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Similarity-preserving linear maps on $B(X)$

by

FANGYAN LU and CHAORAN PENG (Suzhou)

Abstract. Let X be an infinite-dimensional Banach space, and $B(X)$ the algebra of all bounded linear operators on X. Then $\phi : B(X) \to B(X)$ is a bijective similarity-preserving linear map if and only if one of the following holds:

(1) There exist a nonzero complex number c, an invertible bounded operator T in $B(X)$ and a similarity-invariant linear functional h on $B(X)$ with $h(I) \neq -c$ such that $\phi(A) = cTAT^{-1} + h(A)I$ for all $A \in B(X)$.

(2) There exist a nonzero complex number c, an invertible bounded linear operator $T : X^* \to X$ and a similarity-invariant linear functional h on $B(X)$ with $h(I) \neq -c$ such that $\phi(A) = cTA^*T^{-1} + h(A)I$ for all $A \in B(X)$.

1. Introduction. Throughout, X is a complex Banach space with topological dual X^*. We denote by $B(X)$ the set of all bounded linear operators on X. Obviously, $B(X)$ is a Banach algebra with unit I, the identity operator on X. We say that two operators A and B in $B(X)$ are *similar*, denoted by $A \sim B$, if there exists an invertible operator T in $B(X)$ such that $A = TBT^{-1}$. A map $\phi : B(X) \to B(X)$ is said to be *similarity-preserving* if $A \sim B$ implies that $\phi(A) \sim \phi(B)$.

The problem of characterizing similarity-preserving linear maps on the algebras of operators on Hilbert spaces has been studied by many authors and a lot of interesting results were obtained [5–8, 11, 12, 14]. Hiai [5] and Lim [11] proved that if X is a finite-dimensional Hilbert space and $\phi : B(X) \to B(X)$ is a similarity-preserving linear map then ϕ must be of the form either $A \mapsto cTAT^{-1} + d(\operatorname{tr} A)I$ or $A \mapsto cTA^tT^{-1} + d(\operatorname{tr} A)I$ for some complex numbers c, d and some invertible matrix T. Here, $\operatorname{tr} A$ denotes the trace of A. For the infinite-dimensional case, Šemrl [14] recently proved that if X is an infinite-dimensional separable Hilbert space and $\phi : B(X) \to B(X)$ is a similarity-preserving linear map then ϕ must be

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of the form either $A \mapsto cTAT^{-1}$ or $A \mapsto cTA^tT^{-1}$ for some complex number c and some invertible operator T.

One may ask whether results of this nature hold more generally in the Banach space setting. This question seems more difficult. On one hand, Šemrl's proof depends heavily on the surprising result of Davidson and Marcoux [2] that if H is a separable Hilbert space and $B \in B(H)$ is not of the form scalar plus compact, then every $A \in B(H)$ can be written as a linear combination of at most six operators similar to B. On the other hand, if $\mathfrak{J}$ is the James space, then there are nonzero multiplicative linear functionals h on $B(\mathfrak{J})$ (cf. [9]). Then the surjective linear map $\phi : B(\mathfrak{J}) \to B(\mathfrak{J})$ defined by $A \mapsto A + h(A)I$ is similarity-preserving.

The aim of this paper is to characterize similarity-preserving linear maps on the algebras of operators on Banach spaces. We will prove that if X is an infinite-dimensional Banach space and $\phi : B(X) \to B(X)$ is a bijective similarity-preserving linear map then $\phi(A) = c\psi(A) + h(A)I$, where c is a nonzero scalar, ψ is an algebraic isomorphism or anti-isomorphism of $B(X)$, and h is a similarity-invariant linear functional on $B(X)$. Here, a *similarity-invariant* functional h means that $h(A) = h(B)$ whenever $A \sim B$. Obviously, multiplicative linear functionals are similarity-invariant. So, there are nonzero similarity-invariant linear functionals on $B(\mathfrak{J})$. However, by a result of Halmos [4], there are no nonzero similarity-invariant linear functionals on $B(H)$, where H is an infinite-dimensional Hilbert space. Thus, we get Šemrl's result in the nonseparable Hilbert space case.

