The Marcinkiewicz multiplier condition for bilinear operators

by

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Abstract. This article is concerned with the question of whether Marcinkiewicz multipliers on $\mathbb{R}^{2n}$ give rise to bilinear multipliers on $\mathbb{R}^n \times \mathbb{R}^n$. We show that this is not always the case. Moreover, we find necessary and sufficient conditions for such bilinear multipliers to be bounded. These conditions in particular imply that a slight logarithmic modification of the Marcinkiewicz condition gives multipliers for which the corresponding bilinear operators are bounded on products of Lebesgue and Hardy spaces.

1. Introduction. In this article we study bilinear multipliers of Marcinkiewicz type. Recall that a function $\sigma(\xi, \eta) = \sigma(\xi_1, \ldots, \xi_n, \eta_1, \ldots, \eta_n)$ defined away from the coordinate axes on $\mathbb{R}^{2n}$, which satisfies the conditions

$$(1.1) \qquad |\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \le C_{\alpha,\beta} |\xi_1|^{-\alpha_1} \ldots |\xi_n|^{-\alpha_n} |\eta_1|^{-\beta_1} \ldots |\eta_n|^{-\beta_n}$$

for sufficiently large multi-indices $\alpha = (\alpha_1, \ldots, \alpha_n)$ and $\beta = (\beta_1, \ldots, \beta_n)$, is called a *Marcinkiewicz multiplier*. It is a classical result (see for instance [18]) that Marcinkiewicz multipliers give rise to bounded linear operators M_σ from $L_p(\mathbb{R}^{2n})$ into itself for $1 < p < \infty$. Here M_σ is the multiplier operator with symbol σ, that is,

$$M_\sigma(F)(x) = \int_{\mathbb{R}^{2n}} \widehat{F}(\zeta) \sigma(\zeta) e^{2\pi i \langle x, \zeta \rangle} \, d\zeta,$$

where F is a Schwartz function on $\mathbb{R}^{2n}$ and $\widehat{F}(\zeta)$ is the Fourier transform of F, defined by $\widehat{F}(\zeta) = \int_{\mathbb{R}^{2n}} F(x) e^{-2\pi i \langle x, \zeta \rangle} \, dx$. (We will use the notation $\langle x, y \rangle = \sum_{k=1}^m x_k y_k$ for $x = (x_1, \ldots, x_m)$ and $y = (y_1, \ldots, y_m)$ elements of $\mathbb{R}^m$.) The Marcinkiewicz condition (1.1) is less restrictive than the Hörmander–Mikhlin condition

$$(1.2) \qquad |\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi, \eta)| \le C_{\alpha,\beta} (|\xi| + |\eta|)^{-|\alpha| - |\beta|},$$

which is also known to imply boundedness for the linear operator W_σ from $L_p(\mathbb{R}^{2n})$ into itself when $1 < p < \infty$. The advantage of condition (1.2) is that it is supposed to hold for multi-indices up to order $|\alpha| + |\beta| \leq n + 1$ versus up to order $|\alpha| + |\beta| \leq 2n$ for condition (1.1).

In this paper we study bilinear multiplier operators whose symbols satisfy similar conditions. More precisely, we are interested in boundedness properties of bilinear operators

$$W_\sigma(f,g)(x) = \int\limits_{\mathbb{R}^{2n}} \widehat{f}(\xi)\widehat{g}(\eta)\sigma(\xi,\eta)e^{2\pi i\langle x,\xi\rangle}e^{2\pi i\langle x,\eta\rangle}\,d\xi\,d\eta,$$

originally defined for f, g Schwartz functions on $\mathbb{R}^n$ and σ a function on $\mathbb{R}^{2n}$. A well known theorem of Coifman and Meyer [4] says that if the function σ on $\mathbb{R}^{2n}$ satisfies (1.2) for sufficiently large multi-indices α and β, then the bilinear map $W_\sigma(f,g)$ extends to a bounded operator from $L_{p_1}(\mathbb{R}^n) \times L_{p_2}(\mathbb{R}^n)$ into $L_{p_0,\infty}(\mathbb{R}^n)$ when $1 < p_1, p_2 < \infty$, $1/p_1 + 1/p_2 = 1/p_0$ and $p_0 \geq 1$. (Here $L_{p_0,\infty}$ denotes the space weak L_{p_0}.) This result was later extended to the range $1 > p_0 \geq 1/2$ by Grafakos and Torres [9] and Kenig and Stein [11]. The extension into L_{p_0} for $p_0 < 1$ was stimulated by the recent work of Lacey and Thiele [12] who showed that the discontinuous symbol $\sigma(\xi,\eta) = -i\,\mathrm{sgn}(\xi-\eta)$ on $\mathbb{R}^2$ gives rise to a bounded bilinear operator W_σ from $L_{p_1}(\mathbb{R}) \times L_{p_2}(\mathbb{R})$ into $L_{p_0}(\mathbb{R})$ for $2/3 < p_0 < \infty$ when $1 < p_1, p_2 < \infty$ and $1/p_1 + 1/p_2 = 1/p_0$.

In this article we address the question of whether the Marcinkiewicz condition (1.1) on $\mathbb{R}^{2n}$ gives rise to a bounded bilinear operator W_σ on $\mathbb{R}^n \times \mathbb{R}^n$. We answer this question negatively. More precisely, we show that there exist examples of bounded functions $\sigma(\xi,\eta)$ on $\mathbb{R}^n \times \mathbb{R}^n$ which satisfy the stronger condition

$$(1.3) \qquad\qquad |\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \leq C_{\alpha,\beta}|\xi|^{-|\alpha|}|\eta|^{-|\beta|}$$

for all multi-indices α and β, for which the corresponding bilinear operators W_σ do not map $L_{p_1} \times L_{p_2}$ into $L_{p_0,\infty}$ for any triple of exponents satisfying $1/p_1 + 1/p_2 = 1/p_0$ and $1 < p_1, p_2 < \infty$.

We reduce this problem to the study of bilinear operators of the type

$$(1.4) \qquad\qquad (f,g) \mapsto \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{jk}\widetilde{\Delta}_j f\,\widetilde{\Delta}_k g,$$

where a_{jk} is a bounded sequence of scalars depending on σ and $\widetilde{\Delta}_j$ are the Littlewood–Paley operators given by multiplication on the Fourier transform side by a smooth bump supported near the frequency $|\xi| \sim 2^j$. In Section 6, in particular Theorem 6.5, we find a necessary and sufficient condition on the infinite matrix $A = (a_{jk})_{j,k}$ so that the bilinear operator in (1.4) maps $L_{p_1} \times L_{p_2}$ into $L_{p_0,\infty}$. This condition is expressed in terms of an Orlicz space

norm of the sequence $(a_{jk})_{j,k}$. It turns out that this condition is independent of the exponents p_1, p_2, p_0 and depends only on quantities intrinsic to the matrix A (although the actual norm of the operator in (1.4) from $L_{p_1} \times L_{p_2}$ into $L_{p_0,\infty}$ does depend on the indices p_1, p_2, p_0).

The results of Section 6 are transferred in Section 7 to multiplier theorems for bilinear operators. This transference is achieved by using a Fourier expansion of the symbol σ on products of dyadic cubes. Theorem 7.2 is the main result of that section and Theorem 7.3 shows that this theorem is best possible. Theorem 7.2 allows us to derive that the estimates

$$(1.5) \qquad |\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \leq C_{\alpha,\beta} |\xi|^{-|\alpha|} |\eta|^{-|\beta|} \left(\log \left(1 + \left| \log \frac{|\xi|}{|\eta|} \right| \right) \right)^{-\theta}$$

do give rise to a bounded bilinear operator W_σ on products of L_p spaces when $\theta > 1$, while we show that this is not the case when $0 < \theta < 1/2$. We obtain similar results when the expression $(\log(1 + |\log(|\xi|/|\eta|)|))^{-\theta}$ in (1.5) is replaced by the expression

$$\left(\log \left(1 + \left| \log \frac{|\xi|}{|\eta|} \right| \right) \right)^{-1} \left(\log \left(1 + \log \left(1 + \left| \log \frac{|\xi|}{|\eta|} \right| \right) \right) \right)^{-\theta} \qquad \text{for } \theta > 1.$$

We find it more convenient to work with the martingale difference operators Δ_k associated with the σ-algebra of all dyadic cubes of size 2^k in $\mathbb{R}^n$ and later transfer our results to the Littlewood–Paley operators $\widetilde{\Delta}_k$. This point of view is introduced in the next section.

We end this article with a short discussion on paraproducts (see Section 8). These are operators of the type (1.4) for specific sequences $(a_{jk})_{j,k}$ of zeros and ones.

2. A maximal operator. Let $(\Omega, \Sigma, \mathbb{P})$ be any probability space and let $(\Sigma_k)_{k \geq 0}$ be a *filtration*, i.e. an increasing sequence of sub-σ-algebras of Σ. We say that (Σ_k) is a *dyadic filtration* if each Σ_k is atomic and has precisely 2^k atoms each with probability 2^{-k}. We say (Σ_k) is a *2^n-adic filtration* if each Σ_k is atomic with precisely 2^{nk} atoms each with probability 2^{-nk}.

Associated with Σ_k we define the conditional expectation operators $\mathcal{E}_k f = \mathbb{E}(f \mid \Sigma_k)$ and the martingale difference operators $\Delta_k f = \mathcal{E}_k f - \mathcal{E}_{k-1} f$ for $k \geq 1$ and $f \in L_1(\Omega)$.

Let $A = (a_{jk})$ be a complex $M \times N$ matrix, and let $(\Omega, \Sigma, \mathbb{P})$ be a probability space with a dyadic filtration $(\Sigma_k)_{k \geq 0}$. For $1 \leq p < \infty$ we define $h_p(A)$ to be the least constant so that for all $f \in L_p(\Omega)$ we have

$$(2.1) \qquad \left\| \max_{1 \leq j \leq M} \left| \sum_{k=1}^{N} a_{jk} \Delta_k f \right| \right\|_{L_p} \leq h_p(A) \|f\|_{L_p}.$$

We also define the corresponding weak constants, i.e. the least constants so that for all $f \in L_p(\Omega)$ we have

$$(2.2) \qquad \Big\| \max_{1 \le j \le M} \Big| \sum_{k=1}^N a_{jk} \Delta_k f \Big| \Big\|_{L_{p,\infty}} \le h_p^{\mathrm{w}}(A) \|f\|_{L_p}.$$

Finally, for $0 < q < p < \infty$ we define the mixed constants $h_{p,q}(A)$ as the least constants such that for all $f \in L_p(\Omega)$ we have

$$(2.3) \qquad \Big\| \max_{1 \le j \le M} \Big| \sum_{k=1}^N a_{jk} \Delta_k f \Big| \Big\|_{L_q} \le h_{p,q}(A) \|f\|_{L_p}.$$

Note that these definitions are independent of the choice of the probability space and of the dyadic filtration. Indeed, if A is fixed, it suffices to take $f \in L_p(\Sigma_N)$ and hence we can consider a finite probability space with 2^N points and a finite dyadic filtration $(\Sigma_k)_{k=0}^N$. We also note that $h_p(A)$ is the operator norm of the map $T_A : L_p(\Omega) \to L_p(\Omega; \ell_\infty^M)$ defined by

$$T_A f = \Big(\sum_{k=1}^N a_{jk} \Delta_k f \Big)_{j=1}^M.$$

Similarly, $h_p^{\mathrm{w}}(A)$ is the norm of the operator $T_A : L_p \to L_{p,\infty}(\Omega; \ell_\infty^M)$.

Our first result is that all these constants are mutually equivalent, when $1 < p < \infty$:

THEOREM 2.1. *If $1 < p, q < \infty$ then there is a constant $0 < C = C(p,q) < \infty$ such that for all complex $M \times N$ matrices A we have*

$$\frac{1}{C} h_p(A) \le h_q^{\mathrm{w}}(A) \le h_q(A) \le C h_p(A).$$

Proof. It suffices to prove an estimate of the type $h_p(A) \le C h_q^{\mathrm{w}}(A)$ for any choice of $1 < p, q < \infty$. We first prove a weak type $(1,1)$ estimate for T_A, i.e. that $h_1^{\mathrm{w}}(A) \le C h_q^{\mathrm{w}}(A)$. Suppose $f \in L_1$ with $\|f\|_{L_1} = 1$. Then if $\lambda, \gamma > 0$, with $\lambda\gamma > 1$, we can use an appropriate Calderón–Zygmund decomposition to find finite sets $D_1, \ldots, D_m$ so that each D_l is an atom of some Σ_l,

$$\gamma\lambda \le \mathbb{P}(D_l)^{-1} \int_{D_l} |f| \, d\mathbb{P} = \operatorname*{Ave}_{D_l} f \le 2\gamma\lambda,$$

and $|f(\omega)| \le \gamma\lambda$ if $\omega \notin \bigcup_{l=1}^m D_l$. Let

$$g = \sum_{l=1}^m (\operatorname*{Ave}_{D_l} f) \chi_{D_l}$$

and $E = \bigcup_{l=1}^m D_l$. Then $T_A(f\chi_E - g)$ is supported in E and thus

$$(2.4) \qquad \mathbb{P}(\|T_A(f\chi_E - g)\|_{\ell_\infty^M} > \lambda/2) \le \mathbb{P}(E) \le (\gamma\lambda)^{-1}.$$

On the other hand, $\|f - f\chi_E + g\|_{L_\infty} \leq 3\gamma\lambda$ and $\|f - f\chi_E + g\|_{L_1} \leq 1$. Hence $\|f - f\chi_E + g\|_{L_q} \leq 3^{1/q'}(\gamma\lambda)^{1/q'}$ and so

$$(2.5) \qquad \|T_A(f - f\chi_E + g)\|_{L_{q,\infty}(\ell_\infty^M)} \leq h_q^w(A)3^{1/q'}(\gamma\lambda)^{1/q'},$$

which implies that

$$(2.6) \qquad \mathbb{P}(\|T_A(f - f\chi_E + g)\|_{\ell_\infty^M} > \lambda/2) \leq \frac{2^q}{\lambda^q}(h_q^w(A))^q 3^{q-1}(\gamma\lambda)^{q-1}.$$

Selecting $\gamma = 1/h_q^w(A)$ and combining with (2.4) we obtain (for $\lambda > h_q^w(A)$)

$$(2.7) \qquad \lambda\mathbb{P}(\|T_A f\|_{\ell_\infty^M} > \lambda) \leq Ch_q^w(A)$$

where $C = C(p,q)$. This gives the weak-type $(1,1)$ estimate for T_A. Now by the Marcinkiewicz interpolation theorem (applied to the sublinear map $f \mapsto \|T_A f(\omega)\|_{\ell_\infty^M}$) we deduce that $h_p(A) \leq C(p,q)h_q^w(A)$ as long as $1 < p < q$.

We now prove that $h_p(A) \leq C(p,q)h_q^w(A)$ when $1 < q < p < \infty$. We consider the dual map $T_A^* : L_1(\Omega; \ell_1^M) \to L_1$ defined by

$$T_A^* \mathbf{f} = \sum_{j=1}^{M}\sum_{k=1}^{N} a_{jk}\Delta_k f_j$$

where $\mathbf{f}(\omega) = (f_j(\omega))_{j=1}^{M}$. We find that $T_A^* : L_r(\Omega; \ell_1^M) \to L_r$ has norm bounded by $C(q,r)h_q^w(A)$ as long as $1 < r' < q$, i.e. $q' < r < \infty$. Using this r as a starting point, we repeat the argument above to show that $T_A^* : L_1(\Omega; \ell_1^M) \to L_{1,\infty}$ has norm bounded by $Ch_q^w(A)$. The Marcinkiewicz interpolation theorem can again be used to show that $T_A^* : L_{p'}(\Omega, \ell_1^M) \to L_{p'}$ has norm bounded by $Ch_q^w(A)$ for all $1 < p' < r$, and thus in particular when $1 < p' < q'$. Therefore we conclude that $h_p(A) \leq Ch_q^w(A)$ when $1 < q < p < \infty$. ∎

REMARK. From now we will write $h(A) = h_2(A)$ so that each $h_p(A)$ for $1 < p < \infty$ is equivalent to $h(A)$.

It is of some interest to observe that even the corresponding mixed constants are also equivalent to $h(A)$.

