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Entropy pairs of $\mathbb{Z}^2$ and their directional properties

by

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Abstract. Topological and metric entropy pairs of $\mathbb{Z}^2$-actions are defined and their properties are investigated, analogously to $\mathbb{Z}$-actions. In particular, mixing properties are studied in connection with entropy pairs.

1. Introduction. Properties of measurable and topological dynamics have often been studied together. It is known that there are strong similarities and also sharp differences between them. Ergodicity and strong mixing property in ergodic theory correspond respectively to transitivity and strong mixing in topological dynamics. It is well known that a K-system in measurable dynamics (with completely positive entropy) is strongly mixing of all orders. Many of the K-properties are well understood for $\mathbb{Z}$-actions and $\mathbb{Z}^2$-actions [7, 8]. It has been an open question if in the topological setting, there exists a topological property of entropy which implies topological mixing, and topological mixing of all orders.

There is a well known notion of completely positive entropy (CPE) in topological dynamics: every nontrivial factor of a system has positive entropy. There are examples of topological CPE, but without transitivity. This contrasts with measurable dynamics where CPE implies mixing of all orders, hence ergodicity.

The notion of entropy pairs was introduced by F. Blanchard in order to study a topological analogue of the K-mixing property [2]. He introduced the notion of uniformly positive entropy (UPE) of $\mathbb{Z}$-actions, where every pair $(x, y) \in X \times X$ with $x \neq y$ is an entropy pair. By the definition of entropy pairs it is clear that UPE implies CPE. Blanchard showed that UPE implies weak mixing [1]. However he constructed an interesting example which has UPE but is not strongly mixing. There are also weakly mixing flows with CPE, but without UPE [1].

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We investigate the parallel properties for $\mathbb{Z}^2$-actions. We show that UPE also implies weak mixing, but UPE does not necessarily imply strong mixing. That is, there is not much relation between UPE and mixing properties for $\mathbb{Z}^2$-actions.

We define Property P in the $\mathbb{Z}^2$-setting. We prove that Property P implies UPE and we construct an example which has Property P but is not strongly mixing. We will skip the details in the explanation of some of the examples here when they are analogous to those for $\mathbb{Z}$-actions. We construct an example of a $\mathbb{Z}^2$-action which is not strongly mixing but every directional $\mathbb{Z}$-action, $(X, \Phi^{(i,j)})$, is strongly mixing.

Entropy pairs for a $\mathbb{Z}$-action (X, T, μ) were defined in [4]: $(x, y) \in X \times X$ with $x \neq y$ is a *μ-entropy pair* if any measurable partition $\{Q, Q^{c}\}$ such that $x \in \operatorname{int}(Q)$ and $y \in \operatorname{int}(Q^{c})$ has positive entropy. It is shown in [4] that every μ-entropy pair is a topological entropy pair and the converse is true if (X, T) is uniquely ergodic. Moreover it is shown that every topological entropy pair is a μ-entropy pair for some invariant measure μ. The proof requires the study of the relation between entropy of covers and entropy of partitions.

In Sections 4 and 5 we extend the results based on [3] and [4]. We prove the variational principle for entropy pairs for $\mathbb{Z}^2$-actions. First we show that μ-entropy pairs in a topological dynamical system (X, Φ) are always topological entropy pairs for any invariant measure. Moreover, we show that there exists a measure $\mu \in M(X, \Phi)$ such that $E_\mu(X, \Phi) = E(X, \Phi)$. We mention that most of our arguments work for $\mathbb{Z}^n$ for any $n \geq 2$.

The notion of directional entropy was introduced by Milnor [15] to study the Cellular Automaton map together with the Bernoulli shift. Many of its properties are further studied in [6, 13, 17]. Directional systems can be regarded as non-cocompact subgroup actions and hence the directional entropy is a useful tool to investigate zero entropy $\mathbb{Z}^2$-actions. In the last section, we look into the properties of directional entropy pairs for the case of $E(X, \Phi) = \emptyset$. We study the behavior of directional entropy pairs through several examples.

Recently sequence entropy pairs, relative entropy pairs and entropy n-tuples have been introduced and studied in [9, 11, 18]. We believe that most of these notions can be extended to $\mathbb{Z}^n$-actions.

2. Definitions and notations. We consider a topological dynamical system (TDS) (X, Φ), where X is a compact metric space and Φ denotes a $\mathbb{Z}^2$-action. We assume that $\Phi^{(1,0)} = T$ and $\Phi^{(0,1)} = S$ are homeomorphisms. We denote a dynamical system with measurable structure by $(X, \mathcal{B}, \mu, \Phi)$, where

$$\mu(\Phi^{-g} A) = \mu(A) \quad \text{ for all } g \in \mathbb{Z}^2.$$

We now define the measure-theoretic and topological entropies of a $\mathbb{Z}^2$-action.

Let $\{F_n\}$ be an increasing sequence of parallelograms whose module tends to infinity. Throughout the paper we assume that $\{F_n\}$ satisfies the Følner condition, that is,

$$\frac{|\,gF_n \triangle F_n\,|}{|\,F_n\,|} \to 0 \quad \text{for all } g \in \mathbb{Z}^2.$$

1. *Entropy of a partition $\mathcal{P}$ with respect to a measure μ.* Given a measurable partition $\mathcal{P}$ of X with finitely many elements P_i, we define some notions and definitions related to entropy.

Let $H_\mu(\mathcal{P}) = -\sum_i \mu(P_i)\log\mu(P_i)$, where $P_i \in \mathcal{P}$. The *metric entropy* of Φ with respect to the partition $\mathcal{P}$ is defined by

$$h_\mu(\Phi,\mathcal{P}) = \lim_{n\to\infty}\frac{1}{|F_n|}H_\mu\Big(\bigvee_{g\in F_n}\Phi^{-g}\mathcal{P}\Big), \qquad h_\mu(\Phi) = \sup_{\mathcal{P}} h_\mu(\Phi,\mathcal{P}).$$

It is well known that $h_\mu(\Phi,\mathcal{P})$ is independent of the choice of the sequence $\{F_n\}$ (see [16]).

2. *Topological entropy of an open cover $\mathcal{U}$.* Given any cover $\mathcal{U}$ of X, define

$$N(\mathcal{U}) := \min\Big\{|\mathcal{C}| : \bigcup\mathcal{C} \supset X,\, \mathcal{C}\subset\mathcal{U}\Big\},$$

where $\bigcup\mathcal{C}$ denotes the union of all members of $\mathcal{C}$. Then we define $H_{\rm top}(\mathcal{U}) = \log N(\mathcal{U})$ and the *topological entropy* of $\mathcal{U}$ is

$$h_{\rm top}(\Phi,\mathcal{U}) = \lim_{n\to\infty}\frac{1}{|F_n|}H_{\rm top}\Big(\bigvee_{g\in F_n}\Phi^{-g}\mathcal{U}\Big), \qquad h_{\rm top}(\Phi) = \sup_{\mathcal{U}} h_{\rm top}(\Phi,\mathcal{U}).$$

3. *Combinatorial entropy.* Let $\mathcal{U}$ be a finite, not necessarily open, cover of X. The following definition was introduced in [3]. Set

$$H_{\rm c}(\mathcal{U}) = \log N(\mathcal{U})$$

The *combinatorial entropy* of $\mathcal{U}$ is

$$h_{\rm c}(\Phi,\mathcal{U}) = \lim_{n\to\infty}\frac{1}{|F_n|}H_{\rm c}\Big(\bigvee_{g\in F_n}\Phi^{-g}\mathcal{U}\Big).$$

Note that if $\mathcal{U}$ is a finite open cover, then this definition coincides with that of the topological entropy $h_{\rm top}(\Phi,\mathcal{U})$.

We review some of the definitions for the case of $\mathbb{Z}^2$-actions.

Definition 1. (i) A TDS (X,Φ) is *transitive* (ergodicity in measure theory) if for any two nonempty open sets $U, V \subset X$, there is $g\in\mathbb{Z}^2$ such that $\Phi^{-g}U\cap V \neq \emptyset$.

(ii) A TDS (X, Φ) is *weakly mixing* if the cartesian product $(X \times X, \Phi \times \Phi)$ is transitive.
(iii) A TDS (X, Φ) is *strongly mixing* if for any two nonempty open sets $U, V \subset X$, there is $n_0 \in \mathbb{N}$ such that $\Phi^{-g}U \cap V \neq \emptyset$ for $\|g\|_\infty \geq n_0$.

