STUDIA MATHEMATICA 225 (1) (2014)

Two applications of smoothness in $C(K)$ spaces

by

Matías Raja (Murcia)

Abstract. A simple observation about embeddings of smooth Banach spaces into $C(K)$ spaces allows us to construct a parametrization of the separable Banach spaces using closed subsets of the interval $[0,1]$. The same idea is applied to the study of the isometric embedding of ℓ_p spaces into certain $C(K)$ spaces with the additional condition that the functions of the image must be Lipschitz with respect to a fixed finer metric on K. The feasibility of that kind of embeddings is related to Szlenk indices.

1. Introduction. Along the paper all the Banach spaces considered are real. We shall denote by K a compact Hausdorff space, and $C(K)$ will be the Banach space of real continuous functions on K endowed with the supremum norm. As usual, if X is a Banach space we shall denote by B_X its closed unit ball, and by S_X its unit sphere. For any unexplained concept or notation about Banach spaces we refer the reader to [2].

Given a subspace $X \subset C(K)$ and a closed subset $H \subset K$, we shall denote by $X|_H$ the set of restrictions of the functions of X to H, understood as elements of $C(H)$. The map $f \mapsto f|_H$ for $f \in C(K)$ is in general not injective, so any coset is identified with the same function on H. We are ready to state the first result of the paper.

Theorem 1.1. *There exists a closed linear subspace $W \subset C[0,1]$ with the following property: for any separable Banach space X, there exists a closed subset $H \subset [0,1]$ such that X is isometric to $W|_H$.*

The result claims that the range of the mapping that to a closed subset $H \subset [0,1]$ assigns the linear space $W|_H$ covers all the isometry classes of separable Banach spaces. Notice that it provides a sort of "parametrization" of the separable Banach spaces by a quite simple set of indices. The precise description of the family of closed subsets $H \subset [0,1]$ such that $W|_H$ is a Banach space is done in Proposition 2.3. Compare Theorem 1.1 to the

2010 *Mathematics Subject Classification*: Primary 46B04, 46E15; Secondary 54H05.
Key words and phrases: $C(K)$ spaces, ℓ_p spaces, isometric embedding, Szlenk index.

DOI: 10.4064/sm225-1-1

classical Banach–Mazur Theorem [2, Theorem 5.8] about the universality of $C[0,1]$ for the separable Banach spaces.

As a byproduct of the ideas behind the proof of Theorem 1.1, we give an application to the properties of the subspaces of $C(K)$ made up of functions which are Lipschitz with respect to a fixed finer metric defined on K. This topic has been discussed in our papers [4, 5]. It is an easy exercise to prove that if K is a compact metric space, then every closed subspace of $C(K)$ made of Lipschitz functions is finite-dimensional. Therefore, to avoid trivial situations, we shall always consider K equipped with a metric whose induced topology is strictly finer than the original topology on K. A typical scenario for that is a dual ball B_{X^*}, which is compact for the weak* topology, together with the metric d induced by the dual norm on X^*.

The second result that we are going to prove in this note partially solves a question motivated after [5, Proposition 4.14].

Theorem 1.2. *Let $p, q \in [1, \infty)$. The topology τ_p of pointwise convergence turns B_{ℓ_p} into a compact space. On B_{ℓ_p} we also consider the metric d induced by the norm $\|\cdot\|_p$. Then $C(B_{\ell_p})$ contains an isometric copy of ℓ_q made of functions that are Lipschitz for the metric d if and only if $(p-1)(q-1) \geq 1$.*

The isomorphic embedding of ℓ_1 as Lipschitz functions into a $C(K)$ space has been studied in [4] in relation with the *fragmentability* of K. In the case of embeddings of ℓ_p, the "speed of fragmentation" of K, which is understood in terms of the *Szlenk index*, plays a major role in the arguments (see Proposition 3.2).

2. Parametrization of separable Banach spaces. We shall use the notion of *Gâteaux smoothness* of a norm [2, Definition 7.1]. For our purposes it is enough to know that Gâteaux smoothness is equivalent, by the Šmulian Lemma [2, Corollary 7.22], to the uniqueness of norming functionals, that is, the set $\{x^* \in B_{X^*} : x^*(x) = \|x\|\}$ has only one element for every $x \in X \setminus \{0\}$.