THEOREM 2.2. *Suppose $0 < q < p$ and $1 < p < \infty$. Then there is a constant $C = C(p,q)$ so that*

$$\frac{1}{C}h(A) \leq h_{p,q}(A) \leq Ch(A).$$

Proof. This will depend on the following lemma:

LEMMA 2.3. *Suppose $1 \leq p < \infty$ and $0 < q < p$. Then there is a constant $C = C(p,q)$ so that if $r = \min(p,2)$ we have*

$$(2.8) \qquad \|T_A\|_{L_p \to L_{r,\infty}(\ell_\infty^M)} \leq Ch_{p,q}(A).$$

Proof of Lemma 2.3. We may assume $q < r$. The proof is a fairly standard application of Nikishin's theorem (see [16]). Here we use a version given in [17]. It is simplest to consider the case when Ω is finite with $|\Omega| = 2^N$. Consider the map $T_A : L_p \to L_q(\Omega; \ell_\infty^M)$. For each $f \in L_p$ with $\|f\|_{L_p} \le 1$, let $F_f(x) = \|T_A f(x)\|_{\ell_\infty^M}$. For $\|f_j\|_{L_p} \le 1$ with $1 \le j \le J$, $\sum_{j=1}^J |b_j|^r = 1$, and $(\varepsilon_j)_{j=1}^J$ a sequence of independent Bernoulli random variables on some probability space, we have

$$\| \max_{1 \le j \le J} |b_j| F_{f_j} \|_{L_q} \le \mathbb{E}\Big(\Big\| \sum_{j=1}^J \varepsilon_j b_j T_A f_j \Big\|_{L_q(\ell_\infty^M)}\Big) \le C h_{p,q}(A),$$

since L_p has type r. It follows from [17] that there is a function $w \in L_1$, with $\int w\, d\mathbb{P} = 1$, and $w \ge 0$ a.e., such that for any set $E \subset \Omega$,

$$\Big(\int_E F_f^q\, d\mathbb{P}\Big)^{1/q} \le C h_{p,q}(A)\Big(\int_E w\, d\mathbb{P}\Big)^{1/q-1/r}.$$

Now consider the set S of all permutations of Ω which induce permutations of the atoms of each Σ_k for $1 \le k \le N$; there are $2^{2^N - 1}$ such permutations φ. For $\varphi \in S$ we have

$$\Big(\int_E F_{f \circ \varphi}^q\, d\mathbb{P}\Big)^{1/q} \le C h_{p,q}(A)\Big(\int_E w\, d\mathbb{P}\Big)^{1/q-1/r}$$

or equivalently

$$\Big(\int_E F_f^q\, d\mathbb{P}\Big)^{1/q} \le C h_{p,q}(A)\Big(\int_E w \circ \varphi^{-1}\, d\mathbb{P}\Big)^{1/q-1/r}.$$

Raising to the power $(1/q - 1/r)^{-1}$, averaging over S, and then raising to the power $1/q - 1/r$ gives

$$\Big(\int_E F_f^q\, d\mathbb{P}\Big)^{1/q} \le C h_{p,q}(A)\Big(\frac{1}{|S|} \sum_{\varphi \in S} \Big(\int_E w \circ \varphi^{-1}\, d\mathbb{P}\Big)\Big)^{1/q-1/r}.$$

But this implies

$$\Big(\int_E F_f^q\, d\mathbb{P}\Big)^{1/q} \le C h_{p,q}(A)\, \mathbb{P}(E)^{1/q-1/r},$$

which gives the required weak type estimate (2.8). ∎

We now return to the proof of Theorem 2.2. We first observe that we always have $h_{p,q}(A) \le C h_p^{\mathrm{w}}(A)$ since $q < p$. If $1 < p \le 2$, Lemma 2.3 gives $h_p^{\mathrm{w}}(A) \le C h_{p,q}(A)$ and the required conclusion follows from Theorem 2.1. Assume therefore that $p > 2$ and that T_A maps L_p into $L_q(\ell_\infty^M)$ with norm $h_{p,q}(A)$. Fix f with $\|f\|_{L_1} = 1$ and use the Calderón–Zygmund decomposition of Theorem 2.1 to obtain (2.4) as before, but instead of (2.5) the

estimate

$$(2.9) \qquad \|T_A(f - f\chi_E + g)\|_{L_q} \leq h_{p,q}(A)3^{1/p'}(\gamma\lambda)^{1/p'},$$

which implies

$$(2.10) \qquad \mathbb{P}(\|T_A(f - f\chi_E + g)\|_{\ell_\infty^M} > \lambda/2) \leq \frac{2^q}{\lambda^q}(h_{p,q}(A))^q 3^{q/p'}(\gamma\lambda)^{q/p'}.$$

Selecting $\gamma = h_{p,q}(A)^{-s}\lambda^{s-1}$ with $1/s = 1/p' + 1/q$ and combining with (2.4) we obtain

$$(2.11) \qquad \lambda\mathbb{P}(\|T_A f\|_{\ell_\infty^M} > \lambda)^{1/s} \leq Ch_{p,q}(A).$$

This says that T_A maps L_1 into $L_{s,\infty}(\ell_\infty^M)$ with norm at most $Ch_{p,q}(A)$, in particular that T_A maps L_1 into $L_t(\ell_\infty^M)$ as long as $0 < t < s$. Lemma 2.3 implies that T_A maps L_p into $L_{2,\infty}(\ell_\infty^M)$ and also L_1 into $L_{2,\infty}(\ell_\infty^M)$ with norms at most a multiple of $h_{p,q}(A)$. By interpolation it follows that T_A maps L_r into $L_{2,\infty}(\ell_\infty^M) \subset L_{r,\infty}(\ell_\infty^M)$ for $1 \leq r \leq 2$. We conclude that $h_r^{\mathrm{w}}(A) \leq Ch_{p,q}(A)$ for $1 < r < 2$ but since $h_r^{\mathrm{w}}(A)$ is comparable to $h_p^{\mathrm{w}}(A)$, we finally obtain $h_p^{\mathrm{w}}(A) \leq Ch_{p,q}(A)$. Since the converse inequality is always valid when $q < p$, we apply Theorem 2.1 to conclude the proof. ∎

We next prove the elementary observation that, for $1 < p < \infty$, $h(A)$ remains unchanged under interpolation of extra columns or extra rows of zeros.

LEMMA 2.4. *Let A be a complex $M \times N$ matrix and $(m_r)_{r=1}^M$, $(n_s)_{s=1}^N$ be two increasing finite sequences of natural numbers. Suppose $M_1 \geq m_M$ and $N_1 \geq n_N$. Let $B = (b_{jk})$ be the $M_1 \times N_1$ matrix defined by $b_{jk} = a_{rs}$ when $j = m_r$ and $k = n_s$, and $b_{jk} = 0$ otherwise. Then $h(A) = h(B)$.*

Proof. Interpolating extra rows of zeros is trivial, so we can assume $m_r = m$ for all r. For the case of columns, we only need to show that $h(B) \leq h(A)$. We may suppose that Ω is a finite set with 2^{N_1} points and that $(\Sigma_k)_{k=0}^{N_1}$ is a finite dyadic filtration of Ω. It is then possible to write $\Omega = \Omega_1 \times \Omega_2$ where $|\Omega_1| = 2^{N_1-N}$ and $|\Omega_2| = 2^N$, and find a dyadic filtration $(\Sigma_k^{(1)})_{k=0}^{N_1-N}$ of Ω_1 and a dyadic filtration $(\Sigma_k^{(2)})_{k=0}^N$ of Ω_2 so that $\Sigma_k^{(1)} \times \Sigma_k^{(2)} = \Sigma_{n_k}$ for $0 \leq k \leq N$, and $\Sigma_{k+1}^{(1)} \times \Sigma_k^{(2)} = \Sigma_{n_{k+1}-1}$ for $0 \leq k \leq N-1$. Then for $f \in L_2(\Omega_1 \times \Omega_2)$ let $g = \sum_{k=1}^N \Delta_k f$ and note that

$$\Delta_{n_k} f(\omega_1, \omega_2) = \Delta_k^{(2)} g_{\omega_1}(\omega_2),$$

where $g_{\omega_1}(\omega_2) = g(\omega_1, \omega_2)$. Hence

$$\int_{\Omega_2} \sup_j \Big| \sum_{k=1}^N a_{j,n_k} \Delta_{n_k} f(\omega_1, \omega_2) \Big|^2 d\omega_2 \leq h_p(A) \int_{\Omega_2} |g(\omega_1, \omega_2)|^2 d\omega_2.$$

Integrating over Ω_1 gives

$$\Big\| \sup_j \Big| \sum_{k=1}^{N} a_{j,n_k} \Delta_{n_k} f \Big| \Big\|_{L_2} \leq h_p(A)\|g\|_{L_2} \leq h_p(A)\|f\|_{L_2}. \ \blacksquare$$

We can now extend our definitions, replacing dyadic filtrations by 2^n-adic filtrations:

PROPOSITION 2.5. *Suppose* $n \in \mathbb{N}$ *and* $1 < p < \infty$. *Then there is a constant* $C(p,n)$ *with the following property. Let* $(\Omega, \Sigma, \mathbb{P})$ *be a probability space and suppose* $(\Sigma_k)_{k=0}^{\infty}$ *is a* 2^n-*adic filtration. Let* A *be any* $M \times N$ *matrix and let* $h_p(A;n)$ *be the least constant so that*

$$\Big\| \sup_j \Big| \sum_{k=1}^{N} a_{jk} \Delta_k f \Big| \Big\|_{L_p} \leq h_p(A;n)\|f\|_{L_p},$$

and $h_p^{\mathrm{w}}(A;n)$ *be the least constant so that*

$$\Big\| \sup_j \Big| \sum_{k=1}^{N} a_{jk} \Delta_k f \Big| \Big\|_{L_{p,\infty}} \leq h_p^{\mathrm{w}}(A;n)\|f\|_{L_p}.$$

Then $h_p^{\mathrm{w}}(A) \leq h_p^{\mathrm{w}}(A;n)$, $h_p(A) \leq h_p(A;n)$, *and* $h_p^{\mathrm{w}}(A;n) \leq h_p(A;n) \leq Ch(A)$.

Proof. This is essentially trivial; we need only prove that $h_p(A;n) \leq Ch(A)$. To do this note that $h_p(A;n) = h_p(B)$ where B is obtained from A by repeating each column n times. The proposition then follows by the triangle law from Lemma 2.4. $\blacksquare$

3. Estimates for $h(A)$**.** We next turn to the problem of estimating $h(A)$. We shall assume that $(\Omega, \mathbb{P})$ is a fixed probability space with a dyadic filtration $(\Sigma_k)_{k=0}^{\infty}$. Our first estimate is trivial.

PROPOSITION 3.1. *There is a constant* C *so that for any* $M \times N$ *matrix* $A = (a_{jk})$ *we have*

$$h(A) \leq C \sup_{1 \leq j \leq M} \sum_{k=0}^{N} |a_{jk} - a_{j,k+1}|,$$

where we set $a_{j0} = a_{j,N+1} = 0$ *for all* $1 \leq j \leq M$.

Proof. Suppose $f \in L_2$. Summation by parts gives

$$\sum_{k=1}^{N} a_{jk} \Delta_k f = \sum_{k=0}^{N} (a_{jk} - a_{j,k+1})\mathcal{E}_k f,$$

thus

$$\Big| \sum_{k=1}^{N} a_{jk} \Delta_k f \Big| \leq \Big(\sup_{1 \leq j \leq M} \sum_{k=0}^{N} |a_{jk} - a_{j,k+1}| \Big) \sup_k |\mathcal{E}_k f|,$$

and the result follows because of the maximal estimate

$$\| \sup_k |\mathcal{E}_k f| \|_{L_2} \leq C \|f\|_{L_2},$$

proved in [8]. ∎

We next turn to the problem of getting a more delicate estimate. To do this we need the theory of a certain Lorentz space. Let $w = (w_k)_{k=1}^{\infty}$ be a decreasing sequence of positive numbers. We will consider the following two conditions on w:

$$(3.1) \quad \exists C > 0, \ \exists \theta > 0, \quad w_k \leq C \Big(\frac{\log(j+1)}{\log(k+1)} \Big)^{\theta} w_j \quad \text{when } 1 \leq j \leq k,$$

(where throughout this paper log denotes the logarithm with base 2) and

$$(3.2) \qquad\qquad \sum_{k=1}^{\infty} \frac{w_k}{k} < \infty.$$

Roughly speaking, (3.1) means that w_k decays logarithmically while (3.2) implies that it decays reasonably fast. Note $w_k = (\log(k+1))^{-\theta}$ satisfies (3.1) if $\theta > 0$ and (3.2) if $\theta > 1$. The sequence $w_k = (\log(k+1))^{-1}(\log\log(k+2))^{-\theta}$ satisfies both (3.1) and (3.2) when $\theta > 1$.

Now let $d = d(w,1)$ be the Lorentz sequence space of all complex sequences $\mathbf{u} = (u_k)_{k \in \mathbb{Z}}$ such that

$$\|\mathbf{u}\|_d = \sup_{\pi} \sum_{k \in \mathbb{Z}} w_{\pi(k)} |u_k| < \infty,$$

where the supremum is taken over all one-one maps $\pi : \mathbb{Z} \to \mathbb{N}$. The dual of $d(w,1)$ can be naturally identified as the space $d^* = d^*(w,1)$ consisting of all sequences $(v_k)_{k \in \mathbb{Z}}$ so that

$$\sup_{k \in \mathbb{N}} \frac{v_1^* + \ldots + v_k^*}{w_1 + \ldots + w_k} = \|\mathbf{v}\|_{d^*} < \infty$$

where $(v_k^*)_{k=1}^{\infty}$ is the decreasing rearrangement of $(|v_k|)_{k \in \mathbb{Z}}$. We refer to [13], p. 175, for properties of Lorentz spaces. Note that under condition (3.1), $d(w,1)$ is also an Orlicz sequence space (see [13], p. 176).

The following lemma is surely well known to specialists, but we include a proof anyway.

LEMMA 3.2. *Under condition* (3.1), *the Lorentz space* $d(w,1)$ *has cotype two.*

Since $\|f\|_{L_{2m}}^{2m} \leq \|f\|_{L_2}^2 \|f\|_{L_\infty}^{2m-2}$ and since for $m \geq 1$ we have

$$\frac{(2m-1)^{2m}}{(2m)!} \leq \frac{(2m)^{2m}}{(2m)!} \leq e^{2m},$$

it follows from (3.3) that

$$\mathbb{E}(\cosh(\alpha|Sf|) - 1) \leq (\alpha e)^2 \|f\|_{L_2}^2 \sum_{k=0}^{\infty} (\alpha e)^{2k} \|f\|_{L_\infty}^{2k}.$$

In particular, if $\alpha e\|f\|_\infty \leq 1/2$ we have

(3.4) $$\mathbb{E}(\cosh(\alpha|Sf|) - 1) \leq 2e^2\alpha^2\|f\|_{L_2}^2.$$

At this point we return to the Lorentz space $d(w, 1)$. Let us define $\gamma_0 = 0$, $\gamma_1 = 1$, and $\gamma_k = 2^{2^{k-2}}$ for $k \geq 2$. Let $W_k = w_{\gamma_k}$. It will be convenient to normalize condition (3.2) so that we have

(3.5) $$\sum_{k=1}^{\infty} \gamma_k W_k = 1.$$

We also note that (3.1) implies the existence of a constant C such that

(3.6) $$|w_1 + \ldots + w_k| \leq Ckw_k$$

for $k \geq 1$.

Now suppose $\mathbf{f} = (f_j)_{j=1}^M \in L_\infty(\Omega; \mathbb{C}^M)$. Suppose that $\mathbf{f}$ is supported on a measurable set E and satisfies $\|\mathbf{f}(\omega)\|_{d^*} \leq 1$ everywhere. Then we can define a measurable map π from Ω into the set of permutations of $\{1, \ldots, M\}$ so that $|f_{\pi(\omega)(1)}(\omega)| \geq \ldots \geq |f_{\pi(\omega)(M)}(\omega)|$ for all $\omega \in \Omega$. Thus

$$|f_{\pi(\omega)(j)}(\omega)| \leq Cw_j$$

for all $1 \leq j \leq M$. Let $E_{jk} = \{\omega \in E : \pi(\omega)(k) = j\}$ when $j, k \in \{1, \ldots, M\}$ and $E_{jk} = \emptyset$ otherwise. Now for $1 \leq j \leq M$ and $l = 1, 2, \ldots$, let

$$f_j^{(l)} = \sum_{k=\gamma_{l-1}}^{\gamma_l - 1} f_j \chi_{E_{jk}}$$

so that $f_j = \sum_{l=1}^{\infty} f_j^{(l)}$. If $0 < \alpha e \leq 1/(2C)$ we can estimate

$$\mathbb{E}(\cosh(\alpha|Sf_j|) - 1) = \mathbb{E}\Big(\cosh\Big(\Big|\sum_{l=1}^{\infty} \alpha Sf_j^{(l)}\Big|\Big) - 1\Big)$$

$$\leq \mathbb{E}(\max_{l \geq 1}(\cosh(\alpha\gamma_l^{-1}W_l^{-1}|Sf_j^{(l)}|) - 1))$$

$$\leq e^2\alpha^2 \sum_{l=1}^{\infty} \gamma_l^{-2} W_l^{-2} \|f_j^{(l)}\|_{L_2}^2,$$

in view of (3.4) since $\|f_j^{(l)}\|_{L_\infty} \le CW_l$ and $\alpha\gamma_l^{-1}W_l^{-1}\|f_j^{(l)}\|_{L_\infty} \le 1/2$. Thus

$$\mathbb{E}(\cosh(\alpha|Sf_j|) - 1) \le e^2 C^2 \alpha^2 \sum_{l=1}^{\infty} \gamma_l^{-2} \sum_{k=\gamma_{l-1}}^{\gamma_l - 1} \mathbb{P}(E_{jk}).$$

It follows that

$$\mathbb{E}(\cosh(\alpha\|S\mathbf{f}\|_{\ell_\infty}) - 1) \le e^2 C^2 \alpha^2 \sum_{j=1}^{M} \sum_{l=1}^{\infty} \gamma_l^{-2} \sum_{k=\gamma_{l-1}}^{\gamma_l - 1} \mathbb{P}(E_{jk}).$$

Note that for each $k \in \mathbb{N}$, $\sum_{j=1}^{M} \mathbb{P}(E_{jk}) \le \mathbb{P}(E)$. Hence if $\mathbf{f}$ is supported on E with $\|\mathbf{f}(\omega)\|_{d^*} \le 1$ everywhere and $\alpha e < 1/(2C)$, then

$$(3.7) \qquad \mathbb{E}(\cosh(\alpha\|S\mathbf{f}\|_{\ell_\infty}) - 1) \le e^2 C^2 \alpha^2 \sum_{l=1}^{\infty} \gamma_l^{-1} \mathbb{P}(E) = C_1 \alpha^2 \mathbb{P}(E)$$

for a suitable constant C_1. Let us next refine (3.7). For $n \ge 0$, let

$$G_n = \{\omega \in E : 4^{-n-1} < \|\mathbf{f}(\omega)\|_{d^*} \le 4^{-n}\}.$$

Then by (3.7) we have, if $\alpha < (4Ce)^{-1}$,

$$\mathbb{E}(\cosh(2^{n+1}\alpha\|S(\mathbf{f}\chi_{G_n})\|_{\ell_\infty}) - 1) \le C_1 \alpha^2 4^{-n} \mathbb{P}(G_n)$$

and as

$$\mathbb{E}(\cosh(\alpha\|S\mathbf{f}\|_{\ell_\infty}) - 1) \le \mathbb{E}(\sup_{n\ge 0}(\cosh(2^{n+1}\alpha\|S(\mathbf{f}\chi_{G_n})\|_{\ell_\infty}) - 1)),$$

we obtain, under the assumptions $\|\mathbf{f}(\omega)\|_{d^*} \le 1$ everywhere and $\alpha < (4C)^{-1}$,

$$(3.8) \qquad \mathbb{E}(\cosh(\alpha\|S\mathbf{f}\|_{\ell_\infty}) - 1) \le C_1 \alpha^2 \sum_{n=0}^{\infty} 4^{-n} \mathbb{P}(G_n) \le C_2 \mathbb{E}(\|\mathbf{f}\|_{d^*}).$$

If we use a fixed value of α and the estimate $x^2 \le 2(\cosh x - 1)$ we find that

$$\mathbb{E}(\|S\mathbf{f}\|_{\ell_\infty}^2) \le C_3 \mathbb{E}(\|\mathbf{f}\|_{d^*})$$

if $\|\,\|\mathbf{f}\|_{d^*}\|_\infty \le 1$. This in turn gives us, for every $\mathbf{f} \in L_\infty(\Omega; d^*(w,1)^{(M)})$,

$$(3.9) \qquad \mathbb{E}(\|S\mathbf{f}\|_{\ell_\infty}^2) \le C_3 \|\,\|\mathbf{f}\|_{d^*}\|_\infty \mathbb{E}(\|\mathbf{f}\|_{d^*}).$$

Now let $2 < p < \infty$ and fix $\mathbf{f}$ with $\mathbb{E}(\|\mathbf{f}\|_{d^*}^p) = 1$. We set $E_0 = \{\|\mathbf{f}\|_{d^*} \le 1\}$ and $E_n = \{2^{n-1} < \|\mathbf{f}\|_{d^*} \le 2^n\}$ for $n \ge 1$. Applying (3.9) we obtain

$$(\mathbb{E}(\|S\mathbf{f}\|_{\ell_\infty}^2))^{1/2} \le \Big(C_3 \sum_{n=0}^{\infty} 2^n \mathbb{P}(E_n)\mathbb{E}(\|\mathbf{f}\|_{d^*})\Big)^{1/2} \le C_3^{1/2} \sum_{n=0}^{\infty} 2^{n/2}\mathbb{P}(E_n)^{1/2}$$

$$\le C_3^{1/2}\Big(\sum_{n=0}^{\infty} 2^{(2-p)n}\Big)^{1/2}\Big(\sum_{n=0}^{\infty} 2^{np}\mathbb{P}(E_n)\Big)^{1/2} \le C_4,$$

which completes the proof under the assumption $\mathbb{E}(\|\mathbf{f}\|_{d^*}^p) = 1$. The general case follows by scaling. ∎

We now establish our main estimate for $h(A)$.

