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Multiple summing operators on l_p spaces

by

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Abstract. We use the Maurey–Rosenthal factorization theorem to obtain a new characterization of multiple 2-summing operators on a product of l_p spaces. This characterization is used to show that multiple s-summing operators on a product of l_p spaces with values in a Hilbert space are characterized by the boundedness of a natural multilinear functional ($1 \leq s \leq 2$). We use these results to show that there exist many natural multiple s-summing operators $T : l_{4/3} \times l_{4/3} \to l_2$ such that none of the associated linear operators is s-summing ($1 \leq s \leq 2$). Further we show that if $n \geq 2$, there exist natural bounded multilinear operators $T : l_{2n/(n+1)} \times \cdots \times l_{2n/(n+1)} \to l_2$ for which none of the associated multilinear operators is multiple s-summing ($1 \leq s \leq 2$).

1. Introduction and background. The notion of absolutely summing operators was first introduced by A. Grothendieck in his "*Résumé*" [11] under the name "semi-intégrale à droite" and later on, in two other cornerstone papers of A. Pietsch [23] and J. Lindenstrauss and A. Pełczyński [15]. The concept of the absolutely summing operator is a fundamental part of the theory of operator ideals, introduced by A. Pietsch in the linear case. We recommend the reader to consult the celebrated monographs [6, 10, 24, 26, 35].

Motivated by the importance of the theory of absolutely summing operators in recent years, in [3] and independently in [16], this concept was generalized to the multilinear setting via the class of multiple summing operators. Its roots are in the famous paper of H. F. Bohnenblust and E. Hille [2]. Most of the main properties of the linear analogue are true in this new context (see [3, 16, 19, 20, 27, 28, 29, 30, 31]), and several applications were found in different fields.

For instance, in [7] it is applied in vector valued Dirichlet series, in [8] it helps to understand the behavior of unconditionality in tensor products, in [9] it is applied to improve the Bohnenblust–Hille results, and [22] exhibits

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natural connections of this class of operators to problems in mathematical physics.

In this paper we continue the study of multiple summing multilinear operators. We use the Maurey–Rosenthal factorization theorem to obtain a new characterization of multiple 2-summing operators on a product of l_p spaces (Theorem 1). This characterization is used to show that multiple s-summing operators ($1 \leq s \leq 2$) on a product of l_p spaces with values in a Hilbert space are characterized by the boundedness of a natural multilinear functional (Theorem 2). We apply these results to give concrete examples of bilinear and multilinear multiple s-summing operators for which the associated multilinear operators are not multiple s-summing ($1 \leq s \leq 2$) (Proposition 3, Corollaries 5 and 8). As far as we know, these are the first examples of this type.

Now we fix some notation and terminology. Throughout this paper, $X_1, \ldots, X_n$, Y etc. denote Banach spaces over $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$. For X a Banach space, X^* is the dual of X. By $I : X \to X$ we denote the identity operator, $I(x) = x$. For $1 \leq p < \infty$ and $x_1, \ldots, x_m \in X$, we write

$$w_p((x_i)_{1 \leq i \leq m}) = \sup_{\|x^*\| \leq 1} \Big(\sum_{i=1}^{m} |x^*(x_i)|^p \Big)^{1/p}.$$

Let $1 \leq p < \infty$. A bounded linear operator $T : X \to Y$ is *p-summing* if there exists a constant $C \geq 0$ such that for every $x_1, \ldots, x_n \in X$,

$$\Big(\sum_{i=1}^{n} \|T(x_i)\|^p \Big)^{1/p} \leq C w_p((x_i)_{1 \leq i \leq n}),$$

and the *p-summing norm* of T is $\pi_p(T) = \inf\{C \mid C \text{ as above}\}$. We denote by $\Pi_p(X, Y)$ the class of p-summing operators (see [6, 10, 24, 26, 35]).

Let n be a natural number and $1 \leq p < \infty$. A bounded n-linear operator $U : X_1 \times \cdots \times X_n \to Y$ is called *multiple p-summing* if there exists a constant $C \geq 0$ such that for every choice of elements $(x^j_{i_j})_{1 \leq i_j \leq m_j} \subset X_j$ ($1 \leq j \leq n$),

$$\Big(\sum_{i_1, \ldots, i_n = 1}^{m_1, \ldots, m_n} \|U(x^1_{i_1}, \ldots, x^n_{i_n})\|^p \Big)^{1/p} \leq C w_p((x^1_{i_1})_{1 \leq i_1 \leq m_1}) \cdots w_p((x^n_{i_n})_{1 \leq i_n \leq m_n})$$

and the *multiple p-summing norm* of U is $\pi^{\text{mult}}_p(U) = \inf\{C \mid C \text{ as above}\}$. We denote by $\Pi^{\text{mult}}_p(X_1, \ldots, X_n; Y)$, or $\Pi^{\text{mult}}_p(\prod_{i=1}^n X_i; Y)$, the class of all multiple p-summing operators from $X_1 \times \cdots \times X_n = \prod_{i=1}^n X_i$ into Y. We remark that for $n = 1$ we get the definition of p-summing linear operators.

Let $H_1, \ldots, H_n$, H be Hilbert spaces. A bounded multilinear operator $T : H_1 \times \cdots \times H_n \to H$ is said to be *Hilbert–Schmidt* if there is an orthonormal basis $(e^j_{i_j})_{i_j\in I_j}\subset H_j$ $(1\le j\le n)$ such that

$$\|T\|_{\mathrm{HS}}=\Big(\sum_{i_1\in I_1,\dots,i_n\in I_n}\|T(e^1_{i_1},\dots,e^n_{i_n})\|^2\Big)^{1/2}<\infty.$$

By $\mathrm{HS}(H_1,\dots,H_n;H)$ we denote the class of all Hilbert–Schmidt operators $T:H_1\times\dots\times H_n\to H$. We will use the fact that $\mathrm{HS}(H_1,\dots,H_n;H)=\Pi_2^{\mathrm{mult}}(H_1,\dots,H_n;H)$ (see [16], [19]).

For the following theorem, see [10, Corollary 11.16(c)] and [26] for the linear case, and [27, Theorem 10(c)] for the multilinear case,

Coincidence Theorem.

(i) *If X and Y have cotype 2, then*

$$\Pi_s(X,Y)=\Pi_1(X,Y)\quad \textit{for all } 1\le s<\infty.$$

(ii) *If all $X_1,\dots,X_n$ have cotype 2 and Y has also cotype 2, then*

$$\Pi_s^{\mathrm{mult}}(X_1,\dots,X_n;Y)=\Pi_2^{\mathrm{mult}}(X_1,\dots,X_n;Y)\quad \textit{for all } 1\le s\le 2.$$

If $a=(a_n)_{n\in\mathbb{N}}$, $b=(b_n)_{n\in\mathbb{N}}$ are two scalar sequences, we write $ab=(a_nb_n)_{n\in\mathbb{N}}$. For $a=(a_n)_{n\in\mathbb{N}}$ a scalar sequence, we denote by M_a the multiplication operator which acts between two sequence spaces and is defined by $M_a(x)=ax$. As is well known, if $1\le q,p<\infty$ then $M_a:l_q\to l_p$ is well defined if and only if $a\in l_\infty$ for $q\le p$, or $a\in l_r$ for $p<q$, where $1/p=1/q+1/r$.

If $1\le p<\infty$ then p^* denotes the conjugate of p, i.e. $1/p+1/p^*=1$, and $(e_n)_{n\in\mathbb{N}}$ are the standard unit vectors in l_p.

Let $1\le p<\infty$ and X be a Banach space. We write $l_p(X)$ to denote the Banach space of all sequences $(x_n)_{n\in\mathbb{N}}\subset X$ with $\sum_{n=1}^\infty\|x_n\|^p<\infty$, endowed with the norm

$$\|(x_n)_{n\in\mathbb{N}}\|_{l_p(X)}=\Big(\sum_{n=1}^\infty\|x_n\|^p\Big)^{1/p}.$$

We consider the canonical mappings $\sigma_n:X\to l_p(X)$ defined by

$$\sigma_n(x)=(0,\dots,0,\underbrace{x}_{n\mathrm{th}},0,\dots),$$

where n is a natural number.