We use the L^∞ norm to make our proofs simple.

The following definitions and notations were introduced by F. Blanchard [1] for $\mathbb{Z}$-actions, and can be naturally extended to $\mathbb{Z}^2$-actions.

Definition 2. Let (X, Φ) be a TDS. A pair $(x, x') \in X \times X \setminus \triangle$ is called a *topological entropy pair* of (X, Φ) if for every open cover $\mathcal{U} = \{U, V\}$ with $x \in V^{c}$, $x' \in U^{c}$, the entropy $h_{\mathrm{top}}(\mathcal{U})$ is positive. Denote by $E(X, \Phi)$ the set of topological entropy pairs of (X, Φ).

Definition 3. (i) A TDS (X, Φ) has *uniformly positive entropy* (UPE) if $E(X, \Phi) = X \times X \setminus \triangle$. In other words, for any nondense open cover $\{U, V\}$ of X, $h_{\mathrm{top}}(\{U, V\})$ is positive.
(ii) A TDS (X, Φ) has *completely positive entropy* (CPE) if every nontrivial factor of (X, Φ) has positive entropy.

3. Properties of $\mathbb{Z}^2$-actions. The definition of Property P for a $\mathbb{Z}^2$-action, analogous to that in [1], is as follows.

Definition 4. We say that (X, Φ) has *Property P* if for any two nonempty open subsets U_0 and U_1 of X there is an integer N with the following property. For every natural number $k \geq 2$, $i \neq j \in [1, k]$ and every $s = (s(1), \ldots, s(k)) \in \{0, 1\}^k$, if $\|g_i - g_j\|_\infty \equiv 0 \bmod N$, then

$$\bigcap_{i=1}^{k} \Phi^{-g_i} U_{s(i)} \neq \emptyset, \qquad \text{where } g_1 = (0, 0).$$

It is clear that Property P in the case of a $\mathbb{Z}^2$-action implies UPE. We have the following properties whose proofs are similar to those for $\mathbb{Z}$-actions.

Proposition 3.1 ([1]). *Let (X, Φ) be a $\mathbb{Z}^2$-action.*

(i) *UPE implies CPE.*
(ii) *UPE implies weak mixing.*
(iii) *Property P implies UPE.*

Proposition 3.1(i) follows from the definitions of CPE and UPE. But the converse is not true:

Example 1. Let $X = \{0, 1\}^{\mathbb{Z}^2} \cup \{1, 2\}^{\mathbb{Z}^2}$. Since X is not transitive, it is not UPE by Proposition 3.1(ii). We know that a nontrivial factor of a Bernoulli system has positive entropy. Note that every factor of (X, Φ) is the union of those of $\{0,1\}^{\mathbb{Z}^2}$ and $\{1,2\}^{\mathbb{Z}^2}$. Since $\{0,1\}^{\mathbb{Z}^2}$ and $\{1,2\}^{\mathbb{Z}^2}$ are CPE, so is (X,Φ).

Let $\mathcal{W}$ denote the set of all allowed rectangular blocks of X.

Proposition 3.2. *A subshift X on an alphabet A has Property P if for any integer p belonging to some infinite strictly increasing sequence, there exists an integer $N(p)$ such that*

$$\bigcap_{i=0}^{k} \Phi^{-g_i} W_i \neq \emptyset \quad \text{if } \|g_i - g_j\|_\infty \equiv 0 \bmod N(p) \tag{3.1}$$

for arbitrary k and $W_i \in \mathcal{W} \cap A^{p\times p}$ for $0 < i \le k$.

Example 2. We construct a $\mathbb{Z}^2$-action having Property P and therefore UPE, but not strongly mixing.

We modify the construction of Example 5 in [1]. We will choose strictly increasing sequences of positive integers $\{h(n)\}_{n\ge1}$ and $\{g(n)\}_{n\ge1}$. Using these sequences we will construct a decreasing sequence of subshifts of finite type $(X_n)_{n\ge1}$ on $A=\{0,1\}$. This sequence of subshifts converges to a non-strongly mixing subshift X.

First, for $n=1$ choose $g(1)\in 2\mathbb{N}$ and

$$h(1) \in \{1,2,\dots,g(1)-1\} \bmod (1+g(1)).$$

Let $P=\{x : x_{(0,0)}=1\}$ and

$$E_1 = \bigcup_{\{(i,j)\,:\,\|(i,j)\|_\infty = h(1)\}} P\cap \Phi^{-(i,j)}P.$$

We define X_1 by forbidding E_1. Note that X_1 satisfies the condition (3.1) in the above proposition for $p=1$, $N(p)=g(1)$.

For the nth step we choose an even number

$$g(n) \ge \sup\{h(n-1)+1, 2(n-1+g(n-1))+n\}$$

and $h(n)$ satisfying the following conditions:

$$h(n)\in\{n,\dots,g(n)-1\} \bmod (n+g(n)),$$
$$h(n)\in\{n-1,\dots,g(n-1)-1\} \bmod (n-1+g(n-1)),$$
$$\vdots$$
$$h(n)\in\{1,\dots,g(1)-1\} \bmod (1+g(1)).$$

Let

$$E_n = \bigcup_{\{(i,j)\,:\,\|(i,j)\|_\infty = h(n)\}} P\cap \Phi^{-(i,j)}P, \qquad F_n = \bigcup_{j=1}^{n} E_j.$$

We let X_n be the subshift of finite type defined by forbidding F_n. Clearly X_n satisfies (3.1) for $p = k$, $N(p) = g(k)$, $k = 1, \dots, n$. Moreover, X_n is decreasing and nonempty, hence $X = \bigcap X_n$ is nonempty and closed. If $\|(i,j)\|_\infty = h(n)$, then $P \cap \Phi^{(i,j)} P = \emptyset$. Since $\{h(n)\}_{n\ge 1}$ is strictly increasing, X is not strongly mixing.

Note that if (X, Φ) is $\mathbb{Z}^2$-strongly mixing then so is $(X, \Phi^{(i,j)})$ for every $(i,j) \in \mathbb{Z}^2$. However we have the following.

Example 3. There exists a $\mathbb{Z}^2$-action (X, Φ) with the following properties:

(i) $\Phi^{(i,j)}$ is a strongly mixing $\mathbb{Z}$-action for each $(i,j) \in \mathbb{Z}^2$.
(ii) Φ is not strongly mixing.

We modify the construction of the previous example so that it is strongly mixing for every $\mathbb{Z}$-action $\Phi^{(i,j)}$ and each $(i,j) \in \mathbb{Z}^2$.

We choose $g(n)$ and $h(n)$ as in the above example. Let $D_n = P \cap \Phi^{-(1,h(n))} P$ and Y_n be the set defined by forbidding $\bigcup_{j=1}^n D_j$. Then Y_n satisfies (3.1) for $p = 1, \dots, n$ and $N(p) = g(p)$. We let $Y = \bigcap Y_n$. Then Y has Property P, hence UPE. Since $P \cap \Phi^{-(1,h(n))} P = \emptyset$ and $\{h(n)\}_{n\ge 1}$ is strictly increasing, Y is not strongly mixing. However, given a direction (i,j) if there exists n_0 such that $P \cap \Phi^{-n_0(i,j)} P = \emptyset$, then it is clear that $P \cap \Phi^{-n(i,j)} P \neq \emptyset$ for $|n| > n_0$. Hence it is easy to see that $\Phi^{(i,j)}$ is strongly mixing.

4. Measure entropy pairs, topological entropy pairs and relations between them in $\mathbb{Z}^2$. The following Propositions 4.1, 4.2 and 4.5 hold for a $\mathbb{Z}$-action (see [2, 3]). The proofs are independent of the structure of the group.

