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$L^p(\mathbb{R}^n)$ boundedness for the commutator of a homogeneous singular integral operator

by

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Abstract. The commutator of a singular integral operator with homogeneous kernel $\Omega(x)/|x|^n$ is studied, where Ω is homogeneous of degree zero and has mean value zero on the unit sphere. It is proved that $\Omega \in L(\log L)^{k+1}(S^{n-1})$ is a sufficient condition for the kth order commutator to be bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. The corresponding maximal operator is also considered.

1. Introduction. We will work on $\mathbb{R}^n$, $n \ge 2$. Let Ω be a homogeneous function of degree zero with mean value zero on the unit sphere S^{n-1}. Define the homogeneous singular integral operator T by

$$Tf(x) = \text{p.v.} \int_{\mathbb{R}^n} \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy.$$

For a positive integer k and $b \in \mathrm{BMO}(\mathbb{R}^n)$, define the kth order commutator of the operator T and b by

$$T_{b,k}f(x) = \int_{\mathbb{R}^n} (b(x)-b(y))^k \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy, \qquad f \in C_0^\infty(\mathbb{R}^n). \tag{1}$$

Coifman, Rochberg and Weiss [4] showed that if $\Omega \in \mathrm{Lip}_\alpha(S^{n-1})$ $(0<\alpha\le 1)$, then $T_{b,1}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C(n,p)\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}$ for $1 < p < \infty$. By a well-known result of Duoandikoetxea [5] and Watson [10], if $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then for $p > q'$ $(q' = q/(q-1))$ and $w \in A_{p/q'}$, the operator T is bounded on $L^p(\mathbb{R}^n, w(x)\,dx)$ with bound depending only on n, p and the $A_{p/q'}$ constant of w, where A_r is the weight function class of Muckenhoupt (see [9, Chapter V] for the definition and properties of A_r). This together with the Alvarez–Bagby–Kurtz–Pérez boundedness theorem for the commutators of linear operators (see [2, Theorem 2.13]) tells us

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that if $\Omega \in L^q(S^{n-1})$ for some $q > 1$, then $T_{b,k}$ is a bounded operator on $L^p(\mathbb{R}^n)$ for $q' < p < \infty$, and then by standard duality and interpolation argument, it is bounded on $L^p(\mathbb{R}^n)$ for all $1 < p < \infty$. On the other hand, if $\Omega \notin \bigcup_{q>1} L^q(S^{n-1})$, then for any fixed $1 < p, q < \infty$, we do not know whether the operator T is bounded on $L^p(\mathbb{R}^n, w(x)\,dx)$ for all $w \in A_q$, and the Alvarez–Bagby–Kurtz–Pérez theorem does not apply. In this case, the $L^p(\mathbb{R}^n)$ boundedness for $T_{b,k}$ is not known. In [7], we have proved that if Ω satisfies the size condition

$$\sup_{\zeta \in S^{n-1}} \int_{S^{n-1}} |\Omega(\theta)| \log^{\alpha}\left(\frac{1}{|\theta \cdot \zeta|}\right) d\theta < \infty$$

for some $\alpha > k + 1$, then the commutator $T_{b,k}$ is bounded on $L^2(\mathbb{R}^n)$. The purpose of this paper is to give a size condition on Ω which is strictly weaker than $\Omega \in \bigcup_{q>1} L^q(S^{n-1})$ and implies the $L^p(\mathbb{R}^n)$ boundedness of $T_{b,k}$ for all $1 < p < \infty$. Furthermore, we will also consider the $L^p(\mathbb{R}^n)$ boundedness for the corresponding maximal operator defined by

$$(2) \qquad T^*_{b,k} f(x) = \sup_{\varepsilon > 0} \Bigl| \int_{|x-y|>\varepsilon} (b(x) - b(y))^k \frac{\Omega(x-y)}{|x-y|^n} f(y)\,dy \Bigr|.$$

Our main results can be stated as follows.

THEOREM 1. *Let Ω be homogeneous of degree zero and have mean value zero on the unit sphere, k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. If $\Omega \in L(\log L)^{k+1}(S^{n-1})$, that is,*

$$\int_{S^{n-1}} |\Omega(x')| \log^{k+1}(2 + |\Omega(x')|)\,dx' < \infty,$$

then for all $1 < p < \infty$, the commutator $T_{b,k}$ defined by (1) is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.

THEOREM 2. *Let Ω be homogeneous of degree zero and have mean value zero on the unit sphere, k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. If $\Omega \in L(\log L)^{k+1}(S^{n-1})$, then for all $1 < p < \infty$, the operator $T^*_{b,k}$ defined by (2) is bounded on $L^p(\mathbb{R}^n)$ with bound $C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$.*

Some Young functions will be useful in the proof of our theorems. For positive integer k, let

$$a_k(\tau) = \log^k(1 + \tau), \qquad \widetilde{a}_k(\tau) = e^{\tau^{1/k}} - 1.$$

Define the functions Φ_k and Ψ_k by

$$\Phi_k(t) = \int_0^t a_k(\tau)\,d\tau, \qquad \Psi_k(t) = \int_0^t \widetilde{a}_k(\tau)\,d\tau.$$

Then Φ_k and Ψ_k are Young functions and Ψ_k is the complementary Young function of Φ_k. Therefore, for any $0 < t_1, t_2 < \infty$,

$$t_1 t_2 \le \Phi_k(t_1) + \Psi_k(t_2)$$

(see [1, Chap. 8] for details). By a straightforward computation, it follows that

$$\Phi_k(t) \le t \log^k(2+t), \qquad \Psi_k(t) \le t e^{t^{1/k}} \le k^k e^{2t^{1/k}}.$$

Thus, for $0 < t_1, t_2 < \infty$,

$$t_1 t_2^k \le 2^k(\Phi_k(t_1) + \Psi_k((t_2/2)^k)) \le C_k(t_1 \log^k(2+t_1) + e^{t_2}). \tag{3}$$

Throughout this paper, C denotes constants that are independent of the main parameters involved but whose values may differ from line to line. For $p \ge 1$, p' denotes the dual exponent of p, that is, $p' = p/(p-1)$. For a measurable set E, χ_E denotes the characteristic function of E.