We recall (see [17, Proposition 43, p. 68 and Proposition 44, p. 70] or [5])

Maurey–Rosenthal Factorization Theorem. *Let $1\le p<2\le r<\infty$ be such that $1/p=1/2+1/r$ and (Ω,Σ,μ) a measure space. If X has type 2 and Y has cotype 2, then each bounded linear operator $U:X\to L_p(\mu,Y)$ has a factorization of the form*

$$X\xrightarrow{V}L_2(\mu,Y)\xrightarrow{M_g}L_p(\mu,Y)$$

with V bounded linear, $g \in L_r(\mu)$ and $M_g(f) = gf$. Moreover,

$$\|U\| \le \|V\|\,\|g\|_r \le K_{p,2}C_2(Y)T_2(X)\|U\|,$$

where $K_{p,2}$ is the Kahane–Khinchin constant and $T_2(X)$, $C_2(Y)$ are the type 2 constant of X and the cotype 2 constant of Y.

Taking as a measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mathrm{card})$, we get

COROLLARY 1. *Let $1 \le p < 2 \le r < \infty$ be such that $1/p = 1/2 + 1/r$. If X has type 2 and Y has cotype 2, then each bounded linear operator $U : X \to l_p(Y)$ has a factorization of the form*

$$X \xrightarrow{V} l_2(Y) \xrightarrow{M_a} l_p(Y)$$

with V bounded linear and $a \in l_r$. Moreover,

$$\|U\| \le \|V\|\,\|a\|_r \le K_{p,2}C_2(Y)T_2(X)\|U\|.$$

2. Main results. Our first result is a multilinear extension of a well known result in the linear case (see [6, Proposition 11.8, p. 136], [10, Proposition 2.7, p. 39], [35, Proposition 9.7, p. 50]). A proof can be found in [33, Proposition 2.2].

PROPOSITION 1. *Let $1 \le p < \infty$, $1 \le k \le n$ and $T : X_1 \times \cdots \times X_n \to Y$ be a bounded n-linear operator. The following assertions are equivalent:*

(i) *T is multiple p-summing.*
(ii) *The operator*

$$T \circ (S_1, \ldots, S_k, I_{X_{k+1}}, \ldots, I_{X_n}) : l_{p^*} \times \cdots \times l_{p^*} \times X_{k+1} \times \cdots \times X_n \to Y$$

is multiple p-summing for any bounded linear operators $S_j : l_{p^*} \to X_j$ $(\le j \le k)$.

Moreover,

$$\sup_{\|S_1\|,\ldots,\|S_k\| \le 1} \pi_p^{\mathrm{mult}}((T \circ (S_1, \ldots, S_k, I_{X_{k+1}}, \ldots, I_{X_n}))) = \pi_p^{\mathrm{mult}}(T).$$

For $p = 1$ we consider c_0 instead of l_{p^*}.

The following result is the main result of our paper. This result was suggested by Nahoum's theorem [18, p. 4]. Later, we have observed that the same idea was used in a different context by A. Defant and D. Pérez-García in [8, proof of Lemma 4.5].

THEOREM 1. *Let $1 \le k \le n$ and $1 \le p_1, \ldots, p_k < 2 \le r_1, \ldots, r_k < \infty$ be such that $1/p_j = 1/2 + 1/r_j$ for $1 \le j \le k$. Let $X_1, \ldots, X_k$ be Banach spaces of cotype 2, and $X_{k+1}, \ldots, X_n$ be arbitrary Banach spaces. Finally, let*

$$T : l_{p_1}(X_1) \times \cdots \times l_{p_k}(X_k) \times X_{k+1} \times \cdots \times X_n \to Y$$

be a bounded n-linear operator. The following assertions are equivalent:

(i) *T is multiple 2-summing.*
(ii) *The operator*

$$T\circ(M_{a_1},\dots,M_{a_k},I_{X_{k+1}},\dots,I_{X_n}): \\ l_2(X_1)\times\dots\times l_2(X_k)\times X_{k+1}\times\dots\times X_n\to Y$$

is multiple 2-summing for all $a_1\in l_{r_1},\dots,a_k\in l_{r_k}$.

Moreover,

$$\sup_{\|a_1\|_{r_1},\dots,\|a_k\|_{r_k}\le 1}\pi_2^{\mathrm{mult}}(T\circ(M_{a_1},\dots,M_{a_k},I_{X_{k+1}},\dots,I_{X_n})) \\ \le \pi_2^{\mathrm{mult}}(T)\le K_{p_1,2}\cdots K_{p_k,2}C_2(X_1)\cdots C_2(X_k) \\ \times \sup_{\|a_1\|_{r_1},\dots,\|a_k\|_{r_k}\le 1}\pi_2^{\mathrm{mult}}(T\circ(M_{a_1},\dots,M_{a_k},I_{X_{k+1}},\dots,I_{X_n})).$$

Proof. (ii)⇒(i). From (ii) and the uniform boundedness principle,

$$L=\sup_{\|a_1\|_{r_1},\dots,\|a_k\|_{r_k}\le 1}\pi_2^{\mathrm{mult}}(T\circ(M_{a_1},\dots,M_{a_k},I_{X_{k+1}},\dots,I_{X_n}))<\infty.$$

Let $S_j: l_2\to l_{p_j}(X_j)$ be bounded linear operators ($1\le j\le k$). Since l_2 has type 2 (with $T_2(l_2)=1$), $1\le p_j<2$ and X_j has cotype 2 ($1\le j\le k$), from the Maurey–Rosenthal factorization theorem, more precisely from Corollary 1, it follows that there exist bounded linear operators $V_j: l_2\to l_2(X_j)$ and $b_j\in l_{r_j}$ such that

$$S_j=M_{b_j}\circ V_j \quad\text{and}\quad \|V_j\|\,\|b_j\|_r\le K_{p_j,2}C_2(X_j)\|S_j\| \quad \text{for } 1\le j\le k.$$

Note that

$$T\circ(S_1,\dots,S_k,I_{X_{k+1}},\dots,I_{X_n}) \\ =T\circ(M_{b_1},\dots,M_{b_k},I_{X_{k+1}},\dots,I_{X_n})\circ(V_1,\dots,V_k,I_{X_{k+1}},\dots,I_{X_n}).$$

Since by (ii),

$$T\circ(M_{b_1},\dots,M_{b_k},I_{X_{k+1}},\dots,I_{X_n}): l_2(X_1)\times\dots\times l_2(X_k)\times X_{k+1}\times\dots\times X_n\to Y$$

is multiple 2-summing, by the ideal property of the class of multiple 2-summing operators,

$$T\circ(S_1,\dots,S_k,I_{X_{k+1}},\dots,I_{X_n}): l_2(X_1)\times\dots\times l_2(X_k)\times X_{k+1}\times\dots\times X_n\to Y$$

is multiple 2-summing and

$$\pi_2^{\mathrm{mult}}(T\circ(S_1,\dots,S_k,I_{X_{k+1}},\dots,I_{X_n})) \\ \le \pi_2^{\mathrm{mult}}(T\circ(M_{b_1},\dots,M_{b_k},I_{X_{k+1}},\dots,I_{X_n}))\|V_1\|\cdots\|V_k\| \\ \le L\|b_1\|_{r_1}\cdots\|b_k\|_{r_k}\|V_1\|\cdots\|V_k\| \\ \le K_{p_1,2}\cdots K_{p_k,2}C_2(X_1)\cdots C_2(X_k)L\|S_1\|\cdots\|S_k\|.$$

Since $S_1, \dots, S_k$ are arbitrary, Proposition 1 ensures that T is multiple 2-summing and $\pi_2^{\mathrm{mult}}(T) \le K_{p_1,2} \cdots K_{p_k,2} C_2(X_1) \cdots C_2(X_k) L$, which is (i).

(i)⇒(ii). This follows from the ideal property of the class of multiple 2-summing operators. ■

REMARK 1. In the case $k = n$ in Proposition 1(ii) and Theorem 1, the factors $X_{k+1}, \dots, X_n$ do not occur.