2. Preliminary results. For $x \in X$ and $f \in X^*$, we define an operator $x \otimes f$ of rank one by $y \mapsto f(y)x$ for $y \in X$. It is known that every finite rank operator can be written as a sum of finitely many operators of rank one. We denote by $F(X)$ the subspace of all finite rank operators on X, and by $F_0(X)$ the subspace of all trace zero finite rank operators. Note that every trace zero finite rank operator can be written as a linear combination of finitely many nilpotent operators of rank one.

LEMMA 2.1 ([10, Lemma 2.4]). *Let A and B be in $B(X)$. Assume that for every $x \in X$ the vector Ax belongs to the linear span of x and Bx. Then A, B and I are linearly dependent.*

LEMMA 2.2 ([1, Lemma 2.1]). *Let $A \in B(X)$, not of the form scalar plus finite rank. Then for every positive integer n there exist $x_1, \dots, x_n \in X$ such that $x_1, \dots, x_n, Ax_1, \dots, Ax_n$ are linearly independent.*

LEMMA 2.3. *Suppose that X is an infinite-dimensional Banach space. Let $A \in B(X) \setminus \mathbb{C}I$. Then there exist $B_1, B_2 \in F_0(X)$ such that:*

(i) *B_1 and B_2 are linearly independent, and*
(ii) *$A + B_i \sim A$, $i = 1, 2$.*

Proof. We distinguish two cases.

CASE 1: A is not of the form scalar plus finite rank. Then by Lemma 2.2, there are x_1 and x_2 such that x_1, x_2, Ax_1, Ax_2 are linearly independent. Let f_1, f_2 be nonzero functionals in X^* such that $f_1(x_1) = f_2(x_2) = 0$. Then

$$(I-x_i\otimes f_i)A(I+x_i\otimes f_i) = A+Ax_i\otimes f_i-x_i\otimes A^*f_i-f_i(Ax_i)x_i\otimes f_i, \quad i=1,2.$$

Now the operators $Ax_1 \otimes f_1 - x_1 \otimes A^*f_1 - f_1(Ax_1)x_1 \otimes f_1$ and $Ax_2 \otimes f_2 - x_2 \otimes A^*f_2 - f_2(Ax_2)x_2 \otimes f_2$ are as required.

CASE 2: $A = \lambda I + F$, where $\lambda \in \mathbb{C}$ and F is a nonzero finite rank operator. Take $x_1, x_2 \in \ker(F)$ linearly independent and not in the range of F. For $i = 1, 2$, choose $f_i \in X^*$ such that $f_i(x_i) = 0$ but $F^*f_i \neq 0$. Now

$$(I + x_i \otimes f_i)A(I - x_i \otimes f_i) = A + x_i \otimes F^*f_i, \quad i = 1, 2.$$

Obviously, $x_1 \otimes F^*f_1$ and $x_2 \otimes F^*f_2$ are as required. ∎

LEMMA 2.4. *Let $\phi : B(X) \to B(X)$ be a bijective similarity-preserving linear map. Assume that there exists $A \in B(X)$ such that $A \notin \mathbb{C}I + F(X)$ and $\phi(A) = \lambda I + F$ for some $\lambda \in \mathbb{C}$ and some finite rank operator F. Denote $r = \operatorname{rank} F$. Then for every finite rank square-zero operator $B \in B(X)$ we have $\operatorname{rank} \phi(B) \leq 3r$.*

Proof. Let k be any positive integer. By Lemma 2.2 there exist $x_1, \ldots, x_k \in X$ such that $x_1, \ldots, x_k, Ax_1, \ldots, Ax_k$ are linearly independent. Let P be an idempotent operator with range $\operatorname{span}\{x_1, \ldots, x_k\}$. Let

$$C = (I + P)A(I - \tfrac{1}{2}P) \quad \text{and} \quad D = (I + 2P)A(I - \tfrac{2}{3}P).$$

Obviously, both C and D are similar to A. Hence both $\phi(C)$ and $\phi(D)$ are similar to $\phi(A)$, that is, $\phi(C) = T\phi(A)T^{-1}$ and $\phi(D) = S\phi(A)S^{-1}$ for some invertible $T, S \in B(X)$. Since $2C - D - A = \frac{1}{3}(P - I)AP$, the operator $2C - D - A$ is square-zero with range $\operatorname{span}\{PAx_1 - Ax_1, \ldots, PAx_k - Ax_k\}$, which is clearly of dimension k. Now since

$$\begin{aligned}\phi(2C - D - A) &= 2TFT^{-1} + 2\lambda I - SFS^{-1} - \lambda I - \lambda I - F \\ &= 2TFT^{-1} - SFS^{-1} - F,\end{aligned}$$

we see that $\phi(2C - D - A)$ is of rank at most $3r$. Since all square-zero operators of rank k are similar and since k was an arbitrary positive integer, we are done. ∎

