STUDIA MATHEMATICA 225 (1) (2014)

Equivalences involving (p, q)-multi-norms

by

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Abstract. We consider (p, q)-multi-norms and standard t-multi-norms based on Banach spaces of the form $L^r(\Omega)$, and resolve some question about the mutual equivalence of two such multi-norms. We introduce a new multi-norm, called the $[p, q]$-concave multi-norm, and relate it to the standard t-multi-norm.

1. Introduction

1.1. Definitions. A theory of multi-norms based on a normed space E was first introduced by Dales and Polyakov in [10]. We recall the basic definitions of the theory.

We write $\mathbb{N}$ for the set of natural numbers, and set $\mathbb{N}_n = \{1, \dots, n\}$ for $n \in \mathbb{N}$; the collection of permutations of the set $\mathbb{N}_n$ is denoted by $\mathfrak{S}_n$.

Definition 1.1. Let $(E, \|\cdot\|)$ be a complex normed space. A *multi-norm* on the family $\{E^n : n \in \mathbb{N}\}$ is a sequence $(\|\cdot\|_n : n \in \mathbb{N})$ such that $\|\cdot\|_n$ is a norm on E^n for each $n \in \mathbb{N}$, such that $\|x\|_1 = \|x\|$ for each $x \in E$, and such that the following Axioms (A1)–(A4) are satisfied for each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$:

(A1) $\|(x_{\sigma(1)}, \dots, x_{\sigma(n)})\|_n = \|\boldsymbol{x}\|_n$ $(\sigma \in \mathfrak{S}_n)$;
(A2) $\|(\alpha_1 x_1, \dots, \alpha_n x_n)\|_n \le (\max_{i\in\mathbb{N}_n} |\alpha_i|)\|\boldsymbol{x}\|_n$ $(\alpha_1, \dots, \alpha_n \in \mathbb{C})$;
(A3) $\|(x_1, \dots, x_n, 0)\|_{n+1} = \|\boldsymbol{x}\|_n$;
(A4) $\|(x_1, \dots, x_{n-1}, x_n, x_n)\|_{n+1} = \|\boldsymbol{x}\|_n$.

In this case, $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is a *multi-normed space.*

We shall sometimes say that $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm *based on* E; we write $\mathcal{E}_E$ for the family of all multi-norms based on E.

In the case where $(E, \|\cdot\|)$ is a Banach space, each space $(E^n, \|\cdot\|_n)$ is a Banach space, and $((E^n, \|\cdot\|_n) : n \in \mathbb{N})$ is termed a *multi-Banach space.*

2010 *Mathematics Subject Classification*: Primary 46B15; Secondary 46B28, 46B42, 47L10.
Key words and phrases: multi-norms, equivalences, absolutely summing operators, tensor products, regular operators, matrices.

DOI: 10.4064/sm225-1-3

In fact, Axiom (A3) is a consequence of Axioms (A1), (A2), and (A4) [10, Proposition 2.7]; to establish (A4), it suffices to show that

$$\|(x_1, \dots, x_{n-1}, x_n, x_n)\|_{n+1} \leq \|\boldsymbol{x}\|_n$$

for each element $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$.

Many properties of multi-norms were described in [10]; these properties included some strong connections with the theory of absolutely summing operators and with the theory of tensor norms. A study of multi-norms was continued in [8] and [9].

In [8], we explained how multi-norms correspond to certain tensor norms. We recall this briefly; details are given in [8, §3]. We write δ_i for the sequence $(\delta_{i,j} : j \in \mathbb{N})$ for $i \in \mathbb{N}$; c_0 is the Banach space of all complex-valued null sequences.

Definition 1.2. Let E be a normed space. Then a norm $\|\cdot\|$ on $c_0 \otimes E$ is a c_0-*norm* if $\|\delta_1 \otimes x\| = \|x\|$ for each $x \in E$ and if the linear operator $T \otimes I_E$ is bounded on $(c_0 \otimes E, \|\cdot\|)$, with norm at most $\|T\|$, for each compact operator T on E.

We note that a c_0-norm on $c_0 \otimes E$ is a 'reasonable cross-norm' in the sense of [21, §6.1]; see [8, Lemma 3.3].

Suppose that $\|\cdot\|$ is a c_0-norm on $c_0 \otimes E$, and set

$$\|(x_1, \dots, x_n)\|_n = \sum_{i=1}^{n} \delta_i \otimes x_i \quad (x_1, \dots, x_n \in E,\, n \in \mathbb{N}).$$

Then $(\|\cdot\|_n : n \in \mathbb{N})$ is a multi-norm based on E.

A more general and detailed version of the following theorem is given as [8, Theorem 3.4].

Theorem 1.3. *Let E be a normed space. Then the above construction defines a bijection from the family of c_0-norms on $c_0 \otimes E$ onto $\mathcal{E}_E$.* ■

The notion of the equivalence of two multi-norms was given in [10, §2.2.4], as follows.

Definition 1.4. Let $(E, \|\cdot\|)$ be a normed space. Suppose that the two multi-norms $(\|\cdot\|_n^1 : n \in \mathbb{N})$ and $(\|\cdot\|_n^2 : n \in \mathbb{N})$ belong to $\mathcal{E}_E$. Then

$$(\|\cdot\|_n^1) \leq (\|\cdot\|_n^2) \quad \text{if} \quad \|\boldsymbol{x}\|_n^1 \leq \|\boldsymbol{x}\|_n^2 \quad (\boldsymbol{x} \in E^n,\, n \in \mathbb{N}),$$

and $(\|\cdot\|_n^2 : n \in \mathbb{N})$ *dominates* $(\|\cdot\|_n^1 : n \in \mathbb{N})$, written $(\|\cdot\|_n^1) \preccurlyeq (\|\cdot\|_n^2)$, if there is a constant $C > 0$ such that

$$\|\boldsymbol{x}\|_n^1 \leq C\|\boldsymbol{x}\|_n^2 \quad (\boldsymbol{x} \in E^n,\, n \in \mathbb{N}); \tag{1.1}$$

the two multi-norms are *equivalent*, written

$$(\|\cdot\|_n^1 : n \in \mathbb{N}) \cong (\|\cdot\|_n^2 : n \in \mathbb{N}) \quad \text{or} \quad (\|\cdot\|_n^1) \cong (\|\cdot\|_n^2),$$

if each dominates the other.

A main theme of [9] was to determine when two multi-norms based on the same normed space are mutually equivalent. In particular, we discussed in [9] the '(p,q)-multi-norms based on a normed space E', and tried to determine when these multi-norms are mutually equivalent, especially on the Banach spaces of the form $L^r(\Omega)$. The question was resolved for most, but not all, cases. Here we resolve some of the remaining cases, and give simpler proofs of some results already established in [9]. We also consider the question whether a 'standard multi-norm' is ever equivalent to a (p,q)-multi-norm on a space $L^r(\Omega)$. For this, we introduce a new '$[p,q]$-concave multi-norm', and use some theorems of Maurey to show that 'usually' a standard t-multi-norm is not equivalent to any (p,q)-multi-norm on $L^r(\Omega)$. However there are special combinations of p, q, and r when this equivalence does hold, thereby refuting a conjecture of [9].

1.2. Notation. Let E be a normed space. The closed unit ball of E is denoted by $E_{[1]}$, and the dual space of E is E'; the action of $\lambda \in E'$ on $x \in E$ with respect to the duality gives the complex number denoted by $\langle x, \lambda \rangle$. Let E and F be Banach spaces. Then $\mathcal{B}(E,F)$ denotes the Banach space of all bounded linear operators from E to F, with the operator norm.

The standard Banach spaces of all complex-valued sequences on $\mathbb{N}$ that are bounded and r-summable (for $r \geq 1$) are denoted by ℓ^∞ and ℓ^r, respectively; the norms on ℓ^∞ and ℓ^r are denoted by $\|\cdot\|_\infty$ and $\|\cdot\|_r$, respectively, so that c_0 is a closed subspace of ℓ^∞. For $n \in \mathbb{N}$ and $r \in [1,\infty]$, the space $\mathbb{C}^n$ with the ℓ^r-norm is denoted by ℓ^r_n; it is regarded as a subspace of c_0 and ℓ^r by identifying $(x_1,\dots,x_n) \in \mathbb{C}^n$ with $(x_1,\dots,x_n,0,\dots) \in \mathbb{C}^{\mathbb{N}}$. The Banach space of all complex-valued, continuous functions on a compact space K, taken with the uniform norm, is denoted by $C(K)$.

Let Ω be a measure space, and take $r \geq 1$. Then we denote by $L^r(\Omega)$ or $L^r(\Omega,\mu)$ the usual Banach space of complex-valued, r-integrable functions with respect to a positive measure μ on Ω; here

$$\|f\|_r = \Big(\int_\Omega |f(t)|^r \, d\mu(t)\Big)^{1/r} \quad (f \in L^r(\Omega)),$$

and we identify functions which are equal almost everywhere. For each $r > 1$, the conjugate index to r is denoted by r', so that we have $1/r + 1/r' = 1$; we also regard 1 and ∞ as conjugates; throughout we interpret

$$\sum_{i=1}^n |\zeta_i|^{r'} \quad \text{or} \quad \Big(\sum_{i=1}^n |\zeta_i|^{r'}\Big)^{1/r'} \quad \text{as} \quad \max\{|\zeta_1|,\dots,|\zeta_n|\}$$

when $r = 1$. For $r \geq 1$, the dual space of $L^r(\Omega)$ is identified with $L^{r'}(\Omega)$ in the usual manner.

It is standard [1, Proposition 6.4.1] that, in the case where $L^r(\Omega)$ is an infinite-dimensional space, we can regard ℓ^r as a closed, 1-complemented subspace of $L^r(\Omega)$.

Finally in this section, we recall that the generalized Hölder inequality implies the following. Take $q, s, u > 1$ such that $s < q$ and $1/u = 1/s - 1/q$. Then

$$\|(\beta_1, \dots, \beta_n)\|_q = \sup\Big\{\|(\zeta_1\beta_1, \dots, \zeta_n\beta_n)\|_s : \zeta_1, \dots, \zeta_n \in \mathbb{C}, \sum_{j=1}^{n} |\zeta_j|^u \le 1\Big\} \tag{1.2}$$

whenever $n \in \mathbb{N}$ and $\beta_1, \dots, \beta_n \in \mathbb{C}$. Indeed, $1/(u/s) + 1/(q/s) = 1$, and so

$$\begin{aligned}\|(\beta_1, \dots, \beta_n)\|_q &= \|(|\beta_1|^s, \dots, |\beta_n|^s)\|_{q/s}^{1/s}\\ &= \sup\Big\{\Big|\sum_{j=1}^{n} \eta_j|\beta_j|^s\Big|^{1/s} : \sum_{j=1}^{n} |\eta_j|^{u/s} \le 1\Big\}\\ &= \sup\Big\{\Big(\sum_{j=1}^{n} |\zeta_j|^s|\beta_j|^s\Big)^{1/s} : \sum_{j=1}^{n} |\zeta_j|^u \le 1\Big\},\\ &= \sup\Big\{\|(\zeta_1\beta_1, \dots, \zeta_n\beta_n)\|_s : \sum_{j=1}^{\infty} |\zeta_j|^u \le 1\Big\},\end{aligned}$$

giving (1.2).

1.3. The weak p-summing norm. We recall the definition of the weak p-summing norms on a normed space; the following standard definition was given in [10, Definition 4.1.1] and [9, §2.3]. For further discussion, see [11, 13, 14].

Let E be a normed space, and take $p \ge 1$ and $n \in \mathbb{N}$. Following the notation of [10, 8, 14], we define $\mu_{p,n}(\boldsymbol{x})$ for $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$ by

$$\begin{aligned}\mu_{p,n}(\boldsymbol{x}) &= \sup\Big\{\Big(\sum_{i=1}^{n} |\langle x_i, \lambda\rangle|^p\Big)^{1/p} : \lambda \in E'_{[1]}\Big\}\\ &= \sup\{\|(\langle x_1, \lambda\rangle, \dots, \langle x_n, \lambda\rangle)\|_p : \lambda \in E'_{[1]}\}.\end{aligned}$$

Then $\mu_{p,n}$ is the *weak p-summing norm* (at dimension n).

Note that, for all $p \ge 1$, $n \in \mathbb{N}$, and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, we have

$$\mu_{p,n}(\boldsymbol{x}) = \sup\Big\{\Big\|\sum_{j=1}^{n} \zeta_j x_j\Big\| : \zeta_1, \dots, \zeta_n \in \mathbb{C}, \sum_{j=1}^{n} |\zeta_j|^{p'} \le 1\Big\}. \tag{1.3}$$

Let E be a normed space. Take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, and define

$$T_{\boldsymbol{x}} : (\zeta_1, \dots, \zeta_n) \mapsto \sum_{j=1}^{n} \zeta_j x_j, \quad \mathbb{C}^n \to E.$$

It follows from (1.3) that

$$\mu_{p,n}(\boldsymbol{x}) = \|T_{\boldsymbol{x}} : \ell_n^{p'} \to E\| \tag{1.4}$$

for $p \geq 1$; the map $\boldsymbol{x} \mapsto T_{\boldsymbol{x}}$, $(E^n, \mu_{p,n}) \to \mathcal{B}(\ell_n^{p'}, E)$, is an isometric linear isomorphism.