LEMMA 1. *Let $H_1, \dots, H_k$, H be Hilbert spaces and $T : l_2(H_1) \times \cdots \times l_2(H_k) \to H$ a bounded k-linear operator. Then T is Hilbert–Schmidt if and only if all $T \circ (\sigma_{i_1}, \dots, \sigma_{i_k}) : H_1 \times \cdots \times H_k \to H$ are Hilbert–Schmidt and $\sum_{i_1,\dots,i_k=1}^{\infty} \|T \circ (\sigma_{i_1}, \dots, \sigma_{i_k})\|_{\mathrm{HS}}^2 < \infty$. Moreover,*

$$\|T\|_{\mathrm{HS}}^2 = \sum_{i_1,\dots,i_k=1}^{\infty} \|T \circ (\sigma_{i_1}, \dots, \sigma_{i_k})\|_{\mathrm{HS}}^2.$$

Proof. The conclusion follows from the definition of Hilbert–Schmidt operators and the fact that if H is a Hilbert space and $(e_j)_{j\in J}$ is an orthonormal basis in H, then $(\sigma_i(e_j))_{(i,j)\in\mathbb{N}\times J}$ is an orthonormal basis in $l_2(H)$. We omit the details. ■

As an application of Theorem 1 we prove a connection between multiple s-summing operators on a product of l_p spaces with values in a Hilbert space and the boundedness of a natural multilinear functional.

THEOREM 2. *Let k be a natural number and $1 \le p_1, \dots, p_k < 2 \le r_1, \dots, r_k < \infty$ be such that $1/p_j = 1/2 + 1/r_j$ for each $1 \le j \le k$. Let $H_1, \dots, H_k$, H be Hilbert spaces, $T : l_{p_1}(H_1) \times \cdots \times l_{p_k}(H_k) \to H$ a bounded k-linear operator and $1 \le s \le 2$. The following assertions are equivalent:*

(i) *T is multiple 2-summing.*
(ii) *All $T \circ (\sigma_{i_1}, \dots, \sigma_{i_k}) : H_1 \times \cdots \times H_k \to H$ are Hilbert–Schmidt and the k-linear functional $S : l_{r_1/2} \times \cdots \times l_{r_k/2} \to \mathbb{K}$ defined by*

$$S(x_1, \dots, x_k) = \sum_{i_1,\dots,i_k=1}^{\infty} \langle x_1, e_{i_1}\rangle \cdots \langle x_k, e_{i_k}\rangle \|T \circ (\sigma_{i_1}, \dots, \sigma_{i_k})\|_{\mathrm{HS}}^2$$

is bounded.
(iii) *T is multiple s-summing.*

Moreover, $\sqrt{\|S\|} \le \pi_2^{\mathrm{mult}}(T) \le K_{p_1,2} \cdots K_{p_k,2} \sqrt{\|S\|}$.

Proof. (i)⇒(ii). Since T is multiple 2-summing, from the ideal property of the multiple 2-summing operators it follows that all $T \circ (\sigma_{i_1}, \dots, \sigma_{i_k})$ are Hilbert–Schmidt. Let $(a_1, \dots, a_k) \in l_{r_1} \times \cdots \times l_{r_k}$ be such that $\|a_1\|_{r_1}, \dots, \|a_k\|_{r_k} \le 1$. Again by the ideal property of the multiple 2-summing operators, $T \circ (M_{a_1}, \dots, M_{a_k}) : l_2(H_1) \times \cdots \times l_2(H_k) \to H$ is multiple 2-summing and $\pi_2^{\mathrm{mult}}(T \circ (M_{a_1}, \dots, M_{a_k})) \le \pi_2^{\mathrm{mult}}(T)$. Since on Hilbert spaces, the multiple 2-summing operators coincide with the Hilbert–Schmidt operators, from Lemma 1 we deduce

$$(*)\qquad \Big(\sum_{i_1,\ldots,i_k=1}^{\infty} |\langle a_1, e_{i_1}\rangle|^2 \cdots |\langle a_k, e_{i_k}\rangle|^2 \|T\circ(\sigma_{i_1},\ldots,\sigma_{i_k})\|_{\mathrm{HS}}^2\Big)^{1/2} = \pi_2^{\mathrm{mult}}(T\circ(M_{a_1},\ldots,M_{a_k})) \le \pi_2^{\mathrm{mult}}(T).$$

Now let $(x_1,\ldots,x_k) \in l_{r_1/2}\times\cdots\times l_{r_k/2}$ be such that $\|x_1\|_{r_1/2},\ldots,\|x_k\|_{r_k/2} \le 1$. Choose $a_1,\ldots,a_k$ such that

$$|\langle a_1, e_{i_1}\rangle|^2 = |\langle x_1, e_{i_1}\rangle| \quad \text{for each } i_1\in\mathbb{N},$$
$$\vdots$$
$$|\langle a_k, e_{i_k}\rangle|^2 = |\langle x_k, e_{i_k}\rangle| \quad \text{for each } i_k\in\mathbb{N},$$

and note that $\|a_1\|_{r_1} = \|x_1\|_{r_1/2},\ \ldots,\ \|a_k\|_{r_k} = \|x_k\|_{r_k/2}$. Then from $(*)$ we get

$$\sum_{i_1,\ldots,i_k=1}^{\infty} |\langle x_1, e_{i_1}\rangle| \cdots |\langle x_k, e_{i_k}\rangle|\, \|T\circ(\sigma_{i_1},\ldots,\sigma_{i_k})\|_{\mathrm{HS}}^2 \le [\pi_2^{\mathrm{mult}}(T)]^2.$$

This means that $S : l_{r_1/2}\times\cdots\times l_{r_k/2}\to\mathbb{K}$ is bounded k-linear and $\|S\| \le [\pi_2^{\mathrm{mult}}(T)]^2$, proving (ii).

(ii)⇒(i). Since S is bounded k-linear,

$$(**)\qquad \Big|\sum_{i_1,\ldots,i_k=1}^{\infty} \langle x_1, e_{i_1}\rangle \cdots \langle x_k, e_{i_k}\rangle \|T\circ(\sigma_{i_1},\ldots,\sigma_{i_k})\|_{\mathrm{HS}}^2\Big| \le \|S\| \quad \text{for } \|x_1\|_{r_1/2},\ldots,\|x_k\|_{r_k/2}\le 1.$$

Let $a_1\in l_{r_1},\ldots,a_k\in l_{r_k}$ be such that $\|a_1\|_{r_1},\ldots,\|a_k\|_{r_k}\le 1$. Then

$$(|\langle a_1, e_{i_1}\rangle|^2)_{i_1\in\mathbb{N}} \in l_{r_1/2},\ \ldots,\ (|\langle a_k, e_{i_k}\rangle|^2)_{i_k\in\mathbb{N}} \in l_{r_k/2},$$

and from $(**)$ we obtain

$$(***)\qquad \sum_{i_1,\ldots,i_k=1}^{\infty} |\langle a_1, e_{i_1}\rangle|^2 \cdots |\langle a_k, e_{i_k}\rangle|^2 \|T\circ(\sigma_{i_1},\ldots,\sigma_{i_k})\|_{\mathrm{HS}}^2 \le \|S\|.$$

Again since on Hilbert spaces the multiple 2-summing operators coincide with the Hilbert–Schmidt operators, from Lemma 1 we deduce

$$\pi_2^{\mathrm{mult}}(T\circ(M_{a_1},\ldots,M_{a_k})) = \Big(\sum_{i_1,\ldots,i_k=1}^{\infty} \|T\circ(M_{a_1},\ldots,M_{a_k})(\sigma_{i_1},\ldots,\sigma_{i_k})\|_{\mathrm{HS}}^2\Big)^{1/2}$$
$$= \Big(\sum_{i_1,\ldots,i_k=1}^{\infty} |\langle a_1, e_{i_1}\rangle|^2 \cdots |\langle a_k, e_{i_k}\rangle|^2 \|T\circ(\sigma_{i_1},\ldots,\sigma_{i_k})\|_{\mathrm{HS}}^2\Big)^{1/2}.$$

From $(***)$ we see that $T\circ(M_{a_1},\ldots,M_{a_k})$ is multiple 2-summing and

$$\pi_2^{\mathrm{mult}}(T\circ(M_{a_1},\ldots,M_{a_k})) \le \sqrt{\|S\|}.$$

Now Theorem 1 ensures that T is multiple 2-summing and

$$\pi_2^{\mathrm{mult}}(T) \le K_{p_1,2}\cdots K_{p_k,2} \sup_{\|a_1\|_{r_1},\dots,\|a_k\|_{r_k}\le 1} \pi_2^{\mathrm{mult}}(T\circ(M_{a_1},\dots,M_{a_k}))$$
$$\le K_{p_1,2}\cdots K_{p_k,2}\sqrt{\|S\|},$$

proving (i).