Proposition 4.1. *Let (X, Φ) be a $\mathbb{Z}^2$-action.*

(i) *If (X, Φ) has positive entropy, then it has an entropy pair.*
(ii) *For any cover $\mathcal{U} = (U, V)$ of X with $h_{\mathrm{top}}(\mathcal{U}, \Phi) > 0$, there is an entropy pair (x, x'), where $x \in U^{\mathrm{c}}, x' \in V^{\mathrm{c}}$.*
(iii) *$E(X, \Phi) = \emptyset$ if and only if $h_{\mathrm{top}}(\Phi) = 0$.*
(iv) *$E(X, \Phi) \cup \triangle$ is a nonempty closed invariant subset of $X \times X$.*

Proposition 4.2. *Let (X, Φ) and (Y, Σ) be $\mathbb{Z}^2$-actions. Let $\phi : (X, \Phi) \to (Y, \Sigma)$ be a factor map.*

(i) *If $(x, x') \in E(X, \Phi)$ and $y = \phi(x) \neq \phi(x') = y'$, then (y, y') is an entropy pair of (Y, Σ).*
(ii) *Conversely, if $(y, y') \in E(Y, \Sigma)$, then there exists $(x, x') \in X \times X$ such that*

$$\phi(x) = y, \quad \phi(x') = y', \quad (x, x') \in E(X, \Phi).$$

Definition 5. Let (X,Φ) be a $\mathbb{Z}^2$-action and $M(X,\Phi)$ be the set of Φ-invariant measures. Let $\mu \in M(X,\Phi)$. We call a pair $(x,x') \in X\times X$ a *μ-entropy pair* if $h_\mu(\{Q,Q^c\})>0$ for every $Q\in\mathcal{B}$ with $x\in \operatorname{int}(Q)$ and $x'\in \operatorname{int}(Q^c)$. The set of μ-entropy pairs of (X,Φ) is denoted by $E_\mu(X,\Phi)$.

Definition 6. A two-set partition $\{Q,Q^c\}$ is called *replete* if neither $\operatorname{int}(Q)$ nor $\operatorname{int}(Q^c)$ is empty.

We assume that all of our partitions are replete.

Proposition 4.3. *Let (X,Φ) be a TDS, and $\mu\in M(X,\Phi)$.*

(i) *Let A, B be nonempty disjoint closed subsets of X. If $h_\mu(\{Q,Q^c\}) > 0$ for every partition with $A\subset Q$, $B\subset Q^c$, then*

$$(A\times B)\cap E_\mu(X,\Phi)\neq\emptyset.$$

(ii) *$E_\mu(X,\Phi)=\emptyset$ if and only if $h_\mu(\Phi)=0$.*

(iii) *$E_\mu(X,\Phi)\cup\triangle$ is a closed invariant subset of $X\times X$.*

As in the topological case (Proposition 4.2), we have

Proposition 4.4. *Let $\phi:(X,\Phi)\to(Y,\Sigma)$ be a factor map, $\mu\in M(X,\Phi)$ and $\nu=\mu\circ\phi^{-1}$.*

(i) *If $(x,x')\in E_\mu(X,\Phi)$ and $\phi(x)\neq\phi(x')$, then $(\phi(x),\phi(x'))$ is an entropy pair of (Y,Σ).*

(ii) *Conversely, if $(y,y')\in E_\nu(Y,\Sigma)$, then there exists $(x,x')\in X\times X$ such that*

$$\phi(x)=y,\qquad \phi(x')=y',\qquad (x,x')\in E_\mu(X,\Phi).$$

Note that Proposition 4.4(i) directly follows from the definitions. But the proof of (ii) needs a little more work. Although we need the variational principle for open covers of $\mathbb{Z}^2$-actions in Section 5, its proof is similar to that for $\mathbb{Z}$-actions (see [3]). We will just state it here instead of proving.

Let α be a partition of X and set

$$\alpha_\Phi^- = \Big(\bigvee_{i\le -1,\,-\infty\le j\le\infty}\Phi^{-(i,j)}\alpha\Big)\vee\Big(\bigvee_{j\le -1}\Phi^{-(0,j)}\alpha\Big),$$

$$\alpha_{F_n}=\bigvee_{(i,j)\in F_n}\Phi^{-(i,j)}\alpha,\qquad \alpha_\Phi=\bigvee_{(i,j)\in\mathbb{Z}^2}\Phi^{-(i,j)}\alpha.$$

Let

$$\mathcal{P}_{m_k,n_k}(\alpha)=\Big(\bigvee_{i=m_k}^{\infty}\Phi^{-(i,0)}\Big(\bigvee_{j=-\infty}^{\infty}\Phi^{-(0,j)}\alpha\Big)\Big)\vee\Big(\bigvee_{i=0}^{m_k-1}\Phi^{-(i,0)}\Big(\bigvee_{j=n_k}^{\infty}\Phi^{-(0,j)}\alpha\Big)\Big).$$

It is easy to see that $\bigcap_k \mathcal{P}_{m_k,n_k}(\alpha)$ is independent of the chosen sequence $(m_k, n_k) \to (\infty, \infty)$ and it is invariant under $\Phi^{(0,1)}$. Moreover

$$\bigvee_{i=-\infty}^{\infty} \Phi^{-(i,0)}\Big(\bigcap_{k=0}^{\infty} \mathcal{P}_{m_k,n_k}(\alpha)\Big)$$

is the Pinsker algebra relative to α, which is invariant under Φ, that is, the largest σ-algebra $\mathcal{P}_\mu(\alpha)$ for which the entropy of Φ is 0 (see [7]).

For a finite partition α and an integer k, let

$$\mathcal{P}_k(\alpha) = \mathcal{P}_{k,k}(\alpha).$$

Then

$$\mathcal{P}_\mu(\alpha) = \bigvee_{i=-\infty}^{\infty} \Phi^{-(i,0)}\Big(\bigcap_{k=0}^{\infty} \mathcal{P}_k(\alpha)\Big)$$

is the Pinsker σ-algebra relative to α. In particular, if α is a generating partition, then $\mathcal{P}_\mu(\alpha)$ is the Pinsker σ-algebra with respect to μ, denoted by $\mathcal{P}_\mu$ (see [7]).

Proposition 4.5 ([8]). *Let $\mathcal{F}$ be a set of finite partitions of X.*

(i) $h(\alpha, \Phi) \le H(\alpha)$ *for* $\alpha \in \mathcal{F}$.
(ii) $h(\alpha \vee \beta, \Phi) \le h(\alpha, \Phi) + h(\beta, \Phi)$ *for* $\alpha, \beta \in \mathcal{F}$.
(iii) $H_\mu(\alpha_{F_n}) - H_\mu(\beta_{F_n}) = H_\mu(\alpha_{F_n} \mid \beta_{F_n}) - H_\mu(\beta_{F_n} \mid \alpha_{F_n})$.
(iv) $h(\alpha, \Phi) = H(\alpha \mid \alpha_\Phi^-)$ *for* $\alpha \in \mathcal{F}$.
(v) $h(\alpha \vee \beta, \Phi) = h(\alpha, \Phi) + H(\beta \mid \beta_\Phi^- \vee \alpha_\Phi)$ *for* $\alpha, \beta \in \mathcal{F}$.

We denote by $\mathcal{P}_\mu$ the Pinsker σ-algebra of $(X, \mathcal{B}, \Phi, \mu)$.

Definition 7. Let $\mu \in M(X, \Phi)$ and A, B be two disjoint subsets of X. A pair (A, B) has *Property* $S(\mu)$ if $h_\mu(\{Q, Q^{\mathrm{c}}\}) > 0$ for every replete partition $A \subset Q$, $B \subset Q^{\mathrm{c}}$.

Lemma 4.6 ([3, 4]). *Let $\mathcal{U} = \{U, V\}$ be a Borel cover, and set $A = V^{\mathrm{c}}$, $B = U^{\mathrm{c}}$, $C = U \cap V$.*

(i) *The pair (A, B) has Property $S(\mu)$ if and only if*

$$\mathbb{E}(1_A \mid \mathcal{P}_\mu)(x)\mathbb{E}(1_B \mid \mathcal{P}_\mu)(x) \neq 0 \quad \mu\text{-a.e. } x.$$

(ii) *For the measurable partitions $\alpha = \{A, B, C\}$ and $\beta = \{A \cup B, C\}$, we have*

$$0 \le h_\mu(\alpha) - h_\mu(\beta) \le h_{\mathrm{c}}(\mathcal{U}).$$

We denote by $\Phi^{\mathbf{m}}$ an $m\mathbb{Z} \times m\mathbb{Z}$-action.

Lemma 4.7. *The Pinsker σ-algebras of Φ and $\Phi^{\mathbf{m}}$ are the same.*

The above lemma says that for any partition α, $h_\mu(\alpha, \Phi) = 0$ if and only if $h_\mu(\alpha, \Phi^{\mathbf{m}}) = 0$.