1.4. (q,p)-summing operators. Let E and F be Banach spaces, and suppose that $1 \leq p \leq q < \infty$. We recall that an operator $T \in \mathcal{B}(E,F)$ is (q,p)-*summing* if there exists a constant C such that

$$\Big(\sum_{i=1}^n \|Tx_i\|^q\Big)^{1/q} \leq C\,\mu_{p,n}(x_1,\dots,x_n) \quad (x_1,\dots,x_n \in E,\ n \in \mathbb{N}).$$

The smallest such constant C is denoted by $\pi_{q,p}(T)$. The set of these (q,p)-summing operators is denoted by $\Pi_{q,p}(E,F)$; it is a linear subspace of $\mathcal{B}(E,F)$, and $(\Pi_{q,p}(E,F), \pi_{q,p})$ is a Banach space; we write $(\Pi_p(E,F), \pi_p)$ for $(\Pi_{p,p}(E,F), \pi_{p,p})$. The latter space of all *p-summing operators* has been studied by many authors; see [11, 13, 14, 16, 21], for example.

1.5. The maximum and minimum multi-norm. As in [10] and [8], there are a *maximum multi-norm* and *minimum multi-norm* based on a normed space E; they are denoted by $(\|\cdot\|_n^{\max} : n \in \mathbb{N})$ and $(\|\cdot\|_n^{\min} : n \in \mathbb{N})$, respectively, and they are defined by the property that

$$\|\boldsymbol{x}\|_n^{\min} \leq \|\boldsymbol{x}\|_n \leq \|\boldsymbol{x}\|_n^{\max} \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N})$$

for every multi-norm $(\|\cdot\|_n : n \in \mathbb{N})$ based on E. The formula for $\|\cdot\|_n^{\min}$ is

$$\|\boldsymbol{x}\|_n^{\min} = \max_{i \in \mathbb{N}_n} \|x_i\| \quad (\boldsymbol{x} = (x_1,\dots,x_n) \in E^n,\ n \in \mathbb{N}).$$

The dual of $\|\cdot\|_n^{\max}$ is the weak 1-summing norm $\mu_{1,n}$ [10, Theorem 3.33], and hence

$$\|\boldsymbol{x}\|_n^{\max} = \sup\Big\{\Big|\sum_{j=1}^n \langle x_j, \lambda_j\rangle\Big| : \mu_{1,n}(\boldsymbol{\lambda}) \leq 1\Big\}$$

for each $\boldsymbol{x} = (x_1,\dots,x_n) \in E^n$ and $n \in \mathbb{N}$, where the supremum is taken over all $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n) \in (E')^n$.

1.6. The (p,q)-multi-norm. The following definition was first given in [10, §4.1].

DEFINITION 1.5. Let E be a normed space, and take p,q such that $1 \leq p \leq q < \infty$. For each $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1,\dots,x_n) \in E^n$, define

$$\begin{aligned}\|\boldsymbol{x}\|_n^{(p,q)} &= \sup\Big\{\Big(\sum_{j=1}^n |\langle x_j, \lambda_j\rangle|^q\Big)^{1/q} : \mu_{p,n}(\boldsymbol{\lambda}) \leq 1\Big\}\\ &= \sup\big\{\|(\langle x_1,\lambda_1\rangle,\dots,\langle x_n,\lambda_n\rangle)\|_q : \mu_{p,n}(\boldsymbol{\lambda}) \leq 1\big\},\end{aligned}$$

where the supremum is taken over all $\boldsymbol{\lambda} = (\lambda_1,\dots,\lambda_n) \in (E')^n$.

As noted in [10, Theorem 4.1], $(\|\cdot\|_n^{(p,q)} : n \in \mathbb{N})$ is a multi-norm based on E; it is called the (p,q)-*multi-norm*.

Clearly, we have $(\|\cdot\|_n^{(p,q_1)}) \le (\|\cdot\|_n^{(p,q_2)})$ whenever $1 \le p \le q_2 \le q_1$ and $(\|\cdot\|_n^{(p_1,q)}) \le (\|\cdot\|_n^{(p_2,q)})$ whenever $1 \le p_1 \le p_2 \le q$.

Lemma 1.6. *Let E be a normed space, and take p, q_1, q_2 such that*

$$1 \le p \le q_1 < q_2 < \infty.$$

Then

$$\|\boldsymbol{x}\|_n^{(p,q_2)} = \sup\Big\{\|(\zeta_1 x_1, \ldots, \zeta_n x_n)\|_n^{(p,q_1)} : \sum_{j=1}^n |\zeta_j|^u \le 1\Big\}$$

for all $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$ and $n \in \mathbb{N}$, where u is defined by the equation $1/u = 1/q_1 - 1/q_2$.

Proof. The result follows by applying the generalized Hölder inequality (1.2) with $q = q_2$ and $s = q_1$ and with β_i taken to be the value $\langle x_i, \lambda_i\rangle$ for $i \in \mathbb{N}_n$ from the definition of the multi-norms. ∎

A key result from [9, Theorem 2.6] relates (p,q)-multi-norms to the known theory of absolutely summing operators.

Theorem 1.7. *Let E be a normed space, and take p, q such that $1 \le p \le q < \infty$. Then the (p,q)-multi-norm induces the norm on $c_0 \otimes E$ given by embedding $c_0 \otimes E$ into $\Pi_{q,p}(E', c_0)$.* ∎

Indeed, for $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \ldots, x_n) \in E^n$, we have

$$\|\boldsymbol{x}\|_n^{(p,q)} = \pi_{q,p}(T'_{\boldsymbol{x}} : E' \to c_0). \tag{1.5}$$

Further, it is shown in [9, Corollary 2.9] that, for $1 \le p_1 \le q_1 < \infty$ and $1 \le p_2 \le q_2 < \infty$, we have $(\|\cdot\|_n^{(p_1,q_1)}) \cong (\|\cdot\|_n^{(p_2,q_2)})$ if and only if $\Pi_{q_1,p_1}(E', c_0) = \Pi_{q_2,p_2}(E', c_0)$ as subsets of $\mathcal{B}(E', c_0)$.

Let F be a 1-complemented subspace of a Banach space E, and suppose that $1 \le p \le q < \infty$ and $n \in \mathbb{N}$. Then it follows from [10, Proposition 4.3] that the restriction of the norm $\|\cdot\|_n^{(p,q)}$ on E^n to F^n is exactly $\|\cdot\|_n^{(p,q)}$ defined on F^n. In particular, to show that two (p,q)-multi-norms based on an infinite-dimensional space $L^r(\Omega)$ are not equivalent, it suffices to prove this for the corresponding (p,q)-multi-norms based on ℓ^r.

1.7. The standard t-multi-norm. Let (Ω, μ) be a measure space, take $r \ge 1$, and suppose that $r \le t < \infty$. In [10, §4.2] and [8, §6], there is a definition and discussion of the standard t-multi-norm on the Banach space $L^r(\Omega)$. We recall the definition.

Take $n \in \mathbb{N}$. For each ordered partition $\mathbf{X} = (X_1, \ldots, X_n)$ of Ω into measurable subsets and each $f_1, \ldots, f_n \in L^r(\Omega)$, we define

$$r_{\mathbf{X}}((f_1, \ldots, f_n)) = \Big(\sum_{i=1}^{n} \|P_{X_i} f_i\|^t\Big)^{1/t}.$$

Here $P_{X_i} : f \mapsto f|X_i$ is the projection of $L^r(\Omega)$ onto $L^r(X_i)$, and $\|\cdot\|$ is the L^r-norm. Then we define

$$\|(f_1, \ldots, f_n)\|_n^{[t]} = \sup_{\mathbf{X}} r_{\mathbf{X}}((f_1, \ldots, f_n)),$$

where the supremum is taken over all such measurable ordered partitions $\mathbf{X}$. As in [10, §4.2.1], we see that $(\|\cdot\|_n^{[t]} : n \in \mathbb{N})$ is a multi-norm based on $L^r(\Omega)$; it is the *standard t-multi-norm* on $L^r(\Omega)$.

Clearly the norms $\|\cdot\|_n^{[t]}$ decrease as a function of $t \in [r, \infty)$, and so the maximum among these norms is $\|\cdot\|_n^{[r]}$.

For example, by [10, (4.9)], we have

$$\|(f_1, \ldots, f_n)\|_n^{[t]} = (\|f_1\|^t + \cdots + \|f_n\|^t)^{1/t} \quad (n \in \mathbb{N})$$

whenever $f_1, \ldots, f_n$ in $L^r(\Omega)$ have pairwise-disjoint supports, and, in particular,

$$\|(\delta_1, \ldots, \delta_n)\|_n^{[t]} = n^{1/t} \quad (n \in \mathbb{N}), \tag{1.6}$$

where we regard each δ_i as an element of ℓ^r. Further,

$$\|(f_1, \ldots, f_n)\|_n^{[r]} = \|\,|f_1| \vee \cdots \vee |f_n|\,\| \quad (f_1, \ldots, f_n \in L^r(\Omega),\ n \in \mathbb{N}); \tag{1.7}$$

this is equation (4.13) in [10]. Thus $(\|\cdot\|_n^{[r]})$ is the lattice multi-norm on $L^r(\Omega)$; see [10, §4.3].

Let Ω be a measure space, and take $t \geq 1$. By [10, Theorem 4.26], we have $\|\cdot\|_n^{[t]} = \|\cdot\|_n^{(1,t)}$ on $L^1(\Omega)$.

LEMMA 1.8. *Let Ω be a measure space, and take r, t_1, t_2 such that*

$$1 \leq r \leq t_1 < t_2 < \infty.$$

Then

$$\|(f_1, \ldots, f_n)\|_n^{[t_2]} = \sup\Big\{\|(\zeta_1 f_1, \ldots, \zeta_n f_n)\|_n^{[t_1]} : \sum_{j=1}^{n} |\zeta_j|^v \leq 1\Big\}$$

for each $f_1, \ldots, f_n \in L^r(\Omega)$ and $n \in \mathbb{N}$, where v satisfies $1/v = 1/t_1 - 1/t_2$.

Proof. Let $\mathbf{X} = (X_1, \ldots, X_n)$ be an ordered partition of Ω into measurable subsets. Now the generalized Hölder inequality (1.2) with $q = t_2$ and $s = t_1$ and with β_i taken to be the value $\|P_{X_i} f_i\|$ for $i \in \mathbb{N}_n$ shows that

$$r_{\mathbf{X}}((f_1, \ldots, f_n)) = \sup\Big\{r_{\mathbf{X}}((\zeta_1 f_1, \ldots, \zeta_1 f_n)) : \sum_{j=1}^{n} |\zeta_j|^v \leq 1\Big\}$$

for each $f_1, \ldots, f_n \in L^r(\Omega)$ and $n \in \mathbb{N}$. Taking the supremum over all such ordered partitions $\mathbf{X}$ gives the result. ∎

It was conjectured in [9, §3.8] that, whenever $t \geq r > 1$, the standard t-multi-norm on an infinite-dimensional space $L^r(\Omega)$ is never equivalent to a (p,q)-multi-norm based on the same space. In §4, we shall extend the cases for which this is true, but, in §4.3, we shall give a counter-example to this conjecture.

1.8. Earlier results. The basic questions that we are concerned with in this paper are to determine, for a given normed space, when two (p,q)-multi-norms based on that space are mutually equivalent and when a (p,q)-multi-norm is equivalent to a standard t-multi-norm on the space.

Some elementary relations were given in [10]. For example, the following is [10, Theorem 4.6].

THEOREM 1.9. *Let E be a normed space. Then $\|\mathbf{x}\|_n^{(1,1)} = \|\mathbf{x}\|_n^{\max}$ for each $\mathbf{x} \in E^n$ and $n \in \mathbb{N}$, and so $(\|\cdot\|_n^{(1,1)} : n \in \mathbb{N})$ is the maximum multi-norm based on E.* ∎

The mutual equivalence of different (p,q)-multi-norms is discussed more seriously in [9, §3]. The first general result is [9, Theorem 2.11]; it follows immediately from [13, Theorem 10.4] by using the connection between (p,q)-multi-norms and absolutely summing operators given in Theorem 1.7.

THEOREM 1.10. *Let E be a normed space, and suppose that*

$$1 \leq p_1 \leq q_1 < \infty \quad \text{and} \quad 1 \leq p_2 \leq q_2 < \infty.$$

Then $(\|\cdot\|_n^{(p_2,q_2)}) \leq (\|\cdot\|_n^{(p_1,q_1)})$ on E when both $1/p_1 - 1/q_1 \leq 1/p_2 - 1/q_2$ and $q_1 \leq q_2$. ∎

Given a $(\bar{p},\bar{q})$-multi-norm, the following figure illustrates the regions where the (p,q)-multi-norms are definitely smaller and larger than this particular $(\bar{p},\bar{q})$-multi-norm on each space $L^r(\Omega)$. We have not at this stage excluded the possibility that the shaded regions are larger; indeed, we shall show in §4 that the upper area can be larger for certain values of r.