(i)⇔(iii) is the coincidence theorem. ∎

Taking $k=1$ in Theorem 2 we get

Corollary 2. *Let $1\le p<2$, H_1, H be a Hilbert spaces, $T: l_p(H_1)\to H$ a bounded linear operator and $1\le s<\infty$. The following assertions are equivalent:*

(i) *T is 2-summing.*
(ii) *All $T\circ\sigma_i : H_1\to H$ are Hilbert–Schmidt and $(\|T\circ\sigma_i\|_{\mathrm{HS}})_{i\in\mathbb{N}}\in l_{p^*}$.*
(iii) *T is s-summing.*

Moreover,

$$\|(\|T\circ\sigma_i\|_{\mathrm{HS}})_{i\in\mathbb{N}}\|_{p^*} \le \pi_2(T) \le K_{p,2}\|(\|T\circ\sigma_i\|_{\mathrm{HS}})_{i\in\mathbb{N}}\|_{p^*}.$$

We will need the following particular case of Corolary 2, certainly well-known, but for which we do not know an exact reference. Because of its special importance we think that a different proof may be of some interest.

Corollary 3. *Let $1<p<2$, H a Hilbert space, $T: l_p\to H$ a bounded linear operator and $1\le s<\infty$. Then T is s-summing if and only if T is 1-summing if and only if $\sum_{i=1}^{\infty}\|T(e_i)\|^{p^*}<\infty$. Moreover,*

$$\Big(\sum_{i=1}^{\infty}\|T(e_i)\|^{p^*}\Big)^{1/p^*} \le \pi_1(T) \le K_G\Big(\sum_{i=1}^{\infty}\|T(e_i)\|^{p^*}\Big)^{1/p^*},$$

where K_G is the Grothendieck constant.

Proof. Since l_p and H have cotype 2, from the coincidence theorem in the linear case we have $\Pi_s(l_p,H)=\Pi_1(l_p,H)$. Let us define $\lambda=(\|T(e_i)\|)_{i\in\mathbb{N}}$.

If T is 1-summing, then T is 2-summing and $\pi_2(T)\le\pi_1(T)$. Define $2<r<\infty$ by $1/p=1/2+1/r$ and let $a\in l_r$. Since T is 2-summing we infer that $T\circ M_a: l_2\to H$ is 2-summing and $\pi_2(T\circ M_a)\le\pi_2(T)\|M_a\|=\pi_2(T)\|a\|_r$. Since l_2, H are Hilbert spaces,

$$\pi_2(T\circ M_a)=\|T\circ M_a\|_{\mathrm{HS}}=\Big(\sum_{i=1}^{\infty}|a_i|^2\|T(e_i)\|^2\Big)^{1/2}=\|M_\lambda(a)\|_2,$$

and thus $\sup_{\|a\|_r\le 1}\|M_\lambda(a)\|_2\le\pi_2(T)$. By Hölder's inequality we get $\lambda\in l_s$, where $1/2=1/r+1/s$, i.e. $s=p^*$ and $\|\lambda\|_{p^*}\le\pi_2(T)$.

Conversely, if $\lambda \in l_{p^*}$, then T has the factorization $l_p \xrightarrow{M_\lambda} l_1 \xrightarrow{S} H$, where $S : l_1 \to H$ is defined by

$$S(\xi) = \sum_{i=1}^{\infty} \langle \xi, e_i \rangle \frac{T(e_i)}{\|T(e_i)\|}$$

(we use $\frac{0}{0} = 0$) and $\|S\| \le 1$. Since by Grothendieck's theorem, S is 1-summing and $\pi_1(S) \le K_G \|S\|$, we conclude that T is 1-summing and $\pi_1(T) \le \|M_\lambda\| \pi_1(S) \le K_G \|\lambda\|_{p^*}$. ∎

In a 1934 paper, G. H. Hardy and J. E. Littlewood gave necessary conditions for a bilinear functional on $l_p \times l_q$ to be bounded [12, Theorem 5].

THEOREM 3 (Hardy and Littlewood). *Let $1 < p, q < \infty$ be such that $1/p + 1/q < 1$ and define $1/\lambda = 1 - (1/p + 1/q)$. If $a_{ij} \ge 0$ are such that the bilinear functional $B : l_p \times l_q \to \mathbb{K}$ defined by*

$$B(x, y) = \sum_{i,j=1}^{\infty} \langle x, e_i \rangle \langle y, e_j \rangle a_{ij}$$

is bounded, then

$$\Big(\sum_{i,j=1}^{\infty} a_{ij}^{\lambda} \Big)^{1/\lambda} \le \|B\|.$$

From Theorems 2 and 3 we get

COROLLARY 4. *Let $1 < p_1, p_2 < 2$ be such that $1/p_1 + 1/p_2 < 3/2$ and define $1/\lambda = 3 - 2(1/p_1 + 1/p_2)$. Let H_1, H_2, H be a Hilbert spaces, $T : l_{p_1}(H_1) \times l_{p_2}(H_2) \to H$ a bounded bilinear operator and $1 \le s \le 2$. If T is multiple s-summing, then all $T \circ (\sigma_i, \sigma_j) : H_1 \times H_2 \to H$ are Hilbert–Schmidt and*

$$\Big(\sum_{i,j=1}^{\infty} \|T \circ (\sigma_i, \sigma_j)\|_{\mathrm{HS}}^{2\lambda} \Big)^{1/(2\lambda)} \le \pi_2^{\mathrm{mult}}(T).$$

Proof. Define $2 < r_j < \infty$ by $1/p_j = 1/2 + 1/r_j$ for $j = 1, 2$. Note that $2/r_1 + 2/r_2 < 1$ and $1/\lambda = 1 - (2/r_1 + 2/r_2)$. Since T is multiple s-summing, from Theorem 2 it folows that all $T \circ (\sigma_i, \sigma_j) : H_1 \times H_2 \to H$ are Hilbert–Schmidt and the bilinear functional $S : l_{r_1/2} \times l_{r_2/2} \to \mathbb{K}$ defined by

$$S(x, y) = \sum_{i,j=1}^{\infty} \langle x, e_i \rangle \langle y, e_i \rangle \|T(\sigma_i, \sigma_j)\|_{\mathrm{HS}}^2$$

is bounded. The statement now follows from Theorem 3. ∎

3. Examples of multiple summing operators for which no associated multilinear operator is multiple summing. Let $n \ge 2$ be a natural number and $T : X_1 \times \cdots \times X_n \to Y$ a bounded n-linear operator. If

(A, B) is a proper partition of the set $\{1, \ldots, n\}$, i.e. A, B are non-empty, $A \cap B = \emptyset$, $A \cup B = \{1, \ldots, n\}$, we define

$$T^{A,B} : \prod_{i \in A} X_i \to L\Big(\prod_{i \in B} X_i Y\Big),$$
$$T^{A,B}(x_i)_{i \in A}((x_i)_{i \in B}) = T(x_1, \ldots, x_n)$$

(see [9]). For example, in the case $n = 2$ and $T : X \times Y \to Z$ a bounded bilinear operator we have two natural linear operators associated to T denoted by $T_1 : X \to L(Y, Z)$ and $T_2 : Y \to L(X, Z)$, defined by

$$(T_1 x)(y) = T(x, y) \quad \text{and} \quad (T_2 y)(x) = T(x, y).$$

In [16, Proposition 2.5], [20, Proposition 2.5] or [34, proof of Proposition 3.7] it was shown that if $n \geq 2$ is a natural number, $T : X_1 \times \cdots \times X_n \to Y$ a bounded n-linear operator, $1 \leq p < \infty$ and there exists a proper partition (A, B) of $\{1, \ldots, n\}$ such that

$$T^{A,B} : \prod_{i \in A} X_i \to \Pi_p^{\mathrm{mult}}\Big(\prod_{i \in B} X_i, Y\Big)$$

is multiple p-summing, then T is multiple p-summing.

In [20, Remark 3.14] it was shown that if X, Y, Z are infinite-dimensional $\mathcal{L}_\infty$-spaces, then there exists a multiple 2-summing bilinear operator $T : X \times Y \to Z$ such that $T_1 \notin \Pi_2(X, \Pi_2(Y, Z))$.

We will show that well known examples of bilinear and multilinear operators can be adapted to construct multiple s-summing operators for which no associated multilinear operator is multiple s-summing $(1 \leq s \leq 2)$ (see Proposition 3 and Corollaries 5 and 8). As far as we know, these are the first examples of this type. For clarity, we analyze first the bilinear case and then the multilinear case.