LEMMA 4.8. *For any finite partition* α,

$$H(\alpha \mid \alpha_{\Phi}^{-}) = H(\alpha \mid \alpha_{\Phi}^{-} \vee \mathcal{P}_{\mu}).$$

Proof. Let β be a finite $\mathcal{P}_{\mu}$-measurable partition. By Proposition 4.5(v),

$$H(\alpha \vee \beta \mid \alpha_{\Phi}^{-} \vee \beta_{\Phi}^{-}) = H(\alpha \mid \alpha_{\Phi}^{-}) + H(\beta \mid \beta_{\Phi}^{-} \vee \alpha_{\Phi}).$$

By the definition of entropy,

$$H(\alpha \vee \beta \mid \alpha_{\Phi}^{-} \vee \beta_{\Phi}^{-}) = H(\alpha \mid \alpha_{\Phi}^{-} \vee \beta_{\Phi}^{-}) + H(\beta \mid \alpha_{\Phi}^{-} \vee \beta_{\Phi}^{-} \vee \alpha_{\Phi}).$$

Since $\beta \subset \beta_{\Phi}^{-}$,

$$H(\alpha \mid \alpha_{\Phi}^{-} \vee \beta_{\Phi}^{-}) = H(\alpha \mid \alpha_{\Phi}^{-}).$$

If we choose a sequence of finite partitions β_n with $\beta_n \nearrow \mathcal{P}_{\mu}$, then the Martingale Convergence Theorem tells us that

$$H(\alpha \mid \alpha_{\Phi}^{-} \vee \mathcal{P}_{\mu}) = H(\alpha \mid \alpha_{\Phi}^{-}). \ \blacksquare$$

PROPOSITION 4.9 ([4]). *Let $X = (X, \mathcal{B}, \Phi, \mu)$ be a measurable dynamical system. Suppose $\mathcal{U} = \{U, V\}$ is a measurable cover such that every measurable two-set partition $\gamma = \{P, P^{c}\}$ which (as a cover) is finer than $\mathcal{U}$ satisfies $h_{\mu}(\gamma, \Phi) > 0$. Then $h_{c}(\mathcal{U}, \Phi) > 0$.*

Proof. Let α and β be the same partitions as in Lemma 4.6(ii). By that lemma,

$$0 \le h_{\mu}(\alpha, \Phi) - h_{\mu}(\beta, \Phi) \le h_{c}(\mathcal{U}, \Phi). \tag{4.1}$$

Applying (4.1) to the $\Phi^{\mathbf{m}}$-action, for any $\mathbf{m} \in \mathbb{N}$, we get

$$0 \le h_{\mu}(\alpha, \Phi^{\mathbf{m}}) - h_{\mu}(\beta, \Phi^{\mathbf{m}}) \le h_{c}(\mathcal{U}, \Phi^{\mathbf{m}}). \tag{4.2}$$

Now we need to show that $h_{\mu}(\alpha, \Phi^{\mathbf{m}}) - h_{\mu}(\beta, \Phi^{\mathbf{m}}) > 0$ for some $\mathbf{m}$. By Proposition 4.5(iv), Lemma 4.7 and Lemma 4.8, we have

$$\begin{aligned} h_{\mu}(\alpha, \Phi^{\mathbf{m}}) - h_{\mu}(\beta, \Phi^{\mathbf{m}}) &= H_{\mu}(\alpha \mid \alpha_{\Phi^{\mathbf{m}}}^{-}) - H_{\mu}(\beta \mid \beta_{\Phi^{\mathbf{m}}}^{-}) \\ &= H_{\mu}(\alpha \mid \alpha_{\Phi^{\mathbf{m}}}^{-} \vee \mathcal{P}_{\mu}) - H_{\mu}(\beta \mid \beta_{\Phi^{\mathbf{m}}}^{-} \vee \mathcal{P}_{\mu}) \\ &\ge H_{\mu}(\alpha \mid \mathcal{P}_{\mathbf{m}}(\alpha) \vee \mathcal{P}_{\mu}) - H_{\mu}(\beta \mid \beta_{\Phi^{\mathbf{m}}}^{-} \vee \mathcal{P}_{\mu}) \\ &\ge H_{\mu}(\alpha \mid \mathcal{P}_{\mathbf{m}}(\alpha) \vee \mathcal{P}_{\mu}) - H_{\mu}(\beta \mid \mathcal{P}_{\mu}). \end{aligned}$$

Now, we take the limit on both sides; since $\lim_{\mathbf{m}\to\infty} \mathcal{P}_{\mathbf{m}}(\alpha) \subset \bigcap_{\mathbf{m}} \mathcal{P}_{\mathbf{m}}(\alpha) \subset \mathcal{P}_{\mu}$, the Martingale Convergence Theorem implies that $H_{\mu}(\alpha \mid \mathcal{P}_{\mathbf{m}} \vee \mathcal{P}_{\mu}) \to H_{\mu}(\alpha \mid \mathcal{P}_{\mu})$. Hence

$$\lim_{\mathbf{m}\to\infty} h_{\mu}(\alpha, \Phi^{\mathbf{m}}) - h_{\mu}(\beta, \Phi^{\mathbf{m}}) \ge H_{\mu}(\alpha \mid \mathcal{P}_{\mu}) - H_{\mu}(\beta \mid \mathcal{P}_{\mu}). \tag{4.3}$$

Since the set C belongs to both α and β, we have

$$\begin{aligned} &H(\alpha \mid \mathcal{P}_\mu) - H(\beta \mid \mathcal{P}_\mu) \\ &\quad = -\int\Big(\sum_{D\in\alpha} \mathbb{E}(1_D \mid \mathcal{P}_\mu)\log(\mathbb{E}(1_D \mid \mathcal{P}_\mu)) \\ &\qquad - \sum_{D\in\beta} \mathbb{E}(1_D \mid \mathcal{P}_\mu)\log(\mathbb{E}(1_D \mid \mathcal{P}_\mu))\Big)\,d\mu \\ &\quad = -\int(\mathbb{E}(1_A \mid \mathcal{P}_\mu)\log(\mathbb{E}(1_A \mid \mathcal{P}_\mu)) + \mathbb{E}(1_B \mid \mathcal{P}_\mu)\log(\mathbb{E}(1_B \mid \mathcal{P}_\mu)) \\ &\qquad - \mathbb{E}(1_{A\cup B} \mid \mathcal{P}_\mu)\log(\mathbb{E}(1_{A\cup B} \mid \mathcal{P}_\mu)))\,d\mu. \end{aligned}$$

By Lemma 4.6(i) there exists a subset $F \in \mathcal{B}$ with $\mu(F) > 0$ such that for every $x \in F$ both $\mathbb{E}(1_A \mid \mathcal{P}_\mu)(x)$ and $\mathbb{E}(1_B \mid \mathcal{P}_\mu)(x)$ are positive. By the convexity of the function $\phi(x) = -x\log x$, the last integral is positive on F and it follows that

$$H(\alpha \mid \mathcal{P}_\mu) - H(\beta \mid \mathcal{P}_\mu) > 0.$$

From (4.2) and (4.3) we conclude that $h_{\mathrm{c}}(\mathcal{U}, \Phi^{\mathbf{m}}) > 0$ for some $\mathbf{m}$, and therefore also $h_{\mathrm{c}}(\mathcal{U}, \Phi) > 0$. ∎

THEOREM 4.10 ([4]). *Every measure entropy pair is a topological entropy pair.*

Proof. Let $(x, x') \in E_\mu(X, \Phi)$ and $\mathcal{U} = \{U, V\}$ be a measurable cover of X with $x \in \mathrm{int}(U^{\mathrm{c}}), x' \in \mathrm{int}(V^{\mathrm{c}})$. Then every partition which is finer than $\mathcal{U}$ has positive entropy. Since this cover has positive combinatorial entropy by Proposition 4.9, (x, x') is also a topological entropy pair. ∎

5. The variational principle. Let $\mathcal{A}$ denote a set of finite symbols and $\mathcal{W}$ be the set of configurations on the alphabet $\mathcal{A}$ in finite rectangular blocks. Moreover, denote by $\mathcal{W}_{m\times n}$ the set of rectangular configurations of size $m \times n$ on $\mathcal{A}$. Given $W \in \mathcal{W}_{m\times n}$, and a rectangular configuration α of size $k \times l$, we define the map $p(\alpha \mid W)$ on $\mathcal{W}_{m\times n}$ by

$$\begin{aligned} p(\alpha \mid W) = {}& \frac{1}{(m-k+1)(n-l+1)} \\ & \cdot \mathrm{card}\{(i,j) : (i,j) \in [0, m-k+1] \times [0, n-l+1], \\ & \qquad W_{(i+p,j+q)} = \alpha_{(p+1,q+1)},\ (p,q) \in [0,k) \times [0,l)\}, \end{aligned}$$

where $k \le m$ and $l \le n$.