2. Proof of Theorem 1. We begin with some preliminary lemmas.

LEMMA 1. *Let* $\phi \in C_0^\infty(\mathbb{R}^n)$ *be a radial function such that* $\operatorname{supp}\phi \subset \{1/4 \le |\xi| \le 4\}$ *and*

$$\sum_{l\in\mathbb{Z}} \phi^3(2^{-l}\xi) = 1, \qquad |\xi| \ne 0.$$

Define the multiplier operator S_l *by*

$$\widehat{S_l f}(\xi) = \phi(2^{-l}\xi)\widehat{f}(\xi),$$

and S_l^2 *by* $S_l^2 f(x) = S_l(S_l f)(x)$. *For* $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and positive integer* k, *denote by* $S_{l;b,k}$ (*resp.* $S^2_{l;b,k}$) *the* k*th order commutator of* S_l (*resp.* S_l^2). *Then for* $1 < p < \infty$,

(i) $\displaystyle\Big\|\Big(\sum_{l\in\mathbb{Z}} |S_{l;b,k} f|^2\Big)^{1/2}\Big\|_p \le C(n,k,p)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p;$

(ii) $\displaystyle\Big\|\Big(\sum_{l\in\mathbb{Z}} |S^2_{l;b,k} f|^2\Big)^{1/2}\Big\|_p \le C(n,k,p)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p;$

(iii) $\displaystyle\Big\|\sum_{l\in\mathbb{Z}} S_{l;b,k} f_l\Big\|_p \le C(n,k,p)\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\Big\|\Big(\sum_{l\in\mathbb{Z}} |f_l|^2\Big)^{1/2}\Big\|_p.$

Proof. Obviously, (iii) can be deduced from (i) directly. By the weighted Littlewood–Paley theory, we see that for any $1 < p < \infty$ and $w \in A_p$,

$$\Big\|\Big(\sum_{l\in\mathbb{Z}} |S_l f|^2\Big)^{1/2}\Big\|_{p,w} + \Big\|\Big(\sum_{l\in\mathbb{Z}} |S_l^2 f|^2\Big)^{1/2}\Big\|_{p,w} \le C\|f\|_{p,w}.$$

Note that the mappings

$$f \mapsto \{S_l f\}_{l\in\mathbb{Z}}, \qquad f \mapsto \{S_l^2 f\}_{l\in\mathbb{Z}}$$

are linear; then (i) and (ii) follow from the last inequality and Theorem 2.13 of [2].

LEMMA 2. *Let* $m_\delta \in C_0^\infty(\mathbb{R}^n)$ $(0 < \delta < \infty)$ *be a family of multipliers such that* $\operatorname{supp} m_\delta \subset \{\delta/4 \le |\xi| \le 4\delta\}$. *Suppose that for some positive constant* α,

$$\|m_\delta\|_\infty \le C\min\{\delta, \delta^{-\alpha}\}, \qquad \|\nabla m_\delta\|_\infty \le C.$$

Let T_δ *be the multiplier operator defined by*

$$\widehat{T_\delta f}(\xi) = m_\delta(\xi)\widehat{f}(\xi).$$

For $b \in \mathrm{BMO}(\mathbb{R}^n)$ *and positive integer* k, *denote by* $T_{\delta;b,k}$ *the* k*th order commutator of* T_δ. *Then for any fixed* $0 < \nu < 1$, *there exists a positive constant* $C = C(n,k,\nu)$ *such that*

$$\|T_{\delta;b,k}f\|_2 \le C\min\{\delta^\nu, \delta^{-\alpha\nu}\}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2.$$

For the case of $\delta \le 1$, Lemma 2 can be obtained from Lemma 2 of [7]. On the other hand, if $\delta > 1$, Lemma 2 was essentially proved in the proof of Lemma 2.3 of [8].

LEMMA 3. *Let* $\widetilde{\Omega}$ *be homogeneous of degree zero and belong to the space* $L^\infty(S^{n-1})$. *For* $s \ge 1$, *define* $\lambda_{\widetilde{\Omega},s}$ *by*

$$\lambda_{\widetilde{\Omega},s} = \inf\left\{\lambda > 0 : \frac{\|\widetilde{\Omega}\|_1}{\lambda}\log^s\left(2 + \frac{\|\widetilde{\Omega}\|_\infty}{\lambda}\right) \le 1\right\}.$$

Then

(i) *there exists a positive constant* $C = C_{n,s}$ *such that* $C^{-1}\|\widetilde{\Omega}\|_1 \le \lambda_{\widetilde{\Omega},s} \le C\|\widetilde{\Omega}\|_\infty$;

(ii) $\lambda_{\widetilde{\Omega},s} \le C_s((2 + \|\widetilde{\Omega}\|_\infty)^{-1} + \|\widetilde{\Omega}\|_1\log^s(2 + \|\widetilde{\Omega}\|_\infty))$;

(iii) *for any* $1 \le s, t < \infty$, $\lambda^{1/t}_{\widetilde{\Omega},st}\|\widetilde{\Omega}\|_1^{1/t'} \le \lambda_{\widetilde{\Omega},s}$.