THEOREM 3.4. *Let* $w = (w_n)_{n=1}^\infty$ *be a sequence satisfying* (3.1) *and* (3.2). *Then there is a constant* C *so that for any* $M \times N$ *matrix* $A = (a_{kj})_{j,k}$ *we have*

$$h(A) \leq C \max_{1 \leq k \leq N} \|\mathbf{a}_k\|_{d^*}$$

where $\mathbf{a}_k = (a_{kj})_{j=1}^M$. *In particular, we have*

$$h(A) \leq C \max_{j,k} \frac{|a_{jk}|}{w_{|j-k|+1}}.$$

Proof. Suppose $p > 2$ and that A is a matrix satisfying $\max_{1 \leq k \leq N} \|\mathbf{a}_k\|_{d^*} \leq 1$. Consider the operator $T_A : L_p(\Omega) \to L_2(\Omega; \ell_\infty^M)$. The adjoint operator is $T_A^* : L_2(\Omega; \ell_1^M) \to L_{p'}(\Omega)$ given by

$$T_A^*(\mathbf{f}) = \sum_{k=1}^N \langle \Delta_k \mathbf{f}, \mathbf{a}_k \rangle.$$

The dual statement of the result in Proposition 3.3 entails that for any sequence $\varepsilon_1, \ldots, \varepsilon_N$ of ± 1's, we have the estimate

$$(3.10) \qquad \left(\mathbb{E}\left(\left\| \sum_{k=1}^N \varepsilon_k \Delta_k \mathbf{f} \right\|_d^{p'} \right) \right)^{1/p'} \leq C (\mathbb{E}(\|\mathbf{f}\|_{\ell_1}^2))^{1/2}$$

where C depends only on (w_n). Now let $\varepsilon_1, \ldots, \varepsilon_N$ be a sequence of independent Bernoulli random variables on some probability space $(\Omega', \mathbb{P}')$. We use $\mathbb{E}'$ to denote expectations on Ω'. Using Lemma 3.2 we obtain

$$
\begin{aligned}
(\mathbb{E}(\|T_A^* \mathbf{f}\|_d^{p'}))^{1/p'} &\leq C_0 \left(\mathbb{E}\left(\sum_{k=1}^N |\langle \Delta_k \mathbf{f}, \mathbf{a}_k \rangle|^2 \right)^{p'/2} \right)^{1/p'} \\
&\leq C_0 \left(\mathbb{E}\left(\sum_{k=1}^N \|\Delta_k \mathbf{f}\|_d^2 \right)^{p'/2} \right)^{1/p'} \\
&\leq C_1 \left(\mathbb{E}\mathbb{E}'\left(\left\| \sum_{k=1}^N \varepsilon_k \Delta_k \mathbf{f} \right\|_d^{p'} \right) \right)^{1/p'} \leq C_2 (\mathbb{E}\|\mathbf{f}\|_{\ell_1}^2)^{1/2}.
\end{aligned}
$$

This gives $h_{p,2}(A) \leq C_2$, which completes the proof by using Theorem 2.2. ∎

REMARK. Theorem 3.4 implies that given any $\theta > 1$ there is a constant C_θ so that

$$(3.11) \qquad\qquad\qquad h(A) \leq C_\theta$$

whenever $A = (a_{kj})_{j,k}$ is a matrix satisfying

(3.12) $$|a_{jk}| \leq 2(\log(2 + |j - k|))^{-\theta}.$$

We show that this is not the case when $0 < \theta < 1/2$. Let N be any natural number and define $A = (a_{jk})$ to be a $2^N \times N$ matrix given by $a_{jk} = b_{jk}N^{-\theta}$, where $b_{jk} = \pm 1$ and the set $(b_{jk})_{j=1}^{2^N}$ runs through all 2^N choices of signs. Choose f real so that $|\Delta_k f| = 1$ for $1 \leq k \leq N$. Then $\|f\|_{L_2} = \sqrt{N}$. On the other hand,

$$\max_{1 \leq j \leq 2^N} \Big| \sum_{k=1}^{N} a_{jk} \Delta_k f \Big| = N^{1-\theta} \chi_\Omega,$$

which implies that $h(A) \geq N^{1/2-\theta}$. However,

$$|a_{jk}| \leq N^{-\theta} \leq 2(N+1)^{-\theta} \leq 2(\log(2 + |j - k|))^{-\theta}$$

but $h(A) \geq N^{1/2-\theta} \to \infty$ as $N \to \infty$. Thus (3.11) fails when $0 < \theta < 1/2$.

4. The harmonic version of the maximal operator. We shall now fix $n \in \mathbb{N}$ and work with $\mathbb{R}^n$. Let $\mathcal{D}_0$ be the collection of all unit cubes of the form $\prod_{j=1}^{n}[m_j, m_j + 1]$ where $m_j \in \mathbb{Z}$ and let $\mathcal{D}_k$ be the set of all cubes of the form $\prod_{j=1}^{n}[2^{-k}m_j, 2^{-k}(m_j + 1)]$ where $m_j \in \mathbb{Z}$. For $k \in \mathbb{Z}$, let Σ_k denote the σ-algebra generated by the dyadic cubes $\mathcal{D}_k$. We define the corresponding conditional expectation operators

$$\mathcal{E}_k f = \sum_{Q \in \mathcal{D}_k} (\operatorname*{Ave}_{Q} f) \chi_Q$$

for $f \in L_1^{\mathrm{loc}}(\mathbb{R}^n)$ and the martingale difference operators $\Delta_k f = \mathcal{E}_k f - \mathcal{E}_{k-1}f$ for $k \in \mathbb{Z}$.

Now let $A = (a_{jk})_{j,k \in \mathbb{Z}}$ be any infinite complex matrix. We shall call A a c_{00}-*matrix* if it has only finitely many non-zero entries. For a c_{00}-matrix A define $h_p[A; n]$ to be the least constant such that for all $f \in L_p(\mathbb{R}^n)$ we have

(4.1) $$\Big\| \max_{j \in \mathbb{Z}} \Big| \sum_{k \in \mathbb{Z}} a_{jk} \Delta_k f \Big| \Big\|_{L_p} \leq h_p[A; n] \|f\|_{L_p}.$$

Also, let $h_p^{\mathrm{w}}[A; n]$ be the corresponding weak-type constant, i.e. the least constant such that for all $f \in L_p(\mathbb{R}^n)$ we have

(4.2) $$\Big\| \max_{j \in \mathbb{Z}} \Big| \sum_{k=1}^{N} a_{jk} \Delta_k f \Big| \Big\|_{L_{p,\infty}} \leq h_p^{\mathrm{w}}[A; n] \|f\|_{L_p}.$$

The following lemma is easily verified and we omit its proof.

LEMMA 4.1. *Let $h_p^{\mathrm{w}}(A; n)$ and $h_p(A; n)$ be as in Proposition 2.5. For any $1 < p < \infty$ and any infinite c_{00}-matrix A we have $h_p[A; n] = h_p(B; n)$*

and $h_p^{\mathrm{w}}[A;n] = h_p^{\mathrm{w}}(B;n)$, where B is any $M \times N$ matrix of the form $b_{jk} = a_{j+r,k+s}$ for some $r, s \in \mathbb{Z}$ such that $a_{j+r,k+s} = 0$ unless $1 \leq j \leq M$ and $1 \leq k \leq N$.

Now for any infinite matrix A we define

$$h(A) = \sup_N h((a_{j-N,k-N})_{1 \leq j,k \leq 2N}).$$

The following is an immediate consequence of Lemma 4.1 and Proposition 2.5.

COROLLARY 4.2. *For any $1 < p < \infty$ and any $n \in \mathbb{N}$ there is a constant $C = C(p,N)$ so that for any infinite c_{00}-matrix we have*

$$C^{-1}h(A) \leq h_p^{\mathrm{w}}[A;n] \leq h_p[A;n] \leq Ch(A).$$

We now turn to the harmonic model of the maximal operator studied in Section 2. Let $\mathcal{S}(\mathbb{R}^n)$ denote the set of all Schwartz functions on $\mathbb{R}^n$ and for $f \in \mathcal{S}(\mathbb{R}^n)$ let

$$\widehat{f}(\xi) = \int_{\mathbb{R}^n} f(x)e^{-2\pi i \langle \xi, x \rangle} \, dx$$

denote the Fourier transform of f. We will denote by $f^\vee(\xi) = \widehat{f}(-\xi)$ the inverse Fourier transform of f. We shall fix a radial function $\psi \in \mathcal{S}(\mathbb{R}^n)$ whose Fourier transform is real-valued and satisfies $\widehat{\psi}(\xi) = 1$ for $|\xi| \leq 1$ and $\widehat{\psi}(\xi) = 0$ for $|\xi| \geq 2$. We define a Schwartz function ϕ by setting $\widehat{\phi}(\xi) = \widehat{\psi}(\xi) - \widehat{\psi}(2\xi)$. Then $\widehat{\phi}$ is supported in the annulus $2^{-1} \leq |\xi| \leq 2$. We then define $\psi_j(x) = 2^{nj}\psi(2^j x)$ and $\phi_j(x) = 2^{nj}\phi(2^j x)$ for $j \in \mathbb{Z}$. Note that $\widehat{\phi}_j(\xi) = \widehat{\phi}(2^{-j}\xi)$ is supported in the annulus $2^{j-1} \leq |\xi| \leq 2^{j+1}$. We also define operators

$$\widetilde{S}_j f = \psi_j * f \quad \text{and} \quad \widetilde{\Delta}_j f = \phi_j * f$$

for $f \in L_1 + L_\infty$. The $\widetilde{\Delta}_j$'s are called the *Littlewood–Paley operators*. Now if $A = (a_{jk})_{(j,k) \in \mathbb{Z}^2}$ is an infinite c_{00}-matrix and $1 < p < \infty$, we let $\widetilde{h}_p(A)$ be the least constant so that for all $f \in L_p$ we have

$$(4.3) \qquad \left\| \sup_{j \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} a_{jk} \widetilde{\Delta}_k f \right| \right\|_{L_p} \leq \widetilde{h}_p(A) \|f\|_{L_p}.$$

We also define $\widetilde{h}_p^{\mathrm{w}}(A)$ to be the least constant such that for all $f \in L_p$ we have

$$(4.4) \qquad \left\| \sup_{j \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} a_{jk} \widetilde{\Delta}_k f \right| \right\|_{L_{p,\infty}} \leq \widetilde{h}_p^{\mathrm{w}}(A) \|f\|_{L_p}.$$

We now have the following.

LEMMA 4.3. *Suppose $r \in \mathbb{Z}$. If $1 < p < \infty$ and $A = (a_{jk})$ is any infinite c_{00}-matrix, then $\widetilde{h}_p(A) = \widetilde{h}_p(B)$ and $\widetilde{h}_p^{\mathrm{w}}(A) = \widetilde{h}_p^{\mathrm{w}}(B)$, where $B = (b_{jk})$ and $b_{jk} = a_{j,k+r}$.*

Proof. Consider the dilation operator $D_r f(x) = f(2^{-r}x)$. Then we have $D_r^{-1}\widetilde{\Delta}_k D_r f = \widetilde{\Delta}_{k-r}f$ and

$$\left\| \sup_j \left| \sum_k a_{j,k+r}\widetilde{\Delta}_k f \right| \right\|_{L_p} = \left\| \sup_j \left| \sum_k a_{jk}\widetilde{\Delta}_{k-r}f \right| \right\|_{L_p}$$

$$= 2^{-rn/p}\left\| \sup_j \left| \sum_k a_{jk}\widetilde{\Delta}_k D_r f \right| \right\|_{L_p}$$

$$\leq 2^{-rn/p}h_p(A)\|D_r f\|_{L_p} = h_p(A)\|f\|_{L_p},$$

which implies $\widetilde{h}_p(B) \leq \widetilde{h}_p(A)$. Likewise, we obtain $\widetilde{h}_p(A) \leq \widetilde{h}_p(B)$. The corresponding result for the weak type constants follows similarly. ∎

Next we prove that the Littlewood–Paley operators $\widetilde{\Delta}_j$ and the martingale difference operators Δ_k are essentially orthogonal on L_2 when $k \neq j$.

PROPOSITION 4.4. *There exists a constant C so that for every k, j in $\mathbb{Z}$ we have the following estimate on the operator norm of $\Delta_j\widetilde{\Delta}_k : L_2(\mathbb{R}^n) \to L_2(\mathbb{R}^n)$:*

$$(4.5) \qquad\qquad \|\Delta_k\widetilde{\Delta}_j\|_{L_2 \to L_2} \leq C2^{-|j-k|}.$$

Proof. By a simple dilation argument it suffices to prove (4.5) when $k = 0$. In this case we have the estimate

$$\|\Delta_0\widetilde{\Delta}_j\|_{L_2 \to L_2} = \|\mathcal{E}_0\widetilde{\Delta}_j - \mathcal{E}_{-1}\widetilde{\Delta}_j\|_{L_2 \to L_2}$$

$$\leq \|\mathcal{E}_0\widetilde{\Delta}_j - \widetilde{\Delta}_j\|_{L_2 \to L_2} + \|\mathcal{E}_{-1}\widetilde{\Delta}_j - \widetilde{\Delta}_j\|_{L_2 \to L_2}$$

and also by the self-adjointness of the Δ_k's and $\widetilde{\Delta}_j$'s we have

$$\|\Delta_0\widetilde{\Delta}_j\|_{L_2 \to L_2} = \|\widetilde{\Delta}_j\Delta_0\|_{L_2 \to L_2} = \|\widetilde{\Delta}_j\mathcal{E}_0 - \widetilde{\Delta}_j\mathcal{E}_{-1}\|_{L_2 \to L_2}$$

$$\leq \|\widetilde{\Delta}_j\mathcal{E}_0 - \mathcal{E}_0\|_{L_2 \to L_2} + \|\widetilde{\Delta}_j\mathcal{E}_{-1} - \mathcal{E}_0\|_{L_2 \to L_2}.$$

The required estimate (4.5) (when $k = 0$) will be a consequence of the pair of inequalities

$$(4.6) \quad \|\mathcal{E}_0\widetilde{\Delta}_j - \widetilde{\Delta}_j\|_{L_2 \to L_2} + \|\mathcal{E}_{-1}\widetilde{\Delta}_j - \widetilde{\Delta}_j\|_{L_2 \to L_2} \leq C2^j \qquad \text{when } j \leq 0,$$

$$(4.7) \quad \|\widetilde{\Delta}_j\mathcal{E}_0 - \mathcal{E}_0\|_{L_2 \to L_2} + \|\widetilde{\Delta}_j\mathcal{E}_{-1} - \mathcal{E}_0\|_{L_2 \to L_2} \leq C2^{-j} \qquad \text{when } j \geq 0.$$

We start by proving (4.6). We only consider the term $\mathcal{E}_0\widetilde{\Delta}_j - \widetilde{\Delta}_j$ since the term $\mathcal{E}_{-1}\widetilde{\Delta}_j - \widetilde{\Delta}_j$ is similar. Let $f \in L_2(\mathbb{R}^n)$. Then

$$\|\mathcal{E}_0 \widetilde{\Delta}_j f - \widetilde{\Delta}_j f\|^2_{L_2}$$

$$= \sum_{Q \in \mathcal{D}_0} \|f * \phi_j - \operatorname*{Ave}_Q (f * \phi_j)\|^2_{L_2(Q)}$$

$$\leq \sum_{Q \in \mathcal{D}_0} \int_Q \int_Q |(f * \phi_j)(x) - (f * \phi_j)(t)|^2 \, dt \, dx$$

$$\leq \sum_{Q \in \mathcal{D}_0} \int_Q \int_Q \left(\int_{3Q} |f(y)| \cdot |\phi_j(x - y)| \, dy \right)^2 dt \, dx$$

$$+ \sum_{Q \in \mathcal{D}_0} \int_Q \int_Q \left(\int_{3Q} |f(y)| \cdot |\phi_j(t - y)| \, dy \right)^2 dt \, dx$$

$$+ \sum_{Q \in \mathcal{D}_0} \int_Q \int_Q \left(\int_{(3Q)^c} |f(y)| 2^{jn+j} |\nabla \phi(2^j(\xi_{x,t} - y))| \, dy \right)^2 dt \, dx,$$

where $\xi_{x,t}$ lies on the line segment between x and t. It is now easy to see that the sum of the last three expressions above is bounded by

$$C 2^{2jn} \sum_{Q \in \mathcal{D}_0} \int_{3Q} |f(y)|^2 \, dy + C_M 2^{2j} \sum_{Q \in \mathcal{D}_0} \int_Q \left(\int_{\mathbb{R}^n} \frac{2^{jn} |f(y)| \, dy}{(1 + 2^j |x - y|)^M} \right)^2 dx,$$

which is clearly controlled by $C 2^{2j} \|f\|^2_{L_2}$. This estimate is useful when $j \leq 0$.

