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The numerical radius of Lipschitz operators on Banach spaces

by

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Abstract. We study the numerical radius of Lipschitz operators on Banach spaces. We give its basic properties. Our main result is a characterization of finite-dimensional real Banach spaces with Lipschitz numerical index 1. We also explicitly compute the Lipschitz numerical index of some classical Banach spaces.

1. Introduction. O. Toeplitz [T] introduced the concept of numerical range for matrices, and his definition applies equally well to operators on infinite-dimensional Hilbert spaces. In the sixties, the concept of numerical range for operators on general Banach spaces was independently introduced by G. Lumer [L] and F. Bauer [B]. We will use the definition given by F. Bauer.

Let X be a real or complex Banach space. We will denote by $S(X)$ the unit sphere of X, by X^* the Banach space of continuous linear functionals on X, and by $L(X)$ the algebra of bounded linear operators on X. The *numerical range* of an operator $T \in L(X)$ is the subset $V(T)$ of the scalar field defined by

$$V(T) = \{x^*(Tx) : x \in S(X),\, x^* \in S(X^*),\, x^*(x) = 1\}.$$

The *numerical radius* is the seminorm defined on $L(X)$ by

$$\nu(T) = \sup\{|\lambda| : \lambda \in V(T)\}.$$

The *numerical index* of a Banach space X is the constant $n(X)$ defined by

$$n(X) = \inf\{\nu(T) : T \in L(X),\, \|T\| = 1\}$$

A complete survey on this subject can be found in the books of F. Bonsall and J. Duncan [BD1, BD2], and we refer the reader to these books for general information and background.

Recently, the numerical index of Banach spaces has been widely discussed, and the reader is referred to [E, M, MP] for recent developments.

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In [Z], Zarantonello introduced the concept of numerical range of nonlinear Hilbert space operators, and proved that the numerical range contains the spectrum. Verma [V] generalized the results of Zarantonello to the case of nonlinear operators on semi-inner product spaces.

In this paper, we generalize these notions to Lipschitz operators on Banach spaces. First, we give some definitions and notation:

An operator $T : X \to Y$ is called *Lipschitz* if

$$\|Tx - Ty\| \le K\|x - y\|$$

for some constant $K > 0$, and all $x, y \in X$.

Let $\mathrm{Lip}_0(X)$ denote the set of all Lipschitz operators on X which map 0 to 0. The Lipschitz norm of $T \in \mathrm{Lip}_0(X)$, denoted by $\|T\|_L$, is given by

$$\|T\|_L = \sup_{x \ne y} \frac{\|Tx - Ty\|}{\|x - y\|}.$$

Then $(\mathrm{Lip}_0(X), \|\cdot\|_L)$ is a Banach space, To simplify the writing, we will write $\mathrm{Lip}_0(X)$ for $(\mathrm{Lip}_0(X), \|\cdot\|_L)$.

Obviously, $L(X)$ is a subspace of $\mathrm{Lip}_0(X)$.

For each $x \in X$ with $x \ne \theta$, we define

$$D(x) = \{x^* \in X^* : x^*(x) = \|x^*\| \cdot \|x\| = \|x\|^2\}$$

The *numerical range* of a Lipschitz operator $T \in \mathrm{Lip}_0(X)$ is the subset $W(T)$ of the scalar field defined by

$$W(T) = \left\{ \frac{x^*Tx + (x-y)^*(Tx - Ty)}{\|x\|^2 + \|x - y\|^2} : x, y \in X,\ x^* \in D(x), \\ (x-y)^* \in D(x-y) \right\}.$$

From the above definition, we can easily see that $W(T)$ is the union of all the numerical ranges of the Lipschitz operator T in the sense of Verma [V], corresponding to all choices of semi-inner product.