As is known, for any rank one operator $x \otimes f$, $I - x \otimes f$ is invertible if and only if $f(x) \neq 1$. Now, we have a similar result for an operator of rank at most two, which is crucial to proving our main theorem.

PROPOSITION 2.5. *Let $x, y \in X$ and $f, g \in X^*$. Then $I - (x \otimes f + y \otimes g)$ is invertible if and only if $(f(x) - 1)(g(y) - 1) \neq f(y)g(x)$.*

Proof. The statement is trivially true if x and y are linearly dependent. If they are linearly independent, choose a closed subspace $W \subset X$ such that $X = \operatorname{span}\{x, y\} \oplus W$. The corresponding matrix representation of the operator $I - (x \otimes f + y \otimes g)$ is

$$\begin{bmatrix} \begin{bmatrix} 1 - f(x) & -f(y) \\ -g(x) & 1 - g(x) \end{bmatrix} & * \\ 0 & I \end{bmatrix}$$

and it is clear that this operator is invertible if and only if the determinant of the upper left corner is nonzero, that is, $(f(x)-1)(g(y)-1) \neq f(y)g(x)$. ∎

Šemrl [14] proved the following proposition for the Hilbert space case, and his proof is also valid for the Banach space case.

PROPOSITION 2.6. *Let X be an infinite-dimensional Banach space. Let $\phi : B(X) \to B(X)$ be an injective linear map such that $\phi(N)$ is nilpotent of rank one for every rank one nilpotent operator $N \in B(X)$. Then one of the following holds:*

(i) *There exist a nonzero $x \in X$ and an injective linear map τ from $F_0(X)$ to $\{f \in X^* : f(x) = 0\}$ such that $\phi(A) = x \otimes \tau(A)$ for every $A \in F_0(X)$.*
(ii) *There exist a nonzero $f \in X^*$ and an injective linear map δ from $F_0(X)$ to $\{x \in X : f(x) = 0\}$ such that $\phi(A) = \delta(A) \otimes f$ for every $A \in F_0(X)$.*
(iii) *There exist injective linear maps $T : X \to X$ and $S : X^* \to X^*$ such that $\phi(x \otimes f) = Tx \otimes Sf$ for every rank one nilpotent operator $x \otimes f$.*
(iv) *There exist injective linear maps $T : X^* \to X$ and $S : X \to X^*$ such that $\phi(x \otimes f) = Tf \otimes Sx$ for every rank one nilpotent operator $x \otimes f$.*

REMARK 2.7. Consider case (iii) above. If we assume in addition that T and S are bijective, then they are continuous and there exists a scalar c such that $Sf(Tx) = cf(x)$ for all $x \in X$ and $f \in X^*$, which further yields $\phi(A) = cTAT^{-1}$ for every $A \in F_0(X)$. Indeed, fix a nonzero $f \in X^*$, and define $g_f : x \mapsto Sf(Tx)$ for all $x \in X$. From the linearity of S, T, f and the bijectivity of S, T, we see that g_f is linear and nonzero. Noting that $\phi(x \otimes f) = Tx \otimes Sf$ is nilpotent for every nilpotent $x \otimes f$ of rank one, we have $g_f(x) = (Sf)(Tx) = 0$ for every $x \in \ker f$, so $\ker f \subseteq \ker g_f$. Thus, $g_f = c(f)f$, i.e. $Sf(Tx) = c(f)f(x)$ for some nonzero scalar $c(f)$.

To show that $c(f)$ is independent of f, first suppose that f_1 and f_2 in X^* are linearly independent. Then for $x \in X$, we have both

$$S(f_1 + f_2)(Tx) = c(f_1 + f_2)(f_1(x) + f_2(x))$$

and

$$S(f_1 + f_2)(Tx) = c(f_1)f_1(x) + c(f_2)f_2(x).$$

Comparing those two equations, by the linear independence of f_1 and f_2, we get $c(f_1) = c(f_2)$. Now for general f_1 and f_2 in X^*, choose f_3 in X^* such that f_1 and f_3 as well as f_2 and f_3 are linearly independent. Then $c(f_1) = c(f_3) = c(f_2)$.