To explain the main classification result obtained in [9], we refer to some curves $\mathcal{C}_c$ contained in the 'triangle'

$$\mathcal{T} = \{(p,q) : 1 \leq p \leq q < \infty\}.$$

For $c \in [0,1)$, the curve $\mathcal{C}_c$ is

$$\mathcal{C}_c = \left\{(p,q) \in \mathcal{T} : \frac{1}{p} - \frac{1}{q} = c\right\},$$

so that $\mathcal{T}$ is the union of these curves. Note that, for $r > 1$, the curve $\mathcal{C}_{1/r}$ meets the line $p = 1$ at the point $(1, r')$.

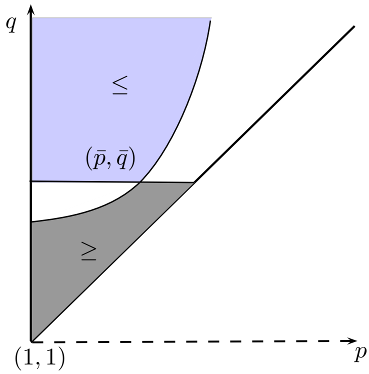

Fig. 1. Regions where the (p,q)-multi-norms are smaller and are larger than a particular $(\bar{p},\bar{q})$-multi-norm

Following [9, §3.2], we say that two points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ in $\mathcal{T}$ are *equivalent for a normed space E* if the corresponding multi-norms $(\|\cdot\|_n^{(p_1,q_1)})$ and $(\|\cdot\|_n^{(p_2,q_2)})$ based on E are equivalent.

The results in [9] on the equivalence of two such points in $\mathcal{T}$ for the Banach space $L^r(\Omega)$ are given in the following cases; here Ω is a measure space, $r \geq 1$, and we suppose that $L^r(\Omega)$ is infinite dimensional.

(I) The case where $r = 1$ is fully resolved in [9, Theorem 3.3].

Indeed, suppose that $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ are in $\mathcal{T}$. In the case where $q_1 \leq q_2$, we have $(\|\cdot\|_n^{(p_2,q_2)}) \preccurlyeq (\|\cdot\|_n^{(p_1,q_1)})$. Thus a necessary condition for the equivalence of P_1 and P_2 on $L^1(\Omega)$ is that $q_1 = q_2$; in this latter case, the points $P_1 = (p_1, q)$ and $P_2 = (p_2, q)$ are equivalent whenever $1 \leq p_1 \leq p_2 < q$, but (p,q) is not equivalent to (q,q) when $1 \leq p < q$.

(II) The case where $r \in (1,2)$ is considered in [9, Theorem 3.16].

(III) The case where $r \geq 2$ is considered in [9, Theorem 3.18].

The latter two cases will be fully described below.

Now take $r > 1$, and set $\overline{r} = \min\{r, 2\}$. We define the set

$$A_r := \left\{(p,q) \in \mathcal{T} : \frac{1}{p} - \frac{1}{q} \geq \frac{1}{\overline{r}}\right\} = \bigcup\{\mathcal{C}_c : c \in [1/\overline{r}, 1)\}.$$

Note that it follows from Theorem 1.10 that $(\|\cdot\|_n^{(p,q)}) \leq (\|\cdot\|_n^{(1,\overline{r}')})$ for each $(p,q) \in A_r$.

The following is [9, Theorem 3.9]. The proof uses Orlicz's theorem and some strong results on tensor norms; we shall give a direct proof of a somewhat more general result in Theorem 2.1, below.

THEOREM 1.11. *Let Ω be a measure space, and take $r > 1$ and $(p,q) \in A_r$. Then $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|^{\min})$ on $L^r(\Omega)$.* ∎

Next, the theorems in [9] show that the two points P_1 and P_2 in $\mathcal{T}$ are not equivalent for $L^r(\Omega)$ (when $L^r(\Omega)$ is an infinite-dimensional space) when at least one point lies outside the region A_r, except perhaps in the following three cases, (A), (B), and (C).

(A): *Both of the points* $P_1 = (p_1, q_1)$ *and* $P_2 = (p_2, q_2)$ *lie on the same curve* $\mathcal{C}_c$, *where* $c \in [0, 1/\overline{r})$ *and, further,* $p_1, p_2 \in [1, r)$ *when* $r < 2$ *and* $p_1, p_2 \in [1, 2]$ *when* $r \geq 2$.

The question whether two such points P_1 and P_2 are indeed equivalent was already resolved in [9, Theorem 3.8] in the special case where $c = 0$: here, $P_1 = (p_1, p_1)$ and $P_2 = (p_2, p_2)$ are equivalent, and the corresponding multi-norms were shown to be equivalent to the maximum multi-norm whenever $p_1, p_2 \in [1, \overline{r})$. Further, in the case where $1 < r < 2$, so that $\overline{r} = r$, the point (r, r) is not equivalent to any point $P = (p, p)$ when $p \in [1, r)$ (this is a result of Kwapień [15, Theorem 7]; see also [3]), and, in the case where $r \geq 2$, so that $\overline{r} = 2$, the point $(2, 2)$ is equivalent to each point $P = (p, p)$ for $p \in [1, 2)$, and hence is equivalent to the maximum multi-norm for $L^r(\Omega)$.

We shall prove in Theorem 2.5 that the above two points P_1 and P_2 specified in case (A) are indeed equivalent whenever $r > 1$. (The case (A) does not arise when $r = 1$.)

The second and third cases that were left open in [9] arise only when $r < 2$ (so that $\overline{r} = r$). Suppose that $c \in [1/2, 1/r)$ and the curve $\mathcal{C}_c$ meets the vertical line $\{(p, q) : p = r\}$ at the point (r, u_c), so that $u_c = r/(1 - cr)$, and consider the horizontal line $\{(p, q) : q = u_c\}$. This line meets the curve $\mathcal{C}_{1/2}$ at the point (x_c, u_c), say, where $x_c = 2u_c/(2 + u_c) = 2r/(2(1 - cr) + r)$, as in [9, §3.5]. Let us denote by L_c the horizontal line segment

$$L_c = \{(p, u_c) : r \leq p \leq x_c\}.$$

(See Figure 3.) Then the following case was also left open in [9].

(B): *Both of the points* $P_1 = (p_1, u_c)$ *and* $P_2 = (p_2, u_c)$ *lie on the line segment* L_c.

Further, the following case was left open.

(C): $P_1 = (p_1, q_1)$ *lies on a curve* $\mathcal{C}_c$, *where* $c \in (0, 1/r)$ *and* $1 \leq p_1 < r$ *and* P_2 *is the point* $(r, r/(1 - cr))$, *which is the left-hand end point of the line* L_c.

We regret that we have not been able to resolve whether P_1 and P_2 are equivalent in case (B); we shall show that we do have equivalence in case (C) whenever $c \in (1/2, 1/r)$, but leave open the case where $0 < c \leq 1/2$.

Two points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ in $\mathcal{T}$ are mutually equivalent for a Banach space E if and only if $\Pi_{q_1,p_1}(E', F) = \Pi_{q_2,p_2}(E', F)$ for every Banach space F [9, Theorem 2.8]. Thus one method of showing that two such points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ are not equivalent for ℓ^r is to show that there is no constant $C > 0$ such that

$$\pi_{q_1,p_1}(I_n : \ell_n^{r'} \to \ell_n^r) \le C\pi_{q_2,p_2}(I_n : \ell_n^{r'} \to \ell_n^r) \quad (n \in \mathbb{N}),$$

where I_n is the identity operator on $\mathbb{C}^n$. For example, it is shown in [3] that

$$\pi_{p,p}(I_n : \ell_n^{r'} \to \ell_n^r) \sim (n \log n)^{1/r} \quad \text{as } n \to \infty$$

for $1 \le p < r < 2$, whereas $\pi_{r,r}(I_n : \ell_n^{r'} \to \ell_n^r) \sim n^{1/r}$ as $n \to \infty$, and so (p,p) is not equivalent to (r,r) whenever $1 \le p < r < 2$. There are several calculations related to these constants $\pi_{q,p}(I_n : \ell_n^{r'} \to \ell_n^r)$ in [5, 12, 19], but it appears that none of them resolve the points that we have left open.

The strongest earlier result about the equivalence of the standard t-multi-norm and a (p,q)-multi-norm on an infinite-dimensional space $L^r(\Omega)$ is given in [9, Theorem 3.22]. It shows that it is possible for a multi-norm $(\|\cdot\|_n^{(p,q)})$ to be equivalent to $(\|\cdot\|_n^{[t]})$ on an infinite-dimensional space $L^r(\Omega)$ only when $1 < r < 2$. Further, if $1 < r < 2$ and $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{[t]})$ on $L^r(\Omega)$, then necessarily $t \ge 2r/(2-r)$, $1/p - 1/q \ge 1/2$, and (p,q) lies on the same curve $\mathcal{D}_c$ (as defined in [9, §3.5]) as (r,t) with $p \le 2t/(2+t)$. Stronger results will be given in §4.

2. Equivalences of (p,q)-multi-norms

2.1. Rademacher functions and Khinchin's inequality. We denote the Rademacher functions defined on $[0,1]$ by r_k for $k \in \mathbb{N}$; see [1, 6.2.1] or [13, p. 10], for example. Then $|r_k(t)| = 1$ ($t \in [0,1]$, $k \in \mathbb{N}$) and

$$\int_0^1 r_i(t) r_j(t)\, dt = 0 \quad (i, j \in \mathbb{N},\, i \ne j).$$

We shall also use a form of Khinchin's inequality (see [1, Theorem 6.2.3] or [22, §I.B.8]): for each $u > 0$, there exist constants A_u and B_u such that

$$\text{(2.1)} \qquad A_u\Big(\sum_{j=1}^n |\alpha_j|^2\Big)^{1/2} \le \Big(\int_0^1 \Big|\sum_{j=1}^n \alpha_j r_j(t)\Big|^u dt\Big)^{1/u} \le B_u\Big(\sum_{j=1}^n |\alpha_j|^2\Big)^{1/2}$$

for all $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ and all $n \in \mathbb{N}$.

A normed space E has *type* u for $1 \le u \le 2$ if there is a constant $K \ge 0$ such that

$$\text{(2.2)} \qquad \Big(\int_0^1 \Big\|\sum_{j=1}^n r_j(t) x_j\Big\|^2 dt\Big)^{1/2} \le K\Big(\sum_{j=1}^n \|x_j\|^u\Big)^{1/u}$$

for all $x_1, \dots, x_n \in E$ and $n \in \mathbb{N}$.

THEOREM 2.1. *Let E be a Banach space with type $u \in [1,2]$, and take $s \in [1,u]$. Then there is a constant $K > 0$ such that*

$$\|\boldsymbol{x}\|_n^{(1,s')} \le K\|\boldsymbol{x}\|_n^{\min} \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N}).$$

Proof. The constant K is defined by equation (2.2).

Take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$, and suppose that $\mu_{1,n}(\boldsymbol{\lambda}) \le 1$, where $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in (E')^n$. Then the following estimates hold; throughout the suprema are taken over all $\zeta_1, \dots, \zeta_n \in \mathbb{C}$ such that $\sum_{j=1}^n |\zeta_j|^s \le 1$:

$$\begin{aligned}\Big(\sum_{j=1}^n |\langle x_j, \lambda_j\rangle|^{s'}\Big)^{1/s'} &= \sup\Big\{\Big|\sum_{j=1}^n \langle \zeta_j x_j, \lambda_j\rangle\Big|\Big\}\\ &= \sup\Big\{\Big|\int_0^1 \Big\langle \sum_{i=1}^n \zeta_i r_i(t) x_i, \sum_{j=1}^n r_j(t)\lambda_j\Big\rangle\, dt\Big|\Big\}\\ &\le \sup\Big\{\int_0^1 \Big\|\sum_{j=1}^n \zeta_j r_j(t) x_j\Big\|\, dt\Big\}\end{aligned}$$

because $\|\sum_{j=1}^n r_j(t)\lambda_j\| \le \mu_{1,n}(\boldsymbol{\lambda})$ by (1.3) (in the case where $p = 1$), and so

$$\begin{aligned}\Big(\sum_{j=1}^n |\langle x_j, \lambda_j\rangle|^{s'}\Big)^{1/s'} &\le \sup\Big\{\Big(\int_0^1 \Big\|\sum_{j=1}^n \zeta_j r_j(t) x_j\Big\|^2 dt\Big)^{1/2}\Big\}\\ &\le K \sup\Big\{\Big(\sum_{j=1}^n \|\zeta_j x_j\|^u\Big)^{1/u}\Big\} \quad \text{by (2.2)}\\ &\le K \max_{j\in\mathbb{N}_n} \|x_j\| \sup\Big\{\Big(\sum_{j=1}^n |\zeta_j|^u\Big)^{1/u}\Big\}\\ &= K \max_{j\in\mathbb{N}_n} \|x_j\|\end{aligned}$$

because $s \le u$.