We now turn to the proof of (4.7). Since $\widetilde{\Delta}_j$ is the difference of two $\widetilde{S}_j$'s, it will suffice to prove (4.7) where $\widetilde{\Delta}_j$ is replaced by $\widetilde{S}_j$. We only work with the term $\widetilde{S}_j \mathcal{E}_0 - \mathcal{E}_0$ since the other term can be treated similarly. We have

$$\|\widetilde{S}_j \mathcal{E}_0 f - \mathcal{E}_0 f\|^2_{L_2} = \left\| \sum_{Q \in \mathcal{D}_0} (\operatorname*{Ave}_Q f)(\psi_j * \chi_Q - \chi_Q) \right\|^2_{L_2}$$

$$\leq 2 \left\| \sum_{Q \in \mathcal{D}_0} (\operatorname*{Ave}_Q f)(\psi_j * \chi_Q - \chi_Q) \chi_{3Q} \right\|^2_{L_2}$$

$$+ 2 \left\| \sum_{Q \in \mathcal{D}_0} (\operatorname*{Ave}_Q f)(\psi_j * \chi_Q) \chi_{(3Q)^c} \right\|^2_{L_2}.$$

Since the functions appearing inside the sum in the first term above have supports with bounded overlap we obtain

$$\left\| \sum_{Q \in \mathcal{D}_0} (\operatorname*{Ave}_Q f)(\psi_j * \chi_Q - \chi_Q) \chi_{3Q} \right\|^2_{L_2} \leq C \sum_{Q \in \mathcal{D}_0} (\operatorname*{Ave}_Q |f|)^2 \|\psi_j * \chi_Q - \chi_Q\|^2_{L_2},$$

and the crucial observation is that

$$\|\psi_j * \chi_Q - \chi_Q\|_{L_2} \leq C 2^{-j},$$

which can be easily checked using the Fourier transform. Therefore we obtain

$$\Big\| \sum_{Q\in\mathcal{D}_0} (\operatorname*{Ave}_Q f)(\psi_j * \chi_Q - \chi_Q)\chi_{3Q} \Big\|_{L_2}^2 \le C2^{-2j}\|f\|_{L_2}^2,$$

and the required conclusion will be proved if we can show that

$$(4.8) \qquad \Big\| \sum_{Q\in\mathcal{D}_0} (\operatorname*{Ave}_Q f)(\psi_j * \chi_Q)\chi_{(3Q)^c} \Big\|_{L_2}^2 \le C2^{-2j}\|f\|_{L_2}^2.$$

We prove (4.8) by using a purely size estimate. Let c_Q be the center of the dyadic cube Q. For $x \notin 3Q$ we have the easy estimate

$$|(\psi_j * \chi_Q)(x)| \le \frac{C_M 2^{jn}}{(1+2^j|x-c_Q|)^M} \le \frac{C_M 2^{jn}}{(1+2^j)^{M/2}} \frac{1}{(1+|x-c_Q|)^{M/2}}$$

since both $2^j \ge 1, |x-c_Q| \ge 1$. We now control the left hand side of (4.8) by

$$2^{j(2n-M)} \sum_{Q\in\mathcal{D}_0} \sum_{Q'\in\mathcal{D}_0} (\operatorname*{Ave}_Q |f|)(\operatorname*{Ave}_{Q'} |f|)$$

$$\times \int_{\mathbb{R}^n} \frac{C_M \, dx}{(1+|x-c_Q|)^{M/2}(1+|x-c_{Q'}|)^{M/2}}$$

$$\le 2^{j(2n-M)} \sum_{Q\in\mathcal{D}_0} \sum_{Q'\in\mathcal{D}_0} \frac{(\operatorname{Ave}_Q |f|)(\operatorname{Ave}_{Q'} |f|)}{(1+|c_Q-c_{Q'}|)^{M/4}}$$

$$\times \int_{\mathbb{R}^n} \frac{C_M \, dx}{(1+|x-c_Q|)^{M/4}(1+|x-c_{Q'}|)^{M/4}}$$

$$\le 2^{j(2n-M)} \sum_{Q\in\mathcal{D}_0} \sum_{Q'\in\mathcal{D}_0} \frac{C'_M}{(1+|c_Q-c_{Q'}|)^{M/4}}\Big(\int_Q + \int_{Q'}\Big)|f(y)|^2 \, dy$$

$$\le C''_M 2^{j(2n-M)} \sum_{Q\in\mathcal{D}_0} \int_Q |f(y)|^2 \, dy = C''_M 2^{j(2n-M)}\|f\|_{L_2}^2.$$

By taking M large enough we obtain (4.8) and thus (4.7). ∎

We have the following result relating $h(A)$ and $\widetilde{h}_p(A)$.

THEOREM 4.5. *For every $1 < p < \infty$, there is a constant C depending only on ψ and p so that for any c_{00}-matrix A we have*

$$\frac{1}{C}h(A) \le \widetilde{h}_p^{\mathrm{w}}(A) \le \widetilde{h}_p(A) \le Ch(A).$$

Proof. Consider the operators V_r, $r \in \mathbb{Z}$, defined by

$$V_r = \sum_{j\in\mathbb{Z}} \Delta_j \widetilde{\Delta}_{j+r}.$$

Then

$$V_r V_r^* = \sum_{j,k} \Delta_j \widetilde{\Delta}_{j+r} \widetilde{\Delta}_{k+r} \Delta_k = \sum_{|j-k|\leq 1} \Delta_j \widetilde{\Delta}_{j+r} \widetilde{\Delta}_{k+r} \Delta_k.$$

Hence by splitting into 3 pieces and using Proposition 4.4 we obtain the estimate

$$\|V_r\|_{L_2 \to L_2} \leq C 2^{-|r|}.$$

Next pick q so that $1 < q < \infty$ and $1/p = \theta/q + (1-\theta)/2$ where $0 < \theta < 1$. Let $(\varepsilon_j)_{j \in \mathbb{Z}}$ be a sequence of independent Bernoulli random variables on some probability space $(\Omega, \mathbb{P})$. Then for $f \in L_q(\Omega)$ we have

$$V_r f = \int_{\Omega} \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \varepsilon_j(\omega) \varepsilon_{k-r}(\omega) \Delta_j \widetilde{\Delta}_k f \, d\mathbb{P}.$$

Averaging now gives

$$\|V_r f\|_{L_q}$$
$$\leq \Big(\max_{\omega} \Big\| \sum_{j \in \mathbb{Z}} \varepsilon_j(\omega) \Delta_j \Big\|_{L_q \to L_q} \Big) \Big(\max_{\omega} \Big\| \sum_{k \in \mathbb{Z}} \varepsilon_{k-r}(\omega) \widetilde{\Delta}_k \Big\|_{L_q \to L_q} \Big) \|f\|_{L_q}.$$

Hence $\|V_r\|_{L_q \to L_q} \leq C$ where C depends only on q and ψ. Similarly $\|V_r^*\|_{L_q \to L_q} \leq C$. By interpolation we obtain $\|V_r\|_{L_p \to L_p}, \|V_r^*\|_{L_p \to L_p} \leq C 2^{-|r|(1-\theta)}$.

Finally, let us write

$$\sup_{j \in \mathbb{Z}} \Big| \sum_{k \in \mathbb{Z}} a_{jk} \widetilde{\Delta}_k f \Big| = \sup_{j \in \mathbb{Z}} \Big| \sum_{k \in \mathbb{Z}} a_{jk} \sum_{r \in \mathbb{Z}} \Delta_{k-r} \widetilde{\Delta}_k f \Big|$$
$$\leq \sum_{r \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} \Big| \sum_{k \in \mathbb{Z}} a_{j,k+r} \Delta_k \widetilde{\Delta}_{k+r} f \Big|.$$

Thus by Proposition 2.5,

$$\Big\| \sup_{j \in \mathbb{Z}} \Big| \sum_{k \in \mathbb{Z}} a_{jk} \widetilde{\Delta}_k f \Big| \Big\|_{L_p} \leq Ch(A) \sum_{r \in \mathbb{Z}} \|V_r f\|_{L_p} \leq Ch_p(A) \|f\|_{L_p}.$$

This shows that $\widetilde{h}_p(A) \leq Ch(A)$.

For the converse direction we use V_r^* and Lemma 4.3. We have

$$\sup_{j \in \mathbb{Z}} \Big| \sum_{k \in \mathbb{Z}} a_{jk} \Delta_k f \Big| \leq \sum_{r \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} \Big| \sum_{k \in \mathbb{Z}} a_{j,k+r} \widetilde{\Delta}_k \Delta_{k+r} f \Big|$$

and so

$$\Big\| \sup_{j \in \mathbb{Z}} \Big| \sum_{k \in \mathbb{Z}} a_{jk} \Delta_k f \Big| \Big\|_{L_{p,\infty}} \leq C \widetilde{h}_p^{\mathrm{w}}(A) \sum_{r \in \mathbb{Z}} \|V_r^* f\|_{L_p},$$

which leads to the estimate $h(A) \leq C \widetilde{h}_p^{\mathrm{w}}(A)$. ∎

We next extend the definition of $\widetilde{h}_p(A)$ to the case when $0 < p \le 1$. For such p's we define $\widetilde{h}_p(A)$ to be the least constant so that for $f \in \mathcal{S}$ we have

$$(4.9) \qquad \left\| \sup_{j \in \mathbb{Z}} \left| \sum_{k \in \mathbb{Z}} a_{jk} \widetilde{\Delta} f \right| \right\|_{L_p} \le C \|f\|_{H_p}.$$

The space H_p that appears on the right of (4.9) when $0 < p \le 1$ is the classical real Hardy space of Fefferman and Stein [7] and the expression $\| \ \|_{H_p}$ is its quasi-norm.

THEOREM 4.6. *If* $0 < p < 1$ *then there is a constant* $C = C(p, \psi)$ *so that* $C^{-1}h(A) \le \widetilde{h}_p(A) \le Ch(A)$.

Proof. First we show the estimate $\widetilde{h}_p(A) \le Ch(A)$. Using the atomic characterization of H_p (cf. [3]), we note that it suffices to get an estimate for a function $f \in \mathcal{S}$ supported in a cube Q so that $|f(x)| \le |Q|^{-1/p}$ for $x \in Q$ and $\int x^\alpha f(x) = 0$ if $|\alpha| \le N = [n(1/p - 1)]$. It is then easy to see that for $x \notin 2Q$,

$$\left| \sum_{k \in \mathbb{Z}} a_{jk} \widetilde{\Delta}_k f(x) \right| \le Ch(A) |x - c_Q|^{-n-N-1},$$

since $|a_{jk}| \le Ch(A)$ for each j, k. (Here $2Q$ is the cube with twice the length and the same center c_Q, as usual.) This gives the estimate

$$\int_{\mathbb{R}^n \setminus 2Q} \sup_j \left| \sum_k a_{jk} \widetilde{\Delta}_k f(x) \right|^p dx \le C^p h(A)^p.$$

On the other hand,

$$\int_{2Q} \sup_j \left| \sum_k a_{jk} \widetilde{\Delta}_k f(x) \right|^p dx \le C|Q|^{1-p/2} h(A)^p \left(\int_Q |f(x)|^2 dx \right)^{p/2}$$

and combining with the previous estimate we obtain $\widetilde{h}_p(A) \le Ch(A)$.

Complex interpolation gives $\widetilde{h}_q(A) \le \widetilde{h}_2(A)^\theta \widetilde{h}_p(A)^{1-\theta}$ when $1 < q < 2$ and $1/q = (1 - \theta)/p + \theta/2$. Since $\widetilde{h}_q(A) \ge C^{-1}h(A)$ we deduce the estimate $\widetilde{h}_p(A) \ge C^{-1}h(A)$. ∎

5. Bilinear operators. Let σ be a bounded measurable function on $\mathbb{R}^n \times \mathbb{R}^n$. For $f, g \in \mathcal{S}(\mathbb{R}^n)$ we define a bilinear operator $W_\sigma(f, g)$ with multiplier σ by setting

$$(5.1) \qquad W_\sigma(f, g)(x) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(\xi, \eta) \widehat{f}(\xi) \widehat{g}(\eta) e^{2\pi i \langle x, \xi + \eta \rangle} \, d\xi \, d\eta.$$

If (5.1) is satisfied we say that σ is the *bilinear symbol* (or *multiplier*) of W_σ. Now suppose $1 < p_1, p_2 < \infty$ and let p_0 be defined by $1/p_0 = 1/p_1 + 1/p_2$. We say that W_σ is *strongly* (p_1, p_2)-*bounded* if W_σ extends to a bounded

bilinear operator $L_{p_1} \times L_{p_2} \to L_{p_0}$. In this case we denote its norm by $\|W_\sigma\|_{L_{p_1} \times L_{p_2} \to L_{p_0}}$ (we define this expression to be ∞ if W_σ is not bounded). Similarly, we say W_σ is *weakly* (p_1, p_2)-*bounded* if it extends to a bounded bilinear operator $L_{p_1} \times L_{p_2} \to L_{p_0,\infty}$ and its norm is then denoted by $\|W_\sigma\|_{L_{p_1} \times L_{p_2} \to L_{p_0,\infty}}$.

We extend these definitions to the case $0 < p_1, p_2 < \infty$ by replacing the L_p spaces by the corresponding Hardy spaces when $0 < p_j \le 1$. In the definition below we set $H_p = L_p$ for $1 < p < \infty$. Given $0 < p_1 < p_2 < \infty$ and p_0 defined by $1/p_0 = 1/p_1 + 1/p_2$, we say that W_σ is *strongly* (p_1, p_2)-*bounded* if it extends to a bounded bilinear operator $H_{p_1} \times H_{p_2} \to L_{p_0}$, and we denote its norm by $\|W_\sigma\|_{H_{p_1} \times H_{p_2} \to L_{p_0}}$. We say that W_σ is *weakly* (p_1, p_2)-*bounded* if it extends to a bounded bilinear operator $H_{p_1} \times H_{p_2} \to L_{p_0,\infty}$, and in this case we denote its norm by $\|W_\sigma\|_{H_{p_1} \times H_{p_2} \to L_{p_0,\infty}}$. Now for a bounded function σ on $\mathbb{R}^n \times \mathbb{R}^n$ and $0 < p_1, p_2 < \infty$ we define its corresponding *strong* and *weak* (p_1, p_2)-*multiplier norm* by

$$\|\sigma\|_{\mathcal{M}_{p_1,p_2}} = \|W_\sigma\|_{H_{p_1} \times H_{p_2} \to L_{p_0}} \quad \text{and} \quad \|\sigma\|_{\mathcal{M}^{\mathrm{w}}_{p_1,p_2}} = \|W_\sigma\|_{H_{p_1} \times H_{p_2} \to L_{p_0,\infty}},$$

where $1/p_0 = 1/p_1 + 1/p_2$.

This definition of multiplier norm is analogous to that in the linear case. If $v \in L_\infty(\mathbb{R}^n)$, then $\|v\|_{\mathcal{M}_p}$ denotes the norm of v as a multiplier from H_p into L_p, that is,

$$\|v\|_{\mathcal{M}_p} = \|M_v\|_{H_p \to L_p}, \quad \text{where} \quad M_v f = (v\widehat{f})^\vee,$$

when $0 < p < \infty$. Next we mention a few properties of multipliers.

PROPOSITION 5.1. *Suppose* $\sigma \in L_\infty(\mathbb{R}^n \times \mathbb{R}^n)$ *and* $0 < p_1, p_2 < \infty$. *Then*:

(i) *If* $\sigma'(\xi, \eta) = \sigma(\xi - \xi_0, \eta - \eta_0)$ *for some fixed* ξ_0, η_0 *then* $\|\sigma'\|_{\mathcal{M}_{p_1,p_2}} = \|\sigma\|_{\mathcal{M}_{p_1,p_2}}$.

(ii) *If* $L : \mathbb{R}^n \to \mathbb{R}^n$ *is an invertible linear operator and* $\sigma_L(\xi, \eta) = \sigma(L\xi, L\eta)$ *then* $\|\sigma_L\|_{\mathcal{M}_{p_1,p_2}} = \|\sigma\|_{\mathcal{M}_{p_1,p_2}}$.

(iii) *If* $\mu, v \in L_\infty(\mathbb{R}^n)$ *and* $\sigma'(\xi, \eta) = \mu(\xi)\sigma(\xi, \eta)v(\eta)$, *then*

$$\|\sigma'\|_{\mathcal{M}_{p_1,p_2}} \le \|\mu\|_{\mathcal{M}_{p_1}} \|\sigma\|_{\mathcal{M}_{p_1,p_2}} \|v\|_{\mathcal{M}_{p_2}}.$$

Proof. For (i) note that

$$W_{\sigma'}(f, g) = e^{2\pi i \langle x, \xi_0 + \eta_0 \rangle} W(e^{-2\pi i \langle x, \xi_0 \rangle} f, e^{-2\pi i \langle x, \eta_0 \rangle} g).$$

For (ii) note that $W_{\sigma_L}(f, g) \circ (L^t)^{-1} = W(f \circ (L^t)^{-1}, g \circ (L^t)^{-1})$. Part (iii) is trivial. ∎

LEMMA 5.2. *Let* $\sigma \in L_\infty(\mathbb{R}^n \times \mathbb{R}^n)$. *Suppose that either* $p_0 \ge 1$, *or* σ *is locally Riemann-integrable (i.e. continuous except on a set of measure zero). Then* $\|\sigma\|_{L_\infty} \le \|\sigma\|_{\mathcal{M}_{p_1,p_2}}$ *whenever* $p_0 = p_1 p_2/(p_1 + p_2)$ *and* $0 < p_1, p_2 < \infty$.

Proof. Suppose that σ is locally Riemann-integrable and let (ξ_0, η_0) be a point of continuity of σ. Then if we put $\sigma'_\lambda(\xi, \eta) = \sigma(\xi_0 + \lambda\xi, \eta_0 + \lambda\eta)$, Proposition 5.1 gives $\|W_{\sigma'_\lambda}\|_{H_{p_1} \times H_{p_2} \to L_{p_0}} = \|W_\sigma\|_{H_{p_1} \times H_{p_2} \to L_{p_0}}$. Now if $f, g \in \mathcal{S}$ it is easy to see that as $\lambda \to 0$ we have convergence in L_2 (and even pointwise) of $W_{\sigma'_\lambda}(f, g)$ to $\sigma(\xi_0, \eta_0)f(x)g(x)$.