Note that $p(\alpha \mid W)$ is the relative frequency of the occurrence of α in W. Using this frequency, we define the entropy of W. Fix a rectangular block $W \in \mathcal{W}_{m\times n}$ and $m > k, n > l$. Let

$$H_{k\times l}(W) = \sum_{\alpha\in\mathcal{A}^{k\times l}} \phi(p(\alpha \mid W)),$$

where $\phi(x) = -x\log x$.

LEMMA 5.1 ([3]). *For any $h > 0$, $\varepsilon > 0$, any integers $k, l \geq 1$ and sufficiently large integers m and n,*

$$\mathrm{card}\{W \in \mathcal{A}^{m\times n} : H_{k\times l}(W) \leq klh\} \leq \exp(mn(h+\varepsilon)).$$

Proof. First assume that $k = l = 1$. We clearly have

$$\mathrm{card}\{W \in \mathcal{A}^{m\times n} : H_{1\times 1}(W) \leq h\} = \sum_{\vec{q}\in Q} \frac{(mn)!}{q_1!\cdots q_s!},$$

where Q is the set of integer vectors $\vec{q} = (q_1, \ldots, q_s) \in \mathbb{N}_+^s$ satisfying

$$\sum_{i=1}^{s} q_i = mn, \quad \sum_{i=1}^{s} \phi\left(\frac{q_i}{mn}\right) \leq h.$$

By Stirling's formula, there exists a constant C_s such that for every $\vec{q} \in Q$,

$$\frac{(mn)!}{q_1!\cdots q_s!} \leq C_s \exp\left(mn\sum_{i=1}^{s}\phi\left(\frac{q_i}{mn}\right)\right) \leq C_s \exp(mnh).$$

Hence

$$\begin{aligned}\mathrm{card}\{W \in \mathcal{A}^{m\times n} : H_{k\times l}(W) \leq h\} &\leq (mn+1)^s C_s \exp(mnh)\\ &\leq \exp(mn(h+\varepsilon))\end{aligned}$$

for sufficiently large m and n, as asserted.

Now we will show that the statement is true for $k > 1$ or $l > 1$ using the above result. For every rectangular block W of size $m\times n$ on the alphabet $\mathcal{A}$, and for an integer pair $(u, v) \in [0, k) \times [0, l)$, we let m_u and n_v be the integer parts of $(m-u)/k, (n-v)/l$ respectively. Given W, we denote by $W^{(u,v)}$ the rectangular block of size $m_u \times n_v$ with the following property: for each $i = 0, \ldots, m_u - 1$ and $j = 0, \ldots, n_v - 1$, $W^{(u,v)}_{i,j}$ is the rectangular subblock of size $k \times l$ of W starting at $(u + ik, v + jl)$. That is, $W^{(u,v)}_{i,j} \in A^{k\times l}$ for $i = 0, \ldots, m_u - 1$, $j = 0, \ldots, n_v - 1$ and $W^{(u,v)}$ is an element of $\mathcal{D}^{m_u\times n_v}$, where $\mathcal{D} = A^{k\times l}$. Since $p(D \mid W^{(u,v)})$ is the relative frequency of D in $W^{(u,v)}$, we have

$$\left|p(D\mid W) - \frac{1}{kl}\sum_{(u,v)\in[0,k)\times[0,l)} p(D\mid W^{(u,v)})\right| \leq \frac{kl}{nm-(k-1)(l-1)}.$$

Since the function $\phi(x) = -x\log x$ is uniformly continuous, for sufficiently large m, n and for every rectangular block of size $m \times n$ on $\mathcal{A}$,

$$\sum_{D\in\mathcal{D}}\left|\phi(p(D\mid W)) - \phi\left(\frac{1}{kl}\sum_{(u,v)\in[0,k)\times[0,l)} p(D\mid W^{(u,v)})\right)\right| \leq \frac{\varepsilon}{2},$$

and by the convexity of ϕ,

$$\frac{1}{kl}\sum_{(u,v)\in[0,k)\times[0,l)}\sum_{D\in\mathcal{D}}\phi(p(D\,|\,W^{(u,v)}))$$
$$\le\sum_{D\in\mathcal{D}}\phi\biggl(\frac{1}{kl}\sum_{(u,v)\in[0,k)\times[0,l)}p(D\,|\,W^{(u,v)})\biggr).$$

Hence

$$\frac{1}{kl}\sum_{(u,v)\in[0,k)\times[0,l)}H_{1\times1}(W^{(u,v)})=\frac{1}{kl}\sum_{(u,v)\in[0,k)\times[0,l)}\sum_{D\in\mathcal{D}}\phi(p(D\,|\,W^{(u,v)}))$$
$$\le\varepsilon/2+\sum_{D\in\mathcal{D}}\phi(p(D\,|\,W))=\varepsilon/2+H_{k\times l}(W).$$

Thus if $H_{k\times l}(W)\le klh$, there exists a pair (u,v) such that

$$H_{1\times1}(W^{(u,v)})\le klh+\varepsilon/2.$$

Now given (u,v) and a word B of size $m_u\times n_v$ on the alphabet $\mathcal{D}$, there exist at most $s^{mn-m_uk\cdot n_vl}$ rectangular blocks W of size $m\times n$ on $\mathcal{A}$ such that $W^{(u,v)}=B$. Thus for sufficiently large m and n, as in the first part of the proof,

$$\begin{aligned}\operatorname{card}\{W\in\mathcal{A}^{m\times n}:H_{k\times l}(W)\le klh\}\\ &\le s^{mn-m_uk\cdot n_vl}\\ &\quad\cdot\sum_{(u,v)\in[0,k)\times[0,l)}\operatorname{card}\{W^{(u,v)}\in\mathcal{D}^{m_u\times n_v}:H_{1\times1}(W^{(u,v)})\le\varepsilon/2+klh\}\\ &\le s^{(k+l)(m+n)}\sum_{(u,v)\in[0,k)\times[0,l)}\exp(m_un_v(\varepsilon+klh))\\ &\le s^{(k+l)(m+n)}kl\exp\biggl(mn\biggl(\frac{\varepsilon}{kl}+h\biggr)\biggr)\le\exp(mn(h+\varepsilon)).\ \blacksquare\end{aligned}$$

We denote by $W(\mathcal{P}_l,M\times N,x)$ a rectangular block name on the alphabet $\{1,\ldots,s\}$ satisfying $W(\mathcal{P}_l,M\times N,x)_g=k$ if $\Phi^{-g}x\in P_k$, for each $g\in[0,M)\times[0,N),1\le k\le s$, where s denotes the number of elements of $\mathcal{P}_l$.

LEMMA 5.2 ([3]). *Let $\mathcal{U}$ be a cover of X, $h=h_{\text{top}}(\mathcal{U},\Phi)$, $K\ge1$ an integer, and $(\mathcal{P}_l:1\le l\le K)$ a finite sequence of partitions of X, all finer than $\mathcal{U}$. For every $\varepsilon>0$ and sufficiently large M and N there exists $x\in X$ such that*

$$H_{m\times n}(W(\mathcal{P}_l,M\times N,x))\ge mn(h-\varepsilon)$$

for every l,m,n with $1\le l,m,n\le K$.