Proof. Obviously, (i) follows directly from the fact that

$$\frac{\|\widetilde{\Omega}\|_1}{\|\widetilde{\Omega}\|_\infty}\log^s\left(2 + \frac{\|\widetilde{\Omega}\|_\infty}{\|\widetilde{\Omega}\|_\infty}\right) \le C|S^{n-1}|$$

and

$$\frac{\|\widetilde{\Omega}\|_1}{\|\widetilde{\Omega}\|_1}\log^s\left(2 + \frac{\|\widetilde{\Omega}\|_\infty}{\|\widetilde{\Omega}\|_1}\right) \ge \log^s(2 + |S^{n-1}|^{-1}).$$

As for (ii), note that

$$\frac{\|\widetilde{\Omega}\|_1}{2^s\|\widetilde{\Omega}\|_1\log^s(2 + \|\widetilde{\Omega}\|_\infty)}\log^s\left(2 + \frac{\|\widetilde{\Omega}\|_\infty}{(2 + \|\widetilde{\Omega}\|_\infty)^{-1}}\right) \le \frac{\|\widetilde{\Omega}\|_1}{2^s\|\widetilde{\Omega}\|_1\log^s(2 + \|\widetilde{\Omega}\|_\infty)}\log^s((2 + \|\widetilde{\Omega}\|_\infty)^2) \le 1.$$

It follows that

$$\lambda_{\widetilde{\Omega},s} \le 2^s((2+\|\widetilde{\Omega}\|_\infty)^{-1} + \|\widetilde{\Omega}\|_1 \log^s(2+\|\widetilde{\Omega}\|_\infty)).$$

To prove (iii), by homogeneity, we may assume that $\lambda_{\widetilde{\Omega},s} = 1$. Then

$$\|\widetilde{\Omega}\|_1 \log^s(2+\|\widetilde{\Omega}\|_\infty) \le 1$$

and so $\|\widetilde{\Omega}\|_1 \le 1$. A trivial computation gives

$$\begin{aligned}\frac{\|\widetilde{\Omega}\|_1}{\|\widetilde{\Omega}\|_1^{-t/t'}} \log^{st}\left(2+\frac{\|\widetilde{\Omega}\|_\infty}{\|\widetilde{\Omega}\|_1^{-t/t'}}\right) &\le \|\widetilde{\Omega}\|_1^{1+t/t'} \log^{st}(2+\|\widetilde{\Omega}\|_\infty)\\ &= (\|\widetilde{\Omega}\|_1 \log^s(2+\|\widetilde{\Omega}\|_\infty))^t \le 1.\end{aligned}$$

This in turn implies $\lambda_{\widetilde{\Omega},st} \le \|\widetilde{\Omega}\|_1^{-t/t'}$, and establishes the desired result.

LEMMA 4. *Let $\widetilde{\Omega}$ be homogeneous of degree zero, k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$. Define the operator $M_{\widetilde{\Omega};b,k}$ by*

$$M_{\widetilde{\Omega};b,k} f(x) = \sup_{r>0} r^{-n} \int\limits_{|x-y|<r} |b(x)-b(y)|^k |\widetilde{\Omega}(x-y) f(y)|\, dy.$$

If $\widetilde{\Omega} \in L^\infty(S^{n-1})$, then the operator $M_{\widetilde{\Omega};b,k}$ is bounded on $L^p(\mathbb{R}^n)$ with bound $C\lambda_{\widetilde{\Omega},k}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$ for all $1<p<\infty$.

Proof. We will employ an observation of Coifman, Rochberg and Weiss (see [4, pp. 620–621]) which shows that certain weighted $L^p(\mathbb{R}^n)$ estimates for linear operators imply the $L^p(\mathbb{R}^n)$ estimates for the corresponding commutators. For each fixed $1<p<\infty$, we claim that there exist two positive constants C_1 and C_2 depending only on n and p such that for real-valued $b \in \mathrm{BMO}(\mathbb{R}^n)$ with $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = C_1$, the operator

$$H(b,f)(x) = \sup_{r>0} r^{-n} \int\limits_{|x-y|<r} e^{b(x)-b(y)} |f(y)|\, dy \tag{4}$$

is bounded on $L^p(\mathbb{R}^n)$ with bound C_2. In fact, by the well-known John–Nirenberg inequality, we know that there exist positive constants A and B such that for any cube Q,

$$\frac{1}{|Q|} \int\limits_Q \exp\left(\frac{|b(x)-b_Q|}{A\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}}\right) dx \le B,$$

where b_Q is the mean value of b on the cube Q. Let $C_1 = (A\max\{p,p'\})^{-1}$. Straightforward computation shows that for real-valued $b \in \mathrm{BMO}(\mathbb{R}^n)$ with $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = C_1$,

$$\frac{1}{|Q|} \int\limits_Q e^{p(b(x)-b_Q)}\, dx \le B, \qquad \frac{1}{|Q|} \int\limits_Q e^{-p'(b(x)-b_Q)}\, dx \le B,$$

and so $e^{pb(x)} \in A_p$ with the A_p constant no more that $C_2 = B^p$ (see also [9, Chap. V]). Therefore, by the weighted $L^p(\mathbb{R}^n)$ estimates with A_p weights for the Hardy–Littlewood maximal operator,

$$\begin{aligned}\|H(b,f)\|_p^p &= \int\limits_{\mathbb{R}^n}\Big(\sup_{r>0} r^{-n}\int\limits_{|x-y|<r} e^{-b(y)}|f(y)|\,dy\Big)^p e^{pb(x)}\,dx\\ &\le C(n,p,C_2)\|f\|_p^p.\end{aligned}$$