If $p_0 \geq 1$ let Q_k be a cube of side 2^{-k} centered at $(0,0)$ in $\mathbb{R}^n \times \mathbb{R}^n$. Let

$$\sigma_k(\xi, \eta) = \frac{1}{|Q_k|} \int\limits_{Q_k} \sigma(\xi + \xi_0, \eta + \eta_0) \, d\xi_0 \, d\eta_0.$$

Proposition 5.1 and the fact that $p_0 \geq 1$ easily imply that $\|W_{\sigma_k}\|_{L_{p_1} \times L_{p_2} \to L_{p_0}} \leq \|W_\sigma\|_{L_{p_1} \times L_{p_2} \to L_{p_0}}$. Since σ_k is continuous we have

$$\|\sigma_k\|_{L_\infty} \leq \|W_\sigma\|_{L_{p_1} \times L_{p_2} \to L_{p_0}}.$$

Taking limits as $k \to \infty$ yields the conclusion. ∎

Next we require a lemma on series in L_p.

LEMMA 5.3. *Let* $0 < p < \infty$. *Suppose that for some sequence* $(f_{jk})_{(j,k)\in\mathbb{Z}^2}$ *of* L_p *functions and for all pairs of sequences* $(\delta_j)_{j\in\mathbb{Z}}, (\delta'_k)_{k\in\mathbb{Z}}$ *with* $\sup_{j\in\mathbb{Z}} |\delta_j| \leq 1$ *and* $\sup_{j\in\mathbb{Z}} |\delta'_j| \leq 1$, *we have*

$$\sup_{N\in\mathbb{N}} \Big\| \sum_{|j|\leq N} \sum_{|k|\leq N} \delta_j \delta'_k f_{jk} \Big\|_{L_p} \leq M.$$

Then there is a constant $C = C(p)$ *such that*

(i) $\sup_{|\delta_j|\leq 1} \|\sum_{j\in\mathbb{Z}} \delta_j f_{jj}\|_{L_p} \leq CM$ *(and the series converges unconditionally)*,

(ii) $\|(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |f_{jk}|^2)^{1/2}\|_{L_p} \leq CM$.

Proof. In fact (ii) follows immediately from Khinchin's inequality by taking two mutually independent sequences $\varepsilon_j, \varepsilon'_k$ of Bernoulli random variables. To obtain (i), take a sequence ε_j of Bernoulli random variables and for any finite subset $\mathcal{F} \subset \mathbb{Z}$ write

(5.2) $$\sum_{j\in\mathcal{F}} \delta_j f_{jj} = \sum_{j\in\mathcal{F}} \sum_{k\in\mathcal{F}} \delta_j \varepsilon_j \varepsilon_k f_{jk} - \sum_{\substack{j,k\in\mathcal{F}\\ j\neq k}} \delta_j \varepsilon_j \varepsilon_k f_{jk}.$$

Now for all $|\delta_j| \leq 1$ (see also [10], proof of Theorem 4.6),

$$\mathbb{E}\Big(\Big\| \sum_{\substack{j,k\in\mathcal{F}\\ j\neq k}} \delta_j \varepsilon_j \varepsilon_k f_{jk} \Big\|_{L_p}^p\Big)^{1/p} \leq C \Big\| \Big(\sum_{\substack{j,k\in\mathcal{F}\\ j<k}} |\delta_j f_{jk} + \delta_k f_{kj}|^2 \Big)^{1/2} \Big\|_{L_p}$$

$$\leq C \Big\| \Big(\sum_{j\in\mathbb{Z}} \sum_{k\in\mathbb{Z}} |f_{jk}|^2 \Big)^{1/2} \Big\|_{L_p} \leq CM$$

by a generalization of Khinchin's inequality due to Bonami [1] and part (ii). The same estimate is also valid for $\sum_{j\in\mathcal{F}}\sum_{k\in\mathcal{F}}\delta_j\varepsilon_j\varepsilon_k f_{jk}$ by our assumptions. These estimates together with (5.2) give (i). $\blacksquare$

We now introduce some notation that will be useful in what follows. For $(j,k)\in\mathbb{Z}$ let $D_{jk}=\{(\xi,\eta):2^{j-1}\leq|\xi|\leq 2^{j+1},\ 2^{k-1}\leq|\eta|\leq 2^{k+1}\}$. Also, for $\theta>0$ let $D_{jk}(\theta)=\{(\xi,\eta):2^{j-\theta}\leq|\xi|\leq 2^{j+\theta},\ 2^{k-\theta}\leq|\eta|\leq 2^{k+\theta}\}$.

PROPOSITION 5.4. *For any* $1<p_1,p_2<\infty$ *there is a constant* $C=C(p_1,p_2)$ *so that whenever* $(\sigma_{jk})_{j,k\in\mathbb{Z}}$ *is a family of bilinear symbols with* $\operatorname{supp}\sigma_{jk}\subset D_{jk}$ *which satisfy*

$$\sup_{|\delta_j|\leq 1}\sup_{|\delta_k'|\leq 1}\Big\|\sum_j\sum_k\delta_j\delta_k'\sigma_{jk}\Big\|_{\mathcal{M}_{p_1,p_2}}\leq M,$$

then the following statements are valid:

(i) *For any scalar sequence* (δ_j) *with* $\sup_j|\delta_j|\leq 1$ *and any* $r\in\mathbb{Z}$ *we have*

$$\Big\|\sum_{j\in\mathbb{Z}}\delta_j\sigma_{j,j+r}\Big\|_{\mathcal{M}_{p_1,p_2}}\leq CM.$$

(ii) *For all* $r\geq 3$ *we have*

$$\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\leq j-r}\sigma_{jk}\Big\|_{\mathcal{M}_{p_1,p_2}}+\Big\|\sum_{k\in\mathbb{Z}}\sum_{j\leq k-r}\sigma_{jk}\Big\|_{\mathcal{M}_{p_1,p_2}}\leq C(1+r^{\max(1/p_0,1)})M.$$

(iii) *For every* $r\geq 3$, $p_0\leq 1$ *and for all* $f,g\in\mathcal{S}$ *we have*

$$\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\leq j-r}W_{\sigma_{jk}}\Big\|_{L_{p_1}\times L_{p_2}\to H_{p_0}}\leq C(1+r^{\max(1/p_0,1)})M,$$

$$\Big\|\sum_{k\in\mathbb{Z}}\sum_{j\leq k-r}W_{\sigma_{jk}}\Big\|_{L_{p_1}\times L_{p_2}\to H_{p_0}}\leq C(1+r^{\max(1/p_0,1)})M.$$

Proof. For simplicity we write $W_{jk}=W_{\sigma_{jk}}$ below. (i) follows directly from Lemma 5.3. To prove (ii) and (iii) it is enough to consider the case $r=3$, since the other cases follow trivially by applying (i) and the case $r=3$. We therefore suppose $r\geq 3$ and establish both (ii) and (iii). An easy calculation gives that for f,g Schwartz, $W_{jk}(f,g)$ has Fourier transform supported in the annulus $2^{j-2}\leq|\zeta|\leq 2^{j+2}$ when $k\leq j-3$. It follows that

$$(5.3)\quad \Big\|\sum_{j\in\mathbb{Z}}\sum_{k\leq j-3}W_{jk}(f,g)\Big\|_{L_{p_0}}\leq\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\leq j-3}W_{jk}(f,g)\Big\|_{H_{p_0}}$$

$$\leq C\Big\|\Big(\sum_{j\in\mathbb{Z}}\Big|\sum_{k\leq j-3}W_{jk}(f,g)\Big|^2\Big)^{1/2}\Big\|_{L_{p_0}}$$

$$\leq C\mathbb{E}\Big(\Big\|\sum_{j\in\mathbb{Z}}\varepsilon_j\sum_{k\leq j-3}W_{jk}(f,g)\Big\|_{L_{p_0}}^{p_0}\Big)^{1/p_0}$$

where as usual $(\varepsilon_j)_{j\in\mathbb{Z}}$ is a sequence of independent Bernoulli random variables. (If $p_0 > 1$ then $H_{p_0} = L_{p_0}$.) We need to control the last term in (5.3).

Our hypothesis gives the estimate

$$(5.4) \qquad \mathbb{E}\Big(\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\varepsilon_j W_{jk}(f,g)\Big\|_{L_{p_0}}^{p_0}\Big)^{1/p_0} \le CM\|f\|_{L_{p_1}}\|g\|_{L_{p_2}},$$

while we can apply (i) to obtain

$$(5.5) \qquad \mathbb{E}\Big(\Big\|\sum_{j\in\mathbb{Z}}\sum_{|k-j|\le 2}\varepsilon_j W_{jk}(f,g)\Big\|_{L_{p_0}}^{p_0}\Big)^{1/p_0} \le CM\|f\|_{L_{p_1}}\|g\|_{L_{p_2}}.$$

It remains to estimate

$$\mathbb{E}\Big(\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\ge j+3}\varepsilon_j W_{jk}(f,g)\Big\|_{L_{p_0}}^{p_0}\Big)^{1/p_0}$$

$$\le \mathbb{E}\Big(\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\ge j+3}\varepsilon_j W_{jk}(f,g)\Big\|_{H_{p_0}}^{p_0}\Big)^{1/p_0}$$

$$\le C\mathbb{E}\Big(\Big\|\sum_{k\in\mathbb{Z}}\sum_{j\le k-3}\varepsilon_j W_{jk}(f,g)\Big\|_{L_{p_0}}^{p_0}\Big)^{1/p_0}$$

$$\le C\mathbb{E}\Big(\Big\|\sum_{k\in\mathbb{Z}}\varepsilon_k'\sum_{j\le k-3}\varepsilon_j W_{jk}(f,g)\Big\|_{L_{p_0}}^{p_0}\Big)^{1/p_0}$$

where ε_k' is a second (independent) sequence of independent Bernoulli random variables. Hence using again Khinchin's inequality we have

$$(5.6) \qquad \mathbb{E}\Big(\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\ge j+3}\varepsilon_j W_{jk}(f,g)\Big\|_{L_{p_0}}^{p_0}\Big)^{1/p_0}$$

$$\le C\Big\|\Big(\sum_{k\in\mathbb{Z}}\sum_{j\le k-3}|W_{jk}(f,g)|^2\Big)^{1/2}\Big\|_{L_{p_0}}$$

$$\le C\Big\|\Big(\sum_{k\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}|W_{jk}(f,g)|^2\Big)^{1/2}\Big\|_{L_{p_0}} \le CM\|f\|_{L_{p_1}}\|g\|_{L_{p_2}}$$

in view of Lemma 5.3. Using (5.4)–(5.6) we obtain

$$\mathbb{E}\Big(\Big\|\sum_{j\in\mathbb{Z}}\varepsilon_j\sum_{k\le j-3}W_{jk}(f,g)\Big\|_{L_{p_0}}^{p_0}\Big)^{1/p_0} \le CM\|f\|_{L_{p_1}}\|g\|_{L_{p_2}},$$

which combined with (5.3) gives the first of the assertions (ii) and (iii) for $r = 3$. The second assertions are derived similarly by symmetry. ∎

We will need one further preliminary lemma.

LEMMA 5.5. *For any* $1 < p_1, p_2 < \infty$ *there is a constant* $C = C(p_1, p_2)$ *such that for any family of symbols* $(\sigma_{jk})_{j,k\in\mathbb{Z}}$ *with* $\operatorname{supp}\sigma_{jk} \subset D_{jk}$ *and for*

any C^∞ functions μ, υ on the annulus $1/4 \leq |\xi| \leq 4$ we have

$$\sup_{|\delta_j|} \sup_{|\delta_k'| \leq 1} \Big\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \delta_j \delta_k' \tau_{jk} \Big\|_{\mathcal{M}_{p_1, p_2}}$$

$$\leq C K_\mu K_\upsilon \sup_{|\delta_j|} \sup_{|\delta_k'| \leq 1} \Big\| \sum_j \sum_k \delta_j \delta_k' \sigma_{jk} \Big\|_{\mathcal{M}_{p_1, p_2}},$$

where $\tau_{jk}(\xi, \eta) = \mu(2^{-j}\xi)\sigma_{jk}(\xi, \eta)\upsilon(2^{-k}\eta)$,

$$K_\mu = \sup_{\substack{|\alpha| \leq m \\ 1/4 \leq |\xi| \leq 4}} \left| \frac{\partial^\alpha \mu}{\partial \xi^\alpha} \right|, \qquad K_\upsilon = \sup_{\substack{|\alpha| \leq m \\ 1/4 \leq |\xi| \leq 4}} \left| \frac{\partial^\alpha \upsilon}{\partial \xi^\alpha} \right|,$$

and $m = [(n+1)/2]$.

Proof. Recalling the definition of ϕ from Section 4 we note that the function

$$\Big(\sum_{l=j-2}^{j+2} \widehat{\phi}_l(\xi) \Big) \Big(\sum_{l=j-2}^{k+2} \widehat{\phi}_l(\eta) \Big)$$

is compactly supported and is equal to 1 on the support of $\sigma_{jk}(\xi, \eta)$. For any sequence δ_j with $\sup |\delta_j| \leq 1$ we observe that

$$(5.7) \qquad \Big\| \Big(\sum_{j \in \mathbb{Z}} \delta_j \mu(2^{-j}\xi) \Big) \Big(\sum_{l=j-2}^{j+2} \widehat{\phi}_l(\xi) \Big) \Big\|_{\mathcal{M}_{p_1}} \leq C K_\mu,$$

$$(5.8) \qquad \Big\| \Big(\sum_{k \in \mathbb{Z}} \delta_k' \mu(2^{-jk}\eta) \Big) \Big(\sum_{l=k-2}^{k+2} \widehat{\phi}_l(\eta) \Big) \Big\|_{\mathcal{M}_{p_2}} \leq C K_\upsilon$$

by the Hörmander multiplier theorem. Let U_{j_1, j_2, k_1, k_2} be the bilinear operator with symbol

$$\Big(\delta_{j_1} \mu(2^{-j_1}\xi) \sum_{l=j_1-2}^{j_1+2} \widehat{\phi}_l(\xi) \Big) \sigma_{j_2, k_2}(\xi, \eta) \Big(\delta_{k_1}' \upsilon(2^{-k_1}\eta) \sum_{l=k_1-2}^{k_1+2} \widehat{\phi}_l(\eta) \Big),$$

for some fixed $|\delta_j|, |\delta_k'| \leq 1$. Let

$$M = \sup_{|\delta_j|} \sup_{|\delta_k'| \leq 1} \Big\| \sum_j \sum_k \delta_j \delta_k' \sigma_{jk} \Big\|_{\mathcal{M}_{p_1, p_2}}$$

and let $(\varepsilon_j), (\varepsilon_k')$ be two sequences of mutually independent Bernoulli random variables. Then for $f, g \in \mathcal{S}$ we have

$$\mathbb{E}\Big(\Big\| \sum_{j_1 \in \mathbb{Z}} \sum_{j_2 \in \mathbb{Z}} \sum_{k_1 \in \mathbb{Z}} \sum_{k_2 \in \mathbb{Z}} \varepsilon_{j_1} \varepsilon_{j_2} \varepsilon_{k_1}' \varepsilon_{k_2}' U_{j_1, j_2, k_1, k_2}(f, g) \Big\|_{L_{p_0}}^{p_0} \Big)^{1/p_0}$$

$$\leq C M K_\mu K_\upsilon \|f\|_{L_{p_1}} \|f\|_{L_{p_2}}$$

by our hypothesis, (5.7), and (5.8). We now use Lemma 5.3 twice to deduce

that

$$\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}U_{j,j,k,k}(f,g)\Big\|_{L_{p_0}} \leq CK_\mu K_\nu M\|f\|_{L_{p_1}}\|g\|_{L_{p_2}}.$$

This proves the required assertion. ∎

6. Bilinear operators and infinite matrices. Recall from Section 4 that $\phi_j(x) = 2^{nj}\phi(2^j x)$ are smooth bumps whose Fourier transforms are supported in the annuli $2^{j-1} \leq |\xi| \leq 2^{j+1}$. In this section we will consider symbols σ of the form

$$(6.1) \qquad \sigma_A(\xi,\eta) = \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{jk}\widehat{\phi}_j(\xi)\widehat{\phi}_k(\eta)$$

where $A = (a_{jk})_{(j,k)\in\mathbb{Z}^2}$ is a bounded infinite matrix. We let $W_A = W_{\sigma_A}$ and $\|A\|_\infty = \sup_{j,k}|a_{jk}|$.

If A is such an infinite matrix we define A_L to be its lower triangle and A_U to be its upper triangle, i.e. $A_L = (a_{jk}\theta_{jk})_{j,k}$ and $A_U = (a_{jk}\theta_{kj})_{j,k}$ where $\theta_{jk} = 1$ if $k < j$ and 0 otherwise. We let A_D be the diagonal $A - A_U - A_L$. Now define

$$(6.2) \qquad H(A) = h(A_L) + h(A_U^t) + \|A\|_\infty$$

Notice that $H(A) \geq \|A\|_\infty$ and that H is a norm on the space of $\{A : H(A) < \infty\}$, which makes it a Banach space.

Our objective will be to show that for any choice of $0 < p_1, p_2 < \infty$ we have $\|W_A\|_{H_{p_1}\times H_{p_2}\to L_{p_0}} \approx H(A)$. This will provide us with an equivalent expression for the norm of the multiplier σ_A defined in (6.1).

We start by proving the simple upper estimate below.

LEMMA 6.1. *If* $0 < p_1, p_2 < \infty$ *there is a constant* $C = C(p_1, p_2)$ *so that for any matrix* A *we have* $\|\sigma_A\|_{\mathcal{M}_{p_1,p_2}} \leq CH(A)$.