The bilinear case. In the next result we give some necessary conditions for the linear operators associated to a bilinear operator defined on a product of l_p spaces with values in a Hilbert space to be multiple s-summing, $1 \leq s \leq 2$.

PROPOSITION 2. *Let $1 < p_1, p_2 < 2$, H be a Hilbert space, $T : l_{p_1} \times l_{p_2} \to H$ a bounded bilinear operator and T_1, T_2 the bounded linear operators associated to T. Let $1 \leq s \leq 2$.*

(i) *If $T_1 : l_{p_1} \to \Pi_s(l_{p_2}, H)$ is s-summing with respect to the s-summing norm on $\Pi_s(l_{p_2}, H)$, then*

$$\sum_{i=1}^{\infty} \Big(\sum_{j=1}^{\infty} \|T(e_i, e_j)\|^{p_2^*}\Big)^{p_1^*/p_2^*} < \infty.$$

Similarly, if $T_2 : l_{p_2} \to \Pi_s(l_{p_1}, H)$ *is s-summing with respect to the s-summing norm on* $\Pi_s(l_{p_1}, H)$*, then*

$$\sum_{j=1}^{\infty}\Big(\sum_{i=1}^{\infty}\|T(e_i,e_j)\|^{p_1^*}\Big)^{p_2^*/p_1^*}<\infty.$$

(ii) *If at least one of the sums in* (i) *in finite, then* T *is multiple s-summing.*

Proof. First we remark that from the coincidence theorem in the linear case, $\Pi_1(l_{p_2}, H) = \Pi_s(l_{p_2}, H)$ for $1 \le s \le 2$ (l_{p_2} has cotype 2, $1 < p_2 < 2$) and $\Pi_s(l_{p_1}, \Pi_s(l_{p_2}, H)) = \Pi_1(l_{p_1}, \Pi_1(l_{p_2}, H))$ (l_{p_1} has cotype 2, $1 < p_1 < 2$).

(i) If $T_1 : l_{p_1} \to \Pi_s(l_{p_2}, H)$ is s-summing with respect to the s-summing norm on $\Pi_s(l_{p_2}, H)$, then $T_1 : l_{p_1} \to \Pi_1(l_{p_2}, H)$ is 1-summing with respect to the 1-summing norm on $\Pi_1(l_{p_2}, H)$. Let $x \in l_{p_1}$. From Corollary 3, $T_1x \in \Pi_1(l_{p_2}, H)$ if and only if

$$\Big(\sum_{j=1}^{\infty}\|(T_1x)(e_j)\|^{p_2^*}\Big)^{1/p_2^*}\le \pi_1(T_1x)\le K_G\Big(\sum_{j=1}^{\infty}\|(T_1x)(e_j)\|^{p_2^*}\Big)^{1/p_2^*}.$$

Thus $T_1 : l_{p_1} \to \Pi_1(l_{p_2}, H)$ is 1-summing with respect to the 1-summing norm on $\Pi_1(l_{p_2}, H)$ if and only if $S : l_{p_1} \to l_{p_2^*}(H)$ defined by $S(x) = (T(x, e_j))_{j\in\mathbb{N}}$ is 1-summing, and in this case $\pi_1(S) \le \pi_1(T_1 : l_{p_1} \to \Pi_1(l_{p_2}, H)) \le K_G\pi_1(S)$. Since S is 1-summing ($T_1 : l_{p_1} \to \Pi_1(l_{p_2}, H)$ is 1-summing by hypothesis), the inclusion theorem (in the linear case) shows that S is p_1^*-summing (linear) with $\pi_{p_1^*}(S) \le \pi_1(S)$. From $w_{p_1^*}(e_n \mid n \in \mathbb{N}; l_{p_1}) = \|I : l_{p_1^*} \to l_{p_1^*}\| = 1$ we get $(\sum_{i=1}^{\infty}\|S(e_i)\|^{p_1^*})^{1/p_1^*} \le \pi_{p_1^*}(S)$, i.e.

$$\Big(\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty}\|T(e_i,e_j)\|^{p_2^*}\Big)^{p_1^*/p_2^*}\Big)^{1/p_1^*}\le \pi_1(T_1 : l_{p_1}\to \Pi_1(l_{p_2},H)).$$

(ii) Suppose, for example, that $\sum_{j=1}^{\infty}(\sum_{i=1}^{\infty}\|T(e_i,e_j)\|^{p_1^*})^{p_2^*/p_1^*} < \infty$. Define $2 < r_2 < \infty$ by $1/p_2 = 1/2 + 1/r_2$ and take $b \in l_{r_2}$. If we show that $T\circ(I, M_b) : l_{p_1}\times l_2 \to H$ is multiple 2-summing then Theorem 1 implies that T is multiple 2-summing and by the coincidence theorem T will be multiple s-summing. But, from [1, Theorem 1], $T \circ (I, M_b)$ is multiple 2-summing, if and only if $[T \circ (I, M_b)]_1 : l_{p_1} \to \mathrm{HS}(l_2, H)$ is multiple 2-summing, and this by Corollary 3 is equivalent to

$$(*)\qquad \sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty}|b_j|^2\|T(e_i,e_j)\|^2\Big)^{p_1^*/2} = \sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty}\|T\circ(I,M_b)(e_i,e_j)\|^2\Big)^{p_1^*/2}<\infty.$$

Since $1/2 = 1/r_2 + 1/p_2^*$, Hölder's inequality yields

$$\Big(\sum_{j=1}^{\infty} |b_j|^2 \|T(e_i, e_j)\|^2\Big)^{1/2} \le \|b\|_{r_2} \Big(\sum_{j=1}^{\infty} \|T(e_i, e_j)\|^{p_2^*}\Big)^{1/p_2^*} \text{ for } i \in \mathbb{N},$$

and thus

$$\sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty} |b_j|^2 \|T(e_i, e_j)\|^2\Big)^{p_1^*/2} \le \|b\|_{r_2} \sum_{i=1}^{\infty}\Big(\sum_{j=1}^{\infty} \|T(e_i, e_j)\|^{p_2^*})^{p_1^*/p_2^*} < \infty,$$

i.e. $(*)$ is satisfied. ∎

Unfortunately we do not know whether the converse of (i) in Proposition 2 is true or not. If it is true (hence (i) and (ii) are equivalent) then (ii) will follow from the already mentioned result in [16, Proposition 2.5], [20, Proposition 2.5] or [34, proof of Proposition 3.7].

Next we indicate a way to construct concrete examples of multiple s-summing bilinear operators, $1 \le s \le 2$, for which the associated linear operators are not s-summing.

We denote by $\mathcal{K}$ the class of all "kernels" $K : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ with the following two properties:

(a) $\sum_{i,j=1}^{\infty} |\langle x, e_i\rangle|\, |\langle y, e_j\rangle| K(i, j) < \infty$ for each $(x, y) \in l_2 \times l_2$.
(b) $\sum_{i,j=1}^{\infty} [K(i, j)]^2 = \infty$.

Note that, from Hilbert's theorem [13], $K : \mathbb{N} \times \mathbb{N} \to [0, \infty)$ defined by $K(i, j) = \frac{1}{i+j}$ satisfies condition (a) and

$$\sum_{i=1}^{\infty} [K(i, j)]^2 = \sum_{i=1}^{\infty} \frac{1}{(i + j)^2} = \sum_{k=j+1}^{\infty} \frac{1}{k^2} \backsim \frac{1}{j} \quad \text{as } j \to \infty,$$

thus $\sum_{i,j=1}^{\infty} [K(i, j)]^2 = \infty$, i.e. K belongs to the class $\mathcal{K}$.

PROPOSITION 3. *Let $K \in \mathcal{K}$. Let $(v_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}} \subset l_2$ be an orthogonal system with $\|v_{ij}\|_2 = \sqrt{K(i, j)}$ for $(i, j) \in \mathbb{N} \times \mathbb{N}$. Let $T : l_{4/3} \times l_{4/3} \to l_2$ be defined by*

$$T(x, y) = \sum_{i,j=1}^{\infty} \langle x, e_i\rangle\langle y, e_j\rangle v_{ij}$$

and $1 \le s \le 2$. Then T is multiple s-summing, but the associated linear operators $T_1, T_2 : l_{4/3} \to \Pi_s(l_{4/3}, l_2)$ are not s-summing.