Proof. We can assume that all the partitions $\mathcal{P}_l$ have the same number of elements s. Let $\mathcal{A} = \{1, \ldots, s\}$. For $1 \le l, m, n \le K$ and $M, N \ge K$, we let

$$\Omega(M \times N, m \times n) = \{W \in \mathcal{A}^{M \times N} : H_{m \times n}(W) < mn(h - \varepsilon)\}.$$

We let $\omega(M \times N, m \times n) = \operatorname{card}(\Omega(M \times N, m \times n))$. By Lemma 5.1, for sufficiently large M and N,

$$\omega(M \times N, m \times n) \le \exp(MN(h - \varepsilon/4)) \quad \text{for all } m, n \le K.$$

Choose M and N so that $K^3 < \exp(MN\varepsilon/8)$. For $1 \le m, n \le K$, let

$$Z(m \times n, l) = \{x \in X : W(\mathcal{P}_l, M \times N, x) \in \Omega(M \times N, m \times n)\}.$$

The set $Z(m \times n, l)$ is the union of $\omega(M \times N, m \times n)$ elements of $(\mathcal{P}_l)_{[0,M) \times [0,N)}$. Note that $(\mathcal{P}_l)_{[0,M) \times [0,N)}$ is finer than the cover $\mathcal{U}_{[0,M) \times [0,N)}$, and hence $Z(m \times n, l)$ is covered by $\omega(M \times N, m \times n)$ elements of $\mathcal{U}_{[0,M) \times [0,N)}$. Finally, $\bigcup_{1 \le l, m, n \le K} Z(m \times n, l)$ is covered by

$$K^3 \omega(M \times N, m \times n) < K^3 \exp(MN(h - \varepsilon/4)) < \exp(MN(h - \varepsilon/8))$$

elements of $\mathcal{U}_{[0,M) \times [0,N)}$. Hence

$$\bigcup_{1 \le l, m, n \le K} Z(m \times n, l) \not\supseteq X.$$

This completes the proof of the lemma. ∎

THEOREM 5.3 (The variational principle for open covers [3]). *Let (X, Φ) be a TDS, and $\mathcal{U}$ an open cover of X. There exists a measure $\mu \in M(X, \Phi)$ such that $h_\mu(\mathcal{P}, \Phi) \ge h_{\mathrm{top}}(\mathcal{U}, \Phi)$ for all Borel partitions $\mathcal{P}$ finer than $\mathcal{U}$.*

Proof. Let $\mathcal{U} = \{U_1, \ldots, U_s\}$ be an open cover of X. It is sufficient to consider Borel partitions $\mathcal{P}$ of X of the form

$$\mathcal{P} = \{P_1, \ldots, P_s\} \quad \text{with } P_i \subset U_i \text{ for } 1 \le i \le s.$$

Assume first X is a Cantor set. The set of partitions finer than $\mathcal{U}$ consisting of clopen sets is countable; we denote it by $\{\mathcal{P}_l : l \ge 1\}$. By Lemma 5.2, there exist sequences of integers (M_K) and (N_K) tending to ∞ and a sequence $\{x_K\}$ of elements of X such that

$$H_{m \times n}(W(\mathcal{P}_l, M_K \times N_K, x_K)) \ge mn(h - 1/K)$$

for every $1 \le m, n, l \le K$. Let

$$\mu_K = \frac{1}{M_K N_K} \sum_{g \in [0, M_K) \times [0, N_K)} \delta_{\Phi^g x_K}.$$

Since the set of probability measures is compact, there exists a subsequence $\{\mu_{K_i}\}$ of $\{\mu_K\}$ that converges weak* to a probability measure μ. The measure μ is clearly Φ-invariant. Fix $m, n > 1$, and let E be an atom of the partition $(P_l)_{[0,M_K)\times[0,N_K)}$, with name $W\in\{1,\dots,s\}^{m\times n}$. For every K one has

$$|\,\mu_K(E)-p(W\,|\,W(\mathcal{P}_l,M_K\times N_K,x_K))\,|\le 2mn/K.$$

For a clopen set E,

$$\mu(E)=\lim_{i\to\infty}\mu_{K_i}(E)=\lim_{i\to\infty}p(W\,|\,W(\mathcal{P}_l,M_{K_i}\times N_{K_i},x_{K_i})),$$

hence

$$\phi(\mu(E))=\lim_{i\to\infty}\phi(p(W\,|\,W(\mathcal{P}_l,M_{K_i}\times N_{K_i},x_{K_i}))),$$

and summing over $W\in\{1,\dots,s\}^{m\times n}$, one gets

$$H_\mu\Big(\bigvee_{g\in[1,m]\times[1,n]}\Phi^{-g}\mathcal{P}_l\Big)=\lim_{i\to\infty}H_{m\times n}(W(\mathcal{P}_l,M_{K_i}\times N_{K_i},x_{K_i}))\ge mnh.$$

Finally, by sending m and n to infinity one obtains $h_\mu(\mathcal{P}_l,\Phi)\ge h$.

Now, as X is a Cantor set, the family $(\mathcal{P}_l)$ of partitions is dense in the collection of Borel partitions with respect to the distance associated with $L^1(\mu)$ (see [10]). Thus, $h_\mu(\mathcal{P}_l)\ge h$ for every partition finer than $\mathcal{U}$.

As in the case of a $\mathbb{Z}$-action, it is known that for a $\mathbb{Z}^2$-action there exists a Cantor set Y and $\pi:Y\to X$ such that $\pi\circ\Sigma=\Phi\circ\pi$. Let $\mathcal{V}=\pi^{-1}(\mathcal{U})=\{\pi^{-1}(U_1),\dots,\pi^{-1}(U_s)\}$ be the pre-image of $\mathcal{U}$ under π. One has $h_{\rm top}(\mathcal{V})=h_{\rm top}(\mathcal{U})=h$. By the first step, there exists $\nu\in M(Y,\Sigma)$ such that $h_\nu(\mathcal{Q},\Phi)\ge h$ for every Borel partition $\mathcal{Q}$ of Y finer than $\mathcal{V}$. Let $\mu=\nu\circ\pi^{-1}$ be the image of ν under π. One has $\mu\in M(X,\Phi)$ and, for every Borel partition $\mathcal{P}$ of X finer than $\mathcal{U}$, $\pi^{-1}(\mathcal{P})$ is a Borel partiton of Y which is finer than $\mathcal{V}$ with

$$h_\nu(\mathcal{P},\Sigma)=h_\nu(\pi^{-1}(\mathcal{P}),\Sigma)\ge h.$$

This completes the proof of the theorem. ∎

THEOREM 5.4 ([3]). *Let (X,Φ) be a topological dynamical system. There exists a measure $\mu\in M(X,\Phi)$ such that*

$$E_\mu(X,\Phi)=E(X,\Phi).$$

Proof. We already know that $E_\mu(X,\Phi)\subset E(X,\Phi)$. We need to prove the other direction. Let $\{(x_n,y_n)\}$ be a countable dense subset of $E(X,\Phi)$. Let U_{x_n,r_n} and V_{y_n,r_n} be closed balls with centers x_n, y_n respectively and radius $r_n=d(x_n,y_n)/4$. Then for each cover $(U^{\rm c}_{X_n,r},V^{\rm c}_{X_n,r})$ there exists a measure $\mu_{n,r}$ that satisfies the conclusion of Theorem 5.3. Let

$$\mu=\sum_n\sum_r 2^{-n-r}\mu_{n,r}.$$

Then μ is as required. ∎

COROLLARY 5.5. *Let (X,Φ) be a uniquely ergodic $\mathbb{Z}^2$-action with a unique invariant measure μ. Then $E(X,\Phi)=E_\mu(X,\Phi)$.*

6. Directional entropy pairs. Let $\mathbf{C}$ be the set of all countable covers of X with finite entropy. Let $\vec{v}=(\eta,\xi)$ be a fixed vector of $\mathbb{R}^2$ and let Γ be the set of bounded subsets of $\mathbb{R}^2$. For a cover $\mathcal{U}\in\mathbf{C}$ we put

$$h_{\mathrm{top}}(\Phi^{\vec{v}},\mathcal{U})=\sup_{B\in\Gamma}\overline{\lim_{t\to\infty}}\,\frac{1}{t}\log N\Big(\bigvee_{g\in B+[0,t)\vec{v}}\Phi^{-g}\mathcal{U}\Big).$$

It is not hard to show that

$$\begin{aligned}h_{\mathrm{top}}(\Phi^{\vec{v}},\mathcal{U})&=\sup_{B\in\Gamma}\lim_{t\to\infty}\frac{1}{t}\log N\Big(\bigvee_{g\in B+[0,t)\vec{v}}\Phi^{-g}\mathcal{U}\Big)\\&=\lim_{m\to\infty}\lim_{t\to\infty}\frac{1}{t}\log N\Big(\bigvee_{g\in R(m,\vec{v},t)}\Phi^{-g}\mathcal{U}\Big),\end{aligned}$$

where

$$R(m,\vec{v},t)=\begin{cases}\{(i,j):0\le j\le[t\xi],-m+j\eta/\xi\le i\le m+j\eta/\xi\} & \text{if } \xi\ne 0,\\ \{(i,j):-m\le j\le m,0\le i\le[t\eta]\} & \text{if } \xi=0.\end{cases}$$

The quantity $h_{\mathrm{top}}(\Phi^{\vec{v}},\mathcal{U})$ is said to be the *directional entropy* of Φ *with respect to* $\mathcal{U}$ *in direction* $\vec{v}$. And the quantity

$$h_{\mathrm{top}}(\Phi^{\vec{v}})=\sup_{\mathcal{U}\in\mathbf{C}}h_{\mathrm{top}}(\Phi^{\vec{v}},\mathcal{U})$$

is said to be the *directional entropy* of Φ *in direction* $\vec{v}$.