Proof. We give the proof in the case $p_1, p_2 > 1$; the only real alteration for the other cases would be to replace the appropriate L_{p_j}-norm with the H_{p_j}-norm and use Theorem 4.6. Suppose $f, g \in \mathcal{S}$ and consider

$$(6.3) \quad W_A(f,g) = \sum_{j\in\mathbb{Z}}\sum_{k\leq j-3} a_{jk}\widetilde{\Delta}_j f\,\widetilde{\Delta}_k g$$

$$+ \sum_{k\in\mathbb{Z}}\sum_{j\leq k-3} a_{jk}\widetilde{\Delta}_j f\,\widetilde{\Delta}_k g + \sum_{j\in\mathbb{Z}}\sum_{k=j-2}^{j+2} a_{jk}\widetilde{\Delta}_j f\,\widetilde{\Delta}_k g.$$

We estimate the first term by noticing that for fixed j the Fourier transform of $\widetilde{\Delta}_j f \sum_{k\leq j-3} a_{jk}\widetilde{\Delta}_k g$ is contained in the set $\{\zeta : 2^{j-2} \leq |\zeta| \leq 2^{j+2}\}$.

Hence if $p_0 > 1$ we have

$$\Big\| \sum_{j \in \mathbb{Z}} \widetilde{\Delta}_j f \sum_{k \leq j-3} a_{jk} \widetilde{\Delta}_k g \Big\|_{L_{p_0}} \leq C \Big\| \Big(\sum_{j \in \mathbb{Z}} |\widetilde{\Delta}_j f|^2 \Big)^{1/2} \Big(\Big| \sum_{k \leq j-3} a_{jk} \widetilde{\Delta}_k \Big|^2 \Big)^{1/2} \Big\|_{L_{p_0}}.$$

If $0 < p_0 \leq 1$ we obtain the same estimate by noticing that

$$\Big\| \sum_{j \in \mathbb{Z}} \widetilde{\Delta}_j f \sum_{k \leq j-3} a_{jk} \widetilde{\Delta}_k g \Big\|_{L_{p_0}} \leq \Big\| \sum_{j \in \mathbb{Z}} \widetilde{\Delta}_j f \sum_{k \leq j-3} a_{jk} \widetilde{\Delta}_k g \Big\|_{H_{p_0}}$$

and using the corresponding square-function estimates in H_{p_0}. Now we have

$$(6.4) \quad \Big\| \Big(\sum_{j \in \mathbb{Z}} |\widetilde{\Delta}_j f|^2 \Big| \sum_{k \leq j-3} a_{jk} \widetilde{\Delta}_k g \Big|^2 \Big)^{1/2} \Big\|_{L_{p_0}}$$

$$\leq \Big\| \Big(\sum_{j \in \mathbb{Z}} |\widetilde{\Delta}_j f|^2 \Big)^{1/2} \sup_{j \in \mathbb{Z}} \Big| \sum_{k \leq j-3} a_{jk} \widetilde{\Delta}_k g \Big| \Big\|_{L_{p_0}}.$$

If we let A_{LL} be the matrix with entries a_{jk} if $k \leq j-3$ and 0 otherwise, then $h(A_{\mathrm{LL}}) \leq h(A_{\mathrm{L}}) + h(B)$ where B is the matrix with entries a_{jk} if $j-2 \leq k \leq j-1$ and 0 otherwise. It is trivial to see that one has the estimate $h(B) \leq 2\|A\|_\infty$ so that $h(A_{\mathrm{LL}}) \leq Ch(A_{\mathrm{L}})$. Hence (6.4) and Theorem 4.5 give

$$\Big\| \sum_{j \in \mathbb{Z}} \sum_{k \leq j-3} a_{jk} \widetilde{\Delta}_j f \widetilde{\Delta}_k g \Big\|_{L_{p_0}} \leq C \Big\| \Big(\sum_{j \in \mathbb{Z}} \widetilde{\Delta}_j f \Big)^{1/2} \Big\|_{L_{p_1}} \Big\| \sup_{j \in \mathbb{Z}} \Big| \sum_{k \in \mathbb{Z}} a_{jk} \widetilde{\Delta}_k g \Big| \Big\|_{L_{p_2}}$$

$$\leq Ch(A_{\mathrm{L}}) \|f\|_{L_{p_1}} \|g\|_{L_{p_2}}.$$

The same argument shows that the third term in (6.3) is controlled by $Ch(A_{\mathrm{U}}^{\mathrm{t}}) \|f\|_{L_{p_1}} \|g\|_{L_{p_2}}$. The middle term in (6.3) is easy. For $-2 \leq r \leq 2$ we have

$$\Big\| \sum_{j \in \mathbb{Z}} a_{j,j+r} \widetilde{\Delta}_j f \widetilde{\Delta}_{j+r} g \Big\|_{L_{p_0}}$$

$$\leq \Big\| \Big(\sum_{j \in \mathbb{Z}} |a_{j,j+r}| \cdot |\widetilde{\Delta}_j f|^2 \Big)^{1/2} \Big\|_{L_{p_1}} \Big\| \Big(\sum_{k \in \mathbb{Z}} |a_{j,j+r}| \cdot |\widetilde{\Delta}_{j+r} g|^2 \Big)^{1/2} \Big\|_{L_{p_2}}$$

$$\leq C \max_j |a_{j,j+r}| \cdot \|f\|_{L_{p_1}} \|g\|_{L_{p_2}}.$$

Combining we obtain the required upper estimate: $\|\sigma_A\|_{\mathcal{M}_{p_1,p_2}} \leq CH(A)$. ∎

To obtain the converse is somewhat more complicated. First we prove a general result which we will use in other situations as well.

PROPOSITION 6.2. *For any $1 < p_1, p_2 < \infty$ with $p_0 = (1/p_1 + 1/p_2)^{-1} \geq 1$, there is a constant $C = C(p_1, p_2)$ with the following property: Whenever $(\sigma_{jk})_{(j,k) \in \mathbb{Z}^2}$ is a family of symbols with $\operatorname{supp} \sigma_{jk} \subset D_{jk}$ which satisfy*

Note that τ'_{jk} is supported on $D_{jk}(3)$. Also, if $2^{j-3} \leq |\xi| \leq 2^{j+3}$ we see that $\tau'_{jk}(\xi, \eta)$ is constant in ξ.

Next let O_n be the orthogonal group of $\mathbb{R}^n$ and let dL denote the Haar measure on this group. Define

$$\tau^{\#}_{jk}(\xi, \eta) = \int\limits_{1/4}^{4} \lambda^{n-1} \int\limits_{O_n} \tau'_{jk}(\lambda L\xi, \lambda L\eta) \, dL \, d\lambda,$$

and let $T^{\#}_{jk}$ be the corresponding bilinear operator. If $(\xi, \eta) \in D_{jk}$ we can compute that

$$\tau^{\#}_{jk}(\xi, \eta) = c2^{nk}|\eta|^{-n}a_{jk}$$

where c is a constant depending only on dimension. On the other hand, since $p_0 \geq 1$, Proposition 5.1(ii) gives

$$\Big\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \delta_j \delta'_k T^{\#}_{jk} \Big\|_{L_{p_1} \times L_{p_2} \to L_{p_0}} \leq CM$$

whenever $|\delta_j|, |\delta'_k| \leq 1$.

Note that $\operatorname{supp} \tau^{\#}_{jk} \subset D_{jk}(6)$. Take $\mathbb{M}_1$ and $\mathbb{M}_2$ to be residue classes modulo 10. Then if we replace δ_j by $\delta_j \chi_{\mathbb{M}_1}(j)$ and δ'_k by $\delta'_k \chi_{\mathbb{M}_2}(k)$ we obtain a bilinear operator whose symbol coincides with $a_{jk}2^{nk}|\eta|^{-n}\delta_j \delta'_k$ on D_{jk} for $(j, k) \in \mathbb{M}_1 \times \mathbb{M}_2$. Using Proposition 5.1(iii) and the multipliers $\sum_{j \in \mathbb{M}_1} \widehat{\phi}_j$ and $\sum_{k \in \mathbb{M}_2} \widehat{\phi}_k$ we find that the bilinear operator V with symbol

$$\sum_{j \in \mathbb{M}_1} \sum_{k \in \mathbb{M}_2} \delta_j \delta'_k 2^{nk}|\eta|^{-n}a_{jk}\widehat{\phi}_j(\xi)\widehat{\phi}_k(\eta),$$

satisfies $\|V\|_{L_{p_1} \times L_{p_2} \to L_{p_0}} \leq CM$. Summing over 100 different pairs of residue classes gives a similar estimate for the symbol

$$\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \delta_j \delta'_k 2^{nk}|\eta|^{-n}a_{jk}\widehat{\phi}_j(\xi)\widehat{\phi}_k(\eta).$$

The last step is to remove the factor $2^{nk}|\eta|^{-n}$. But this can be done by using Lemma 5.5 since $|\eta|^{-n}$ is C^{∞} on $1/4 \leq |\eta| \leq 4$. ∎

We will use this result to make an important estimate on the effect of translation in the computation of $\|W_A\|_{L_{p_1} \times L_{p_2} \to L_{p_0}}$. Let us define $A^{[r,s]}$ to be the matrix $(a_{j+r,k+s})_{j,k}$.

LEMMA 6.3. (i) *There is a constant C so that for all matrices A we have*

$$\|\sigma_{A^{[r,s]}}\|_{\mathcal{M}_{2,2}} \leq C^{|r-s|}\|\sigma_A\|_{\mathcal{M}_{2,2}}.$$

(ii) *For all $1 < p_1, p_2 < \infty$ with $p_0 = p_1 p_2/(p_1 + p_2) \geq 1$, there is a constant $C = C(p_1, p_2)$ so that if $|\delta_j|, |\delta'_k| \leq 1$ then*

$$\Big\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \delta_j \delta_k' a_{jk} \widehat{\phi}(2^{-j}\xi) \widehat{\phi}(2^{-k}\eta) \Big\|_{\mathcal{M}_{p_1,p_2}} \leq C \|\sigma_A\|_{\mathcal{M}_{p_1,p_2}},$$

i.e. $\|\sigma_D\|_{M_{p_1,p_2}} \leq C\|\sigma_A\|_{\mathcal{M}_{p_1,p_2}}$, *where* $D = (d_{jk})_{j,k} = (\delta_j \delta_k' a_{jk})_{j,k}$.

Proof. It is clear from Proposition 5.1 that for any $r \in \mathbb{Z}$ we have

$$\|W_{A^{[r,r]}}\|_{L_2 \times L_2 \to L_1} = \|W_A\|_{L_2 \times L_2 \to L_1}.$$

Thus it suffices to consider the case $r = 0$ and $s = \pm 1$ and establish a bound in this case. To do this we consider the symbols

$$\sigma_{jk}(\xi,\eta) = \sigma_A(\xi,\eta)\mu(2^{-j}\xi)\upsilon(2^{-k}\eta)\widehat{\phi}_j(\xi)\widehat{\phi}_k(\eta),$$

where μ, υ are C^∞-functions satisfying $|\mu(\xi)|, |\upsilon(\eta)| \leq 1$ for all ξ, η. Since $\|\sum_{j \in \mathbb{Z}} \delta_j \mu(2^{-j}\xi)\widehat{\phi}_j(\xi)\|_{\mathcal{M}_2}$ is bounded by 3 whenever $\sup_j |\delta_j| \leq 1$, and there is a similar bound for $\sum_{k \in \mathbb{Z}} \delta_k' \upsilon(2^{-k}\eta)\widehat{\phi}_k(\eta)$, we have an immediate estimate:

$$\Big\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \delta_j \delta_k' W_{\sigma_{jk}} \Big\|_{L_2 \times L_2 \to L_1} \leq 9\|W_A\|_{L_2 \times L_2 \to L_1}.$$

Now let

$$b_{jk} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma_{jk}(2^j\xi, 2^k\eta) \, d\xi \, d\eta.$$

Then we can compute

$$b_{jk} = \sum_{r=-1}^{1} \sum_{s=-1}^{1} c_{rs} a_{j+r,k+s}$$

where

$$c_{rs} = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \mu(\xi)\upsilon(\eta)\widehat{\phi}_{-r}(\xi)\widehat{\phi}_{-s}(\eta)\widehat{\phi}_0(\xi)\widehat{\phi}_0(\eta) \, d\xi \, d\eta.$$

Since the functions $\widehat{\phi}_r$ for $-1 \leq r \leq 1$ are linearly independent on the support of $\widehat{\phi}_0$ we can use the above estimate for a linear combination of a finite number of choices of υ and ξ so that $c_{rs} = 0$ except when $r = 0$ and $s = 1$, so that $B = cA^{[0,1]}$ for some fixed constant $c \neq 0$. By Proposition 6.2 we have $\|W_B\|_{L_2 \times L_2 \to L_1} \leq C\|W_A\|_{L_2 \times L_2 \to L_1}$. This and the similar argument for the case $s = -1$ gives the result (i).

For (ii) we observe that the above argument actually also yields a bound on $\|W_D\|_{L_2 \times L_2 \to L_1}$ when $D = (d_{jk}) = (\delta_j \delta_k' b_{jk})$ (since $\delta_j \delta_k' \sigma_{jk}$ also satisfies the hypotheses of Proposition 6.2). By choosing a similar linear combination we can then ensure that $b_{jk} = ca_{jk}$ and obtain the desired result. ∎

The next step is to consider a discrete model of the bilinear operator W_{σ_A}. We restrict ourselves to $p_1 = p_2 = 2$ for this, although our calculations

can be done in more generality. If A is a c_{00}-matrix we define $V_A : L_2 \times L_2 \to L_1$ by

$$V_A(f, g) = \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} a_{jk} \Delta_j f \Delta_k g,$$

where Δ_j are the martingale difference operators as defined in Section 4. We then have

LEMMA 6.4. *There is a constant C so that if A is a (strictly) lower-triangular matrix we have $h(A) \le C\|V_A\|_{L_2 \times L_2 \to L_1}$.*

Proof. This is a stopping time argument. Suppose $f \in L_2$ with $\|f\|_{L_2} = 1$. Note that for each j the function $f_j = \sum_{k \in \mathbb{Z}} a_{jk} \Delta_k f$ is Σ_{j-1}-measurable where Σ_{j-1} is the σ-algebra generated by the dyadic cubes in $\mathcal{D}_{j-1}$. Fix $\lambda > 0$. For each j let $\mathcal{Q}_j$ be the collection of cubes $Q \in \mathcal{D}_{j-1}$ so that $|f_j| > \lambda$ on Q and for each $j_1 < j$ we have $|f_{j_1}| \le \lambda$ on Q. It is not difficult to see that

$$\{x : \max_{j \in \mathbb{Z}} |f_j(x)| > \lambda\} = \bigcup_{j \in \mathbb{Z}} \bigcup_{Q \in \mathcal{Q}_j} Q$$

and this is a disjoint union. Also note the left-hand side has finite measure.

For each j let u_j be a Σ_j-measurable function such that $|u_j| = 1$ everywhere and $\mathcal{E}_{j-1} u_j = 0$. Let

$$g = \sum_{j \in \mathbb{Z}} u_j \sum_{Q \in \mathcal{Q}_j} \chi_Q.$$

Then

$$\|g\|_{L_2}^2 = |\{x : \max_{j \in \mathbb{Z}} |f_j(x)|\}|$$

and

$$V_A(f, g) = \sum_{j \in \mathbb{Z}} f_j \Delta_j g = \sum_{j \in \mathbb{Z}} f_j u_j \sum_{Q \in \mathcal{Q}_j} \chi_Q.$$

Hence

$$|V_A(f, g)| \ge \lambda \chi_{(\max_j |f_j| > \lambda)}$$

so that we have

$$\lambda |\{\max_j |f_j| > \lambda\}| \le \|V_A\|_{L_2 \times L_2 \to L_1}.$$

This implies that $h_2^{\mathrm{w}}(A) \le \|V_A\|_{L_2 \times L_2 \to L_1}$ and the result follows from Theorem 2.1. ∎

We are now ready for the main result:

THEOREM 6.5. *Suppose $0 < p_1, p_2 < \infty$. Then there is a constant $C = C(p_1, p_2)$ so that for any infinite matrix A we have*

$$\frac{1}{C} H(A) \le \|\sigma_A\|_{\mathcal{M}_{p_1, p_2}^{\mathrm{w}}} \le \|\sigma_A\|_{\mathcal{M}_{p_1, p_2}} \le CH(A).$$

Proof. The upper bound is proved in Lemma 6.1 so we only need to prove the lower bound. It suffices to prove the results for the case when A is a c_{00}-matrix. We start by considering the case $p_1 = p_2 = 2$, when A is strictly lower-triangular.

In this case let us estimate the norm of the discrete model V_A. In fact,

$$
\begin{aligned}
V_A(f,g) &= \sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{jk}\Delta_j f \Delta_k g = \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}}\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{jk}\widetilde{\Delta}_{j-r}\Delta_j f \widetilde{\Delta}_{k-s}\Delta_k g \\
&= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} a_{j+r,k+s}\widetilde{\Delta}_j \Delta_{j+r} f \widetilde{\Delta}_k \Delta_{k+s} g \\
&= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} W_{A^{[r,s]}}\Big(\sum_{j\in\mathbb{Z}}\widetilde{\Delta}_j \Delta_{j+r} f, \sum_{k\in\mathbb{Z}}\widetilde{\Delta}_k \Delta_{k+s} g\Big) \\
&= \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} W_{A^{[r,s]}}(V_{-r}^* f, V_{-s}^* g),
\end{aligned}
$$

where V_r is defined in the proof of Theorem 4.5. Using Proposition 4.4 we obtain

$$
\|V_A\|_{L_2\times L_2\to L_1} \le C \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} 2^{-|r|-|s|}\|W_{A^{[r,s]}}\|_{L_2\times L_2\to L_1}.
$$

(All these quantities are finite since A has only finitely many non-zero entries, and so there is a uniform bound on $W_{A^{[r,s]}}$.)