Now we can prove Lemma 4. Without loss of generality, we may assume that $\lambda_{\widetilde{\Omega},k} = 1$. It is obvious that

$$\|\widetilde{\Omega}\|_1 \log^k(2 + \|\widetilde{\Omega}\|_\infty) \le 1.$$

Let $\widetilde{\Phi}_k(t) = t\log^k(2+t)$ for $t > 0$. Then

$$\|\widetilde{\Phi}_k(|\widetilde{\Omega}|)\|_1 \le 1.$$

We may also assume that b is real-valued and $\|b\|_{\mathrm{BMO}(\mathbb{R}^n)} = C_1$. By the inequality (3), we have

$$\begin{aligned}M_{\widetilde{\Omega};b,k}f(x) &\le \sup_{r>0} r^{-n}\int\limits_{|x-y|<r}\widetilde{\Phi}_k(|\widetilde{\Omega}(x-y)|)|f(y)|\,dy\\ &\quad + C\sup_{r>0} r^{-n}\int\limits_{|x-y|<r} e^{|b(x)-b(y)|}|f(y)|\,dy\\ &\le \sup_{r>0} r^{-n}\int\limits_{|x-y|<r}\widetilde{\Phi}_k(|\widetilde{\Omega}(x-y)|)|f(y)|\,dy\\ &\quad + C\sup_{r>0} r^{-n}\int\limits_{|x-y|<r} e^{b(x)-b(y)}|f(y)|\,dy\\ &\quad + C\sup_{r>0} r^{-n}\int\limits_{|x-y|<r} e^{b(y)-b(x)}|f(y)|\,dy\\ &= \mathrm{I}(f)(x) + \mathrm{II}(f)(x) + \mathrm{III}(f)(x).\end{aligned}$$

Our claim says that

$$\|\mathrm{II}(f)\|_p \le C\|f\|_p, \qquad \|\mathrm{III}(f)\|_p \le C\|f\|_p.$$

On the other hand, the method of rotation of Calderón and Zygmund [3] states that

$$\|\mathrm{I}(f)\|_p \le C\|\widetilde{\Phi}_k(|\widetilde{\Omega}|)\|_1\|f\|_p \le C\|f\|_p.$$

Therefore,

$$\|M_{\widetilde{\Omega};b,k}f\|_p \le C\|f\|_p.$$

This completes the proof of Lemma 4.

LEMMA 5. *Let k be a positive integer and $b \in \mathrm{BMO}(\mathbb{R}^n)$, $\widetilde{\Omega}$ be homogeneous of degree zero and belong to $L^\infty(S^{n-1})$. For $j \in \mathbb{Z}$, let $\sigma_j(x) =$*

$|x|^{-n}\widetilde{\Omega}(x)\chi_{\{2^j<|x|\le 2^{j+1}\}}(x)$. Denote by U_j the convolution operator whose kernel is σ_j, and $U_{j;b,k}$ the kth order commutator of U_j. Then

$$\Big\|\Big(\sum_{j\in\mathbb{Z}}|U_{j;b,k}f_j|^2\Big)^{1/2}\Big\|_p\le C_{k,p}\lambda_{\widetilde{\Omega},k}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^2\Big)^{1/2}\Big\|_p \tag{5}$$

for any $1<p<\infty$.

Proof. By standard duality and interpolation argument, it suffices to consider the case $2<p<\infty$. Let

$$\widetilde{U}_{j;b,2k}f(x)=\int_{\mathbb{R}^n}|b(x)-b(y)|^{2k}|\sigma_j(x-y)|\,|f(y)|\,dy.$$

Note that

$$|U_{j,b,k}f(x)|^2\le C\|\widetilde{\Omega}\|_1\widetilde{U}_{j;b,2k}(|f|^2)(x).$$

It follows from (iii) of Lemma 3 that for $2<p<\infty$,

$$\begin{aligned}\Big\|\Big(\sum_{j\in\mathbb{Z}}|U_{j;b,k}f_j|^2\Big)^{1/2}\Big\|_p^2&=\sup_{\|h\|_{(p/2)'}\le 1}\Big|\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}|U_{j,b,k}f_j(x)|^2h(x)\,dx\Big|\\&\le C\|\widetilde{\Omega}\|_1\sup_{\|h\|_{(p/2)'}\le 1}\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}\widetilde{U}_{j;b,2k}(|f_j|^2)(x)|h(x)|\,dx\\&\le\|\widetilde{\Omega}\|_1\sup_{\|h\|_{(p/2)'}\le 1}\int_{\mathbb{R}^n}\sum_{j\in\mathbb{Z}}|f_j(x)|^2M_{\widetilde{\Omega};b,2k}h(x)\,dx\\&\le C\|\widetilde{\Omega}\|_1\sup_{\|h\|_{(p/2)'}\le 1}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^2\Big)^{1/2}\Big\|_p^2\|M_{\widetilde{\Omega};b,2k}h\|_{(p/2)'}\\&\le C\|b\|^{2k}_{\mathrm{BMO}(\mathbb{R}^n)}\|\widetilde{\Omega}\|_1\lambda_{\widetilde{\Omega},2k}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^2\Big)^{1/2}\Big\|_p^2.\\&\le C\|b\|^{2k}_{\mathrm{BMO}(\mathbb{R}^n)}\lambda^2_{\widetilde{\Omega},k}\Big\|\Big(\sum_{j\in\mathbb{Z}}|f_j|^2\Big)^{1/2}\Big\|_p^2.\end{aligned}$$