The following result was first noticed by Donoghue [1] under stronger hypotheses and used for the construction of Peano-type filling curves.

Lemma 2.1. *Let X be an infinite-dimensional Banach space endowed with a Gâteaux smooth norm and let $J : X \to C(K)$ be an isometric embedding. Then*

$$B_{X^*} = J^*(K) \cup (-J^*(K)),$$

where J^* denotes the adjoint map from $C(K)^*$ into X^*.

Proof. Let $NA \subset S_{X^*}$ the set of norm-one attaining functionals. Given $x \in X$ and its corresponding norm attaining functional $x^* \in NA$, we have

$$\{y^* \in B_{X^*} : |y^*(x)| = \|x\|\} = \{x^*, -x^*\},$$

since the norm is Gâteaux smooth. The function $J(x)$ attains its norm at some $t \in K$, and so, since J is an isometry, $\|x\| = |J(x)(t)|$. It follows that $J^*(t) \in \{x^*, -x^*\}$. Since $x \in X$ was arbitrary, we have

$$NA \subset J^*(K) \cup (-J^*(K)).$$

Now $\overline{NA} = S_{X^*}$ by the Bishop–Phelps Theorem [2, Theorem 7.41]. As X is infinite-dimensional,

$$B_{X^*} = \overline{S_{X^*}}^{w^*} = \overline{NA}^{w^*} \subset J^*(K) \cup (-J^*(K)),$$

finishing the proof, since the other inclusion is trivial. ∎

The above lemma has a simpler proof—skipping the use of the Bishop–Phelps theorem—if we make the stronger assumption that X^* is strictly convex. Note that every separable Banach space has an equivalent Gâteaux norm, which can be obtained by a strictly convex dual renorming of its dual [2, Corollary 7.23].

Proof of Theorem 1.1. Let W be the space ℓ_1 with a Gâteaux smooth equivalent norm. By the Banach–Mazur Theorem [2, Theorem 5.8] we may find W isometrically inside $C[0,1]$. Let J be the inclusion mapping of W into $C[0,1]$.

Given a separable Banach space X, there is an onto linear operator $T : W \to X$ with $\|T\| \leq 1$, since every separable Banach space is isometric to a quotient of ℓ_1 [2, Theorem 5.1]. For the adjoint operator we have $\|T^*\| = \|T\| \leq 1$ and thus

$$T^*(B_{X^*}) \subset B_{W^*} = J^*([0,1]) \cup (-J^*([0,1])).$$

Take $H = \{t \in [0,1] : J^*(t) \in T^*(B_{X^*})\}$. Obviously,

$$T^*(B_{X^*}) = J^*(H) \cup (-J^*(H)).$$

Given any $w \in W$, we have

$$\|T(w)\| = \sup_{x^* \in B_{X^*}} T^*(x^*)(w) = \sup_{w^* \in T^*(B_{X^*})} w^*(w) = \sup_{t \in H} |J(w)(t)|.$$

This implies that X is isometric to $W|_H$. ∎

Remark 2.2. Given a closed subspace $W \subset C(K)$ and a closed subset $H \subset K$, in general $W|_H$ is not a closed subspace of $C(H)$. As a matter of fact, in the proof of Theorem 1.1 we may suppose that X is the range of a bounded linear operator defined on ℓ_1 (or any separable Banach space) in order to obtain an isometry onto a linear space of the form $W|_H$.

The parametrization of the class of separable Banach spaces provided by Theorem 1.1 will be completed with a suitable description of the set of indices. We shall denote by $\mathcal{F}(K)$ the family of nonempty closed subsets of a metrizable compact space K. Endowed with the *Vietoris topology*, $\mathcal{F}(K)$ becomes a metrizable compact space, and its associated Borel σ-algebra coincides with the *Effros Borel structure*. Recall that the Vietoris topology of $\mathcal{F}(K)$ is generated by the sets of the form $\{H \in \mathcal{F}(K) : H \subset U\}$ and $\{H \in \mathcal{F}(K) : H \cap U \neq \emptyset\}$ where $U \subset K$ is open. We address the reader to [6] for additional definitions and more information about these topics.