Proof. Observe that $v_{ij} = \sqrt{K(i, j)}\, e_{2^i 3^j}$ is an orthogonal system in l_2 which satisfies the conditions stated. (More generally, if $\sigma : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ is an injective mapping then $v_{ij} = \sqrt{K(i, j)} e_{\sigma(i,j)}$ is an orthogonal system in l_2 which satisfies these conditions.)

From hypothesis (a), for each $(a,b) \in l_2 \times l_2$ the series

$$\sum_{i,j=1}^{\infty} |\langle a, e_i\rangle|\,|\langle b, e_j\rangle| K(i,j) < \infty. \tag{1}$$

Let $(x,y) \in l_{4/3} \times l_{4/3}$. Since $(v_{ij})_{(i,j)\in\mathbb{N}\times\mathbb{N}}$ is an orthogonal system in l_2, the series $\sum_{i,j=1}^{\infty} \langle x, e_i\rangle\langle y, e_j\rangle v_{ij}$ is norm convergent in l_2 if and only if the numerical series $\sum_{i,j=1}^{\infty} |\langle x, e_i\rangle|^2 |\langle y, e_j\rangle|^2 \|v_{ij}\|^2 < \infty$, i.e.

$$\sum_{i,j=1}^{\infty} |\langle x, e_i\rangle|^2 |\langle y, e_j\rangle|^2 K(i,j) < \infty. \tag{2}$$

Since $x \in l_{4/3}$, we have $a = (|\langle x, e_i\rangle|^2)_{i\in\mathbb{N}} \in l_{2/3} \subset l_2$ and similarly $b = (|\langle y, e_j\rangle|^2)_{j\in\mathbb{N}} \in l_2$. Then from (1) we deduce (2) and hence T is well defined.

From Theorem 2, T is multiple s-summing if and only if T is multiple 2-summing if and only if the bilinear functional $S : l_2 \times l_2 \to \mathbb{C}$ given by

$$S(x,y) = \sum_{i,j=1}^{\infty} \langle x, e_i\rangle\langle y, e_j\rangle \|v_{ij}\|^2 = \sum_{i,j=1}^{\infty} \langle x, e_i\rangle\langle y, e_j\rangle K(i,j)$$

is bounded. Again, by hypothesis (a) this is true; $p_1 = p_2 = 4/3$ and from $1/p_1 = 1/2 + 1/r_1$, $1/p_2 = 1/2 + 1/r_2$ we get $r_1 = r_2 = 4$.

Further, by hypothesis (b),

$$\sum_{i,j=1}^{\infty} \|T(e_i, e_j)\|^4 = \sum_{i,j=1}^{\infty} \|v_{ij}\|^4 = \sum_{i,j=1}^{\infty} [K(i,j)]^2 = \infty. \tag{3}$$

Now, if one of $T_1, T_2 : l_{4/3} \to \Pi_s(l_{4/3}, l_2)$ is s-summing, then (see the remark at the beginning of the proof of Proposition 2(i)), one of $T_1, T_2 : l_{4/3} \to \Pi_1(l_{4/3}, l_2)$ is 1-summing and by Proposition 2(i), $\sum_{i,j=1}^{\infty} \|T(e_i, e_j)\|^4 < \infty$, which contradicts (3). ∎

For example, from Proposition 3 it follows that the bilinear operator $T : l_{4/3} \times l_{4/3} \to l_2$ defined by

$$T(x,y) = \sum_{i,j=1}^{\infty} \frac{1}{\sqrt{i+j}} \langle x, e_i\rangle\langle y, e_j\rangle e_{2^i 3^j}$$

is multiple s-summing, but neither of the associated linear operators $T_1, T_2 : l_{4/3} \to \Pi_s(l_{4/3}, l_2)$ is s-summing for $1 \le s \le 2$.

We will need the bilinear multiplication operator.

Let $1 \le p, q, r < \infty$, let $a = (a_n)_{n\in\mathbb{N}}$ be such that $a \in l_\infty$ if $1/p \le 1/q + 1/r$, or $a \in l_s$ if $1/p > 1/q + 1/r$, where $1/p = 1/q + 1/r + 1/s$, and let $B_a : l_r \times l_q \to l_p$ be the bilinear multiplication operator

$$B_a(x,y) = axy.$$

We give necessary and sufficient conditions for a bilinear multiplication operator to be multiple s-summing and the associated linear operators to be s-summing, $1 \le s \le 2$.

PROPOSITION 4. *Let $1 < p_1, p_2 < 2$, $a \in l_\infty$, $B_a : l_{p_1} \times l_{p_2} \to l_2$ be the multiplication operator and $1 \le s \le 2$.*

(i) *If $3/2 \le 1/p_1 + 1/p_2$, then B_a is multiple s-summing.*
(ii) *If $3/2 > 1/p_1 + 1/p_2$, then B_a is multiple s-summing if and only if $a \in l_u$, where $1/u = 3/2 - (1/p_1 + 1/p_2)$.*
(iii) *$(B_a)_1 : l_{p_1} \to \Pi_s(l_{p_2}, l_2)$ is s-summing if and only if $a \in l_{p_1^*}$; similarly $(B_a)_2 : l_{p_2} \to \Pi_s(l_{p_1}, l_2)$ is s-summing if and only if $a \in l_{p_2^*}$.*

Proof. (i) and (ii) are particular cases of the next Proposition 5(b).

(iii) We need

CLAIM. *Let $1 < p < 2 \le q \le \infty$, $a \in l_\infty$ and $M_a : l_p \to l_q$. Then M_a is 1-summing if and only if it is 2-summing if and only if $a \in l_{p^*}$.*

This follows from the results in [32]. For the sake of completeness we give a direct proof.

If M_a is 1-summing, then M_a is p^*-summing and since $w_{p^*}(e_n; l_p) = \|I : l_{p^*} \to l_{p^*}\| = 1$, we have $(\|M_a(e_n)\|)_{n\in\mathbb{N}} \in l_{p^*}$, i.e. $a \in l_{p^*}$.

Conversely, if $a \in l_{p^*}$, then $M_a : l_p \to l_q$ has the factorization

$$l_p \xrightarrow{M_a} l_1 \overset{J}{\hookrightarrow} l_2 \hookrightarrow l_q.$$

Since by Grothendieck's theorem $l_1 \overset{J}{\hookrightarrow} l_2$ is 1-summing, we deduce that M_a is 1-summing.

We have observed that $(B_a)_1 : l_{p_1} \to \Pi_s(l_{p_2}, l_2)$ is s-summing if and only if $(B_a)_1 : l_{p_1} \to \Pi_1(l_{p_2}, l_2)$ is 1-summing (see the beginning of the proof of Proposition 2). Let $x \in l_{p_1}$. From Corollary 3, $(B_a)_1 x \in \Pi_1(l_{p_2}, H)$ if and only if $\sum_{j=1}^\infty \|B_a(x, e_j)\|^{p_2^*} < \infty$, and in this case

$$\Big(\sum_{j=1}^\infty \|B_a(x, e_j)\|^{p_2^*}\Big)^{1/p_2^*} \le \pi_1((B_a)_1 x) \le K_G\Big(\sum_{j=1}^\infty \|B_a(x, e_j)\|^{p_2^*}\Big)^{1/p_2^*}.$$

Since $B_a(x, e_j) = a_j \langle x, e_j\rangle e_j$,

$$\|M_a(x)\|_{p_2^*} \le \pi_1((B_a)_1 x) \le K_G \|M_a(x)\|_{p_2^*}.$$

Thus $(B_a)_1 : l_{p_1} \to \Pi_1(l_{p_2}, l_2)$ is 1-summing with respect to the 1-summing norm on $\Pi_1(l_{p_2}, l_2)$ if and only if $M_a : l_{p_1} \to l_{p_2^*}$ is 1-summing. By the Claim this is equivalent to $a \in l_{p_1^*}$. ■

Taking $p_1 = p_2 = p$ in Proposition 4 we again get concrete examples of multiple s-summing bilinear operators for which the associated linear operators are not s-summing, for $1 \le s \le 2$.