The result follows. ■

2.2. Calculations for the spaces $L^r(\Omega)$. We now make some calculations that are specific to the Banach space $L^r(\Omega)$. Again, for $r \ge 1$, we set $\overline{r} = \min\{r, 2\}$.

The first result is a reprise of Theorem 1.11 with a more elementary proof; it follows immediately from Theorem 2.1 because a space $L^r(\Omega)$, for $r \ge 1$, has type $\min\{r, 2\}$ [13, Corollary 11.7(a)].

THEOREM 2.2. *Let Ω be a measure space, and take $r > 1$ and $(p,q) \in A_r$. Then $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|^{\min})$ on $L^r(\Omega)$.* ■

We shall use the following elementary calculation, given in [9, (2.5)], concerning (p,q)-multi-norms based on ℓ^r, where $r \ge 1$. Recall that, for each

$k \in \mathbb{N}$, we write δ_k for the sequence $(\delta_{j,k} : j \in \mathbb{N})$. Indeed, for each $(p,q) \in \mathcal{T}$ and each $n \in \mathbb{N}$, we have

$$\Delta_n(p,q) = \begin{cases} n^{1/r+1/q-1/p} & \text{when } p < r \text{ and } 1/p - 1/q \le 1/r, \\ 1 & \text{when } 1/p - 1/q > 1/r, \\ n^{1/q} & \text{when } p \ge r, \end{cases} \tag{2.3}$$

where $\Delta_n(p,q) = \|(\delta_1, \ldots, \delta_n)\|_n^{(p,q)}$ for $(p,q) \in \mathcal{T}$.

The next result is a simple part of [9, Theorem 3.11]; it follows by inspecting the proof of that theorem.

PROPOSITION 2.3. *Let Ω be a measure space such that $L^r(\Omega)$ is infinite dimensional, where $r > 1$. Suppose that $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ lie on curves $\mathcal{C}_{c_1}$ and $\mathcal{C}_{c_2}$, respectively, where $c_2 < \min\{c_1, 1/\overline{r}\}$ and $p_1, p_2 \in [1, \overline{r}]$. Then it is not the case that $(\|\cdot\|_n^{(p_2,q_2)}) \preccurlyeq (\|\cdot\|_n^{(p_1,q_1)})$, and so P_1 and P_2 are not equivalent for $L^r(\Omega)$.* ■

The next lemma is essentially the 'factorization theorem' given as [13, Lemma 2.23], combined with results related to Grothendieck's constant, K_G.

LEMMA 2.4. *Let $F = L^s(\Omega)$, where Ω is a measure space and $s \ge 1$. Take $u > s$ and $u = 2$ in the cases where $s > 2$ and $s \in [1,2]$, respectively. Then there is a constant $K_u > 0$ with the property that, for each $n \in \mathbb{N}$ and each $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in F^n$ with $\mu_{1,n}(\boldsymbol{\lambda}) = 1$, there exist $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$ and $\boldsymbol{\nu} = (\nu_1, \ldots, \nu_n) \in F^n$ such that:*

(i) $\lambda_j = \zeta_j \nu_j$ $(j \in \mathbb{N}_n)$;
(ii) $\sum_{j=1}^n |\zeta_j|^u \le 1$;
(iii) $\mu_{u',n}(\boldsymbol{\nu}) \le K_u$.

In the case where $s \in [1,2]$, we can take $K_u = K_G$.

Proof. First, suppose that $s \in [1,2]$. By [13, Theorem 3.7], each operator $T \in \mathcal{B}(\ell^\infty, F)$ is 2-summing, with $\pi_2(T) \le K_G\|T\|$ $(T \in \mathcal{B}(\ell^\infty, F))$. Second, suppose that $s > 2$, and take $u > s$. By [13, Corollary 10.10], each operator $T \in \mathcal{B}(\ell^\infty, F)$ is u-summing, and so there is a constant K_u (depending on u) such that $\pi_u(T) \le K_u\|T\|$ $(T \in \mathcal{B}(\ell^\infty, F))$.

Now take $n \in \mathbb{N}$ and $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_n) \in F^n$ with $\mu_{1,n}(\boldsymbol{\lambda}) = 1$, and define an operator $T_{\boldsymbol{\lambda}} \in \mathcal{B}(\ell^\infty, F)$ by requiring that $T_{\boldsymbol{\lambda}}(\delta_j) = \lambda_j$ $(j \in \mathbb{N}_n)$ and $T_{\boldsymbol{\lambda}}(\delta_j) = 0$ $(j > n)$. We note that $\|T_{\boldsymbol{\lambda}}\| = \mu_{1,n}(\boldsymbol{\lambda}) = 1$ by (1.4), and so, in each case, T is u-summing, with $\pi_u(T_{\boldsymbol{\lambda}}) \le K_u$.

We now use [13, Lemma 2.23] (taking $r = 1$ in that result) to see that there exist $\zeta_1, \ldots, \zeta_n \in \mathbb{C}$ and $\boldsymbol{\nu} \in F^n$ with the required properties. ■

2.3. The open case (A). The following result resolves the first open case, (A), specified on page 38.

THEOREM 2.5. *Let Ω be a measure space, and take $r > 1$. Consider two points $P_1 = (p_1, q_1)$ and $P_2 = (p_2, q_2)$ in $\mathcal{T}$ lying on the same curve $\mathcal{C}_c$ with $0 \le c < 1$. Suppose, further, that $p_1, p_2 \in [1, r)$ in the case where $1 < r < 2$ and $p_1, p_2 \in [1, 2]$ in the case where $r \ge 2$. Then P_1 and P_2 are equivalent for $L^r(\Omega)$.*

Proof. We set $E = L^r(\Omega)$, $s = r'$, and $F = E' = L^s(\Omega)$.

Take $p < r$ in the case where $1 < r < 2$ and $p = 2$ when $r \ge 2$. We shall first show that there is a constant $K_p > 0$ such that

$$\|\boldsymbol{x}\|_n^{(1,1)} \le K_p \|\boldsymbol{x}\|_n^{(p,p)} \quad (\boldsymbol{x} \in E^n,\ n \in \mathbb{N}). \tag{2.4}$$

Indeed, take $u = p' > s$ when $1 < r < 2$ and $u = 2$ when $r \ge 2$. Let K_p be the constant K_u specified in Lemma 2.4, and take $n \in \mathbb{N}$ and $\boldsymbol{\lambda} = (\lambda_1, \dots, \lambda_n) \in F^n$ with $\mu_{1,n}(\boldsymbol{\lambda}) = 1$; we adopt the notation of the factorization in Lemma 2.4. Take $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$. Then

$$\sum_{j=1}^n |\langle x_j, \lambda_j \rangle| = \sum_{j=1}^n |\langle x_j, \zeta_j \nu_j \rangle| = \sum_{j=1}^n |\zeta_j|\, |\langle x_j, \nu_j \rangle| \le \Big(\sum_{j=1}^n |\langle x_j, \nu_j \rangle|^{u'} \Big)^{1/u'}$$

by Hölder's inequality, noting that $\sum_{j=1}^n |\zeta_j|^u \le 1$, and so

$$\sum_{j=1}^n |\langle x_j, \lambda_j \rangle| \le \Big(\sum_{j=1}^n |\langle x_j, \nu_j \rangle|^p \Big)^{1/p} \le \|\boldsymbol{x}\|_n^{(p,p)} \mu_{p,n}(\boldsymbol{\nu}) \le K_p \|\boldsymbol{x}\|_n^{(p,p)},$$

giving (2.4). This covers the case where $c = 0$.

For the case where $c > 0$, consider a point $P = (p_0, q_0)$ which lies on a curve $\mathcal{C}_{1/v}$, where $v > 1$, and is such that $p_0 \in [1, r)$ in the case where $1 < r < 2$ and $p_0 \in [1, 2]$ in the case where $r \ge 2$; we recall that $(1, v')$ is a point of $\mathcal{C}_{1/v}$. It follows from Theorem 1.10 that it suffices to prove that $(\|\cdot\|_n^{(1,v')}) \preccurlyeq (\|\cdot\|_n^{(p_0,q_0)})$. Again take $n \in \mathbb{N}$ and $\boldsymbol{x} = (x_1, \dots, x_n) \in E^n$.

By Lemma 1.6 with $p = s = 1$ and $q = v'$, we have

$$\|\boldsymbol{x}\|_n^{(1,v')} = \sup\Big\{ \|(\zeta_1 x_1, \dots, \zeta_n x_n)\|_n^{(1,1)} : \sum_{j=1}^n |\zeta_j|^v \le 1 \Big\}.$$

By (2.4),

$$\|\boldsymbol{x}\|_n^{(1,v')} \le K_{p_0} \sup\Big\{ \|(\zeta_1 x_1, \dots, \zeta_n x_n)\|_n^{(p_0,p_0)} : \sum_{j=1}^n |\zeta_j|^v \le 1 \Big\}.$$

However, again by Lemma 1.6, now with $s = p_0$ and $q = q_0$, we have

$$\|\boldsymbol{x}\|_n^{(p_0,q_0)} = \sup\Big\{ \|(\zeta_1 x_1, \dots, \zeta_n x_n)\|_n^{(p_0,p_0)} : \sum_{j=1}^n |\zeta_j|^v \le 1 \Big\}$$

because $1/v = 1/p_0 - 1/q_0$. Thus $(\|\cdot\|_n^{(1,v')}) \preccurlyeq (\|\cdot\|_n^{(p_0,q_0)})$, as required. ■

It remains to be decided whether $P = (r, r/(1-cr)) = (r, u_c)$ is equivalent to $(1, 1/(1-c))$ when $1 < r < 2$; we shall discuss this further later.

We summarize the situation in the case where $r \geq 2$, where we have a full solution to the question concerning the equivalence of (p,q)-multi-norms.

THEOREM 2.6. *Let Ω be a measure space such that $E := L^r(\Omega)$ is an infinite-dimensional space, where $r \geq 2$. Then the triangle $\mathcal{T}$ is decomposed into the following (mutually disjoint) equivalence classes:*

(i) *the region $\mathcal{T}_{\min} := A_r = \{(p,q) \in \mathcal{T} : 1/p - 1/q \geq 1/2\}$;*
(ii) *the curves $\mathcal{T}_c := \{(p,q) \in \mathcal{C}_c : 1 \leq p \leq 2\}$ for $c \in (0, 1/2)$;*
(iii) *the line segment $\mathcal{T}_{\max} := \{(p,p) : 1 \leq p \leq 2\}$;*
(iv) *the singletons $\mathcal{T}_{(p,q)} := \{(p,q)\}$ for $(p,q) \in \mathcal{T}$ with $p > 2$.*

Moreover:

(v) *there is a constant $K > 0$ such that*
$$\|\cdot\|_n^{\min} \leq \|\cdot\|_n^{(p,q)} \leq \|\cdot\|_n^{(1,2)} \leq K\|\cdot\|_n^{\min} \quad (n \in \mathbb{N}),$$
and so the (p,q)-multi-norm is equivalent to the minimum multi-norm for E for each $(p,q) \in \mathcal{T}_{\min}$;
(vi) *for each $c \in (0, 1/2)$ and each $(p,q) \in \mathcal{T}_c$, we have*
$$\|\cdot\|_n^{(2,2/(1-2c))} \leq \|\cdot\|_n^{(p,q)} \leq \|\cdot\|_n^{(1,1/(1-c))} \leq K_G\|\cdot\|_n^{(2,2/(1-2c))} \quad (n \in \mathbb{N});$$
(vii) *for each $(p,p) \in \mathcal{T}_{\max}$, the (p,p)-multi-norm is equivalent to the maximum multi-norm for E, and the $(1,1)$-multi-norm is equal to the maximum multi-norm.*

Proof. It follows from Theorem 2.2 that $\mathcal{T}_{\min}$ is an equivalence class and that clause (v) holds. By Theorems 1.9 and 2.5, $\mathcal{T}_c$ is an equivalence class for each $c \in [0, 1/2)$ and clause (vi) holds, noting that the constant in (2.4) can be taken to be K_G because $s = r' \in [1, 2]$.

It remains to show that there are no other equivalences than those specified above. Again it is sufficient to prove the result for the space ℓ^r. This was established in [9, Theorem 3.18] with the help of Khinchin's inequalities and classical results about Schatten classes. ∎

We now summarize the situation in the case where $1 < r < 2$. Most of the result is contained in [9, Theorem 3.16]; this is combined with the new information given in Theorem 2.5. Clause (vii) will be extended in Proposition 4.10.