PROPOSITION 2.3. *Let K be a compact metric space and let $W \subset C(K)$ be a closed subspace. Then the set*

$$D = \{H \in \mathcal{F}(K) : W|_H \text{ is Banach}\}$$

is Borel with respect to the Vietoris topology on $\mathcal{F}(K)$.

Proof. Fix a dense sequence $(f_k)_{k\in\mathbb{N}} \subset W$. The subsets of K defined by

$$U(m,k,j) = \{x \in K : |f_k(x) - f_j(x)| < 1/m\},$$
$$V(n,m,k,j) = \{x \in K : \|f_j\| < n|f_k(x)| + 1/m\}$$

for $n,m,k,j \in \mathbb{N}$ are open. Lemma 2.4 below applied to the restriction operator $T_H : W \to C(H)$, which is defined as $T_H(f) = f|_H$ for $H \in \mathcal{F}(K)$, implies that

$$D = \bigcup_{n\in\mathbb{N}} \bigcap_{m,k\in\mathbb{N}} \bigcup_{j\in\mathbb{N}} \{H \in \mathcal{F}(K) : H \subset U(m,k,j),\ H \cap V(n,m,k,j) \neq \emptyset\}.$$

Hence D is a $\mathcal{G}_{\delta\sigma}$ set in the Vietoris topology, and so it is Borel. ∎

LEMMA 2.4. *Let X and Y be separable Banach spaces and let $(x_k)_{k\in\mathbb{N}} \subset X$ be a dense sequence. Then $T(X)$ is closed in Y if and only if there is $\beta > 0$ such that, for every $\varepsilon > 0$ and every $k \in \mathbb{N}$, there is $j \in \mathbb{N}$ with the property that $\|T(x_k) - T(x_j)\| < \varepsilon$ and $\|x_j\| < \beta\|T(x_k)\| + \varepsilon$.*

Proof. If $T(X)$ is closed in Y, then T is open onto $T(X)$ by the open mapping principle [2, Theorem 2.25]. Hence there is $\beta > 0$ such that for every $y \in T(X)$, there is $x \in X$ such that $T(x) = y$ and $\|x\| \le \beta\|y\|$. Now set $y = T(x_k)$ and find $j \in \mathbb{N}$ such that $\|x - x_j\| < \min\{\varepsilon, \|T\|^{-1}\varepsilon\}$. Then $\|T(x_k) - T(x_j)\| < \varepsilon$ and $\|x_j\| \le \|x\| + \|x - x_j\| < \beta\|T(x_k)\| + \varepsilon$.

For the converse, let $y \in Z := \overline{T(X)}$ with $\|y\| \le 1$. Find a subsequence $(x_{k_n})_{n\in\mathbb{N}}$ such that $y = \lim_n x_{k_n}$. We may assume $\|T(x_{k_n})\| < 2$ for all $n \in \mathbb{N}$. Find, according to our assumption, x_{j_n} such that $\|x_{j_n}\| \le 2\beta + 1/n$ with $\|T(x_{k_n}) - T(x_{j_n})\| < 1/n$. Then $\lim_n T(x_{j_n}) = y$ and $\|x_{j_n}\| < \alpha := 2\beta + 1$. This shows that $B_Z \subset \overline{T(\alpha B_X)}$. By [2, Lemma 2.24], we get $\lambda B_Z \subset T(\alpha B_X)$ for some $\lambda \in (0,1)$, and so T is an open mapping from X into Z. This shows, in particular, that $T(X)$ is closed. ∎

3. Smooth subspaces and finite Szlenk indices. We need to introduce several notions. In all that follows, we shall consider a pair (K,d)

consisting of a compact space K and a metric d on K whose induced topology is strictly finer than the original topology on K. Let 'diam' denote the diameter measured with respect to d. For any subset $A \subset K$ consider the *derived set*

$$\langle A\rangle'_\varepsilon = \{x \in A : \operatorname{diam}(A \cap U) \geq \varepsilon \text{ for each neighbourhood of } x\}.$$