It follows that we have an estimate (for a suitable C_0)

$$
(6.5) \qquad h(A) \le C_0 \sum_{r\in\mathbb{Z}}\sum_{s\in\mathbb{Z}} 2^{-|r|-|s|}\|W_{A^{[r,s]}}\|_{L_2\times L_2\to L_1}.
$$

Next we estimate $H(A^{[r,s]})$. If $s \ge r$ it is clear that A remains lower-triangular and the invariance properties of $h(A)$ imply that $H(A^{[r,s]}) \le H(A)$. If $s < r$ then it is easy to estimate

$$
h(A_{\mathrm{L}}^{[r,s]}) \le h(A_{\mathrm{L}}) + (r-s)\|A\|_\infty
$$

and

$$
h((A_{\mathrm{U}}^{[r,s]})^{\mathrm{t}}) \le (r-s)\|A\|_\infty.
$$

We deduce that

$$
H(A^{[r,s]}) \le h(A) + |r-s|\cdot\|A\|_\infty
$$

for all r,s. Thus for a suitable constant C_0 we have

$$
(6.6) \qquad \|W_{A^{[r,s]}}\|_{L_2\times L_2\to L_1} \le C_1(1+|r-s|)h(A).
$$

Now we may pick an integer N large enough so that

$$
C_1 C_0 \sum_{|r|>N}\sum_{|s|>N} (1+|r-s|)2^{-|r|-|s|} \le \frac{1}{2}.
$$

Then we can combine (6.5) and (6.6) to obtain

$$(6.7) \qquad h(A) \leq C_2 \sum_{|r| \leq N} \sum_{|s| \leq N} \|W_{A^{[r,s]}}\|_{L_2 \times L_2 \to L_1}.$$

At this point Lemma 6.3 gives the conclusion that

$$h(A) \leq C\|W_A\|_{L_2 \times L_2 \to L_1}.$$

Now suppose A is arbitrary. If we let W_{jk} be the bilinear operator with symbol $a_{jk}\widehat{\phi}_j(\xi)\widehat{\phi}_k(\eta)$, Lemma 6.3(ii) implies that we can use Proposition 5.4(ii) to deduce that $\|W_{A_L}\|_{L_2 \times L_2 \to L_1} \leq C\|W_A\|_{L_2 \times L_2 \to L_1}$ for some absolute constant C. Thus the above argument yields $h(A_L) \leq C\|W_A\|_{L_2 \times L_2 \to L_1}$. Similarly $h(A_U^t) \leq C\|W_A\|_{L_2 \times L_2 \to L_1}$ and Lemma 5.2 is enough to show that $\|A\|_\infty \leq C\|W_A\|_{L_2 \times L_2 \to L_1}$. Combining these we have the estimate

$$H(A) \leq C\|W_A\|_{L_2 \times L_2 \to L_1}.$$

The proof is completed by a simple interpolation technique. We will argue first that an estimate of the type

$$(6.8) \qquad H(A) \leq C(p_1, p_2)\|\sigma_A\|_{\mathcal{M}_{p_1, p_2}}$$

for some fixed $1 < p_1, p_2 < \infty$ implies the estimate

$$(6.9) \qquad H(A) \leq C(q, p_2)\|\sigma_A\|_{\mathcal{M}^w_{p_1, q}}$$

for every $1 < q < \infty$. We only need to consider the first case and $q \neq p_2$ (when $q = p_2$ one repeats the step). Then we may find $1 < r < \infty$ and $0 < \theta < 1$ so that

$$\frac{1}{p_2} = \frac{1 - \theta}{q} + \frac{\theta}{r}.$$

The Marcinkiewicz interpolation theorem yields

$$(6.10) \qquad \|\sigma_A\|_{\mathcal{M}_{p_1, p_2}} \leq C(p_1, p_2, \theta)(\|\sigma_A\|_{\mathcal{M}^w_{p_1, q}})^{1-\theta}(\|\sigma_A\|_{\mathcal{M}_{p_2, r}})^\theta.$$

Since $\|\sigma_A\|_{\mathcal{M}_{p_2, r}} \leq C(p_2, r)H(A)$, using (6.10) and (6.8) we obtain estimate (6.9) as required (recall that we assume A is a c_{00}-matrix so that all these quantities are finite).

Repeated use of this argument starting from $p_1 = p_2 = 2$ gives the theorem in the cases $1 < p_1, p_2 < \infty$.

Finally, in the case where either $p_1 \leq 1$ or $p_2 \leq 1$ (or both) one can use complex interpolation to deduce

$$\|\sigma_A\|_{\mathcal{M}^w_{q_1, q_2}} \leq C(\|\sigma_A\|_{\mathcal{M}^w_{p_1, p_2}})^{1-\theta}(\|\sigma_A\|_{\mathcal{M}_{2,2}})^\theta$$

where $q_1, q_2 > 1$ and

$$\frac{1}{q_1} = \frac{1 - \theta}{p_1} + \frac{\theta}{2}, \qquad \frac{1}{q_2} = \frac{1 - \theta}{p_2} + \frac{\theta}{2}.$$

This clearly extends the lower estimate to the cases $p_1, p_2 \leq 1$. ∎

7. Applications to bilinear multipliers. We will now consider the boundedness of the bilinear operator W_σ under conditions of Marcinkiewicz type on the symbol σ. We will say that a symbol σ is C^N if it is C^N on the set $\{(\xi,\eta): |\xi|,|\eta| > 0\}$. We first give an example to show that conditions (1.3) for a function σ on $\mathbb{R}^{2n}$ do not imply boundedness for the corresponding bilinear map on $\mathbb{R}^n \times \mathbb{R}^n$.

EXAMPLE. There is a C^∞ symbol σ so that for every pair of multi-indices (α,β) there is a constant $C_{\alpha,\beta}$ so that

$$(7.1) \qquad |\xi|^{|\alpha|}|\eta|^{|\beta|}|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta}$$

but W_σ is not of weak type (p_1,p_2) for any $0 < p_1,p_2 < \infty$.

Indeed, if we let A be a bounded infinite matrix and $\sigma(\xi,\eta) = \sigma_A(\xi,\eta)$, then σ satisfies the condition (7.1). However, W_A is of weak type (p_1,p_2) if and only if $H(A) < \infty$ by Theorem 6.5. At the end of Section 3 we showed that there are examples (with A lower-triangular) where $H(A) = \infty$.

In fact, more is true. It is shown that the condition $0 < \theta < 1/2$ in (3.12) is insufficient to give a bound on $h(A)$ or $H(A)$ when A is lower-triangular. This means that if $0 < \theta < 1/2$ we can construct a symbol σ which is C^∞, with W_σ not of weak type (p_1,p_2) for any $0 < p_1,p_2 < \infty$ and such that for each pair of multi-indices (α,β) there is a constant $C_{\alpha,\beta}$ with

$$(7.2) \qquad |\xi|^{|\alpha|}|\eta|^{|\beta|}|\partial_\xi^\alpha \partial_\eta^\beta \sigma(\xi,\eta)| \le C_{\alpha,\beta}(\log(1 + |\log|\xi|/|\eta||))^{-\theta}$$

but W_σ is not of weak type (p_1,p_2) for any $p_1,p_2 > 0$.

These examples indicate that the Marcinkiewicz-type conditions (7.1) need to be modified if they are to imply boundedness for bilinear operators on $\mathbb{R}^n \times \mathbb{R}^n$.

In order to formulate some general results, let us introduce the following notation. For $\sigma \in L_\infty$ we define

$$(7.3) \qquad \|\sigma\|_H = \sup_{1 \le |\xi| \le 2} \sup_{1 \le |\eta| \le 2} H((\sigma(2^j\xi,2^k\eta))_{j,k}).$$

If σ is of class C^N we define

$$(7.4) \qquad \|\sigma\|_H^{(N)} = \sum_{|\alpha|\le N} \| |\xi|^{|\alpha|}\partial_\xi^\alpha\sigma\|_H + \sum_{|\beta|\le N} \| |\eta|^{|\beta|}\partial_\eta^\beta\sigma\|_H.$$

It will also be useful to define in this case

$$(7.5) \qquad \|\sigma\|_{\mathcal{M}_{p_1,p_2}}^{(N)} = \sum_{|\alpha|\le N} \| |\xi|^{|\alpha|}\partial_\xi^\alpha\sigma\|_{\mathcal{M}_{p_1,p_2}} + \sum_{|\beta|\le N} \| |\eta|^{|\beta|}\partial_\eta^\beta\sigma\|_{\mathcal{M}_{p_1,p_2}}.$$

Now consider an arbitrary L^∞ symbol σ of class C^{n+1}. Let

$$(7.6) \qquad \sigma_{jk}(\xi,\eta) = \sigma(\xi,\eta)\widehat{\phi}(2^{-j}\xi)\widehat{\phi}(2^{-k}\eta).$$

Set $\widehat{\zeta}(\xi) = \widehat{\phi}_{-2}(\xi) + \widehat{\phi}_{-3}(\xi) + \widehat{\phi}_{-4}(\xi)$. Then $\widehat{\zeta}$ is equal to 1 on the annulus $1/16 \leq |\xi| \leq 1/4$ and vanishes off the annulus $1/32 \leq |\xi| \leq 1/2$. Thus the function $\widehat{\zeta}(\xi)\widehat{\zeta}(\eta)$ is supported in the unit cube $[0,1]^{2n}$ and is equal to one on the support of

$$(\xi, \eta) \mapsto \sigma_{jk}(2^{j+3}\xi, 2^{k+3}\eta),$$

which is also contained in $[0,1]^{2n}$. Inspired by [5], we expand the function above in a Fourier series on $[0,1]^{2n}$. We have

$$\sigma_{jk}(2^{j+3}\xi, 2^{k+3}\eta) = \sum_{\nu \in \mathbb{Z}^n} \sum_{\varrho \in \mathbb{Z}^n} a_{jk}(\nu, \varrho) e^{2\pi i(\langle \xi, \nu \rangle + \langle \eta, \varrho \rangle)} \widehat{\zeta}(\xi)\widehat{\zeta}(\eta),$$

where for $(\nu, \varrho) \in \mathbb{Z}^n \times \mathbb{Z}^n$ we set

$$(7.7) \qquad a_{jk}(\nu, \varrho) = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} \sigma(2^{j+3}t, 2^{k+3}s)\widehat{\phi}(8t)\widehat{\phi}(8s)e^{-2\pi i(\langle t, \nu \rangle + \langle s, \varrho \rangle)} \, dt \, ds.$$

We will denote by $A(\nu, \varrho)$ the matrix with entries $a_{jk}(\nu, \varrho)$. Now setting

$$(7.8) \qquad \tau^{\nu, \varrho}(\xi, \eta)$$
$$= \Big(\sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} a_{jk}(\nu, \varrho) e^{(\pi i/4)(2^{-j}\langle \xi, \nu \rangle + 2^{-k}\langle \eta, \varrho \rangle)} \Big) \widehat{\zeta}(2^{-j-3}\xi)\widehat{\zeta}(2^{-k-3}\eta),$$

we can write a symbol σ of class C^{n+1} as

$$(7.9) \qquad \sigma(\xi, \eta) = \sum_{\nu \in \mathbb{Z}^n} \sum_{\varrho \in \mathbb{Z}^n} \tau^{\nu, \varrho}(\xi, \eta).$$

In the next lemma we obtain some elementary estimates based on this expansion.

LEMMA 7.1. *Suppose $0 < p_1, p_2 < \infty$ and $1/p_0 = 1/p_1 + 1/p_2$. Then:*

(i) *There is a constant $C = C(p_1, p_2)$ so that for any (ν, ϱ),*

$$\|\tau^{\nu, \varrho}\|_{\mathcal{M}_{p_1, p_2}} \leq C(1 + |\nu| + |\varrho|)^{2m} H(A(\nu, \varrho))$$

where $m = [(n+1)/2]$.

(ii) *There is a constant $C = C(N, p_1, p_2)$ such that if σ is of class C^N, and $|\nu| + |\varrho| > 0$, then*

$$H(A(\nu, \varrho)) \leq C(1 + |\nu| + |\varrho|)^{-N}\|\sigma\|_H^{(N)},$$

while

$$H(A(0,0)) \leq C\|\sigma\|_H.$$

(iii) *If $p_0 \geq 1$ and σ is of class C^N then there is a constant $C = C(N, p_1, p_2)$ such that*

$$H(A(\nu, \varrho)) \leq C(1 + |\nu| + |\varrho|)^{2m-N}\|\sigma\|_{\mathcal{M}_{p_1, p_2}}^{(N)}.$$

Proof. Observe that $\widehat{\zeta}(2^{-j-3}\xi) = \widehat{\phi}(2^{-j-1}\xi) + \widehat{\phi}(2^{-j}\xi) + \widehat{\phi}(2^{-j+1}\xi)$ and therefore $\tau^{\nu,\varrho}(\xi,\eta)$ is the sum of nine terms of the form

$$\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}} a_{j,k}(\nu,\varrho)(e^{(\pi i/4)\langle 2^{-j}\xi,\nu\rangle}\widehat{\phi}(2^{-j-r}\xi))(e^{(\pi i/4)\langle 2^{-k}\eta,\varrho\rangle}\widehat{\phi}(2^{-k-s}\eta))$$

where $r,s\in\{-1,0,+1\}$. We now use Lemmas 5.5, 6.3(ii) and 6.1 in that order to obtain

$$\|\tau^{\nu,\varrho}\|_{\mathcal{M}_{p_1,p_2}} \le C(1+|\nu|)^m(1+|\varrho|)^m H(A(\nu,\varrho))$$

where $m = [(n+1)/2]$. This proves (i).

For (ii) note that if $|\alpha|,|\beta|\le N$ integration by parts gives

(7.10)　$a_{jk}(\nu,\varrho)$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_\xi^\alpha(\sigma(2^{j+3}\xi,2^{k+3}\eta)\widehat{\phi}(8\xi)\widehat{\phi}(8\eta))\frac{e^{-2\pi i(\langle\xi,\nu\rangle+\langle\eta,\varrho\rangle)}}{(-2\pi i\nu)^\alpha}\,d\xi\,d\eta,$$

(7.11)　$a_{jk}(\nu,\varrho)$

$$= \int_{\mathbb{R}^n}\int_{\mathbb{R}^n} \partial_\eta^\beta(\sigma(2^{j+3}\xi,2^{k+3}\eta)\widehat{\phi}(8\xi)\widehat{\phi}(8\eta))\frac{e^{-2\pi i(\langle\xi,\nu\rangle+\langle\eta,\varrho\rangle)}}{(-2\pi i\varrho)^\beta}\,d\xi\,d\eta,$$

provided $\nu_1^{\alpha_1}\ldots\nu_n^{\alpha_n}$ and $\varrho_1^{\beta_1}\ldots\varrho_n^{\beta_n}$ are non-zero.

Now using the fact that H is a norm it is easy to see that by choosing an appropriate α or β for each pair $(\nu,\varrho)\ne(0,0)$ one obtains the estimate

$$H(A(\nu,\varrho)) \le C(N,p_1,p_2)(1+|\nu|+|\varrho|)^{-N}\|\sigma\|_H^{(N)}.$$

If $(\nu,\varrho)=(0,0)$ the same estimate follows directly from (7.7).

Finally we turn to (iii). For fixed δ_j,δ_k' with $\sup|\delta_j|,\sup|\delta_k'|\le 1$ define $\mu(\xi) = \sum_{j\in\mathbb{Z}}\delta_j\widehat{\phi}_j(\xi)$ and $\upsilon(\eta) = \sum_{k\in\mathbb{Z}}\delta_k'\widehat{\phi}_j(\eta)$. Then it follows from Lemma 5.5 that for any multi-indices α,α' we have

$$\||\xi|^{|\alpha|+|\alpha'|}\partial_\xi^\alpha\mu(\xi)\partial_\xi^{\alpha'}\sigma(\xi,\eta)\upsilon(\eta)\|_{\mathcal{M}_{p_1,p_2}} \le C(\alpha,\alpha')\|\sigma\|_{\mathcal{M}_{p_1,p_2}}^{(|\alpha'|)}.$$

This implies that for fixed N and any α with $|\alpha|=N$ we have

(7.12)　$$\sup_{|\delta_j|\le1}\sup_{|\delta_k'|\le1}\||\xi|^N\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\delta_j\delta_k'\partial_\xi^\alpha\sigma_{jk}(\xi,\eta)\|_{\mathcal{M}_{p_1,p_2}} \le C(N)\|\sigma\|_{\mathcal{M}_{p_1,p_2}}^{(N)}.$$

Now we either use (7.7) if $(\nu,\varrho)=(0,0)$ or refer back to Proposition 6.2 (7.10) or (7.11), according to the values of ν or ϱ, when $(\nu,\varrho)\ne(0,0)$. For example, if $N=|\nu|\ge|\varrho|$ and the lth entry of ν has maximal size N, then

$$\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}a_{jk}(\nu,\varrho)\widehat{\phi}_j(\xi)\widehat{\phi}_k(\eta)\Big\|_{\mathcal{M}_{p_1,p_2}}$$

$$\le C\sup_{|\delta_j|\le1}\sup_{|\delta_k'|\le1}\Big\|\sum_{j\in\mathbb{Z}}\sum_{k\in\mathbb{Z}}\delta_j\delta_k'2^{jN}\frac{\partial^N}{\partial\xi_l^N}\sigma_{jk}(\xi,\eta)\frac{e^{-2\pi i(\langle 2^{-j}\xi,\nu\rangle+\langle 2^{-k}\eta,\varrho\rangle)}}{(-2\pi i\nu_l)^N}\Big\|_{\mathcal{M}_{p_1,p_2}}.$$

Now by Lemma 5.5 we can estimate the last expression above by

$$C(1 + |\nu| + |\varrho|)^{2m-N} \sup_{|\delta_j| \leq 1} \sup_{|\delta_k'| \leq 1} \left\| \sum_{j \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} \delta_j \delta_k' |\xi|^N \frac{\partial^N}{\partial \xi_l^N} \sigma_{jk}(\xi, \eta) \right\|_{\mathcal{M}_{p_1, p_2}}.$$

Using (7.12) we obtain (iii). ∎

Let us state the main result of this section.

THEOREM 7.2. *Suppose $0 < p_1, p_2 < \infty$ and $1/p_0 = 1/p_1 + 1/p_2$. Let $N = 2n + 1$ if $p_0 \geq 1$ and $N = n + 2 + [n/p_0]$ if $p_0 < 1$. Then for any C^N symbol σ such that $\|\sigma\|_H^{(N)} < \infty$ we have $\|\sigma\|_{\mathcal{M}_{p_1, p_2}} < \infty$. Furthermore, there is a constant $C = C(p_1, p_2)$ so that $\|\sigma\|_{\mathcal{M}_{p_1, p_2}} \leq C\|\sigma\|_H^{(N)}$.*

Proof. This follows directly from Lemma 7.1 and (7.9). Indeed, we have

$$\|\tau^{\nu, \varrho}\|_{\mathcal{M}_{p_1, p_2}} \leq C(1 + |\nu| + |\varrho|)^{2m-N}.$$

If $t = \min(p_0, 1)$ we have

$$\|\sigma\|_{\mathcal{M}_{p_1, p_2}} \leq C \Big(\sum_{\nu \in \mathbb{Z}} \sum_{\varrho \in \mathbb{Z}} (1 + |\nu| + |\varrho|)^{(2m-N)t} \Big)^{1/t} \|\sigma\|_H^{(N)}.$$

Since $(N - 2m)t > n$ this gives the result. ∎

We next show that in a certain sense the preceding theorem is best possible.