THEOREM 2.7. *Let Ω be a measure space such that $E := L^r(\Omega)$ is an infinite-dimensional space, where $1 < r < 2$. Then the triangle $\mathcal{T}$ is decomposed into the following (mutually disjoint) sets. Further, two points in distinct sets are not equivalent, and each specified set is an equivalence class, except possibly as noted:*

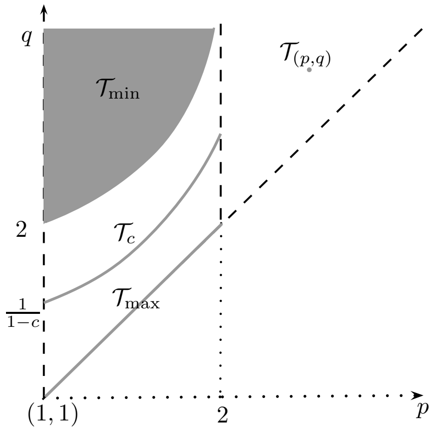

Fig. 2. The various mutually disjoint equivalence classes of (p, q)-multi-norms on $L^r(\Omega)$ for $r \geq 2$

(i) *the region* $\mathcal{T}_{\min} := A_r = \{(p, q) \in \mathcal{T} : 1/p - 1/q \geq 1/r\}$;
(ii) *the curves* $\mathcal{T}_c := \{(p, q) \in \mathcal{C}_c : 1 \leq p \leq r\} \cup \{(p, u_c) : r \leq p \leq x_c\}$, *where* $1/r - 1/u_c = c$ *and* $1/x_c - 1/u_c = 1/2$ *for some* $c \in (1/2, 1/r)$;
(iii) *the curves* $\mathcal{T}_c := \{(p, q) \in \mathcal{C}_c : 1 \leq p \leq r\}$ *for some* $c \in (0, 1/2]$;
(iv) *the line segment* $\mathcal{T}_{\max} := \{(p, p) : 1 \leq p < r\}$;
(v) *the singletons* $\mathcal{T}_{(p,q)} := \{(p, q)\}$ *for* $(p, q) \in \mathcal{T}$ *with either* $p = q = r$ *or both* $p > r$ *and* $1/p - 1/q < 1/2$.

Moreover:

(vi) *there is a constant* $K > 0$ *such that*

$$\|\cdot\|_n^{\min} \leq \|\cdot\|_n^{(p,q)} \leq \|\cdot\|_n^{(1,r')} \leq K\|\cdot\|_n^{\min} \quad (n \in \mathbb{N}),$$

and so the (p, q)*-multi-norm is equivalent to the minimum multi-norm for* E *for each* $(p, q) \in \mathcal{T}_{\min}$;
(vii) *in* $\mathcal{T}_c$ *for* $c \in (0, 1/r)$, *the* (p, q)*-multi-norms with* $1 \leq p < r$ *are all equivalent to the* $(1, 1/(1 - c))$*-multi-norm, but we cannot say whether any two* (p, q)*-multi-norms on the horizontal segment* L_c (*when* $c > 1/2$) *are mutually equivalent, or whether the* (r, u_c)*-multi-norm is equivalent to the* $(1, 1/(1 - c))$*-multi-norm;*
(viii) *for each* $(p, p) \in \mathcal{T}_{\max}$, *the* (p, p)*-multi-norm is equivalent to the maximum multi-norm for* E, *and the* $(1, 1)$*-multi-norm is equal to the maximum multi-norm.* ■

3. The $[p, q]$-concave multi-norms on Banach lattices. In this section, we shall introduce a new class of multi-norms on general Banach lattices, and relate some of them to standard t-multi-norms: these multi-norms are of interest in their own right, and also will help us to settle at least one of

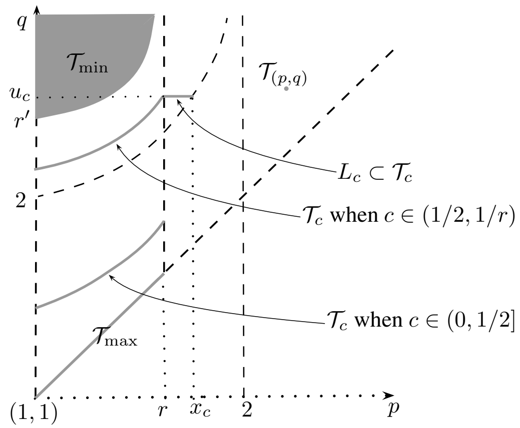

Fig. 3. The various mutually inequivalent sets of (p,q)-multi-norms on $L^r(\Omega)$ for $1 < r < 2$

the above questions about the equivalence of the (p,q)-multi-norms and to resolve the conjecture on the equivalence of (p,q)- and standard t-multi-norms on ℓ^r.

Let $(L, \|\cdot\|)$ be a (complex) Banach lattice. A summary of all necessary background in Banach lattice theory is given in [10, §1.3].

Throughout, L' denotes the dual Banach lattice to L. We write $|x|$ for the modulus of an element $x \in L$. Take $n \in \mathbb{N}$ and an n-tuple $(x_1, \dots, x_n)$ in L^n. Recall that, for each $p \geq 1$, we can define the element $(\sum_{j=1}^n |x_j|^p)^{1/p} \in L$ by the Krivine calculus, and that

$$\Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p} = \sup\Big\{\Big|\sum_{j=1}^n \zeta_j x_j\Big| : \zeta_1, \dots, \zeta_n \in \mathbb{C},\ \sum_{j=1}^n |\zeta_j|^{p'} \leq 1\Big\},$$

where the supremum is taken in the Banach lattice sense; for more details, see [10] and [17, II.1.d], although only real Banach lattices were considered in the latter source. In fact, it can be seen that

$$\begin{aligned}\Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p} &= \sup\Big\{\Re\Big(\sum_{j=1}^n \zeta_j x_j\Big) : \zeta_1, \dots, \zeta_n \in \mathbb{C},\ \sum_{j=1}^n |\zeta_j|^{p'} \leq 1\Big\}\\ &= \sup\Big\{\sum_{j=1}^n |\zeta_j x_j| : \zeta_1, \dots, \zeta_n \in \mathbb{C},\ \sum_{j=1}^n |\zeta_j|^{p'} \leq 1\Big\}.\end{aligned}$$

It is also obvious that

$$\mu_{p,n}(x_1, \dots, x_n) \leq \Big\|\Big(\sum_{j=1}^n |x_j|^p\Big)^{1/p}\Big\|, \tag{3.1}$$

with equality whenever L is a $C(K)$-space.

Definition 3.1. Let $(L, \|\cdot\|)$ be a Banach lattice, and take $p, q \geq 1$ and $n \in \mathbb{N}$. For each $\boldsymbol{x} \in L^n$, define

$$\|\boldsymbol{x}\|_n^{[p,q]} = \sup\Big\{\Big(\sum_{j=1}^n |\langle x_j, \lambda_j\rangle|^q\Big)^{1/q} : \Big\|\Big(\sum_{j=1}^n |\lambda_j|^p\Big)^{1/p}\Big\| \leq 1\Big\},$$

where $\lambda_1, \ldots, \lambda_n \in L'$. Then $\|\cdot\|_n^{[p,q]}$ is the nth $[p,q]$-*concave norm* on L^n.

Clearly, we have $(\|\cdot\|_n^{[p,q_1]}) \leq (\|\cdot\|_n^{[p,q_2]})$ when $1 \leq p \leq q_2 \leq q_1$ and $(\|\cdot\|_n^{[p_1,q]}) \leq (\|\cdot\|_n^{[p_2,q]})$ when $1 \leq p_1 \leq p_2 \leq q$.

We shall prove that $(\|\cdot\|_n^{[p,q]} : n \in \mathbb{N})$ is a multi-norm on L whenever $1 \leq p \leq q < \infty$, and then we shall call the sequence $(\|\cdot\|_n^{[p,q]} : n \in \mathbb{N})$ *the* $[p,q]$-*concave multi-norm on* L. For the remainder of this section, we suppose that $L = (L, \|\cdot\|)$ is a Banach lattice.

Lemma 3.2. *Suppose that* $1 \leq p \leq q_1 < q_2 < \infty$. *Then*

$$\|\boldsymbol{x}\|_n^{[p,q_2]} = \sup\Big\{\|(\zeta_1 x_1, \ldots, \zeta_n x_n)\|_n^{[p,q_1]} : \sum_{j=1}^n |\zeta_j|^u \leq 1\Big\}$$

for each $\boldsymbol{x} = (x_1, \ldots, x_n) \in L^n$ *and* $n \in \mathbb{N}$, *where* u *satisfies the equation* $1/u = 1/q_1 - 1/q_2$.

Proof. This is essentially the same as the proof of Lemma 1.6. ■

Following the argument in [2, Proposition 3], we obtain the following basic result.

Proposition 3.3. *Suppose that* $1 \leq p \leq q < \infty$, *and let* $\sigma : \mathbb{N}_n \to \mathbb{N}_n$ *be any map. Denote by* $i_1, \ldots, i_m$ *the distinct elements of* $\sigma(\mathbb{N}_n)$. *Then*

$$\|(x_{\sigma(1)}, \ldots, x_{\sigma(n)})\|_n^{[p,q]} \leq \|(x_{i_1}, \ldots, x_{i_m})\|_m^{[p,q]} \quad (x_1, \ldots, x_n \in L).$$

Proof. Let $\lambda_1, \ldots, \lambda_n \in L'$ with $\|(\sum_{j=1}^n |\lambda_j|^p)^{1/p}\| \leq 1$. Then

$$\sum_{j=1}^n |\langle x_{\sigma(j)}, \lambda_j\rangle|^q = \sum_{k=1}^m \sum_{\sigma(j)=i_k} |\langle x_{\sigma(j)}, \lambda_j\rangle|^q \leq \sum_{k=1}^m \Big(\sum_{\sigma(j)=i_k} |\langle x_{\sigma(j)}, \lambda_j\rangle|^p\Big)^{q/p}$$
$$= \sum_{k=1}^m \Big|\sum_{\sigma(j)=i_k} \langle x_{\sigma(j)}, \lambda_j\rangle \zeta_j\Big|^q$$

for some $\zeta_j \in \mathbb{C}$ with $\sum_{\sigma(j)=i_k} |\zeta_j|^{p'} \leq 1$, and so

$$\sum_{j=1}^n |\langle x_{\sigma(j)}, \lambda_j\rangle|^q = \sum_{k=1}^m |\langle x_{i_k}, \mu_k\rangle|^q,$$

where $\mu_k = \sum_{\sigma(j)=i_k} \zeta_j \lambda_j \in L'$.

We see that, for all $\alpha_1,\dots,\alpha_m\in\mathbb{C}$ with $\sum_{k=1}^n|\alpha_k|^{p'}\le 1$, we have

$$\Big|\sum_{k=1}^m \alpha_k\mu_k\Big| = \Big|\sum_{k=1}^m\sum_{\sigma(j)=i_k}\alpha_k\zeta_j\lambda_j\Big| \le \Big(\sum_{j=1}^n|\lambda_j|^p\Big)^{1/p}$$

because $\sum_{k=1}^m\sum_{\sigma(j)=i_k}|\alpha_k\zeta_j|^{p'}\le\sum_{k=1}^n|\alpha_k|^{p'}\le 1$. It follows that

$$\Big(\sum_{k=1}^m|\mu_k|^p\Big)^{1/p}\le\Big(\sum_{j=1}^n|\lambda_j|^p\Big)^{1/p},$$

and so $\|(\sum_{k=1}^m|\mu_k|^p)^{1/p}\|\le 1$.

The result now follows. ∎

THEOREM 3.4. *Let $(L,\|\cdot\|)$ be a Banach lattice. Then the sequence*

$$(\|\cdot\|_n^{[p,q]} : n\in\mathbb{N})$$

is a multi-norm based on L whenever $1\le p\le q<\infty$.

Proof. The multi-norm axioms follow easily, using Proposition 3.3. ∎

Let E be a Banach space, and suppose that $1\le p\le q<\infty$. Recall from [13, p. 330] that a bounded linear operator $T: L\to E$ is (q,p)-*concave* if there is a constant $C>0$ such that

$$\Big(\sum_{j=1}^n\|Tx_j\|^q\Big)^{1/q}\le C\Big\|\Big(\sum_{j=1}^n|x_j|^p\Big)^{1/p}\Big\|\quad(x_1,\dots,x_n\in L,\ n\in\mathbb{N});$$

the least such constant C is denoted by $K_{q,p}(T)$. We write $\mathcal{C}_{q,p}(L,E)$ for the space of (q,p)-concave operators; $\mathcal{C}_{q,p}(L,E)$ is a Banach space with respect to the norm $K_{q,p}(\cdot)$. The Banach lattice L is (q,p)-*concave* if the identity operator $I_L : L\to L$ is (q,p)-concave.

PROPOSITION 3.5. *Let L be a Banach lattice, and take p,q such that $1\le p\le q<\infty$. Then L' is (q,p)-concave if and only if the $[p,q]$-concave multi-norm is equivalent to the minimum multi-norm on L.*

Proof. Suppose first that L' is (q,p)-concave, so that $C:=K_{q,p}(I_L)<\infty$. Then, for each $n\in\mathbb{N}$, $x_1,\dots,x_n\in L$, and $\lambda_1,\dots,\lambda_n\in L'$, we have

$$\begin{aligned}\Big(\sum_{j=1}^n|\langle x_j,\lambda_j\rangle|^q\Big)^{1/q}&\le\max_{j\in\mathbb{N}_n}\|x_j\|\cdot\Big(\sum_{j=1}^n\|\lambda_j\|^q\Big)^{1/q}\\&\le C\max_{j\in\mathbb{N}_n}\|x_j\|\cdot\Big\|\Big(\sum_{j=1}^n|\lambda_j|^p\Big)^{1/p}\Big\|.\end{aligned}$$

Hence $\|(x_1,\dots,x_n)\|_n^{[p,q]}\le C\max_{j\in\mathbb{N}_n}\|x_j\|=C\|(x_1,\dots,x_n)\|_n^{\min}$.