Thus we have got a nonzero scalar c such that

$$Sf(Tx) = cf(x) \tag{2.1}$$

for all $x \in X$ and $f \in X^*$. If a sequence $\{x_n\}$ in X tends to zero and $\{Tx_n\}$ tends to y, then by (2.1) we have $y = 0$, and hence T is bounded by the Closed Graph Theorem. Similarly, S is also bounded. By the bijectivity of T, we can write (2.1) as $Sf(x) = cf(T^{-1}x)$. From this, $\phi(x\otimes f) = cT(x\otimes f)T^{-1}$ for every nilpotent $x \otimes f$ of rank one, and by linearity $\phi(A) = cTAT^{-1}$ for every $A \in F_0(X)$.

In case (iv) above, if we assume in addition that T and S are bijective, then, in a similar way, we get $(Sx)(Tf) = cf(x)$ for all $x \in X$ and $f \in X^*$, where c is a nonzero complex number. This equation moreover shows that S and T are bounded. Further by the bijectivity of T, $(Sx)y = c(T^{-1}y)(x)$ for all $x, y \in X$. Thus,

$$\begin{aligned}\phi(x \otimes f)y = (Tf \otimes Sx)y = c(Tf \otimes T^{-1}y)x = cT(f \otimes x^{**})T^{-1}y \\ = cT(x \otimes f)^*T^{-1}y\end{aligned}$$

for all $y \in X$, $x \in X$ and $f \in X^*$ with $f(x) = 0$, so $\phi(x\otimes f) = cT(x\otimes f)^*T^{-1}$ for all $x \in X$ and $f \in X^*$, and by linearity $\phi(A) = cTA^*T^{-1}$ for every $A \in F_0(X)$.

3. The main result and proof. Our main result reads as follows.

THEOREM 3.1. *Let X be an infinite-dimensional Banach space. Then $\phi : B(X) \to B(X)$ ϕ is a similarity-preserving bijective linear map if and only if one of the following holds:*

(1) *There exist a nonzero complex number c, an invertible bounded operator T in $B(X)$ and a similarity-invariant linear functional h on $B(X)$ with $h(I) \neq -c$ such that $\phi(A) = cTAT^{-1} + h(A)I$ for all $A \in B(X)$.*
(2) *There exist a nonzero complex number c, an invertible bounded linear operator $T : X^* \to X$ and a similarity-invariant linear functional h on $B(X)$ with $h(I) \neq -c$ such that $\phi(A) = cTA^*T^{-1} + h(A)I$ for all $A \in B(X)$.*

Proof. Sufficiency. It is easy to see that ϕ is a similarity-preserving linear map. We show the bijectivity of ϕ under (1) only. If $\phi(A) = 0$ for some $A \in B(X)$, i.e., $cTAT^{-1} + h(A)I = 0$, then $cA + h(A)I = 0$ and hence $(c + h(I))h(A) = 0$. Since $c + h(I) \neq 0$, it follows that $h(A) = 0$. Hence $A = 0$. This proves the injectivity of ϕ. The surjectivity can be seen from

$$\phi\Big(\frac{1}{c}T^{-1}AT - \frac{h(A)}{(c+h(I))c}I\Big) = A - \frac{h(A)}{c+h(I)}I + \frac{h(A)}{c}I - \frac{h(A)h(I)}{(c+h(I))c}I = A$$

for $A \in B(X)$.

Necessity. Let us first prove that $\phi(I) = \mu I$ for some nonzero $\mu \in \mathbb{C}$. By surjectivity there exists $A \in B(X)$ such that $\phi(A) = I$. We have to show that A is a scalar operator. If not, then it is easy to find $B \neq A$ such that $A \sim B$. Then $\phi(A) = I \sim \phi(B)$, and so $\phi(B) = I$, contradicting the injectivity of ϕ.