We define entropy pairs for directional systems for TDS. For a given direction $\vec{v}$ a pair $(x,x')\in X\times X$ is called a $\vec{v}$*-entropy pair* if every non-dense cover $\mathcal{U}=(U,V)$ with $x\in\operatorname{int}(U^{\mathrm{c}})$ and $x'\in\operatorname{int}(V^{\mathrm{c}})$ has positive (possibly ∞) entropy for this direction $\vec{v}$. Denote by $E(X,\Phi^{\vec{v}})$ the set of all $\vec{v}$-entropy pairs. They have the following properties, similar to those of entropy pairs of Φ.

PROPOSITION 6.1. *Let (X,Φ) be a TDS and $\vec{v}$ be a direction vector. Then*

$$h_{\mathrm{top}}(\Phi^{\vec{v}})=0\quad\textit{iff}\quad E(X,\Phi^{\vec{v}})=\emptyset.$$

PROPOSITION 6.2. *Let $\phi:(X,\Phi)\to(Y,\Sigma)$ and $\phi\circ\Phi^{(i,j)}=\Sigma^{(i,j)}\circ\phi$ for each $(i,j)\in\mathbb{Z}^2$.*

(i) *If $(x,x')\in E(X,\Phi^{\vec{v}})$ and $\phi(x)\ne\phi(x')$, then $(\phi(x),\phi(x'))$ is an entropy pair of $(Y,\Sigma^{\vec{v}})$.*

(ii) *Conversely, if $(y,y')\in E(Y,\Sigma^{\vec{v}})$, then there exists $(x,x')\in X\times X\setminus\triangle$ such that*

$$\phi(x)=y,\qquad\phi(x')=y',\qquad(x,x')\in E(X,\Phi^{\vec{v}}).$$

If $E(X,\Phi^{\vec{v}})\cup\triangle=X\times X$ for a given $\vec{v}$, we say that (X,Φ) has $\vec{v}$*-uniformly positive entropy* ($\vec{v}$-UPE).

We will give an example with uniformly positive entropy as a $\mathbb{Z}^2$-action, hence (X,Φ) is $\vec{v}$-UPE for every direction $\vec{v}$ (Example 4).

We now define the sequence entropy of a $\mathbb{Z}^2$-action. Let $A=\{a_n\}_{n=1}^{\infty}$ be a sequence of elements of $\mathbb{Z}^2$. For a cover $\mathcal{U}\in\mathbf{C}$ we put

$$h_A(\Phi,\mathcal{U})=\overline{\lim_{n\to\infty}}\,\frac{1}{n}H\Big(\bigvee_{i=1}^{n}\Phi^{-a_i}\mathcal{U}\Big).$$

The quantity

$$h_A(\Phi)=\sup_{\mathcal{U}\in\mathbf{C}}h_A(\Phi,\mathcal{U})$$

is said to be the *sequence entropy* of Φ along A.

The following proposition explains the relation between the sequence entropy and directional entropy of $\mathbb{Z}^2$-actions. It is known for measurable dynamical systems [13].

PROPOSITION 6.3. *For a direction* $\vec{v}=(\eta,\xi)$:

(i) $h_{\mathrm{top}}(\Phi^{\vec{v}})=|\xi|h_A(\Phi)$, *where* $A=([n\eta/\xi],n)$ *if* $\xi\neq 0$,
(ii) $h_{\mathrm{top}}(\Phi^{\vec{v}})=|\eta|h_A(\Phi)$, *where* $A=([n\eta],0)$ *if* $\xi=0$.

We note that Proposition 6.3 also holds for an irrational direction $\vec{v}=(\eta,\xi)$ if $A=([[n\eta/\xi]],n)$, where $[[t]]$ denotes the nearest integer to $t\in\mathbb{R}\setminus\mathbb{Q}$ (cf. [13, 14]).

DEFINITION 8. For a given sequence A a pair $(x,x')\in X\times X$ is called an A-*sequence entropy pair* if every nondense open cover $\mathcal{U}=(U,V)$ with $x\in\operatorname{int}(U^{\mathrm{c}})$ and $x'\in\operatorname{int}(V^{\mathrm{c}})$ has positive entropy along A. Denote by $SE_A(X,\Phi)$ the set of sequence entropy pairs along A.

PROPOSITION 6.4. *Let* $A=\{a_n\}=\{([[n\eta]],n)\}$ *and* $B=\{b_n\}=\{([n\eta],n)\}$ *be sequences in* $\mathbb{Z}^2$ *and suppose that* η *is irrational. Then*

$$SE_A(X,\Phi)=SE_B(X,\Phi).$$

Proof. It is enough to show that for any nondense open cover $\mathcal{U}=(U,V)$, $h_A(\Phi,\mathcal{U})$ is positive if and only if $h_B(\Phi,\mathcal{U})$ is positive. Let $C_n=A_n\cap B_n$, where $A_n=\{a_1,\dots,a_n\}$ and $B_n=\{b_1,\dots,b_n\}$. Clearly $\lim C_n=C$ is a subsequence of A and B. Since η is irrational, the positivity of $h_A(\Phi,\mathcal{U})$ or $h_B(\Phi,\mathcal{U})$ is equivalent to the positivity of $h_C(\Phi,\mathcal{U})$. ■

REMARK 1. It is not clear yet that for a given nondense open cover $\mathcal{U}$, $h_A(\Phi,\mathcal{U})=h_B(\Phi,\mathcal{U})$.

PROPOSITION 6.5. *For a direction* $\vec{v}=(\eta,\xi)$ *and a sequence* $A=([n\eta/\xi],n)$,

$$E(X,\Phi^{\vec{v}})=SE_A(X,\Phi).$$

Proof. We simply note that for each cover $\mathcal{U}$, $h(\Phi^{\vec{v}},\mathcal{U})$ is positive if and only if $h_A(\Phi,\mathcal{U})$ is positive. ■

DEFINITION 9. For a TDS (X,Φ), $(x_1,x_2) \in X\times X\setminus\triangle$ is a *weakly mixing pair* if for every open neighborhood U_i of $x_i, i = 1, 2$, there is $(m,n) \in \mathbb{N}\times\mathbb{N}$ such that $U_1 \cap \Phi^{-(m,n)}U_1 \neq \emptyset$ and $U_1 \cap \Phi^{-(m,n)}U_2 \neq \emptyset$. We denote by $WM(X,\Phi)$ the set of weakly mixing pairs.

We can define sequence entropy pairs along A analogously to the $\mathbb{Z}$-case. Namely, a pair (x, y) is a *sequence entropy pair* for a $\mathbb{Z}^2$-action if for any open cover $\mathcal{U}$ there is a sequence $\{(i_n, j_n)\}$ in $\mathbb{Z}^2$ such that the entropy of the cover along the sequence is nonzero. We have the following lemma whose proof is similar to one for $\mathbb{Z}$-actions [12].

LEMMA 6.6. *Every sequence entropy pair is a weakly mixing pair in $\mathbb{Z}^2$.*

PROPOSITION 6.7. *Every directional entropy pair is a weakly mixing pair.*

Proof. A directional entropy is a sequence entropy for a $\mathbb{Z}^2$-action. Now apply Lemma 6.6. ■

COROLLARY 6.8. *If (X,Φ) has $\vec{v}$-uniformly positive entropy for a direction $\vec{v}$, then $WM(X,\Phi) = X \times X \setminus \triangle$, that is, (X,Φ) is weakly mixing.*

7. Examples. We will give some examples and also find their entropy pairs or directional entropy pairs. The 2-dimensional golden mean shift which is an analogue to the 1-dimensional golden mean shift is a nontrivial example having UPE.