Conversely, suppose that the $[p,q]$-concave multi-norm is equivalent to the minimum multi-norm on L, so that there is a constant $C > 0$ such that

$$\|(x_1,\dots,x_n)\|_n^{[p,q]} \leq C\|(x_1,\dots,x_n)\|_n^{\min} \quad (x_1,\dots,x_n \in L,\ n \in \mathbb{N}).$$

Let $\lambda_1,\dots,\lambda_n \in L'$. Take $\eta > 1$ and $j \in \mathbb{N}_n$, and choose $x_j \in L$ with $\|x_j\| = 1$ and such that $\|\lambda_j\| \leq \eta|\langle x_j,\lambda_j\rangle|$. Then

$$\begin{aligned}\Big(\sum_{j=1}^n \|\lambda_j\|^q\Big)^{1/q} &\leq \eta\Big(\sum_{j=1}^n |\langle x_j,\lambda_j\rangle|^q\Big)^{1/q}\\ &\leq \eta\|(x_1,\dots,x_n)\|_n^{[p,q]}\cdot\Big\|\Big(\sum_{j=1}^n|\lambda_j|^p\Big)^{1/p}\Big\|\\ &\leq C\eta\Big\|\Big(\sum_{j=1}^n|\lambda_j|^p\Big)^{1/p}\Big\|.\end{aligned}$$

Thus L' is (q,p)-concave, with $K_{q,p}(L) \leq C$. ∎

Note that we simply say 'p-concave' for '(p,p)-concave'; in the case where $p = 1$, '$(q,1)$-concave' is also called 'having a lower q-estimate' in [17, II.1.f].

Let E be a Banach space. By theorems of Maurey (see [18] and [13, Corollaries 16.6 and 16.7]), we have

$$\mathcal{C}_{q,p}(L,E) = \mathcal{C}_{q,1}(L,E) \subset \mathcal{C}_{r,r}(L,E)$$

whenever $1 \leq p < q < r < \infty$, and

$$\mathcal{C}_{q,1}(L,E) = \Pi_{q,1}(L,E) \quad \text{whenever } q > 2.$$

The proof of [13, Corollary 16.7] also gives the inclusion

$$\mathcal{C}_{2,2}(L,E) \subset \Pi_{2,1}(L,E).$$

We also have the following more elementary inclusion, which follows immediately from the definitions and inequality (3.1):

$$\Pi_{q,p}(L,E) \subset \mathcal{C}_{q,p}(L,E) \quad \text{with} \quad K_{q,p}(T) \leq \pi_{q,p}(T) \quad (T \in \Pi_{q,p}(L,E))$$

whenever $1 \leq p < q < \infty$; moreover, $\Pi_{q,p}(C(K),E) = \mathcal{C}_{q,p}(C(K),E)$ with $K_{q,p}(T) = \pi_{q,p}(T)$ $(T \in \Pi_{q,p}(C(K),E))$ for a compact space K.

We remark also that, by [13, Theorems 10.4 and 16.5], the inclusion

$$\mathcal{C}_{q_1,p_1}(L,E) \subset \mathcal{C}_{q_2,p_2}(L,E)$$

holds, with $K_{p_2,q_2}(T) \leq K_{p_1,q_1}(T)$ $(T \in \mathcal{C}_{q_1,p_1}(L,E))$ whenever we have $1 \leq p_1 \leq q_1 < \infty$, $1 \leq p_2 \leq q_2 < \infty$, and both $1/p_1 - 1/q_1 \leq 1/p_2 - 1/q_2$ and $q_1 \leq q_2$.

The following result is similar to equation (1.5).

THEOREM 3.6. *Let L be a Banach lattice, and take p,q such that $1 \leq p \leq q < \infty$. Then*

$$\|\boldsymbol{x}\|_n^{[p,q]} = K_{q,p}(T'_{\boldsymbol{x}} : L' \to \ell_n^\infty) \quad (\boldsymbol{x} \in L^n,\ n \in \mathbb{N}).$$

Proof. Set $\boldsymbol{x} = (x_1, \dots, x_n)$ and $K_{q,p} = K_{q,p}(T'_{\boldsymbol{x}} : L' \to \ell_n^\infty)$.

We see that

$$\begin{aligned} K_{q,p} &= \sup\Big\{\Big(\sum_{j=1}^n \|T'_{\boldsymbol{x}}\lambda_j\|_{\ell_n^\infty}^q\Big)^{1/q} : \Big\|\Big(\sum_{j=1}^n |\lambda_j|^p\Big)^{1/p}\Big\| \leq 1\Big\} \\ &= \sup\Big\{\Big(\sum_{j=1}^n \sup_{k\in\mathbb{N}_n} |\langle x_k, \lambda_j\rangle|^q\Big)^{1/q} : \Big\|\Big(\sum_{j=1}^n |\lambda_j|^p\Big)^{1/p}\Big\| \leq 1\Big\} \\ &\geq \sup\Big\{\Big(\sum_{j=1}^n |\langle x_j, \lambda_j\rangle|^q\Big)^{1/q} : \Big\|\Big(\sum_{j=1}^n |\lambda_j|^p\Big)^{1/p}\Big\| \leq 1\Big\} \\ &= \|(x_1, \dots, x_n)\|_n^{[p,q]}, \end{aligned}$$

where $\lambda_1, \dots, \lambda_n \in L'$. In particular, this gives $\|\boldsymbol{x}\|_n^{[p,q]} \leq K_{q,p}$.

On the other hand, take $\lambda_1, \dots, \lambda_n \in L'$ with $\|(\sum_{j=1}^n |\lambda_j|^p)^{1/p}\| \leq 1$. For each $j \in \mathbb{N}_n$, let $k_j \in \mathbb{N}_n$ be such that $\sup_{k\in\mathbb{N}_n} |\langle x_k, \lambda_j\rangle| = |\langle x_{k_j}, \lambda_j\rangle|$, and set $\sigma(j) = k_j$. Then we see that

$$\Big(\sum_{j=1}^n \sup_{k\in\mathbb{N}_n} |\langle x_k, \lambda_j\rangle|^q\Big)^{1/q} \leq \|(x_{\sigma(1)}, \dots, x_{\sigma(n)})\|_n^{[p,q]} \leq \|\boldsymbol{x}\|_n^{[p,q]}.$$

Hence $K_{q,p} \leq \|\boldsymbol{x}\|_n^{[p,q]}$. ■

Consequently, we have the following conclusions.

COROLLARY 3.7. *Let L be a Banach lattice, and consider multi-norms based on L. Then:*

(i) $(\|\cdot\|_n^{[p_2,q_2]}) \leq (\|\cdot\|_n^{[p_1,q_1]})$ *whenever we have* $1 \leq p_1 \leq q_1 < \infty$ *and* $1 \leq p_2 \leq q_2 < \infty$ *and both* $1/p_1 - 1/q_1 \leq 1/p_2 - 1/q_2$ *and* $q_1 \leq q_2$*;*
(ii) $(\|\cdot\|_n^{[p,q]}) \leq (\|\cdot\|_n^{(p,q)})$ *whenever* $1 \leq p \leq q < \infty$*;*
(iii) $(\|\cdot\|_n^{[p,q]}) \cong (\|\cdot\|_n^{[1,q]}) \succcurlyeq (\|\cdot\|_n^{[r,r]})$ *whenever* $1 \leq p < q < r < \infty$*;*
(iv) $(\|\cdot\|_n^{[1,q]}) \cong (\|\cdot\|_n^{(1,q)})$ *in the case where* $q > 2$*;*
(v) $(\|\cdot\|_n^{(1,2)}) \preccurlyeq (\|\cdot\|_n^{[2,2]})$. ■

PROPOSITION 3.8. *Let E be a Banach space, and take $r \geq 1$. Then the map*

$$T \mapsto (T(\delta_j)), \qquad \mathcal{C}_{1,1}(\ell^{r'}, E) \to \ell^r(E),$$

is an isometric isomorphism.

Proof. Take $T \in \mathcal{C}_{1,1}(\ell^{r'}, E)$. For each $n \in \mathbb{N}$, there are $\alpha_1, \dots, \alpha_n \in \mathbb{C}$ with

$$\sum_{j=1}^{n} |\alpha_j|^{r'} \le 1 \quad \text{and} \quad \Big(\sum_{j=1}^{n} \|T(\delta_j)\|^r\Big)^{1/r} = \sum_{j=1}^{n} \|T(\alpha_j \delta_j)\|.$$

Therefore

$$\Big(\sum_{j=1}^{n} \|T(\delta_j)\|^r\Big)^{1/r} \le K_{1,1}(T) \Big\| \sum_{j=1}^{n} |\alpha_j \delta_j| \Big\|_{\ell^{r'}} = K_{1,1}(T).$$

Conversely, take $\boldsymbol{x} = (x_j) \in \ell^r(E)$, and set $T(\delta_j) = x_j$ ($j \in \mathbb{N}$); extend T to be a linear map from c_{00} into E. Then, for each $n \in \mathbb{N}$ and each $f_1, \dots, f_n \in c_{00}$, we see that

$$\begin{aligned} \sum_{k=1}^{n} \|T(f_k)\| &\le \sum_{k=1}^{n} \sum_{j=1}^{\infty} |f_k(j)| \, \|T(\delta_j)\| = \sum_{j=1}^{\infty} \sum_{k=1}^{n} |f_k(j)| \, \|x_j\| \\ &\le \Big(\sum_{j=1}^{\infty} \Big(\sum_{k=1}^{n} |f_k(j)|\Big)^{r'}\Big)^{1/r'} \Big(\sum_{j=1}^{\infty} \|x_j\|^r\Big)^{1/r} \\ &= \Big\| \sum_{k=1}^{n} |f_k| \Big\|_{\ell^{r'}} \|\boldsymbol{x}\|_{\ell^r(E)}. \end{aligned}$$

Thus T extends uniquely to an operator in $\mathcal{C}_{1,1}(\ell^{r'}, E)$ with the 1-concave norm at most $\|\boldsymbol{x}\|_{\ell^r(E)}$. ∎

We can now give a key relationship between a standard t-multi-norm and certain concave multi-norms.

THEOREM 3.9. *Suppose that $1 \le r \le t < \infty$, and set $1/v = 1/r - 1/t$. Then the standard t-multi-norm is equal to the $[1, v']$-concave multi-norm on ℓ^r.*

Proof. By Lemmas 1.8 and 3.2, it is sufficient to consider only the case where $r = t$, so that $v' = 1$. Thus we need to show that

$$\|\boldsymbol{x}\|_n^{[1,1]} = \|\boldsymbol{x}\|_n^{[r]} \quad (\boldsymbol{x} = (x_1, \dots, x_n) \in (\ell^r)^n,\ n \in \mathbb{N}).$$

However, we have seen that

$$\begin{aligned} \|\boldsymbol{x}\|_n^{[1,1]} &= K_{1,1}(T'_{\boldsymbol{x}} : \ell^{r'} \to \ell_n^\infty) = \Big(\sum_{j=1}^{n} \|T'_{\boldsymbol{x}}(\delta_j)\|^r\Big)^{1/r} \\ &= \| \, |x_1| \vee \cdots \vee |x_n| \, \|_{\ell^r}, \end{aligned}$$

and this gives the result. ∎

4. Equivalence of the standard t-multi-norm and a (p, q)-multi-norm

4.1. Notation. We now consider when a standard t-multi-norm is equivalent to a (p, q)-multi-norm on an infinite-dimensional space $L^r(\Omega)$. In fact, this problem clearly divides into two separate questions: determine when $(\|\cdot\|_n^{[t]}) \preccurlyeq (\|\cdot\|_n^{(p,q)})$ and when $(\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[t]})$.

We define two new subsets of the triangle $\mathcal{T}$: for $1 \le r \le t$, we set

$$B_{r,t} = \{(p,q) \in \mathcal{T} : 1/p - 1/q \le 1/r - 1/t,\ q \le t\}$$

and

$$C_{r,t} = \{(p,q) \in \mathcal{T} : 1/p - 1/q \ge 1/r - 1/t\} \cup \{(p,q) \in \mathcal{T} : q \ge t\},$$

so that $B_{r,t}$ and $C_{r,t}$ intersect in the curve

$$L_{r,t} := \{(p,q) \in \mathcal{T} : 1/p - 1/q = 1/r - 1/t,\ p \le r\} \cup \{(p,t) \in \mathcal{T} : r \le p \le t\}.$$

Further, we set $B_r = B_{r,r} = \{(p,p) : 1 \le p \le r\}$ and $C_r = C_{r,r} = \mathcal{T}$. Note that

$$B_{1,t} = \{(p,q) \in \mathcal{T} : q \le t\} \quad \text{and} \quad C_{1,t} = \{(p,q) \in \mathcal{T} : q \ge t\}.$$

The answer to the first question is easy.