By iteration, the sets $\langle A\rangle^\gamma_\varepsilon$ are defined for any ordinal γ, taking intersection in the case of limit ordinals. The Szlenk indices of K (with respect to d) are ordinal numbers defined by

$$\mathrm{Sz}(K,\varepsilon) = \inf\{\gamma : \langle K\rangle^\gamma_\varepsilon = \emptyset\}$$

if such an ordinal γ exists, otherwise we say that $\mathrm{Sz}(K,\varepsilon) = \infty$ (beyond ordinals). We say that K *has Szlenk index at most* ω if $\mathrm{Sz}(K,\varepsilon) < \omega$ for every $\varepsilon > 0$. For instance, the closed balls of superreflexive Banach spaces endowed with the weak topology have Szlenk index at most ω with respect to the norm metric. Note that the standard Szlenk index of a Banach space X is defined dually as $\sup_{\varepsilon>0} \mathrm{Sz}(B_{X^*},\varepsilon)$ and it has many applications in isomorphic theory of Banach spaces (see [3]). The "bitopological" version of the Szlenk index that we will use here has been studied in [5].

Finally, $L(K,d)$ stands for the set of real functions defined on K which are Lipschitz with respect to the metric d. If d is lower semicontinuous, then $C(K) \cap L(K,d)$ separates the points of K.

The next lemma contains the properties of the Szlenk index that we shall use here.

LEMMA 3.1. *Let (K,d) be a compact space together with an associated metric.*

(a) $\mathrm{Sz}(K,\varepsilon) \leq \max\{\mathrm{Sz}(A_i,\varepsilon/2) : i = 1,\dots,n\}$ *whenever $A_i \subset K$ are closed with $K = \bigcup_{i=1}^n A_i$ and $\varepsilon > 0$.*

(b) *Let $(\tilde{K},\tilde{d})$ be a compact space with an associated metric such that there exists a continuous surjection of K onto $\tilde{K}$ which is Lipschitz for the two metrics. Then there exists $a > 0$ such that* $\mathrm{Sz}(\tilde{K},\varepsilon) \leq \mathrm{Sz}(K,a\varepsilon)$ *for any $\varepsilon > 0$.*

Hint to the proof. Replacing the diameter by the *Kuratowski measure of noncompactness* in the definition of the derived set above, we obtain a new ordinal index denoted by $\mathrm{Sk}(K,\varepsilon)$ (see the details in [5]). The relation between the functions Sk and Sz is given by the inequality

$$\mathrm{Sz}(K,2\varepsilon) \leq \mathrm{Sk}(K,\varepsilon) \leq \mathrm{Sz}(K,\varepsilon).$$

Statement (a) follows from the fact that $\mathrm{Sk}(K,\varepsilon) = \max_{1\leq i\leq n} \mathrm{Sk}(A_i,\varepsilon)$ [5, Proposition 2.5]. On the other hand, (b) follows from [5, Corollary 2.11], saying that $\mathrm{Sk}(\tilde{K},\varepsilon) \leq \mathrm{Sk}(K,\varepsilon/\lambda)$ where λ the Lipschitz constant of the surjection. ∎

This result is an improvement of [5, Theorem 4.4] under stronger assumptions.

Proposition 3.2. *Let (K, d) have Szlenk index at most ω. If X is a Banach space endowed with a Gâteaux smooth norm which embeds isometrically into $C(K)$ as a subset of $L(K, d)$, then*

$$\mathrm{Sz}(B_{X^*}, \varepsilon) \leq \mathrm{Sz}(K, c\,\varepsilon)$$

for some $c > 0$ and every $\varepsilon > 0$.

Proof. Without loss of generality we may assume that X is of infinite dimension. Let $J : X \to C(K)$ be the embedding and $J^* : C(K)^* \to X^*$ its adjoint. A suitable use of the Baire category theorem implies that there is a common Lipschitz bound $\lambda > 0$ for all the functions of $J(B_X)$. The set $J^*(K)$ is a weak* compact subset of X^* such that $B_{X^*} = J^*(K) \cup (-J^*(K))$ by Lemma 2.1. We claim that $J^*(K)$ is also a Lipschitz image of K. Indeed, if $x \in B_X$ and $t_1, t_2 \in K$ then

$$|J^*(t_1)(x) - J^*(t_2)(x)| = |J(x)(t_1) - J(x)(t_2)| \leq \lambda d(t_1, t_2).$$