The *numerical radius* of a Lipschitz operator T is defined on $\mathrm{Lip}_0(X)$ by

$$\omega(T) = \sup\{|\lambda| : \lambda \in W(T)\}.$$

The *Lipschitz numerical index* of the Banach space X is the constant $m(X)$ defined by

$$m(X) = \inf\{\omega(T) : T \in \mathrm{Lip}_0(X),\ \|T\|_L = 1\}.$$

The organization of the paper is the following: In Section 2, we give some basic properties of the numerical radius of Lipschitz operators on Banach spaces. In Section 3, we give our main result which gives a characterization of finite-dimensional real Banach spaces with Lipschitz numerical index 1.

In Section 4, we explicitly compute the Lipschitz numerical index of some classical Banach spaces. Finally, we list some open problems in Section 5.

2. Some basic properties. First, let us give some properties of the numerical radius of Lipschitz operators on the Banach space X.

PROPOSITION 2.1.

(i) *For any* $T \in \mathrm{Lip}_0(X)$, $\omega(T) \le \|T\|_L$.
(ii) $\omega(\alpha T) = |\alpha|\omega(T)$ *for* $T \in \mathrm{Lip}_0(X)$ *and* $\alpha \in \mathbb{R}$.
(iii) $\omega(T_1 + T_2) \le \omega(T_1) + \omega(T_2)$ *for* $T_1, T_2 \in \mathrm{Lip}_0(X)$.

Proof. For any $x, y \in X$ and $x^* \in D(x)$, $(x-y)^* \in D(x-y)$, we have

$$\left|\frac{x^*(Tx) + (x-y)^*(Tx - Ty)}{\|x\|^2 + \|x-y\|^2}\right| \le \frac{\|x^*\|\,\|T\|_L\|x\| + \|(x-y)^*\|\,\|T\|_L\|x-y\|}{\|x\|^2 + \|x-y\|^2} = \|T\|_L.$$

This proves (i).

Assertion (ii) being obvious, only (iii) needs proof. For any $x, y \in X$ and $x^* \in D(x)$, $(x-y)^* \in D(x-y)$, we have

$$\begin{aligned}&\left|\frac{x^*((T_1+T_2)(x)) + (x-y)^*((T_1+T_2)(x) - (T_1+T_2)(y))}{\|x\|^2 + \|x-y\|^2}\right| \\ &\quad \le \left|\frac{x^*(T_1x) + (x-y)^*(T_1x - T_1y)}{\|x\|^2 + \|x-y\|^2}\right| + \left|\frac{x^*(T_2x) + (x-y)^*(T_2x - T_2y)}{\|x\|^2 + \|x-y\|^2}\right| \\ &\quad \le \omega(T_1) + \omega(T_2).\end{aligned}$$

So $\omega(T_1 + T_2) \le \omega(T_1) + \omega(T_2)$. ∎

REMARK. From the above proposition, we see that ω is a seminorm on $\mathrm{Lip}_0(X)$.

PROPOSITION 2.2. *For any Banach space X, if $T \in L(X)$, then $\omega(T) = \nu(T)$.*

Proof. From the definition of $\omega(T)$, fixing $y = 0$, we have

$$\begin{aligned}\omega(T) &= \sup\left\{\left|\frac{x^*(Tx) + (x-y)^*(Tx - Ty)}{\|x\|^2 + \|x-y\|^2}\right| : x, y \in X,\right. \\ &\qquad\qquad \left. x^* \in D(x),\ (x-y)^* \in D(x-y)\right\} \\ &\ge \sup\left\{\frac{x^*(Tx)}{\|x\|^2} : x \in X,\ x^* \in D(x)\right\} = \nu(T).\end{aligned}$$

For the converse, since T is linear, it follows that for any $\varepsilon > 0$, there exist $x_0, y_0 \in X$ and $x_0^* \in D(x_0)$, $(x_0 - y_0)^* \in D(x_0 - y_0)$ such that

$$\begin{aligned}\omega(T) - \varepsilon &\le \left|\frac{x_0^*(Tx_0) + (x_0 - y_0)^*(Tx_0 - Ty_0)}{\|x_0\|^2 + \|x_0 - y_0\|^2}\right| \\ &\le \left|\frac{x_0^*(Tx_0) + (x_0 - y_0)^*(T(x_0 - y_0))}{\|x_0\|^2 + \|x_0 - y_0\|^2}\right|.\end{aligned}$$