Proof of Theorem 1. Let $\phi\in C_0^\infty(\mathbb{R}^n)$ be a radial function such that $0\le\phi\le 1$, $\operatorname{supp}\phi\subset\{1/4\le|\xi|\le 4\}$ and

$$\sum_{l\in\mathbb{Z}}\phi^3(2^{-l}\xi)=1,\qquad |\xi|\ne 0.$$

Define the multiplier operator S_l by

$$\widehat{S_lf}(\xi)=\phi(2^{-l}\xi)\widehat{f}(\xi).$$

Write

$$K_j(x)=\frac{\Omega(x)}{|x|^n}\,\chi_{\{2^j<|x|\le 2^{j+1}\}}(x).$$

Set

$$m_j(\xi) = \widehat{K}_j(\xi), \quad m_j^l(\xi) = m_j(\xi)\phi(2^{j-l}\xi).$$

Define the operator T_j^l by

$$\widehat{T_j^l f}(\xi) = m_j^l(\xi)\widehat{f}(\xi).$$

Let

$$V_l f(x) = \sum_{j\in\mathbb{Z}}((S_{l-j}T_j^l S_{l-j})_{b,k}f)(x).$$

We know from [7, p. 65] that for $f, h \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} h(x)T_{b,k}f(x)\,dx = \int_{\mathbb{R}^n} h(x)\sum_{l\in\mathbb{Z}} V_l f(x)\,dx.$$

Therefore,

$$\|T_{b,k}f\|_p \le \sum_{l\le 0}\|V_l f\|_p + \sum_{l>0}\|V_l f\|_p.$$

We first consider the term $\sum_{l\le 0}\|V_l f\|_p$. We claim that V_l satisfies the crude estimate

$$\|V_l f\|_p \le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p, \quad l\in\mathbb{Z},\ 2<p<\infty. \tag{6}$$

In fact, let $E_0 = \{x' \in S^{n-1} : |\Omega(x')| \le 2\}$ and $E_d = \{x' \in S^{n-1} : 2^d < |\Omega(x')| \le 2^{d+1}\}$ for positive integer d. Denote by Ω_d the restriction of Ω to E_d, that is, $\Omega_d(x') = \Omega(x')\chi_{E_d}(x')$. Our hypothesis on Ω now shows that $\sum_{d\ge 1} d^{k+1}\|\Omega_d\|_1 < \infty$. Let

$$K_{j,d}(x) = \frac{\Omega_d(x)}{|x|^n}\,\chi_{\{2^j<|x|\le 2^{j+1}\}}(x)$$

and

$$m_{j,d}(\xi) = \widehat{K}_{j,d}(\xi), \quad m_{j,d}^l(\xi) = m_{j,d}(\xi)\phi(2^{j-l}\xi).$$

Define the operator $T_{j,d}^l$ by

$$\widehat{T_{j,d}^l f}(\xi) = m_{j,d}^l(\xi)\widehat{f}(\xi),$$

and the operator $V_{l,d}$ by

$$V_{l,d}f(x) = \sum_{j\in\mathbb{Z}}((S_{l-j}T_{j,d}^l S_{l-j})_{b,k}f)(x).$$

With the aid of the formula

$$(b(x)-b(y))^k = \sum_{m=0}^{k} C_k^m (b(x)-b(z))^m (b(z)-b(y))^{k-m}, \quad x,y,z\in\mathbb{R}^n,$$

straightforward computation shows that for $f, h \in C_0^\infty(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} h(x) V_{l,d} f(x)\, dx = \sum_{m=0}^{k} C_k^m \int_{\mathbb{R}^n} h(x) \sum_{j\in\mathbb{Z}} S_{l-j;b,k-m}((T_{j,d}^l S_{l-j})_{b,m} f)(x)\, dx.$$

Lemma 1 now tells us that

$$\|V_{l,d} f\|_p \le C \sum_{m=0}^{k} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{k-m} \Big\| \Big(\sum_{j\in\mathbb{Z}} |(T_{j,d}^l S_{l-j})_{b,m} f|^2 \Big)^{1/2} \Big\|_p, \quad 1<p<\infty.$$

Set

$$T_{j,d} h(x) = K_{j,d} * h(x).$$

For each m with $0 \le m \le k$, write

$$(T_{j,d}^l S_{l-j})_{b,m} f(x) = \sum_{i=0}^{m} C_m^i T_{j,d;b,i}(S_{l-j;b,m-i}^2 f)(x).$$

By Lemmas 1 and 5, we have

$$\begin{aligned} \Big\| \Big(\sum_{j\in\mathbb{Z}} |(T_{j,d}^l S_{l-j} f)_{b,m}|^2 \Big)^{1/2} \Big\|_p & \\ &\le C \sum_{i=0}^{m} \lambda_{\Omega_d,i} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{i} \Big\| \Big(\sum_{j\in\mathbb{Z}} |S_{l-j;b,m-i}^2 f|^2 \Big)^{1/2} \Big\|_p \\ &\le C \lambda_{\Omega_d,m} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{m} \|f\|_p, \quad 1<p<\infty. \end{aligned}$$