Taking the supremum over $x \in B_X$ we get $\|J^*(t_1) - J^*(t_2)\| \leq \lambda\, d(t_1, t_2)$. Then $\mathrm{Sz}(J^*(K), \varepsilon) \leq \mathrm{Sz}(K, a\varepsilon)$ by Lemma 3.1(b). Applying now Lemma 3.1(a), we have $\mathrm{Sz}(B_{X^*}, \varepsilon) \leq \mathrm{Sz}(J^*(K), \varepsilon/2)$ and the conclusion of the proof is straightforward. ∎

Remark 3.3. If (K, d) has Szlenk index at most ω, and the Banach space X embeds isomorphically into $C(K)$ as a subset of $L(K, d)$, then B_{X^*} has Szlenk index at most ω by [5, Theorem 4.4]. In particular, X^* admits an equivalent locally uniformly rotund dual norm [3, Theorem 13] and therefore X is Fréchet smoothable.

Proof of Theorem 1.2. Let q' denote the conjugate exponent of q, that is, $1/q + 1/q' = 1$. Clearly, the inequality $(p-1)(q-1) \geq 1$ is equivalent to $q' \leq p$. Consider the Mazur mapping $\varphi_{p,q'} : B_{\ell_p} \to B_{\ell_{q'}}$ defined by

$$\varphi_{p,q'}((x_n)_{n\in\mathbb{N}}) := (\mathrm{sign}(x_n)|x_n|^{p/q'})_{n\in\mathbb{N}},$$

which is Lipschitz for $q' \leq p$ (see [2, proof of Theorem 12.50]). The natural embedding of ℓ_q into $C(B_{\ell_{q'}})$ composed with the Mazur mapping will provide an isometric embedding of ℓ_q as Lipschitz functions. On the other hand, if X is a Gâteaux smooth subspace of $C(B_{\ell_p})$, then

$$\mathrm{Sz}(B_{X^*}, \varepsilon) \leq \mathrm{Sz}(B_{\ell_p}, c\varepsilon) \leq c\varepsilon^{-p}$$

for some $c > 0$ by Proposition 3.2, and so ℓ_q does not embed as Lipschitz functions if $q' \in (p, \infty)$ because $\mathrm{Sz}(B_{\ell_{q'}}, \varepsilon) \geq a\varepsilon^{-q'}$ for some $a > 0$ (see [5, Example 4.10]). If $q' = \infty$, then $\mathrm{Sz}(B_{\ell_\infty}) = \infty$ since ℓ_1 is not Asplund (see [3, Theorem 2]), and so ℓ_1 does not embed as Lipschitz functions into $C(B_{\ell_p})$. ∎

Acknowledgements. I want to express my gratitude to Gilles Godefroy for pointing out Remark 2.2 and suggesting the study of the Borel complexity of the set D in Proposition 2.3. I am also grateful to the anonymous referee for the helpful suggestions to improve the exposition style. This research was partly supported by the MEC/FEDER project MTM2011-25377.

References

[1] W. F. Donoghue, *Continuous function spaces isometric to a Hilbert space*, Proc. Amer. Math. Soc. 8 (1957), 1–2.

[2] M. Fabian, P. Habala, P. Hájek, V. Montesinos and V. Zizler, *Banach Space Theory. The Basis for Linear and Nonlinear Analysis*, CMS Books in Math., Springer, New York, 2011.

[3] G. Lancien, *A survey on the Szlenk index and some of its applications*, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. 100 (2006), 209–235.

[4] M. Raja, *Embedding ℓ_1 as Lipschitz functions*, Proc. Amer. Math. Soc. 133 (2005), 2395–2400.

[5] M. Raja, *Compact spaces of Szlenk index ω*, J. Math. Anal. Appl. 391 (2012), 496–509.

[6] S. M. Srivastava, *A Course on Borel Sets*, Grad. Texts in Math. 180, Springer, New York, 1998.

Matías Raja
Departamento de Matemáticas
Universidad de Murcia
Campus de Espinardo
30100 Espinardo, Murcia, Spain
E-mail: matias@um.es

Received February 1, 2014
Revised version December 8, 2014 (7910)