Without loss of generality, we may assume that

$$\left|\frac{x_0^*(Tx_0)}{\|x_0\|^2}\right| \ge \left|\frac{(x_0 - y_0)^*(Tx_0 - Ty_0)}{\|x_0 - y_0\|^2}\right|.$$

Then

$$\begin{aligned}&\left|\frac{x_0^*(Tx_0) + (x_0 - y_0)^*(T(x_0 - y_0))}{\|x_0\|^2 + \|x_0 - y_0\|^2}\right| \\ &\qquad \le \frac{|x_0^*(Tx_0)| + |(x_0 - y_0)^*(T(x_0 - y_0))|}{\|x_0\|^2 + \|x_0 - y_0\|^2} \\ &\qquad = \frac{\|x_0\|^2 \left|\frac{x_0^*(Tx_0)}{\|x_0\|^2}\right| + \|x_0 - y_0\|^2 \left|\frac{(x_0 - y_0)^*(T(x_0 - y_0))}{\|x_0 - y_0\|^2}\right|}{\|x_0\|^2 + \|x_0 - y_0\|^2} \\ &\qquad \le \left|\frac{x_0^*(Tx_0)}{\|x_0\|^2}\right| \le \nu(T). \ \blacksquare\end{aligned}$$

Because for any Banach space X, $L(X)$ is a subspace of $\mathrm{Lip}_0(X)$, using Proposition 2.2, we can easily get the following:

Proposition 2.3. *For any Banach space X, we have $m(X) \le n(X)$.*

3. Lipschitz numerical index on finite-dimensional Banach spaces. In [MG], C. McGregor gave the following theorem which characterizes finite-dimensional Banach spaces with numerical index 1:

Theorem 3.1. *Let X be a finite-dimensional normed linear space over $\mathbb{R}$ or $\mathbb{C}$. Then the following are equivalent:*

(i) *$n(X) = 1$,*
(ii) *for all $x \in \mathrm{ext}(B(X))$ and all $x^* \in \mathrm{ext}(B(X^*))$, $|x^*(x)| = 1$,*
(iii) *for all $x \in \mathrm{ext}(B(X))$ and all $y \in S(X)$, there exists a scalar λ with $|\lambda| = 1$ such that $D(\lambda x) \cap D(y) \ne \emptyset$.*

Here $B(X)$ is the closed unit ball of X, and $\mathrm{ext}(A)$ is the set of extreme points of a convex set $A \subset X$.

More concretely, a finite-dimensional normed space X has numerical index 1 if and only if

$$|x^*(x)| = 1, \quad \forall x \in \mathrm{ext}(B(X)) \text{ and } \forall x^* \in \mathrm{ext}(B(X^*)).$$

Moreover, if a finite-dimensional real normed space X has numerical index 1, then for any $x_1, x_2 \in \operatorname{ext}(B(X))$, $x_1 \neq x_2$, there exists $x^* \in \operatorname{ext}(B(X^*))$ such that $x^*(x_1) \cdot x^*(x_2) = -1$ and $\|x^*(x_1)\| = \|x^*(x_2)\| = 1$. So

$$\|x_1 - x_2\| \geq |x^*(x_1 - x_2)| = 2.$$

Since $S(X)$ is compact, it follows that $\operatorname{ext}(B(X))$ is finite.

The aim of this section is to give a characterization of finite-dimensional real Banach spaces with Lipschitz numerical index 1.

The following lemma is immediate from Theorem 3.1, so we omit the proof.