Consequently,

$$\|V_{l,d} f\|_p \le C \lambda_{\Omega_d,k} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{k} \|f\|_p, \quad 1<p<\infty. \tag{7}$$

This together with (ii) of Lemma 3 shows that

$$\|V_l f\|_p \le \sum_{d=0}^{\infty} \|V_{l,d} f\|_p \le C \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{k} \|f\|_p, \quad 1<p<\infty,$$

and establishes our claim (6). Now our goal is to obtain a refined $L^2(\mathbb{R}^n)$ estimate for V_l, i.e., we want to show that there exists a positive constant $\nu = \nu_n > 0$ such that

$$\|V_l f\|_2 \le C 2^{\nu l} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{k} \|f\|_2, \quad l \le 0. \tag{8}$$

If we can do this, interpolating the inequalities (6) and (8) yields

$$\|V_l f\|_p \le C 2^{\widetilde{\nu} l} \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{k} \|f\|_p, \quad l \le 0,\ 1<p<\infty, \tag{9}$$

where $\widetilde{\nu} = \widetilde{\nu}_{n,p} > 0$. So,

$$\sum_{l\le 0} \|V_l f\|_p \le C \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{k} \|f\|_p.$$

To prove (9), let $\widetilde{T}_j^l$ be the operator defined by

$$\widehat{\widetilde{T}_j^l f}(\xi) = m_j^l(2^{-j}\xi)\widehat{f}(\xi).$$

By the vanishing moment and integrability of Ω, we have

$$|\widehat{K}_j(\xi)| \le C|2^j\xi|, \quad \|\nabla \widehat{K}_j\|_\infty \le C2^j.$$

Thus,

$$\|m_j^l(2^{-j}\cdot)\|_\infty \le C2^l, \quad \|\nabla m_j^l(2^{-j}\cdot)\|_\infty \le C.$$

This via Lemma 2 says that for any fixed $l \le 0$, $0 < \nu < 1$ and positive integer i,

$$\|\widetilde{T}_{j;b,i}^l f\|_2 \le C2^{\nu l}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^i \|f\|_2,$$

which by dilation-invariance implies

$$\|T_{j;b,i}^l f\|_2 \le C2^{\nu l}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^i \|f\|_2. \tag{10}$$

On the other hand, the Plancherel theorem tells us that

$$\|T_j^l f\|_2 \le C2^l\|f\|_2. \tag{11}$$

Write

$$(T_j^l S_{l-j} f)_{b,m} f(x) = \sum_{i=0}^m C_m^i T_{j;b,i}^l (S_{l-j;b,m-i} f)(x).$$

It follows from (10), (11) and Lemma 1 that

$$\begin{aligned}\Big\|\Big(\sum_{j\in\mathbb{Z}} |(T_j^l S_{l-j} f)_{b,m}|^2\Big)^{1/2}\Big\|_2^2 &\le C2^{2\nu l}\sum_{i=0}^m \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{2i} \sum_{j\in\mathbb{Z}} \|S_{l-j;b,m-i} f\|_2^2 \\ &\le C2^{2\nu l}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{2m}\|f\|_2^2, \quad l \le 0.\end{aligned}$$

Therefore, by a familiar argument involving Lemma 1, we can obtain

$$\begin{aligned}\|V_l f\|_2 &\le C\sum_{m=0}^k \|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^{k-m} \Big\|\Big(\sum_{j\in\mathbb{Z}} |(T_j^l S_{l-j})_{b,m} f|^2\Big)^{1/2}\Big\|_2 \\ &\le C2^{\nu l}\|b\|_{\mathrm{BMO}(\mathbb{R}^n)}^k \|f\|_2, \quad l \le 0.\end{aligned}$$

Now we turn our attention to the term $\sum_{l>0}\|V_l f\|_p$. By the well-known estimate of Duoandikoetxea and Rubio de Francia [6], we know that there exists a positive constant β such that

$$|\widehat{K}_{j,d}(\xi)| \le C\|\Omega_d\|_\infty \min\{1, |2^j\xi|^{-\beta}\}, \quad \|\nabla \widehat{K}_{j,d}\|_\infty \le C2^j\|\Omega_d\|_1.$$

This gives

$$\|m_{j,d}^l\|_\infty \le C2^{-\beta l}\|\Omega_d\|_\infty, \quad \|\nabla m_{j,d}^l\|_\infty \le 2^j\|\Omega_d\|_\infty.$$

Invoking Lemma 2 again, as in the proof of (10) and (11), we see that there exists some constant $0 < \gamma < 1$ such that for non-negative integer m,

$$\|T^l_{j,d;b,m}\|_2 \le C\|\Omega_d\|_\infty 2^{-\gamma l}\|b\|^m_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2.$$

Similarly to (8), we can obtain

$$\|V_{l,d}f\|_2 \le C\sum_{m=0}^{k}\|b\|^{k-m}_{\mathrm{BMO}(\mathbb{R}^n)}\Big\|\Big(\sum_{j\in\mathbb{Z}}|(T^l_{j,d}S_{l-j}f)_{b,m}|^2\Big)^{1/2}\Big\|_2 \tag{12}$$

$$\le C\|\Omega_d\|_\infty 2^{-\gamma l}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_2.$$

Interpolating (7) and (12) shows that for $\widetilde{\gamma} = \widetilde{\gamma}_{n,p} > 0$,

$$\|V_{l,d}f\|_p \le C\|\Omega_d\|_\infty 2^{-\widetilde{\gamma} l}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p, \quad 1 < p < \infty. \tag{13}$$