Corollary 5. *Let $1 < p < 2$, $a \in l_\infty$, $B_a : l_p \times l_p \to l_2$ be the multiplication operator and $1 \le s \le 2$.*

(i) *If $1 < p \le 4/3$, then B_a is multiple s-summing.*
(ii) *If $4/3 < p < 2$, then B_a is multiple s-summing if and only if $a \in l_{2p/(3p-2)}$.*
(iii) *$(B_a)_1, (B_a)_2 : l_p \to \Pi_s(l_p, l_2)$ are s-summing if and only if $a \in l_{p^*}$.*
(iv) *If $1 < p \le 4/3$ and $a \in l_\infty$ but $a \notin l_{p^*}$, then B_a is multiple s-summing and none of $(B_a)_1, (B_a)_2 : l_p \to \Pi_s(l_p, l_2)$ is s-summing.*
(v) *If $4/3 < p < 2$ and $a \in l_{2p/(3p-2)}$ but $a \notin l_{p/(p-1)}$, then B_a is multiple s-summing and none of $(B_a)_1, (B_a)_2 : l_p \to \Pi_s(l_p, l_2)$ is s-summing.*

The multilinear case. For convenience, we denote by M_a both the linear multiplication operator defined by $M_a(x) = ax$ and the multilinear operator defined by $M_a(x_1, \ldots, x_n) = ax_1 \cdots x_n$. This will cause no confusion since the context will make it clear when we are in the linear case and when with in the multilinear case.

For brevity, we write $X \times \overset{(k)}{\cdots} \times X$ for $\underbrace{X \times \cdots \times X}_{k \text{ times}}$, and similarly $X, \overset{(k)}{\cdots}, X$ for $\underbrace{X, \ldots, X}_{k \text{ times}}$.

Proposition 5.

(a) *Let n be a natural number, $2 \le q \le \infty$, $a \in l_\infty$, $M_a : l_2 \times \overset{(n)}{\cdots} \times l_2 \to l_q$ the multiplication operator and $1 \le s \le 2$. Then M_a is multiple s-summing if and only if $a \in l_2$.*

(b) *Let n be a natural number, $1 < p_1, \ldots, p_n < 2 \le q \le \infty$, $a \in l_\infty$, $M_a : l_{p_1} \times \cdots \times l_{p_n} \to l_q$ the multiplication operator and $1 \le s \le 2$.*
   (i) *If $(n+1)/2 \le 1/p_1 + \cdots + 1/p_n$, then M_a is multiple s-summing.*
   (ii) *If $(n+1)/2 > 1/p_1 + \cdots + 1/p_n$, then M_a is multiple s-summing if and only if $a \in l_u$, where $1/u = (n+1)/2 - (1/p_1 + \cdots + 1/p_n)$.*

(c) *Let n be a natural number, $1 < p < 2 \le q \le \infty$, $a \in l_\infty$, $M_a : l_p \times \overset{(n)}{\cdots} \times l_p \to l_q$ the multiplication operator and $1 \le s \le 2$.*
   (i) *If $1 < p \le 2n/(n+1)$, then M_a is multiple s-summing.*
   (ii) *If $2n/(n+1) < p < 2$, then M_a is multiple s-summing if and only if $a \in l_u$, where $1/u = (n+1)/2 - n/p$.*

Proof. (a) Suppose that M_a is multiple 2-summing. Since $w_2(e_k \mid k \in \mathbb{N}; l_2) = \|I : l_2 \to l_2\| = 1$ we get $(\|M_a(e_k, \ldots, e_k)\|)_{k \in \mathbb{N}} \in l_2$, i.e. $a \in l_2$.

Conversely, if $a \in l_2$ then $M_a : l_2 \times \cdots \times l_2 \to l_2$ is Hilbert–Schmidt, thus multiple 2-summing and hence ($l_2 \hookrightarrow l_q$, $2 \le q \le \infty$), $M_a : l_2 \times \cdots \times l_2 \to l_q$ is multiple 2-summing. Note that by the coincidence theorem, M_a is multiple s-summing if and only if it is multiple 2-summing.

(b) Define $2 < r_j < \infty$ by $1/p_j = 1/2 + 1/r_j$ for each $1 \le j \le n$. From Theorem 1, M_a is multiple s-summing if and only if it is multiple 2-summing if and only if for each $(a_1, \ldots, a_n) \in l_{r_1} \times \cdots \times l_{r_n}$ the operator $M_{aa_1\cdots a_n} = M_a \circ (M_{a_1}, \ldots, M_{a_n}) : l_2 \times \cdots \times l_2 \to l_q$ is multiple 2-summing. By (a) this is equivalent to $aa_1 \cdots a_n \in l_2$, i.e. $M_a : l_{r_1} \times \cdots \times l_{r_n} \to l_2$ is well defined. Then this is equivalent to $a \in l_\infty$ if $1/2 \le 1/r_1 + \cdots + 1/r_n$, and $a \in l_u$ where $1/2 = 1/r_1 + \cdots + 1/r_n + 1/u$ if $1/2 > 1/r_1 + \cdots + 1/r_n$, as claimed.

(c) This is a particular case of (b). ∎

In the rest of the paper, for simplicity, if $n \ge 2$ is a natural number, $1 \le k \le n-1$ and $T : X_1 \times \cdots \times X_n \to Y$ is a bounded n-linear operator, and $A = \{1, \ldots, k\}$, $B = \{k+1, \ldots, n\}$, we denote $T^{A,B} = \widetilde{T} : X_1 \times \cdots \times X_k \to L(X_{k+1}, \ldots, X_n; Y)$, thus

$$\widetilde{T}(x_1, \ldots, x_k)(x_{k+1}, \ldots, x_n) = T(x_1, \ldots, x_k, x_{k+1}, \ldots, x_n).$$

Proposition 6. *Let $n \ge 2$ be a natural number, $1 \le k \le n-1$, $1 < p_1, \ldots, p_n < 2$, $a \in l_\infty$, $M_a : l_{p_1} \times \cdots \times l_{p_n} \to l_2$ the multiplication operator and $1 \le s \le 2$.*

(i) *If*

$$\frac{k+1}{2} \le \frac{1}{p_1} + \cdots + \frac{1}{p_k} \quad \text{and} \quad \frac{n-k+1}{2} \le \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_n},$$

then $\widetilde{M_a} : l_{p_1} \times \cdots \times l_{p_k} \to \Pi_s^{\mathrm{mult}}(l_{p_{k+1}}, \ldots, l_{p_n}; l_2)$ is multiple s-summing.

(ii) *If*

$$\frac{n-k+1}{2} \le \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_n} \quad \text{and} \quad \frac{k+1}{2} > \frac{1}{p_1} + \cdots + \frac{1}{p_k},$$

then $\widetilde{M_a} : l_{p_1} \times \cdots \times l_{p_k} \to \Pi_s^{\mathrm{mult}}(l_{p_{k+1}}, \ldots, l_{p_n}; l_2)$ is multiple s-summing if and only if $a \in l_u$, where

$$\frac{1}{u} = \frac{k+1}{2} - \left(\frac{1}{p_1} + \cdots + \frac{1}{p_k}\right).$$

(iii) *If*

$$\frac{n-k+1}{2} > \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_n} \quad \text{and} \quad \frac{k+1}{2} \le \frac{1}{p_1} + \cdots + \frac{1}{p_k},$$

then $\widetilde{M_a} : l_{p_1} \times \cdots \times l_{p_k} \to \Pi_s^{\mathrm{mult}}(l_{p_{k+1}}, \ldots, l_{p_n}; l_2)$ is multiple s-summing.

(iv) *If*

$$\frac{n-k+1}{2} > \frac{1}{p_{k+1}} + \cdots + \frac{1}{p_n} \quad \text{and} \quad \frac{k+1}{2} > \frac{1}{p_1} + \cdots + \frac{1}{p_k},$$

then $\widetilde{M_a} : l_{p_1} \times \cdots \times l_{p_k} \to \Pi_s^{\mathrm{mult}}(l_{p_{k+1}}, \ldots, l_{p_n}; l_2)$ is multiple s-summing if and only if $a \in l_u$, where u is as in (ii).

Proof. Let $(x_1,\ldots,x_k)\in l_{p_1}\times\cdots\times l_{p_k}$. Then

$$M_{ax_1\cdots x_k}=\widetilde{M}_a(x_1,\ldots,x_k):l_{p_{k+1}}\times\cdots\times l_{p_n}\to l_2$$

and by Proposition 5(b) two situations are possible, denoted by (a) and (b) below.

