{i=0}^\infty)$, with $L_1\in Q$, $\varrho_0\in\{0,1\}$, $\varphi_0\in X$,

$$L_i : X^{i-1}\to Q \quad (i=2,3,\dots)$$

and

$$\varrho_i : X^i\to\{0,1\},\quad \varphi_i : X^i\to X \quad (i=1,2,\dots)$$

being arbitrary mappings. To each $f\in C(Q,X)$, we associate a sequence $(t_i)_{i=1}^\infty$ with $t_i\in Q$ as follows:

$$t_1=L_1,\quad t_i=L_i(f(t_1),\dots,f(t_{i-1}))\quad (i\ge 2).$$

Define $\operatorname{card}(A,f)$, the cardinality of A at input f, to be 0 if $\varrho_0=1$. If $\varrho_0=0$, let $\operatorname{card}(A,f)$ be the first integer $n\ge 1$ with $\varrho_n(f(t_1),\dots,f(t_n))=1$ if there is such an n, and $\operatorname{card}(A,f)=\infty$ otherwise. For $f\in C(Q,X)$ with $\operatorname{card}(A,f)<\infty$ we define the output Af of algorithm A at input f as

$$Af=\begin{cases}\varphi_0 & \text{if } n=0,\\ \varphi_n(f(t_1),\dots,f(t_n)) & \text{if } n\ge 1.\end{cases}$$

Let $r\in\mathbb{N}_0$. Given $n\in\mathbb{N}_0$, we let $\mathcal{A}_n^{\det}(B_{C^r(Q,X)},X)$ be the set of those $A\in\mathcal{A}^{\det}(C(Q,X),X)$ for which

$$\max_{f\in B_{C^r(Q,X)}}\operatorname{card}(A,f)\le n.$$

The *error of* $A\in\mathcal{A}_n^{\det}(B_{C^r(Q,X)},X)$ *as an approximation of* S^X is defined as

$$e(S^X,A,B_{C^r(Q,X)})=\sup_{f\in B_{C^r(Q,X)}}\|S^Xf-Af\|.$$

The *deterministic nth minimal error* of S^X is defined for $n\in\mathbb{N}_0$ as

$$e_n^{\det}(S^X,B_{C^r(Q,X)})=\inf_{A\in\mathcal{A}_n^{\det}(B_{C^r(Q,X)})}e(S^X,A,B_{C^r(Q,X)}).$$

It follows that no deterministic algorithm that uses at most n function values can have an error smaller than $e_n^{\det}(S^X,B_{C^r(Q,X)})$.

Next we introduce the class of randomized adaptive algorithms of varying cardinality, $\mathcal{A}_n^{\mathrm{ran}}(B_{C^r(Q,X)},X)$, consisting of tuples $A=((\Omega,\Sigma,\mathbb{P}),(A_\omega)_{\omega\in\Omega})$, where $(\Omega,\Sigma,\mathbb{P})$ is a probability space, $A_\omega\in\mathcal{A}^{\det}(C(Q,X),X)$ for all $\omega\in\Omega$, and for each $f\in B_{C^r(Q,X)}$ the mapping $\Omega\ni\omega\mapsto\operatorname{card}(A_\omega,f)$ is Σ-measurable and satisfies $\mathbb{E}\operatorname{card}(A_\omega,f)\le n$. Moreover, the mapping $\Omega\ni\omega\mapsto A_\omega f\in X$ is Σ-to-Borel measurable and essentially separably valued, i.e., there is a separable subspace $X_0\subseteq X$ such that $A_\omega f\in X_0$ for $\mathbb{P}$-almost all $\omega\in\Omega$. The *error of* $A\in\mathcal{A}_n^{\mathrm{ran}}(C(Q,X),X)$ *in approximating* S^X *on*

$B_{C^r(Q,X)}$ is defined as

$$e(S^X, A, B_{C^r(Q,X)}) = \sup_{f \in B_{C^r(Q,X)}} \mathbb{E}\, \|S^X f - A_\omega f\|,$$

and the *randomized nth minimal error* of S^X as

$$e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}) = \inf_{A \in \mathcal{A}_n^{\mathrm{ran}}(B_{C^r(Q,X)})} e(S^X, A, B_{C^r(Q,X)}).$$

Consequently, no randomized algorithm that uses (on the average) at most n function values has an error smaller than $e_n^{\mathrm{ran}}(S^X, B_{C^r(Q,X)}, X)$.