Next we will show that ϕ maps nilpotents of rank one into nilpotents of rank one. Choose a rank one operator $B \in B(X)$ and let $A \in B(X)$ with $\phi(A) = B$. Clearly, A is not a scalar operator. Thus, we can find a vector x such that x and Ax are linearly independent. Consequently, there exists $f \in X^*$ such that $f(x) = 0$ and $f(Ax) = 1$. Set $N = x \otimes f$. Then $N^2 = 0$ and $NAN = N$. For every $\lambda \in \mathbb{C}$ we have $\phi((I+\lambda N)A(I-\lambda N)) \sim B$. Thus

$$\phi((I + \lambda N)A(I - \lambda N)) = -\lambda^2\phi(N) + \lambda\phi(AN - NA) + B$$

is of rank one for every complex λ. Dividing by λ^2, sending λ to infinity, and applying the fact that the set of all operators of rank at most one is closed, we arrive at $\operatorname{rank}\phi(N) = 1$. Moreover, $N \sim 2N$, and therefore $\phi(N) \sim 2\phi(N)$. As $\phi(N)$ is of rank one, it has to be nilpotent. Since all nilpotents of rank one are similar, we conclude that ϕ maps nilpotents of rank one into nilpotents of rank one. It follows that

$$\phi(F_0(X)) \subseteq F_0(X).$$

Now we prove that if $A \in B(X)$ is not of the form scalar plus finite rank, then $\phi(A)$ is not of that form either. Assume on the contrary that there exists $A \in B(X)$ such that $A \notin \mathbb{C}I + F(X)$ and $\phi(A) = \lambda I + F$ for some $\lambda \in \mathbb{C}$ and some finite rank operator F. Then, by Lemma 2.4, there exists an integer M such that for every finite rank square-zero operator $B \in B(X)$ we have $\operatorname{rank}\phi(B) \leq M$. Since ϕ maps the set of nilpotents of rank one into itself, we can apply Proposition 2.6. Assume that case (iii) there holds. It is known that every square-zero operator B of rank m, where m is any positive integer larger than M, can be written as $B = \sum_{k=1}^m x_k \otimes f_k$ where $x_1, \dots, x_m$ are linearly independent, $f_1, \dots, f_m$ are linearly independent and $f_k(x_k) = 0, k = 1, \dots, m$. But then, since T and S are injective, the operator $\phi(B) = \sum_{k=1}^m (Tx_k) \otimes (Sf_k)$ is of rank $m > M$, a contradiction. In a similar way we prove that (iv) cannot occur. So, we have either (i) or (ii). We will just consider (i) since the proof in the other case goes through in almost the same way. Thus, there are a nonzero $x \in X$ and a linear map $\tau : F_0(X) \to \{f \in X^* : f(x) = 0\}$ such that $\phi(C) = x \otimes \tau(C)$ for every $C \in F_0(X)$. Let $y \in X$ be linearly independent of x. Choose a nonzero $g \in X^*$. By surjectivity, there exists $D \in B(X)$ such that $\phi(D) = y \otimes g$. According to the previous step, D is not a scalar operator. Applying Lemma 2.3 we find linearly independent $N_1, N_2 \in F_0(X)$ such that both $D + N_1$ and $D + N_2$ are similar to D. It follows that $y \otimes g + x \otimes \tau(N_1) \sim y \otimes g$. Now, since $y \otimes g + x \otimes \tau(N_1)$ is of rank one and y and x are linearly independent, the functionals g and $\tau(N_1)$ are linearly dependent. In the same way we prove that g and $\tau(N_2)$ are linearly dependent. But then $\phi(N_1)$ and $\phi(N_2)$ are linearly dependent, contradicting the bijectivity of ϕ. By now we have proved that the set of operators that are not of the form scalar plus finite rank is invariant under ϕ.