THEOREM 7.3. *Suppose $1 < p_1, p_2 < \infty$ and $1/p_0 = 1/p_1 + 1/p_2 \leq 1$. Suppose σ is a C^∞ symbol. Then the following are equivalent:*

(i) $\|\sigma\|_{\mathcal{M}_{p_1, p_2}}^{(N)} < \infty$ *for every $N \geq 0$.*
(ii) $\|\sigma\|_H^{(N)} < \infty$ *for every $N \geq 0$.*

Proof. Assume (i); then it follows from Lemma 7.1 that for any $N > 0$ we have an estimate $H(A(\nu, \varrho)) \leq C_N(1 + |\nu| + |\varrho|)^{-N}$. Now it is clear from the definition and from Theorem 6.5 and Lemma 6.3 that we have an estimate

$$\| \, |\xi|^{|\alpha|} \partial_\xi^\alpha \tau^{\nu, \varrho} \|_H \leq C_\alpha (1 + |\nu|)^{|\alpha|} H(A(\nu, \varrho)).$$

Hence we can easily deduce that

$$\| \, |\xi|^\alpha \partial_\xi^\alpha \sigma \|_H < \infty$$

for each multi-index α. Repeating the same reasoning with the second variable η gives (ii).

Now assume (ii). Then for any multi-indices α, β one can easily see by differentiation that (ii) is satisfied by the symbols $|\xi|^{|\alpha|} \partial_\xi^\alpha \sigma$ and $|\eta|^{|\beta|} \partial_\eta^\beta \sigma$ in place of σ. Applying Theorem 7.2 gives (i). ∎

Now let us recast Theorem 7.2 in terms of estimates on the symbol σ using the results of Section 3.

THEOREM 7.4. *Suppose* $0 < p_1, p_2 < \infty$ *and* $1/p_0 = 1/p_1 + 1/p_2$. *Let* $N = 2n+1$ *if* $p_0 \geq 1$ *and* $N = n+2+[n/p_0]$ *if* $p_0 < 1$. *Suppose that* $\theta > 1$ *and* σ *is a* C^N *symbol such that for any multi-indices* α, β *with* $0 \leq |\alpha| \leq N$ *and* $0 \leq |\beta| \leq N$ *there exist constants* C_α, C_β *with*

$$(7.13) \qquad |\xi|^{|\alpha|}|\partial_\xi^\alpha \sigma(\xi, \eta)| \leq C_\alpha (\log(1 + |\log |\xi|/|\eta| |))^{-\theta},$$

$$(7.14) \qquad |\eta|^{|\beta|}|\partial_\eta^\beta \sigma(\xi, \eta)| \leq C_\beta (\log(1 + |\log |\xi|/|\eta| |))^{-\theta}.$$

Then $\|\sigma\|_{\mathcal{M}_{p_1,p_2}} < \infty$.

REMARK. We have already seen in (7.2) that this is false when $0 < \theta < 1/2$. However the arguments of Section 3 show that we can improve (7.13) and (7.14) somewhat. For example, we can replace $(\log(1 + |\log |\xi|/|\eta| |))^{-\theta}$ where $\theta > 1$ by $(\log(1 + |\log |\xi|/|\eta| |))^{-1}(\log(1 + \log(1 + |\log |\xi|/|\eta| |)))^{-\gamma}$ where $\gamma > 1$.

Proof of Theorem 7.4. This follows immediately from Theorems 7.2 and 3.4 which yield the estimate

$$H(A) \leq C \sup_{j,k} |a_{jk}|/w_{|j-k|+1}$$

with $w_k = (\log(1 + k))^{-\theta}$. ∎

It is possible to "mix and match" the estimates in Section 3: for example, in the following theorem we remove the conditions for $|\alpha|, |\beta| = 0$ but insist on a stronger condition for $|\alpha| = |\beta| = 1$:

THEOREM 7.5. *Suppose* $0 < p_1, p_2 < \infty$ *and* $1/p_0 = 1/p_1 + 1/p_2$. *Let* $N = 2n+1$ *if* $p_0 \geq 1$ *and* $N = n+2+[n/p_0]$ *if* $p_0 < 1$. *Suppose that* $\theta > 1$ *and* σ *is a* C^N *symbol which satisfies conditions* (7.13) *and* (7.14) *for* $2 \leq |\alpha|, |\beta| \leq N$, *and if* $|\alpha| = |\beta| = 1$,

$$(7.15) \qquad |\xi|^{|\alpha|}|\partial_\xi^\alpha \sigma(\xi, \eta)| \leq C_\alpha (1 + |\log |\xi|/|\eta| |)^{-\theta},$$

$$(7.16) \qquad |\eta|^{|\beta|}|\partial_\eta^\beta \sigma(\xi, \eta)| \leq C_\beta (1 + |\log |\xi|/|\eta| |)^{-\theta}.$$

Then $\|\sigma\|_{\mathcal{M}_{p_1,p_2}} < \infty$.

Proof. It is only necessary to show that $\|\sigma\|_H < \infty$. Note first that Proposition 3.1 can be used to give the following estimate for any infinite matrix:

$$H(A) \leq C \Big(\|A\|_\infty + \sup_j \sum_{k<j} |a_{j,k} - a_{j,k+1}| + \sup_k \sum_{j<k} |a_{j,k} - a_{j+1,k}| \Big).$$

Now suppose $1 \leq |\xi|, |\eta| \leq 2$. Then if $k < j$,

$$|\sigma(2^j\xi, 2^k\eta) - \sigma(2^j\xi, 2^{k+1}\eta)| \leq Ck^{-\theta}$$

by (7.16). Combining with a similar estimate from (7.15) gives the theorem. ∎

We conclude this section with a theorem of the type of Theorem 7.2 for operators on L_1.

THEOREM 7.6. *Suppose $N = 2n + 3$ and that σ is a C^N symbol with $\|\sigma\|_H^{(N)} < \infty$. Then $W_\sigma : L_1 \times L_1 \to L_{1/2,\infty}$ is bounded.*

Proof. Let Q be the cube $\{x : \max_k |x_k| \leq 1\}$ and consider the bilinear operator $W_{\sigma,Q}(f,g) = \chi_Q W_\sigma(f,g)$. We will show that if $r < 1/2$ is such that $n+2+[n/(2r)] = N$, then $W_{\sigma,Q} : L_1(2Q) \times L_1(2Q) \to L_r(Q)$ is bounded and $\|W_{\sigma,Q}\| \leq C\|\sigma\|_H^{(N)}$ where C is a constant depending only on dimension.

Suppose that $f, g \in \mathcal{S}$ are functions with support contained in $2Q$ and such that $\int f(x)\,dx = \int g(x)\,dx = 0$. Then $f, g \in H_{2r}$ with $\|f\|_{H_{2r}} \leq C\|f\|_{L_1}$ and $\|g\|_{H_{2r}} \leq C\|g\|_{L_1}$. Applying Theorem 7.2 we obtain

$$(7.17) \qquad \|W_\sigma(f,g)\|_{L_r} \leq C\|\sigma\|_H^{(N)}\|f\|_{L_1}\|g\|_{L_1}$$

where C is an absolute constant. It follows that W_σ extends unambiguously to any $f, g \in L_1(2Q)$ with $\int f(x)\,dx = \int g(x)\,dx = 0$ and (7.17) holds.

Next fix $\psi \in \mathcal{S}$ so that $\int \psi(x)\,dx = 1$ and ψ has support contained in Q. Now for any $f, g \in L_1(3Q)$ let $f_0 = f - (\int f(x)\,dx)\psi$ and $g_0 = g - (\int g(x)\,dx)\psi$. Then (7.17) gives

$$\|W_{\sigma,Q}(f_0,g_0)\|_{L_r} \leq C\|\sigma\|_H^{(N)}\|f\|_{L_1}\|g\|_{L_1}.$$

We also note that $\|W_{\sigma,Q}(\psi,\psi)\|_{L_r} \leq C\|\sigma\|_H^{(N)}$. Now consider the linear map $Tf = W_\sigma(f,\psi)$. Since $\psi \in L_2$ we see that, if $1/s = 1/(2r) + 1/2$, then $T : H_{2r} \to L_s$ is bounded with norm controlled by $C\|\sigma\|_H^{(N)}$ (again using Theorem 7.2). Hence since $r < s$,

$$\|W_{\sigma,Q}(f_0,\psi)\|_{L_r} \leq C\|\sigma\|_H^{(N)}\|f\|_{L_1}.$$

Similarly,

$$\|W_{\sigma,Q}(\psi,g_0)\|_{L_r} \leq C\|\sigma\|_H^{(N)}\|g\|_{L_1}.$$

Combining these estimates gives

$$(7.18) \qquad \|W_{\sigma,Q}(f,g)\|_{L_r} \leq C\|\sigma\|_H^{(N)}\|f\|_{L_1}\|g\|_{L_1}.$$

We now use a Nikishin type argument as earlier in Lemma 2.3. Suppose $(f_j)_{j=1}^J$ and $(g_j)_{j=1}^J$ satisfy $\|f_j\|_{L_1}, \|g_j\|_{L_1} \leq 1$ and that $\sum_{j=1}^J |b_j|^{1/2} = 1$. Then if $(\varepsilon_j)_{j=1}^J$ and $(\varepsilon_j')_{j=1}^J$ are two independent sequences of Bernoulli random variables we have

$$\Big(\mathbb{E}\Big(\Big\|\sum_{j=1}^J \sum_{k=1}^J \varepsilon_j \varepsilon_k' |b_j|^{1/2} |b_k|^{1/2} W_{\sigma,Q}(f_j,g_k)\Big\|_{L_r}^r\Big)\Big)^{1/r} \leq C\|\sigma\|_H^{(N)}.$$

Again by using the result of Bonami [1], we obtain an estimate

$$\Big\|\Big(\sum_{j=1}^J \sum_{k=1}^J |b_j| \cdot |b_k| \cdot |W_{\sigma,Q}(f_j,g_k)|^2\Big)^{1/2}\Big\|_{L_r} \leq C\|\sigma\|_H^{(N)}.$$

Extracting the diagonal gives

$$\Big\| \max_{1 \le j \le J} |b_j| \cdot |W_{\sigma,Q}(f_j, g_j)| \Big\|_{L_r} \le C \|\sigma\|_H^{(N)}.$$

We now use [17] as before. There is a weight function $w \in L_1(Q)$ with $w \ge 0$ a.e. and $\int w(x)\,dx = 1$ so that for any $f, g \in L_1(3Q)$ with $\|f\|_{L_1}, \|g\|_{L_1} \le 1$ and any measurable $E \subset Q$ we have

$$\Big(\int_E |W_\sigma(f, g)|^r \, dx \Big)^{1/r} \le C\|\sigma\|_H^{(N)} \Big(\int_E w(x) \, dx \Big)^{1/r - 2}.$$

Now suppose f, g are supported in Q and $\lambda > 0$. Let $E = \{x \in Q : |W_\sigma(f, g)| > \lambda\}$. Then the above equation yields

(7.19) $$\lambda |E|^{1/r} \le C\|\sigma\|_H^{(N)} \Big(\int_E w(x) \, dx \Big)^{1/r - 2}.$$

On the other hand, if we apply (7.19) to $f_t(x) = f(x - t)$ where $t \in Q$ and note that $W_\sigma(f_t, g) = (W_\sigma(f, g))_t$ we also obtain

$$\lambda |E \cap (Q + t)|^{1/r} \le C\|\sigma\|_H^{(N)} \Big(\int_E w(x - t) \, dx \Big)^{1/r - 2}.$$

Raising to the power $(1/r - 2)^{-1}$ and averaging gives

$$\lambda |E|^{1/r} \le C\|\sigma\|_H^{(N)} |E|^{1/r - 2}.$$

Thus $W_{\sigma,Q}$ maps $L_1(2Q) \times L_1(2Q)$ into $L_{1/2,\infty}(Q)$ with norm at most $C\|\sigma\|_H^{(N)}$.

Now let $\lambda > 1$. If we define $\sigma_\lambda(\xi, \eta) = \sigma(\lambda^{-1}\xi, \lambda^{-1}\eta)$, then we have $\|\sigma_\lambda\|_H^{(N)} = \|\sigma\|_H^{(N)}$ and we can apply this result to σ_λ. Notice that $W_{\sigma_\lambda}(f, g)(x) = W_\sigma(f_\lambda, g_\lambda)(\lambda x)$ where $f_\lambda(x) = f(\lambda x)$ and $g_\lambda(x) = g(\lambda x)$. This implies that for any $\lambda > 0$ we have the estimate

$$\|\chi_{\lambda Q} W_\sigma(f, g)\|_{L_{1/2,\infty}} \le C\|\sigma\|_H^{(N)} \|f\|_{L_1} \|g\|_{L_1}$$

for f, g supported in λQ. Letting $\lambda \to \infty$ gives the result. ∎

8. Discussion on paraproducts. Paraproducts are bilinear operators of the type σ_A for some specific upper (or lower) triangular matrices A of zeros and ones. Paraproducts are important tools which have been used on several occasions in harmonic analysis, such as in the proof of the $T1$ theorem of David and Journé [6]. We define the lower and upper paraproducts as the bilinear operators $\Pi_{\rm L}$ and $\Pi_{\rm U}$ with symbols

$$\tau_{\rm L}(\xi, \eta) = \sum_{j \in \mathbb{Z}} \sum_{k \le j - 3} \widehat{\phi}_j(\xi)\widehat{\phi}_k(\eta) \quad \text{and} \quad \tau_{\rm U}(\xi, \eta) = \sum_{k \in \mathbb{Z}} \sum_{j \le k - 3} \widehat{\phi}_j(\xi)\widehat{\phi}_k(\eta)$$

respectively. It is easy to see that $\|\tau_{\mathrm{L}}\|_{\mathcal{M}_{p_1,p_2}}, \|\tau_{\mathrm{U}}\|_{\mathcal{M}_{p_1,p_2}} < \infty$ for all $0 < p_1, p_2 < \infty$. This can be deduced in several ways, e.g. from Proposition 5.4 using Lemma 6.3 or directly from Theorem 7.2 and Proposition 3.1. We conclude that for all $0 < p, q < \infty$, Π_{L} maps $H_{p_1} \times H_{p_2}$ to H_{p_0} when $1/p_1 + 1/p_2 = 1/p_0$ and to $H_q = L_q$ when $1 < q < \infty$. We now turn to some endpoint cases regarding the paraproduct operator Π_{L}.

PROPOSITION 8.1. *Let $0 < q < \infty$. Then the paraproduct operator Π_{L} is bounded on the following products of spaces*:

(1) $\mathrm{BMO} \times H_q(\mathbb{R}^n) \to H_q(\mathbb{R}^n)$, *where* $H_q = L_q$ *when* $1 < q < \infty$;
(2) $\mathrm{BMO} \times H_1(\mathbb{R}^n) \to L_1(\mathbb{R}^n)$;
(3) $\mathrm{BMO} \times L_\infty(\mathbb{R}^n) \to \mathrm{BMO}$;
(4) $H_q(\mathbb{R}^n) \times L_\infty(\mathbb{R}^n) \to H_q(\mathbb{R}^n)$, *where* $H_q = L_q$ *when* $1 < q < \infty$;
(5) $L_1(\mathbb{R}^n) \times L_\infty(\mathbb{R}^n) \to L_{1,\infty}(\mathbb{R}^n)$;
(6) $\mathrm{BMO} \times L_1(\mathbb{R}^n) \to L_{1,\infty}(\mathbb{R}^n)$;
(7) $L_1(\mathbb{R}^n) \times L_1(\mathbb{R}^n) \to L_{1/2,\infty}(\mathbb{R}^n)$.

Proof. Statement (1) is a classical result on paraproducts when $1 < q < \infty$ and we refer the reader to [19], p. 303 for a proof. Note that for a fixed $f \in \mathrm{BMO}$, the map $g \mapsto \Pi_{\mathrm{L}}(f, g)$ is a Calderón–Zygmund singular integral. The extension of (1) to H_q for $q \leq 1$ is a consequence of the fact that if a convolution type singular integral operator maps $L_2 \to L_2$ with bound a multiple of $\|f\|_{\mathrm{BMO}}$, then it also maps H_q into itself with bound a multiple of this constant. (2) follows from a similar observation while (3) is a dual statement to (2). To prove (4) set $\widetilde{S}_j g = \sum_{k \leq j-3} \widetilde{\Delta}_k g$. We have $\Pi_{\mathrm{L}}(f, g) = \sum_{j \in \mathbb{Z}} \widetilde{\Delta}_j f \widetilde{S}_j g$ and the Fourier transform of $\widetilde{\Delta}_j f \widetilde{S}_j g$ is supported in the annulus $2^{j-2} \leq |\xi| \leq 2^{j+2}$. It follows that

$$\|\Pi_{\mathrm{L}}(f, g)\|_{H_q} \leq C \left\| \left(\sum_{j \in \mathbb{Z}} |\widetilde{\Delta}_j f \widetilde{S}_j g|^2 \right)^{1/2} \right\|_{H_q} \leq \|f\|_{H_q} \|Mg\|_{L_\infty},$$

where M is the Hardy–Littlewood maximal operator, which is certainly bounded on L_∞. To prove (5) we freeze g and look at the linear operator $f \mapsto \Pi_{\mathrm{L}}(f, g)$ whose kernel is $K(x, y) = \sum_{j \in \mathbb{Z}} \phi_j(x - y) S_j(g)(x)$. It is easy to see that

$$|\nabla_y K(x, y)| \leq C \|g\|_{L_\infty} |x - y|^{-n-1}.$$

This estimate together with the fact that the linear operator $f \mapsto \Pi_{\mathrm{L}}(f, g)$ maps L_2 into L_2 entails that $f \mapsto \Pi_{\mathrm{L}}(f, g)$ maps L_1 into $L_{1,\infty}$ using the Calderón–Zygmund decomposition. This proves (5). To obtain (6) we use (1) (with $q = 2$) and we apply the Calderón–Zygmund decomposition to the operator $g \mapsto \Pi_{\mathrm{L}}(f, g)$ for fixed $f \in \mathrm{BMO}$. Finally, (7) is a consequence of Theorem 7.6. ∎

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