Let N be a large positive integer such that $N > 2\widetilde{\gamma}^{-1}$. Combining (7) and (13) gives

$$\begin{aligned}\sum_{l>0}\|V_lf\|_p &\le \sum_{l>0}\|V_{l,0}f\|_p + \sum_{d>0}\sum_{0<l\le Nd}\|V_{l,d}f\|_p + \sum_{d>0}\sum_{l>Nd}\|V_{l,d}f\|_p \\ &\le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{l>0}2^{-\mu l}\|f\|_p + C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{d>0}d\lambda_{\Omega_d,k}\|f\|_p \\ &\quad + C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{d>0}2^d\sum_{l>Nd}2^{-\mu l}\|f\|_p \\ &\le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p + C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{d>0}d\lambda_{\Omega_d,k}\|f\|_p \\ &\le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p.\end{aligned}$$

This completes the proof of Theorem 1.

Proof of Theorem 2. We shall carry out the argument by induction on the order k. If $k = 0$, Theorem 2 is the remarkable result of Calderón and Zygmund [3]. Now let k be a positive integer, and assume that the assertion is true for all integers m with $0 \le m \le k-1$. Let K_j, $K_{j,d}$, Ω_d and the operator $T_{j,d}$ be the same as in the proof of Theorem 1. Define

$$T_{j;b,m}f(x) = \int\limits_{2^j<|x-y|\le 2^{j+1}}(b(x)-b(y))^m\,\frac{\Omega(x-y)}{|x-y|^n}\,f(y)\,dy.$$

Write

$$\begin{aligned}M_{\Omega;b,k}f(x) &= \sup_{r>0} r^{-n}\int\limits_{|x-y|<r}|b(x)-b(y)|^k|\Omega(x-y)|\,|f(y)|\,dy \\ &\le \sum_{d=0}^{\infty}M_{\Omega_d;b,k}f(x).\end{aligned}$$

Lemma 4 now tells us that for all $1 < p < \infty$,

$$\|M_{\Omega;b,k}f\|_p \le \|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)} \sum_{d=0}^{\infty} \lambda_{\Omega_d,k}\|f\|_p \le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p.$$

Thus, it suffices to consider the $L^p(\mathbb{R}^n)$ norm of $\sup_{l\in\mathbb{Z}} |\sum_{j=l}^{\infty} T_{j;b,k}f(x)|$. Take $\eta \in \mathcal{S}(\mathbb{R}^n)$ such that $\eta(x) \equiv 1$ when $|x| \le 1$. Let $\Phi_l \in \mathcal{S}(\mathbb{R}^n)$ be such that $\widehat{\Phi}_l(\xi) = \eta(2^l\xi)$. Denote by G_l the convolution operator whose kernel is Φ_l and G_l^j the convolution operator whose kernel is $K_j - \Phi_l * K_j$. Write

$$\begin{aligned}\sum_{j=l}^{\infty} T_{j;b,k}f(x) &= \Phi_l * \Big(T_{b,k}f - \sum_{j=-\infty}^{l-1} T_{j;b,k}f\Big)(x)\\ &\quad + \Big(\sum_{j=l}^{\infty} T_{j;b,k}f(x) - \Phi_l * \Big(\sum_{j=l}^{\infty} T_{j;b,k}f\Big)(x)\Big)\\ &= \mathrm{I}_l(f)(x) + \mathrm{II}_l(f)(x).\end{aligned}$$

Define the operator

$$M_{b,k}h(x) = \sup_{r>0} r^{-n} \int_{|x-y|<r} |b(x) - b(y)|^k |h(y)|\, dy.$$

Observe that

$$\Big|\Phi_l * \sum_{j=\infty}^{l-1} K_l(x)\Big| \le C2^{-nl}/(1 + |2^{-l}x|)^{n+1}$$

(see [6]) and

$$\begin{aligned}&\Phi_l * \Big(\sum_{j=-\infty}^{l-1} T_{j;b,k}f\Big)(x)\\ &\quad = \Big(\Phi_l * \sum_{j=-\infty}^{l-1} K_j\Big)_{b,k} f(x) - \sum_{m=0}^{k-1} C_k^m G_{l;b,k-m}\Big(\sum_{j=-\infty}^{l-1} T_{j;b,m}f\Big)(x).\end{aligned}$$

It follows that

$$\begin{aligned}\sup_{l\in\mathbb{Z}} |\mathrm{I}_l(f)(x)| \le &\sum_{m=0}^{k-1} (M_{b,k-m}(T_{b,m}f)(x) + M_{b,k-m}(T^*_{b,m}f)(x))\\ &+ CM_{b,k}f(x) + CM(T_{b,k}f)(x).\end{aligned}$$

This shows that $\sup_{l\in\mathbb{Z}} |\mathrm{I}_l(f)(x)|$ is pointwise bounded by a function whose $L^p(\mathbb{R}^n)$ norm is no more than $C_{n,p}\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}$ for all $1 < p < \infty$. To estimate

$\sup_{l\in\mathbb{Z}} |\mathrm{II}_l(f)(x)|$, write

$$\begin{aligned}\mathrm{II}_l f(x) &= \sum_{j=l}^{\infty} T_{j;b,k} f(x) - \Big(\varPhi_l * \sum_{j=l}^{\infty} T_j\Big)_{b,k} f(x)\\ &\quad - \sum_{m=0}^{k-1} C_k^m G_{l;b,k-m}\Big(\sum_{j=l}^{\infty} T_{j;b,m} f\Big)(x)\\ &= \sum_{j=l}^{\infty} G^j_{l;b,k} f(x) - \sum_{m=0}^{k-1} C_k^m G_{l;b,k-m}\Big(\sum_{j=l}^{\infty} T_{j;b,m} f\Big)(x)\end{aligned}$$