Theorem 4.1. *Let Ω be a measure space such that $L^r(\Omega)$ is infinite dimensional, where $r \ge 1$. Then $(\|\cdot\|_n^{[t]}) \preccurlyeq (\|\cdot\|_n^{(p,q)})$ for $L^r(\Omega)$ if and only if $(p,q) \in B_{r,t}$.*

Proof. Let S be the set of points $(p,q) \in \mathcal{T}$ with $(\|\cdot\|_n^{[t]}) \preccurlyeq (\|\cdot\|_n^{(p,q)})$.

By [10, Theorem 4.22], $(\|\cdot\|_n^{[t]}) \le (\|\cdot\|_n^{(r,t)})$, and so $(r,t) \in S$. By Theorem 1.10, we increase $(\|\cdot\|_n^{(p,q)})$ when we move from (r,t) to any point $(p,q) \in \mathcal{T}$ with $1/p - 1/q \le 1/r - 1/t$ and $q \le t$, and so $B_{r,t} \subset S$.

Conversely, let $(p,q) \in S$. In the case where $p \ge r$, we have seen that $\Delta_n(p,q) = n^{1/q}$ ($n \in \mathbb{N}$), and so, by (1.6), we also have $q \le t$ In the case where $p \in [1,r)$, by (2.3) and (1.6) again, we must have $1/p - 1/q \le 1/r - 1/t$, which implies also that $q \le t$. Thus in both cases $(p,q) \in B_{r,t}$, and so $S \subset B_{r,t}$. ■

We now consider the second question.

Definition 4.2. Let Ω be a measure space, set $E = L^r(\Omega)$, where $r \ge 1$, and take $t \ge r$. Then define

$$D_{r,t} = \{(p,q) \in \mathcal{T} : (\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[t]}) \text{ on } E\},$$

with $D_r = D_{r,r}$.

Note that $D_{r,t_2} \subset D_{r,t_1}$ whenever $r \le t_1 \le t_2$, and hence, in particular, $D_{r,t} \subset D_r$ whenever $t \ge r$. It is clear that $A_r \subset D_{r,t}$ for $t \ge r \ge 1$ because $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{\min})$ when $(p,q) \in A_r$ by Theorem 2.2. By comparing the values of $\|(\delta_1,\dots,\delta_n)\|_n^{(p,q)}$ and $\|(\delta_1,\dots,\delta_n)\|_n^{[t]}$ given in (2.3) and (1.6), we see that $D_{r,t} \subset C_{r,t}$ for $t \ge r$.

We now work on the spaces ℓ^r, where $r \ge 1$.

4.2. The case where $r = 1$. We first give a full solution to our questions in the case where $r = 1$. Recall that we have $(\|\cdot\|_n^{[1]}) = (\|\cdot\|_n^{(1,1)}) = (\|\cdot\|_n^{\max})$ on ℓ^1, and so $D_{1,1} = \mathcal{T}$.

Proposition 4.3. *Take $t > 1$. Then*

$$D_{1,t} = \{(p,q) : q \geq \max\{t,p\}\} \setminus \{(t,t)\} = C_{1,t} \setminus \{(t,t)\}.$$

Proof. We know that

$$D_{1,t} \subset C_{1,t} = \{(p,q) : q \geq \max\{t,p\}\}.$$

Also, it is proved in [10, Theorem 4.26] that $(\|\cdot\|_n^{[q]}) = (\|\cdot\|_n^{(1,q)})$ on ℓ^1 for each $q \geq 1$, and so $(1,t) \in D_{1,t}$. By [8, Theorem 5.6] (which depends on [20, Corollary 2.5], cf. [13, Theorem 10.9]), we have $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{(1,q)})$ for $1 \leq p < q$, and so $(p,t) \in D_{1,t}$ for $1 \leq p < t$.

Take $(p,q) \in \mathcal{T}$. It follows from the previous paragraph and Theorem 1.10 that $(p,q) \in D_{1,t}$ whenever $q \geq t$ and $q > p$. It remains to consider the case where $q = p$. If $q = p > t$, then, by [8, Theorem 5.6] again, we have

$$(\|\cdot\|_n^{(p,p)}) \preccurlyeq (\|\cdot\|_n^{(1,t)}) = (\|\cdot\|_n^{[t]}),$$

and so $(p,p) \in D_{1,t}$. On the other hand, in the case where $p = q = t$, we certainly have $(\|\cdot\|_n^{(1,t)}) \leq (\|\cdot\|_n^{(t,t)})$. However, by [9, Theorem 3.2], $(\|\cdot\|_n^{(1,t)}) \not\cong (\|\cdot\|_n^{(t,t)})$, and so it follows that $(\|\cdot\|_n^{(t,t)}) \not\preccurlyeq (\|\cdot\|_n^{(1,t)}) = (\|\cdot\|_n^{[t]})$. Thus $(t,t) \notin D_{1,t}$. ∎

Theorem 4.4. *Suppose that $t \geq 1$ and $1 \leq p \leq q < \infty$. Then*

$$(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{[t]})$$

on the space ℓ^1 if and only if $p = q = t = 1$ or $p < q = t$.

Proof. This follows from Theorem 4.1 and Proposition 4.3. ∎

4.3. The case where $r > 1$. We now turn to the case where $r > 1$.

Lemma 4.5. *Take $t \geq r > 1$ and $1 \leq p \leq q < \infty$, and consider the space ℓ^r. Then*

$$A_r \subset D_{r,t} \subset \left\{(p,q) \in C_{r,t} : \frac{1}{p} - \frac{1}{q} \geq \frac{1}{2}\right\} \subsetneq C_{r,t}.$$

Proof. Let $n \in \mathbb{N}$. As shown in the proof of [9, Theorem 3.22], there exists an element $\boldsymbol{g} = (g_1, \dots, g_n) \in (\ell^r)^n$ such that $\|\boldsymbol{g}\|_n^{[t]} \leq 1$ and

$$\|\boldsymbol{g}\|_n^{(p,q)} \sim \|(\delta_1, \dots, \delta_n)\|_n^{(p,q)} \quad \text{as } n \to \infty,$$

where we are now regarding $\delta_1, \dots, \delta_n$ as elements of ℓ^2. Now suppose that $1/p - 1/q < 1/2$. Then it follows from (2.3) that $\|(\delta_1, \dots, \delta_n)\|_n^{(p,q)} \geq n^\alpha$, where $\alpha = \min\{1/2 + 1/q - 1/p, 1/q\} > 0$. Hence $(p,q) \notin D_{r,t}$. ∎

The following theorem, which is essentially [9, Theorem 3.22], determines fully the relation between the multi-norms $(\|\cdot\|_n^{(p,q)})$ and $(\|\cdot\|_n^{[t]})$ on the space ℓ^r in the case where $r \geq 2$.

THEOREM 4.6. *Suppose that $t \geq r \geq 2$ and $1 \leq p \leq q < \infty$, and consider the space ℓ^r. Then $(\|\cdot\|^{(p,q)}) \preccurlyeq (\|\cdot\|^{[t]})$ if and only if $1/p - 1/q \geq 1/2$, and $(\|\cdot\|^{[t]}) \preccurlyeq (\|\cdot\|^{(p,q)})$ if and only if $(p,q) \in B_{r,t}$. In particular, $(\|\cdot\|_n^{(p,q)})$ and $(\|\cdot\|_n^{[t]})$ are not equivalent on ℓ^r for any $(p,q) \in \mathcal{T}$ and any $t \geq r$.*

Proof. Since $r \geq 2$, the set A_r is equal to $\{(p,q) \in \mathcal{T} : 1/p - 1/q \geq 1/2\}$, giving the first clause. The second clause is Theorem 4.1. ■

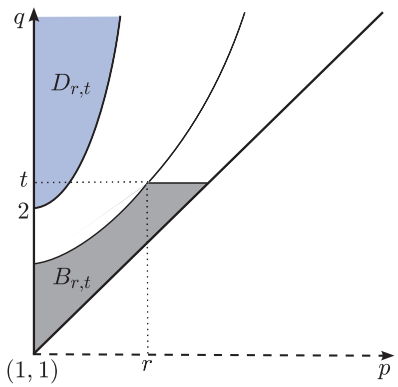

Fig. 4. The sets $B_{r,t}$ and $D_{r,t}$ for $r \geq 2$

It remains to consider the case where $1 < r < 2$, and again it is this case that is the more difficult. Throughout we fix $t \geq r$ and define v by

$$\frac{1}{v} = \frac{1}{r} - \frac{1}{t},$$

taking $v = \infty$ when $t = r$.

PROPOSITION 4.7. *Suppose that $r \in (1,2)$, $t \geq r$, and $1 \leq p \leq q < \infty$. Then:*

(i) *$(p,q) \in D_{r,t}$ whenever $1/p - 1/q \geq 1/v$ and $v < 2$;*
(ii) *$(p,q) \in D_{r,t}$ whenever $1/p - 1/q > 1/2$ and $2 \leq v < \infty$;*
(iii) *$(p,q) \in D_{r,t}$ whenever $1/p - 1/q \geq 1/2$ and $v = \infty$.*

Proof. (i) By Theorem 1.10, it suffices to show that $(\|\cdot\|_n^{(1,v')}) \preccurlyeq (\|\cdot\|_n^{[t]})$. By Theorem 3.9, $(\|\cdot\|_n^{[t]}) = (\|\cdot\|_n^{[1,v']})$. Also it follows from Corollary 3.7(iv) that $(\|\cdot\|_n^{(1,v')}) \cong (\|\cdot\|_n^{[1,v']})$, where we note that $v' > 2$.

(ii) By Theorem 1.10, it suffices to show that $(\|\cdot\|_n^{(1,u)}) \preccurlyeq (\|\cdot\|_n^{[t]})$ whenever $u > 2$. But now

$$(\|\cdot\|_n^{[t]}) = (\|\cdot\|_n^{[1,v']}) \geq (\|\cdot\|_n^{[1,u]}) \cong (\|\cdot\|_n^{(1,u)}) \quad \text{on } \ell^r,$$

as required.

(iii) By Corollary 3.7(v), we have $(\|\cdot\|_n^{(1,2)}) \preccurlyeq (\|\cdot\|_n^{[2,2]})$; by Corollary 3.7(i), we have $(\|\cdot\|_n^{[2,2]}) \leq (\|\cdot\|_n^{[1,1]})$; by Theorem 3.9, $(\|\cdot\|_n^{[1,1]}) = (\|\cdot\|_n^{[t]})$. This gives the stated result. ■

We interpret the above proposition in Figures 5 and 6 below.

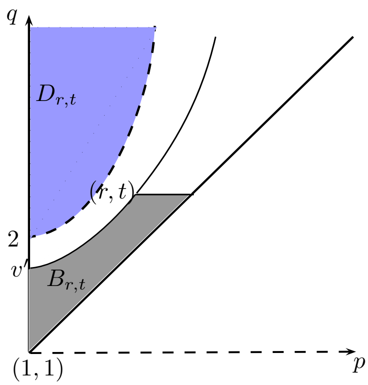

Fig. 5. The set $B_{r,t}$ and (the possible range for) the set $D_{r,t}$ when $1 < r < 2$, $t \geq r$, and $1/r - 1/t \leq 1/2$. When $r \geq 2$, the set $D_{r,t}$ contains the dotted line.

It follows from Figure 5 that, in the case where $1 \leq r \leq t$ and $v > 2$, the multi-norms $(\|\cdot\|_n^{(p,q)})$ are never equivalent to the multi-norm $(\|\cdot\|_n^{[t]})$, as remarked on page 39.

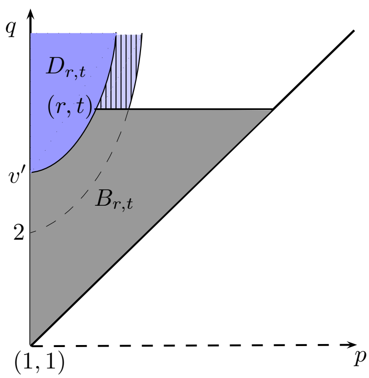

Fig. 6. The set $B_{r,t}$ and (the possible range for) the set $D_{r,t}$ when $1 < r < 2$, $t \geq r$, and $1/r - 1/t > 1/2$

Corollary 4.8. *Suppose that $r > 1$ and that $1 \le p \le q < \infty$. Then $(\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[r]})$ on ℓ^r if and only if $1/p - 1/q \ge 1/2$.*

Proof. Suppose that $(p,q) \in D_r$. Then $1/p - 1/q \ge 1/2$ by Lemma 4.5.

Suppose that $1/p - 1/q \ge 1/2$. Then $(p,q) \in D_r$ on ℓ^r: this follows from Theorem 4.6 when $r \ge 2$ and from Proposition 4.7(iii) when $r \in (1,2)$. ∎

Thus $A_r \subset D_{r,t} \subset D_r = A_2$ and $D_{r,t} \subset C_{r,t}$.

We now have the following counter to the conjecture in [9, §3.8] on the equivalence of (p,q)-multi-norms and standard t-multi-norms.