(a) If $(n-k+1)/2\le 1/p_{k+1}+\cdots+1/p_n$ then $M_{ax_1\cdots x_k}$ is multiple s-summing and for some constants $c,C>0$,

$$c\|ax_1\cdots x_k\|_\infty\le\pi_s^{\mathrm{mult}}(\widetilde{M}_a(x_1,\ldots,x_k))=\pi_s^{\mathrm{mult}}(M_{ax_1\cdots x_k})\le C\|ax_1\cdots x_k\|_\infty.$$

Thus, in this case $\widetilde{M}_a:l_{p_1}\times\cdots\times l_{p_k}\to\Pi_s^{\mathrm{mult}}(l_{p_{k+1}},\ldots,l_{p_n};l_2)$ is multiple s-summing if and only if $M_a:l_{p_1}\times\cdots\times l_{p_k}\to l_\infty$ is multiple s-summing.

By Proposition 5(b) we get:

(i) If $(k+1)/2\le 1/p_1+\cdots+1/p_k$, then

$$\widetilde{M}_a:l_{p_1}\times\cdots\times l_{p_k}\to\Pi_s^{\mathrm{mult}}(l_{p_{k+1}},\ldots,l_{p_n};l_2)$$

is multiple s-summing.

(ii) If $(k+1)/2>1/p_1+\cdots+1/p_k$, then $\widetilde{M}_a:l_{p_1}\times\cdots\times l_{p_k}\to\Pi_s^{\mathrm{mult}}(l_{p_{k+1}},\ldots,l_{p_n};l_2)$ is multiple s-summing if and only if $a\in l_u$, where $1/u=(k+1)/2-(1/p_1+\cdots+1/p_k)$.

(b) If $(n-k+1)/2>1/p_{k+1}+\cdots+1/p_n$, then $M_{ax_1\cdots x_k}$ is multiple s-summing if and only if $a\in l_t$, where $1/t=(n-k+1)/2-(1/p_{k+1}+\cdots+1/p_n)$ and for some constants $c,C>0$,

$$c\|ax_1\cdots x_k\|_t\le\pi_s^{\mathrm{mult}}(\widetilde{M}_a(x_1,\ldots,x_k))=\pi_s^{\mathrm{mult}}(M_{ax_1\cdots x_k})\le C\|ax_1\cdots x_k\|_t.$$

Thus in this case $\widetilde{M}_a:l_{p_1}\times\cdots\times l_{p_k}\to\Pi_s^{\mathrm{mult}}(l_{p_{k+1}},\ldots,l_{p_n};l_2)$ is multiple s-summing if and only if $M_a:l_{p_1}\times\cdots\times l_{p_k}\to l_t$ is multiple s-summing.

By Proposition 5(b) ($t>2$ is obvious) this is equivalent to:

(iii) If $(k+1)/2\le 1/p_1+\cdots+1/p_k$, then $\widetilde{M}_a:l_{p_1}\times\cdots\times l_{p_k}\to\Pi_s^{\mathrm{mult}}(l_{p_{k+1}},\ldots,l_{p_n};l_2)$ is multiple s-summing.

(iv) If $(k+1)/2>1/p_1+\cdots+1/p_k$, then $\widetilde{M}_a:l_{p_1}\times\cdots\times l_{p_k}\to\Pi_s^{\mathrm{mult}}(l_{p_{k+1}},\ldots,l_{p_n};l_2)$ is multiple s-summing if and only if $a\in l_u$, with u as stated. ■

We will need the following particular case of Proposition 6.

Corollary 6. *Let $n\ge 2$ be a natural number, $1\le k\le n-1$, $1<p<2$, $a\in l_\infty$, $M_a:l_p\times\overset{(n)}{\cdots}\times l_p\to l_2$ the multiplication operator and $1\le s\le 2$.*

(i) *If*

$$p\le\frac{2k}{k+1}\quad\text{and}\quad p\le\frac{2(n-k)}{n-k+1},$$

then $\widetilde{M}_a:l_p\times\overset{(k)}{\cdots}\times l_p\to\Pi_s^{\mathrm{mult}}(l_p,\overset{(n-k)}{\ldots},l_p;l_2)$ is multiple s-summing.

(ii) *If*
$$\frac{2k}{k+1} < p \le \frac{2(n-k)}{n-k+1},$$
then $\widetilde{M}_a : l_p \times \overset{(k)}{\cdots} \times l_p \to \Pi_s^{\mathrm{mult}}(l_p, \overset{(n-k)}{\ldots}, l_p; l_2)$ *is multiple* s*-summing if and only if* $a \in l_u$*, where* $1/u = (k+1)/2 - k/p$.

(iii) *If*
$$\frac{2(n-k)}{n-k+1} < p \le \frac{2k}{k+1},$$
then $\widetilde{M}_a : l_p \times \overset{(k)}{\cdots} \times l_p \to \Pi_s^{\mathrm{mult}}(l_p, \overset{(n-k)}{\ldots}, l_p; l_2)$ *is multiple* s*-summing.*

(iv) *If*
$$p > \frac{2k}{k+1} \quad \textit{and} \quad p > \frac{2(n-k)}{n-k+1},$$
then $\widetilde{M}_a : l_p \times \overset{(k)}{\cdots} \times l_p \to \Pi_s^{\mathrm{mult}}(l_p, \overset{(n-k)}{\ldots}, l_p; l_2)$ *is multiple* s*-summing if and only if* $a \in l_u$*, where* u *is as in* (ii).

Taking $p = \frac{2n}{n+1}$ in Corollary 6 we get (only (iv) can occur and $u = \frac{2n}{n-k}$)

COROLLARY 7. *Let* $n \ge 2$ *be a natural number,* $1 \le k \le n-1$, $a \in l_\infty$, $M_a : l_{2n/(n+1)} \times \overset{(n)}{\cdots} \times l_{2n/(n+1)} \to l_2$ *the multiplication operator and* $1 \le s \le 2$*. Then*
$$\widetilde{M}_a : l_{2n/(n+1)} \times \overset{(k)}{\cdots} \times l_{2n/(n+1)} \to \Pi_s^{\mathrm{mult}}(l_{2n/(n+1)}, \overset{(n-k)}{\ldots}, l_{2n/(n+1)}; l_2)$$
is multiple s*-summing if and only if* $a \in l_{2n/(n-k)}$.

Now we can give an example of a multiple s-summing n-linear operator with the property that for each proper partition of $\{1, \ldots, n\}$ the natural associated multilinear operators are not multiple s-summing, $1 \le s \le 2$. As far as we know, this is the first example of this kind.

COROLLARY 8. *Let* $n \ge 2$ *be a natural number,* $1 \le s \le 2$, $a \in l_\infty$ *but* $a \notin l_{2n}$*, and* $M_a : l_{2n/(n+1)} \times \overset{(n)}{\cdots} \times l_{2n/(n+1)} \to l_2$ *the multiplication operator. Then:*

(i) M_a *is multiple* s*-summing;*

(ii) *for each proper partition* (A, B) *of* $\{1, \ldots, n\}$ *the operator*
$$M_a^{A,B} : \prod_{i\in A} l_{2n/(n+1)} \to \Pi_s^{\mathrm{mult}}\Big(\prod_{i\in B} l_{2n/(n+1)}, l_2\Big)$$
is not multiple s*-summing.*

Proof. (i) follows from Proposition 5(c)(i).

(ii) Suppose that there exists a proper partition (A, B) of $\{1, \ldots, n\}$ such that
$$(*) \qquad M_a^{A,B} : \prod_{i\in A} l_{2n/(n+1)} \to \Pi_s^{\mathrm{mult}}\Big(\prod_{i\in B} l_{2n/(n+1)}, l_2\Big)$$

is multiple s-summing. Denote $\mathrm{card}(A) = k$ and thus $\mathrm{card}(B) = n - k$ $(1 \le k \le n-1)$. Then $(*)$ asserts that

$$\widetilde{M}_a : l_{2n/(n+1)} \times \overset{(k)}{\cdots} \times l_{2n/(n+1)} \to \Pi_s^{\mathrm{mult}}(l_{2n/(n+1)}, \overset{(n-k)}{\cdots}, l_{2n/(n+1)}; l_2)$$

is multiple s-summing. From Corollary 7 we get $a \in l_{2n/(n-k)}$. Since $2n/(n-k) \le 2n$ for $1 \le k \le n-1$ we have $l_{2n/(n-k)} \subset l_{2n}$ and thus $a \in l_{2n}$, which contradicts the hypothesis $a \notin l_{2n}$. ∎

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