For each $0 \le m \le k-1$, it is easy to see that

$$\sup_{l\in\mathbb{Z}} \Big| G_{l;b,k-m}\Big(\sum_{j=l}^{\infty} T_{j;b,m} f\Big)(x)\Big| \le C M_{b,k-m}(T^*_{b,m} f)(x).$$

Thus, the proof of Theorem 2 can be reduced to estimating the $L^p(\mathbb{R}^n)$ norm for the term $\sup_{l\in\mathbb{Z}} |\sum_{j=l}^{\infty} G^j_{l;b,k} f(x)|$. Denote by $G_l^{j,d}$ the convolution operator whose kernel is $K_{j,d} - \varPhi_l * K_{j,d}$. Let N_1 be a positive integer which will be chosen later. Write

$$\begin{aligned}\sup_{l\in\mathbb{Z}} \Big|\sum_{j=l}^{\infty} G^j_{l;b,k} f(x)\Big| &\le \sum_{j=0}^{\infty} \sup_{l\in\mathbb{Z}} |G^l_{l-j;b,k} f(x)|\\ &\le \sum_{d>0} \sum_{0<j\le N_1 d} \sup_{l\in\mathbb{Z}} |G^{l,d}_{l-j;b,k} f(x)| + \sum_{j=0}^{\infty} \sup_{l\in\mathbb{Z}} |G^{l,0}_{l-j;b,k} f(x)|\\ &\quad + \sum_{d>0} \sum_{j>N_1 d} \sup_{l\in\mathbb{Z}} |G^{l,d}_{l-j;b,k} f(x)|.\end{aligned}$$

Employing Lemma 4, we have

$$\begin{aligned}&\sum_{d>0} \sum_{0<j\le N_1 d} \| \sup_{l\in\mathbb{Z}} |G^{l,d}_{l-j;b,k} f| \|_p\\ &\qquad \le C \sum_{d>0} \sum_{0<j\le N_1 d} \|M_{\Omega_d;b,k} f\|_p\\ &\qquad\quad + \sum_{m=0}^{k} \sum_{d>0} \sum_{0<j\le N_1 d} \|M_{b,m}(M_{\Omega_d;b,k-m} f)\|_p\\ &\qquad \le C \sum_{d>0} d\lambda_{\Omega_d,k} \|f\|_p \le C\|f\|_p.\end{aligned}$$

Now trivial computation gives

$$|\widehat{K}_{l,d}(\xi) - \widehat{\varPhi_{l-j} * K_{l,d}}(\xi)| \le C\|\Omega_d\|_\infty \min\{2^{-j}|2^l\xi|, |2^l\xi|^{-\mu}\},$$

with $\mu = \mu_n > 0$. This via the Plancherel theorem shows that for some $\widetilde{\mu} > 0$,

$$\|\sup_{l\in\mathbb{Z}} |G^{l,d}_{l-j}h|\,\|_2 \le \Big\|\Big(\sum_{l\in\mathbb{Z}} |G^{l,d}_{l-j}h|^2\Big)^{1/2}\Big\|_2 \le C2^{-\widetilde{\mu}j}\|\Omega_d\|_\infty\|h\|_2. \tag{14}$$

On the other hand, it is easy to see that for each fixed $1 < p < \infty$ and $w \in A_p$,

$$\|\sup_{l\in\mathbb{Z}} |G^{l,d}_{l-j}h|\,\|_{p,w} \le C\|\Omega_d\|_\infty\|h\|_{p,w}, \tag{15}$$

and the constant C depending only on n, p and the A_p constant of w. Interpolating the inequalities (14) and (15) with change of measures implies that for each $1 < p < \infty$ and $w \in A_p$,

$$\|\sup_{l\in\mathbb{Z}} |G^{l,d}_{l-j}h|\,\|_{p,w} \le C2^{-\delta j}\|\Omega_d\|_\infty\|h\|_{p,w}. \tag{15}$$

Since the mapping $f \mapsto \{G^{l,d}_{l-j}f\}_{l\in\mathbb{Z}}$ is linear, applying Theorem 2.13 of [2], we can obtain

$$\|\sup_{l\in\mathbb{Z}} |G^{l,d}_{l-j,b,k}h|\,\|_p \le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}2^{-\delta j}\|\Omega_d\|_\infty\|h\|_p.$$

Let $N_1 > 2\delta^{-1}$. We conclude the proof of Theorem 2 by noting that

$$\sum_{j=0}^{\infty}\|\sup_{l\in\mathbb{Z}} |G^{l,0}_{l-j;b,k}f|\,\|_p \le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{j=0}^{\infty}2^{-\delta j}\|f\|_p \le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p$$

and

$$\begin{aligned}\sum_{d>0}\sum_{j>N_1 d}\sup_{l\in\mathbb{Z}}\|\,|G^{l,d}_{l-j;b,k}f|\,\|_p &\le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\sum_{d>0}2^d\sum_{j>N_1 d}^{\infty}2^{-\delta j}\|f\|_p\\ &\le C\|b\|^k_{\mathrm{BMO}(\mathbb{R}^n)}\|f\|_p.\end{aligned}$$

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