Theorem 4.9. *Suppose that $1 < r < 2$, that $t \ge r$, and that $1 \le p \le q < \infty$, and consider the space ℓ^r. Suppose further that $1/r - 1/t > 1/2$. Then $(\|\cdot\|_n^{(p,q)}) \cong (\|\cdot\|_n^{[t]})$ whenever*

$$\frac{1}{p} - \frac{1}{q} = \frac{1}{r} - \frac{1}{t} \quad \textit{and} \quad 1 \le p \le r.$$

Proof. Take v as above, so that $v < 2$ and $1/p - 1/q = 1/v$. By Proposition 4.7(i), $(p,q) \in D_{r,t}$, and, by Theorem 4.1, $(p,q) \in B_{r,t}$ whenever $1 \le p \le r$. ∎

In fact, in the case specified in the above theorem, we know that

$$\left\{(p,q) \in \mathcal{T} : \frac{1}{p} - \frac{1}{q} \ge \frac{1}{r} - \frac{1}{t}\right\} \subset D_{r,t} \subset \left\{(p,q) \in C_{r,t} : \frac{1}{p} - \frac{1}{q} \ge \frac{1}{2}\right\},$$

but this is all that we know; if we could resolve case (B) above positively, we would know that

$$D_{r,t} = \left\{(p,q) \in C_{r,t} : \frac{1}{p} - \frac{1}{q} \ge \frac{1}{2}\right\}.$$

The above theory does allow us to improve clause (vii) of Theorem 2.7. We recall that $u_c = r/(1-cr)$.

Proposition 4.10. *Suppose that $1 < r < 2$, and consider the space ℓ^r. Suppose further that $1/2 < c < 1/r$. Then the points $(1, 1/(1-c))$ and (r, u_c) are equivalent, and there is a constant K such that*

$$\|\cdot\|_n^{(r,u_c)} \le \|\cdot\|_n^{(p,q)} \le \|\cdot\|_n^{(1,1/(1-c))} \le K\|\cdot\|_n^{(r,u_c)} \quad (n \in \mathbb{N})$$

whenever $(p,q) \in \mathcal{C}_c$ and $1 \le p \le r$.

Proof. The new information is that $(\|\cdot\|_n^{(r,u_c)}) \cong (\|\cdot\|_n^{[u_c]}) \cong (\|\cdot\|_n^{(1,1/(1-c))})$ by Theorem 4.9. ∎

5. Regular operators. The above results actually have the following interesting consequence concerning the regularity of operators from ℓ^r into ℓ^q.

For a sequence $\alpha = (\alpha_j) \in \mathbb{C}^{\mathbb{N}}$, we set $|\alpha|$ to be the sequence $(|\alpha_j|)$; we say that $\alpha \ge 0$ whenever $\alpha_j \ge 0$ $(j \in \mathbb{N})$. Take $r, q \ge 1$ and $T \in \mathcal{B}(\ell^r, \ell^q)$. Then

T specifies an infinite matrix $(T_{i,j} : i, j \in \mathbb{N})$, where $T_{i,j} = (T\delta_j)_i$ $(i, j \in \mathbb{N})$. The matrix $(|T_{i,j}|)$ then specifies a linear map $|T|$ from ℓ^r to $\mathbb{C}^{\mathbb{N}}$. Another way to define $|T|$ is as follows. A map $T \in \mathcal{B}(\ell^r, \ell^q)$ is *positive* if $T\alpha \geq 0$ in ℓ^q whenever $\alpha \geq 0$ in ℓ^r, and T is *regular* if it is a linear combination of positive operators; the collection of regular operators from ℓ^r to ℓ^q is denoted by $\mathcal{B}_r(\ell^r, \ell^q)$. Thus $T \in \mathcal{B}_r(\ell^r, \ell^q)$ if and only if $|T| \in \mathcal{B}(\ell^r, \ell^q)$. In fact, T is regular if and only if it is order-bounded [10, Theorem 1.31]. For $T \in \mathcal{B}_r(\ell^r, \ell^q)$, we define $|T|$ by

$$|T|(u) = \sup\{|Tz| : |z| \leq u\} \quad (u \geq 0),$$

and extend T linearly. For a summary of properties of the space $\mathcal{B}_r(\ell^r, \ell^q)$ and its connections with 'multi-bounded operators', see [10, §§1.3.4, 6.4.1].

It is well-known that $\mathcal{B}_r(\ell^r, \ell^q) \subsetneq \mathcal{B}(\ell^r, \ell^q)$ when $1 < r, q < \infty$ (cf. [6], where more general results are proved).

THEOREM 5.1. *Take $r \geq 1$. Then the following conditions on $(p, q) \in \mathcal{T}$ are equivalent:*

(a) $(\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[r]})$ *on* ℓ^r*;*

(b) *there exists a constant $C > 0$ such that*

$$\| \, |A| : \ell_m^r \to \ell_n^q \| \leq C \|A : \ell_m^r \to \ell_n^p\|$$

for all $m, n \in \mathbb{N}$ and every $n \times m$ matrix A;

(c) $T \in \mathcal{B}_r(\ell^r, \ell^q)$ *whenever* $T \in \mathcal{B}(\ell^r, \ell^p)$.

Proof. We set $s = r'$.

(a)$\Leftrightarrow$(b) From the definition, we see that $(\|\cdot\|_n^{(p,q)}) \preccurlyeq (\|\cdot\|_n^{[r]})$ on ℓ^r if and only if there is a constant $C > 0$ such that, for all $n \in \mathbb{N}$, all $f_1, \ldots, f_n \in \ell^r$, and all $\lambda_1, \ldots, \lambda_n \in \ell^s$, we have

$$\Big(\sum_{j=1}^{n} |\langle f_j, \lambda_j \rangle|^q\Big)^{1/q} \leq C\mu_{p,n}(\lambda_1, \ldots, \lambda_n)\|(f_1, \ldots, f_n)\|_n^{[r]}.$$

Set $f = |f_1| \vee \cdots \vee |f_n|$. Then $f \in (\ell^r)^+$ and $\|(f_1, \ldots, f_n)\|_n^{[r]} = \|f\|$. So the statement above is equivalent to the condition that there is a constant $C > 0$ such that, for all $n \in \mathbb{N}$, $f \in (\ell^r)^+$, and $\lambda_1, \ldots, \lambda_n \in \ell^s$, we have

$$\sup\Big\{\Big(\sum_{j=1}^{n} |\langle f_j, \lambda_j \rangle|^q\Big)^{1/q} : f_1, \ldots, f_n \in \ell^r \text{ with } |f_1| \vee \cdots \vee |f_n| = f\Big\} \leq C\mu_{p,n}(\lambda_1, \ldots, \lambda_n)\|f\|.$$

Since the supremum above is attained when $|f_1| = \cdots = |f_n| = f$ and when each $f_j\lambda_j$ is a positive sequence, this inequality can be rewritten as

$$\Big(\sum_{j=1}^{n} \langle f, |\lambda_j| \rangle^q\Big)^{1/q} \leq C\mu_{p,n}(\lambda_1, \ldots, \lambda_n)\|f\|$$

for all $n \in \mathbb{N}$, $f \in (\ell^r)^+$, and $\lambda_1, \ldots, \lambda_n \in \ell^s$.

By a standard approximation argument, we can reduce the above further by requiring that the preceding inequality hold for all $m,n\in\mathbb{N}$, $f\in(\ell^r_m)^+$, and $\lambda_1,\dots,\lambda_n\in\ell^s_m$.

In the latter case, we set $\lambda_j=(\lambda_{1,j},\lambda_{2,j},\dots,\lambda_{m,j})$ for $j\in\mathbb{N}_n$ and set $f=(\alpha_1,\dots,\alpha_m)$. Then the preceding inequality becomes

$$\Big(\sum_{j=1}^{n}\Big(\sum_{i=1}^{m}\alpha_i|\lambda_{i,j}|\Big)^q\Big)^{1/q}\le C\mu_{p,n}(\lambda_1,\dots,\lambda_n)\|(\alpha_i)\|_{\ell^r}$$

for all $m,n\in\mathbb{N}$, $(\alpha_i)\in(\ell^r_m)^+$ and $\lambda_1,\dots,\lambda_n\in\ell^s_m$.

As usual, $(\lambda_{i,j}:i\in\mathbb{N}_m,\ j\in\mathbb{N}_n)$ forms an $m\times n$ matrix, say Λ, whose columns are the vectors $\lambda_1,\dots,\lambda_n$. The above argument shows that $(\|\cdot\|^{(p,q)}_n)\preccurlyeq(\|\cdot\|^{[r]}_n)$ on ℓ^r if and only if there is a constant $C>0$ such that, for every $m\times n$ matrix Λ, we have

$$\|\,|\Lambda|^t:\ell^r_m\to\ell^q_n\,\|\le C\|\Lambda:\ell^{p'}_n\to\ell^s_m\|,$$

where M^t is the transpose of a matrix M and we are using (1.4). In other words, the condition in (a) is equivalent to the existence of a constant $C>0$ such that

$$\|\,|A|:\ell^r_m\to\ell^q_n\|\le C\|A:\ell^r_m\to\ell^p_n\|$$

for all $m,n\in\mathbb{N}$ and every $n\times m$ matrix A.

This establishes the equivalence of (a) and (b).

(b)⇒(c) Clearly, (b) implies that $|A|\in\mathcal{B}(\ell^r,\ell^q)$ whenever $A\in\mathcal{B}(\ell^r,\ell^p)$, and hence that $A\in\mathcal{B}_r(\ell^r,\ell^q)$ whenever $A\in\mathcal{B}(\ell^r,\ell^p)$.

(c)⇒(b) Assume towards a contradiction that (b) does not hold. Then there exists a sequence (A_n) of finite-dimensional matrices such that

$$\|\,|A_n|:\ell^r_*\to\ell^q_*\|\ge n$$

whereas $\|A_n:\ell^r_*\to\ell^p_*\|\le 1$, where $*$ represents suitable indices. Now set

$$A:=A_1\oplus A_2\oplus\cdots,$$

so that A is the block-diagonal matrix where the blocks are the finite-dimensional matrices A_n. Then $A\in\mathcal{B}(\ell^r,\ell^p)$, but $|A|\notin\mathcal{B}(\ell^r,\ell^q)$. Hence (c) fails, the required contradiction. ∎

The discussion above leads to the following result, possibly new, about matrices.

Corollary 5.2. *Take $r>1$ and $1\le p\le q<\infty$. Then there exists a constant $C>0$ such that*

$$\|\,|A|:\ell^r_m\to\ell^q_n\|\le C\|A:\ell^r_m\to\ell^p_n\| \tag{5.1}$$

for all $m,n\in\mathbb{N}$ and every $n\times m$ matrix A if and only if $1/p-1/q\ge1/2$.

Proof. This follows from the equivalence of (a) and (b) in the above proposition and Corollary 4.8. ∎

In terms of operators, we similarly have:

COROLLARY 5.3. *Take $r > 1$ and $1 \leq p \leq q < \infty$. Then $T \in \mathcal{B}_r(\ell^r, \ell^q)$ for every operator $T \in \mathcal{B}(\ell^r, \ell^p)$ if and only if $1/p - 1/q \geq 1/2$.* ■

One implication of Corollary 5.2 was already known (in a stronger form) by a result of G. Bennett. Indeed, by [4, Proposition 3.2], there exist a constant K and, for each $m, n \in \mathbb{N}$, an $n \times m$ matrix A whose entries are all ± 1 such that

$$\|A : \ell_m^r \to \ell_n^p\| \leq K \max\{n^{1/p} m^{(1/2-1/r)^+}, m^{1/r'} n^{(1/p-1/2)^+}\}.$$

It is easy to see that

$$\| |A| : \ell_m^r \to \ell_n^q\| = n^{1/q} m^{1/r'},$$

and so

$$\frac{\|A : \ell_m^r \to \ell_n^q\|}{\| |A| : \ell_m^r \to \ell_n^p\|} \leq K \max\{n^{1/p-1/q}/m^{1/r'-(1/2-1/r)^+}, n^{(1/p-1/2)^+-1/q}\}.$$

Now suppose that $1/p - 1/q < 1/2$. Then $(1/p - 1/2)^+ - 1/q < 0$ and $1/r' - (1/2 - 1/r)^+ > 0$, and so the right-hand side of the above inequality is $K \max\{n^{1/p-1/q} m^{-\alpha}, n^{-\beta}\}$ for some $\alpha, \beta > 0$ which depend on only p, q, and r, and this expression can be made arbitrarily small by making a suitable choice first of $n \in \mathbb{N}$ and then of $m \in \mathbb{N}$. Thus, for a matrix A of the above restricted form, there is no constant $C > 0$ such that (5.1) holds.

Acknowledgements. The first two authors met in Pretoria, South Africa, in December, 2011, at a conference on 'Operator algebra and functional analysis'; they are grateful for the invitations to this conference from Jan Fourie, Louis Labuschagne, and Anton Ströh. The second author is grateful to the London Mathematical Society for financial support that allowed him to visit South Africa. The first author was supported by project MTM2011-23164 of the Ministerio de Ciencia e Innovación, Spain. The second and third authors were supported by a Fast Start Marsden Fund (the Royal Society of New Zealand) awarded to H. L. Pham.

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Received May 8, 2014 (7960)