We have

$$\phi(F_0(X)) \subseteq F_0(X) \subseteq F(X) \subseteq F(X) + \mathbb{C}I \subseteq \phi(F(X) + \mathbb{C}I),$$

the last inclusion being just a reformulation of the previous step. Let P be any idempotent of rank one. Then $F(X) = F_0(X) \oplus \mathbb{C}P$. Indeed, let $C \in F(X)$. Then $C = (\operatorname{tr} C)P + (C - (\operatorname{tr} C)P)$ and $C - (\operatorname{tr} C)P$ is a trace zero operator. Hence, $F_0(X)$ is a subspace of codimension 1 in $F(X)$. Also, $F(X)$ is of codimension 1 in $F(X) + \mathbb{C}I$. By bijectivity, $\phi(F_0(X))$ is of codimension 2 in $\phi(F(X) + \mathbb{C}I)$. It follows that

$$\phi(F_0(X)) = F_0(X) \quad \text{and} \quad \phi(F(X) + \mathbb{C}I) = F(X) + \mathbb{C}I.$$

We apply Proposition 2.6 once again. Because $\phi(F_0(X)) = F_0(X)$ we have either case (iii) or (iv) with T and S bijective. Thus, by the remark following Proposition 2.6, either $\phi(A) = cTAT^{-1}$ for all $A \in F_0(X)$, or $\phi(A) = cTA^*T^{-1}$ for all $A \in F_0(X)$.

We consider just the case $\phi(A) = cTAT^{-1}$ for all $A \in F_0(X)$ since the other case is similar. Composing ϕ with a similarity transformation, and then multiplying it by c^{-1}, we may assume that

$$\phi(A) = A$$

for every finite rank trace zero operator A. We will show that there exists a linear functional h on $B(X)$ such that

$$\phi(A) = A + h(A)I \qquad \text{for every } A \in B(X).$$

To this end, let $A \in B(X)$ with $A \notin \mathbb{C}I$ and set $B = \phi(A)$. We may assume that B is invertible, for if not, we replace A by $\lambda I + A$ with an appropriate scalar λ since $\phi(\mathbb{C}I) = \mathbb{C}I$. Let $x \in X$ and $f \in X^*$ be such that $f(x) = f(B^{-1}x) = 0$. For all $\alpha \in \mathbb{C}$, we have

$$(I + \alpha x \otimes f)A(I - \alpha x \otimes f) = A + F_\alpha,$$

where $F_\alpha = \alpha x \otimes A^* f - \alpha Ax \otimes f - \alpha^2 f(Ax)x \otimes f$ is a trace zero operator with rank at most two. Since $A \sim A + F_\alpha$, it follows that $B \sim B + F_\alpha$. Hence $I + B^{-1}F_\alpha = B^{-1}(B + F_\alpha)$ is invertible. Applying Proposition 2.5 to $I + B^{-1}F_\alpha$ and seeing that $f(B^{-1}x) = 0$, we have

$$f(AB^{-1}x)f(B^{-1}Ax)\alpha^2 - (f(AB^{-1}x) - f(B^{-1}Ax))\alpha - 1 \neq 0$$

for every $\alpha \in \mathbb{C}$. Thus, $f(AB^{-1}x)f(B^{-1}Ax) = 0$ and $f(AB^{-1}x) - f(B^{-1}Ax) = 0$. It follows that $f(AB^{-1}x) = 0$. Since x, f are arbitrary vectors satisfying $f(x) = f(B^{-1}x) = 0$, it follows from Lemma 2.1 that I, B^{-1}, AB^{-1} are linearly dependent, i.e., there exist complex μ_1, μ_2, μ_3, not all zero, such that $\mu_1 I + \mu_2 B^{-1} + \mu_3 AB^{-1} = 0$. Since $A \notin \mathbb{C}I$, it follows that

$$B = g(A)A + h(A)I \quad \text{for some scalars } g(A), h(A).$$

We will show that $g(A) = 1$ for all A. Indeed, choose $C \in F_0(X)$ such that A, C and I are linearly independent. Then $\phi(A + C) = g(A + C)(A + C) + h(A + C)I$ and $\phi(A + C) = \phi(A) + \phi(C) = g(A)A + h(A)I + C$. Comparing those two equations and using the linear independence of the operators involved, we get $g(A) = g(A + C) = 1$.