EXAMPLE 4 (Golden mean shift). Let $A = \{0, 1\}$, and X be the subset of arrays in $A^{\mathbb{Z}^2}$ such that there are never two 1's adjacent either horizontally or vertically. This is called the 2-dimensional *golden mean shift*. We want to show that this is a UPE system. Since every cover (or partition) can be approximated by a union of rectangular configurations (resp. cylinder sets) it is enough to show that a cover of cylinder sets has positive entropy. Now we take two clopen sets U and V which have different configurations with size $N \times N$. Let $\mathcal{U} = (U^{c}, V^{c})$ and K_N be the number of square configurations which are allowed in X of size $N \times N$. We note that rectangular blocks are independent unless they are adjacent. Hence for integers $m > N$ and $n > N$, it is not hard to see that the cardinality of a minimal subcover of $\bigvee_{(i,j)\in[0,m]\times[0,n]} \Phi^{-(i,j)}\mathcal{U}$ is at least $K_N^{[\frac{m-N}{N+1}][\frac{n-N}{N+1}]}$. We are interested in positivity of entropy, and not in its exact value. We have

$$h_{\mathrm{top}}(\Phi,\mathcal{U}) \geq \lim_{n\to\infty}\lim_{m\to\infty}\frac{1}{nm}\left[\frac{m-N}{N+1}\right]\left[\frac{n-N}{N+1}\right]\log K_N > 0.$$

This holds for all N, hence every cover has positive entropy, and UPE follows.

Ledrappier's example has zero entropy as a $\mathbb{Z}^2$-action, therefore $E(X,\Phi) = \emptyset$. However it has the following property.

EXAMPLE 5 (Ledrappier's example). Let $A = \{0,1\}$ and

$$X = \{x \in A^{\mathbb{Z}^2} : x_{i,j} + x_{i,j+1} + x_{i+1,j} = 0(\mathrm{mod} 2),\ \forall (i,j) \in \mathbb{Z}^2\}.$$

We will show that every two-set open cover has positive entropy in every direction $\vec{v} = (i,j)$. Let $\mathcal{U} = (U^c, V^c)$ be the same cover as in Example 4 and K_N be the number of square configurations which are allowed in X with size $N \times N$. It is not hard to see that the cardinality of a minimal subcover of $\bigvee_{g \in R(m,\vec{v},t)} \Phi^{-g}\mathcal{U}$ is at least

$$\begin{cases} 2^m 2^{(i+j)(t-2)} & \text{if } 0 \le j/i < \infty, \\ 2^m 2^{j(t-2)} & \text{if } -\infty < j/i \le -1, \\ 2^m 2^{-i(t-2)} & \text{otherwise.} \end{cases}$$

Therefore $h(\Phi^{\vec{v}}, \mathcal{U}) > 0$. Since this is true for all N, every open cover has positive entropy, hence the system has UPE for all directions.

EXAMPLE 6. Let $X = \{0,1\}^{\mathbb{Z}}$ and $Y = [0,1)$. Let T be the shift map in X and $S(x) = \alpha x$ an irrational rotation by α in Y. We define $\Phi : X \times Y \to X \times Y$ by

$$\Phi^{(i,j)}(x,y) = T^i S^j (x,y) = (T^i x, \alpha^j y).$$

It is obvious that $E(X \times Y, \Phi^{(0,1)}) = \emptyset$, because (Y,S) has zero entropy. Note that (Y, S^j) is minimal and (X, T^i) is UPE for any i and j. By Proposition 3.1 in [5],

$$E(X \times Y, \Phi^{(i,j)}) \cup \triangle_{X \times Y} = \{((x,y),(x',y)) : x, x' \in X,\ y \in Y\}.$$

For an irrational direction $\vec{v}$ it is not hard to show that the set of entropy pairs of $E(X \times Y, \Phi^{\vec{v}})$ is the same as in the rational case.

The following example shows different behaviors of directional entropy pairs in the case that Y is not minimal.

EXAMPLE 7. Let (X,T) have UPE and Y be the compact space $\mathbb{Z} \cup \{\infty\}$ with S being translation by 1 on Y. Let $\Phi : X \times Y \to X \times Y$ be defined by

$$\Phi^{(i,j)}(x,n) = (T^i x, n+j).$$

Then the set of directional entropy pairs is

$$E(X \times Y, \Phi^{\vec{v}}) \cup \triangle_{X \times Y} = \begin{cases} \emptyset & \text{if } \vec{v} = (0,1), \\ \{((x,y),(x',y)) : x, x' \in X,\ y \in Y\} & \text{if } \vec{v} = (1,0), \\ (X \times \{\infty\}) \times (X \times \{\infty\}) & \text{otherwise.} \end{cases}$$

We will prove the third case. Suppose $\mathcal{U} = (U, V)$ is a standard cover of X and $\vec{v} = (p, q)$, $pq \neq 0$, $i \in Y \setminus \{\infty\}$. Let $U' = (U^{c} \times \{i\})^{c}$, $V' = (V^{c} \times \{i\})^{c}$ and $\mathcal{R} = (U', V')$. We will show that the cover $\mathcal{R}$ has zero entropy for every $i \in \mathbb{Z}$. Note that for $(x, n) \in X \times Y$,

$$(x, n) \in (T^p S^q)^{-k}(U' \cap V') \quad \text{whenever } qk + n \neq i.$$

If $qk + n = i$ for some k, then either

$$(x, n) \in (T^p S^q)^{-k} U' \quad \text{or} \quad (x, n) \in (T^p S^q)^{-k} V'.$$

Let $\mathcal{R}_m = \bigvee_{j=0}^{m-1} (T^p S^q)^{-j} \mathcal{R}$. If n satisfies $qk+n = i$ and $(x, n) \in (T^p S^q)^{-k} U'$ then

$$(x, n) \in U' \cap (T^p S^q)^{-1} U' \cap (T^p S^q)^{-2} U' \cap \cdots \cap (T^p S^q)^{-m+1} U'.$$

Otherwise

$$(x, n) \in V' \cap (T^p S^q)^{-1} V' \cap (T^p S^q)^{-2} V' \cap \cdots \cap (T^p S^q)^{-m+1} V'.$$

Hence the cardinality of a minimal subcover of $\mathcal{R}_m$ is 2, and therefore $((x, i), (x', i))$ is not a directional entropy pair for any $i \in Y \setminus \{\infty\}$. We take an open cover $\mathcal{Q}^N = ((U^{c} \times [N, -N])^{c}, (V^{c} \times [N, -N])^{c})$. It is not hard to see that the cardinality of a minimal subcover of $\mathcal{Q}_m^N = \bigvee_{j=0}^{m-1} (T^p S^q)^{-j} \mathcal{Q}^N$ is the same as the cardinality of a minimal subcover of $\bigvee_{j=0}^{m-1} (T^p)^{-j} \mathcal{U}$ on X, hence $h(T^p S^q, \mathcal{Q}^N) > 0$ for all N and $((x, \infty), (x', \infty)) \in E(X \times Y, \Phi^{\vec{v}})$ for all $x' \neq x \in X$.

The case of an irrational direction is the same as that of a rational direction. Hence

$$E(X \times Y, \Phi^{\vec{v}}) \cup \triangle_{X \times Y} = (X \times \{\infty\}) \times (X \times \{\infty\}).$$

Example 8 ([6]). Let $T : X \to X$ be an expansive homeomorphism with $h(T) > 0$. There is a natural $\mathbb{Z}^2$-action α on $X \times \mathbb{Z}$ given by

$$\Phi^{(i,j)}((x, n)) = (T^i x, j + n).$$

Let $Y = (X \times \mathbb{Z}) \cup \{\infty\}$ be the one-point compactification of $X \times \mathbb{Z}$. It is not hard to see that the set of directional entropy pairs is $E(Y, \Phi^{\vec{v}}) = \emptyset$ except $\vec{v} = (\xi, 0)$ with $\xi \neq 0$ (cf. Example 7).

Remark 2. Let $X = [0, 1)$ and $\Phi : X \to X$ be defined by $\Phi^{(i,j)}(x) = \alpha^i \beta^j x$, where α and β are two irrational numbers. Clearly none of the directional systems has entropy pairs.

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