Define for $\varepsilon > 0$ the *information complexity* as

$$n_\varepsilon^{\mathrm{ran}}(S, B_{C^r(Q,X)}) = \min\{n \in \mathbb{N}_0 : e_n^{\mathrm{ran}}(S, B_{C^r(Q,X)}) \le \varepsilon\}$$

if there is such an n, and $n_\varepsilon^{\mathrm{ran}}(S, B_{C^r(Q,X)}) = \infty$ if there is no such n. Thus, if $n_\varepsilon^{\mathrm{ran}}(S, B_{C^r(Q,X)}) < \infty$, it follows that any algorithm with error $\le \varepsilon$ needs at least $n_\varepsilon^{\mathrm{ran}}(S, B_{C^r(Q,X)})$ function values, while $n_\varepsilon^{\mathrm{ran}}(S, B_{C^r(Q,X)}) = \infty$ means that no algorithm at all has error $\le \varepsilon$. The information complexity is essentially the inverse function of the nth minimal error. So determining the latter means determining the information complexity of the problem.

Let us also mention the subclasses consisting of quadrature formulas. Let $n \ge 1$. A mapping $A : C(Q,X) \to X$ is called a *deterministic quadrature formula with n nodes* if there are $t_i \in Q$ and $a_i \in \mathbb{K}$ $(1 \le i \le n)$ such that

$$Af = \sum_{i=1}^{n} a_i f(t_i) \quad (f \in C(Q,X)).$$

In terms of the definition of $\mathcal{A}^{\mathrm{det}}(C(Q,X),X)$ this means that the functions L_i and ϱ_i are constant, $\varrho_0 = \varrho_1 = \cdots = \varrho_{n-1} = 0$, $\varrho_n = 1$, and φ_n has the form $\varphi_n(x_1,\dots,x_n) = \sum_{i=1}^n a_i x_i$. Clearly, $A \in \mathcal{A}_n^{\mathrm{det}}(B_{C^r(Q,X)}, X)$.

A tuple $A = ((\Omega, \Sigma, \mathbb{P}), (A_\omega)_{\omega\in\Omega})$ is called a *randomized quadrature with n nodes* if there exist random variables $t_i : \Omega \to Q$ and $a_i : \Omega \to \mathbb{K}$ $(1 \le i \le n)$ with

$$A_\omega f = \sum_{i=1}^{n} a_i(\omega) f(t_i(\omega)) \quad (f \in C(Q,X),\, \omega \in \Omega).$$

For each such A we have $A \in \mathcal{A}_n^{\mathrm{ran}}(B_{C^r(Q,X)}, X)$. Finally we note that the algorithms $A_{n,\omega}^{r,X}$ defined in (5) and (6) are quadratures. Indeed, for $A_{n,\omega}^{0,X}$ given by (5) this is obvious. For $r \ge 1$ we represent $P_k^{r,X} \in \mathscr{L}(C(Q,X))$ as

$$P_k^{r,X} f = \sum_{j=1}^{M} f(u_j)\psi_j(t)$$

with $M \le ck^d$, $u_j \in Q$, $\psi_j \in C(Q)$ $(1 \le i \le M)$, and obtain, setting $b_j = \int_Q \psi_j(t)\,dt$,

$$(16)\quad \begin{aligned} A^{r,X}_{n,\omega} f &= S^X(P^{r,X}_k f) + A^{0,X}_{n,\omega}(f - P^{r,X}_k f) \\ &= \sum_{j=1}^M b_j f(u_j) + \frac{1}{n}\sum_{i=1}^n \big(f(\xi_i(\omega)) - (P^{r,X}_k f)(\xi_i(\omega))\big) \\ &= \sum_{j=1}^M b_j f(u_j) + \frac{1}{n}\sum_{i=1}^n f(\xi_i(\omega)) - \sum_{j=1}^M \Big(\frac{1}{n}\sum_{i=1}^n \psi_j(\xi_i(\omega))\Big) f(u_j). \end{aligned}$$

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Stefan Heinrich
Department of Computer Science
University of Kaiserslautern
D-67653 Kaiserslautern, Germany
E-mail: heinrich@informatik.uni-kl.de

Aicke Hinrichs
Institute of Analysis
Johannes Kepler University Linz
Altenberger Str. 69
A-4040 Linz, Austria
E-mail: aicke.hinrichs@jku.at

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