It is easily seen that the linearity of ϕ gives the linearity of h. On the other hand, from $I + h(I)I = \phi(I) \neq 0$, we see that $h(I) \neq -1$. Now we show that h is similarity-invariant. Suppose that $A \sim B$ in $B(X)$. It follows that $\phi(A) \sim \phi(B)$. Thus $A = T_1 B T_1^{-1}$ and $\phi(A) = T_2\phi(B)T_2^{-1}$ for some invertible T_1 and T_2 in $B(X)$. So, $A + h(A)I = T_2(B + h(B)I)T_2^{-1}$ and hence $T_1BT_1^{-1} - T_2BT_2^{-1} = A - T_2BT_2^{-1} = (h(B) - h(A))I$. Assume on the contrary that $h(A) \neq h(B)$. Choose a complex number λ such that $\lambda I - T_1BT_1^{-1}$ is not invertible. Then $(\lambda + h(A) - h(B))I - T_2BT_2^{-1} = \lambda I - T_1BT_1^{-1}$ is not invertible either. Thus, $\lambda + h(A) - h(B)$ is in the spectrum of $T_2BT_2^{-1}$ and hence in the spectrum of $T_1BT_1^{-1}$ because of the similarity of $T_1BT_1^{-1}$ and $T_2BT_2^{-1}$. By induction, for every positive integer k, $\lambda + k(h(A) - h(B))$ is in the spectrum of $T_1BT_1^{-1}$. However, this is impossible for the bounded operator $T_1BT_1^{-1}$. So $h(A) = h(B)$. ∎

Finally, we investigate similarity-invariant linear functionals. It turns out that they are closely related to commutators. Recall that the *commutator* of A and B in $B(X)$ is $[A, B] = AB - BA$.

Proposition 3.2. *Let $\mathcal{I} \subseteq B(X)$ be a two-sided ideal in $B(X)$ and let h be a linear functional on $\mathcal{I}$. Then h is similarity-invariant if and only if $h([A, B]) = 0$ for each $A \in \mathcal{I}$ and $B \in B(X)$.*

Proof. Suppose first that h vanishes at each commutator $[A, B]$, where $A \in \mathcal{I}$ and $B \in B(X)$, so $h(AB) = h(BA)$. In particular, if B is invertible, then $h(BAB^{-1}) = h(B^{-1}(BA)) = h(A)$, hence h is similarity-invariant.

Conversely, suppose h is similarity-invariant. Let $A \in \mathcal{I}$ and $B \in B(X)$. Write $B = B_1 + B_2$, where B_1 is an invertible operator in $B(X)$ and B_2 is a multiple of the identity operator on X. Since $B_1(AB_1)B_1^{-1} = B_1A$, it follows that $h([A, B_1]) = 0$. Hence, $h([A, B]) = h([A, B_1]) + h([A, B_2]) = 0 + 0 = 0$. ∎

For more information on commutators on operator ideals we refer to [3].

Remark 3.3. Since every operator on an infinite-dimensional Hilbert space H is the sum of two commutators [4], it follows from Proposition 3.2 that there are no nonzero similarity-invariant linear functionals on $B(H)$. Combining this and Theorem 3.1, one can extend Šemrl's result in [14] to nonseparable Hilbert spaces.

The following example shows that a similarity-invariant linear functional is not necessarily continuous and even if it is continuous it does not have to be multiplicative.

Example 3.4. Read [13] constructs a Banach space $\mathfrak{R}$ with the following properties:

(1) The closed ideal $W(\mathfrak{R})$ of weakly compact operators on $\mathfrak{R}$ has infinite codimension in $B(\mathfrak{R})$.
(2) The linear subspace of $B(\mathfrak{R})$ spanned by its commutators is contained in $W(\mathfrak{R})$.

Now there obviously exists a nonzero continuous linear functional $\tilde{h}$ on the quotient Banach algebra $B(\mathfrak{R})/W(\mathfrak{R})$ such that $\tilde{h}(\tilde{I}) = 0$. Extending $\tilde{h}$ to $B(\mathfrak{R})$, we get a nonzero continuous linear functional h vanishing at I and on $W(\mathfrak{R})$. In particular, h vanishes at each commutator. It follows from the proposition above that h is similarity-invariant. However, h is not multiplicative since $h(I) = 0$.

Also, there exists a discontinuous linear functional $\tilde{w}$ on $B(\mathfrak{R})/W(\mathfrak{R})$. Extending $\tilde{w}$ to $B(\mathfrak{R})$, we get a discontinuous linear functional w which vanishes on $W(\mathfrak{R})$. In particular, w vanishes at each commutator and so it is similarity-invariant.

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Fangyan Lu, Chaoran Peng
Department of Mathematics
Soochow University
Suzhou 215006, P.R. China
E-mail: fylu@suda.edu.cn

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