LEMMA 3.2. *Let X be a finite-dimensional Banach space. If $n(X) = 1$, then for any $x \in X$ and any $x_0 \in \operatorname{ext}(B(X))$,*

$$\sup_{x_0^* \in D(x_0)} |x_0^*(x)| = \|x\|.$$

For any Banach space X, we define $B_\delta(X) = \{x \in X : \|x\| \leq \delta\}$.

LEMMA 3.3. *Let X be a finite-dimensional real Banach space with $n(X) = 1$. If there exists $\delta_X > 0$ such that for any $x \in S(X)$ there exists $\hat{x} \in \operatorname{ext}(B_{\delta_x}(X))$ satisfying $\|x - \hat{x}\| = 1 - \delta_x$, where $\delta_x \geq \delta_X$, then $m(X) = 1$.*

Proof. Fix $T \in \operatorname{Lip}_0(X)$ with $\|T\|_L = 1$. Then for any $\varepsilon > 0$, there exist $x, y \in X$ such that

$$\|Tx - Ty\| \geq (1 - \varepsilon)\|x - y\|.$$

By hypothesis, there exists $z \in \operatorname{ext}(B_{\|x-y\|\delta_{x-y}}(X))$ such that

$$\|x + z - y\| = \|x - y\|(1 - \delta_{x-y})$$

where $\delta_{x-y} \geq \delta_X$.

Because $\|T(x+z) - Ty\| \leq \|x + z - y\| = \|x - y\|(1 - \delta_{x-y})$ and $z \in \operatorname{ext}(B_{\|x-y\|\delta_{x-y}}(X))$, using Lemma 3.2, we obtain

$$\sup_{(x+z-x)^* \in D(x+z-x)} \left| \frac{(x+z-x)^*(T(x+z) - Tx)}{\|z\|^2} \right| = \frac{\|T(x+z) - Tx\|}{\|z\|}$$
$$\geq \frac{\|Tx - Ty\| - \|T(x+z) - Ty\|}{\|z\|} \geq \frac{(1-\varepsilon)\|x - y\| - \|x - y\|(1 - \delta_{x-y})}{\|x - y\|\delta_{x-y}}$$
$$= 1 - \frac{\varepsilon}{\delta_{x-y}} \geq 1 - \frac{\varepsilon}{\delta_X}.$$

Since ε is arbitrary, it follows that $m(X) = 1$. ∎

MAIN THEOREM 3.4. *Let X be a finite-dimensional real Banach space. Then $m(X) = 1$ if and only if $n(X) = 1$.*

Proof. If $m(X) = 1$, by Proposition 2.3 we obtain $1 = m(X) \leq n(X) \leq 1$. So $n(X) = 1$.

For the converse, if $n(X) = 1$ and X is a finite-dimensional real Banach space, then $\operatorname{ext}(B(X))$ is finite, say $\operatorname{card}\operatorname{ext}(B(X)) = n_0$. We claim that X satisfies the condition of Lemma 3.3 with $\delta_X = 1/n_0$.

For any $x \in S(X)$, since X is finite-dimensional real Banach space, we have $x = \sum_{i=1}^{n_0} \alpha_i x_i$, where $\sum_{i=1}^{n_0} \alpha_i = 1$, $\alpha_i \geq 0$ and $\operatorname{ext}(B(X)) = \{x_1, \ldots, x_{n_0}\}$. So there exists $1 \leq i_0 \leq n_0$ such that $\alpha_{i_0} \geq 1/n_0$.

Let $x_0 = \alpha_{i_0} x_{i_0} \in \operatorname{ext}(B_{\alpha_{i_0}}(X))$. Then

$$\|x - x_0\| = \Big\| \sum_{i \neq i_0} \alpha_i x_i \Big\| \leq 1 - \alpha_{i_0}.$$

Because

$$1 = \|x\| \leq \|x - x_0\| + \|x_0\| \leq (1 - \alpha_{i_0}) + \alpha_{i_0} = 1,$$

we have $\|x - x_0\| = 1 - \alpha_{i_0}$ as claimed. So $m(X) = 1$ by Lemma 3.3. ∎

4. Lipschitz numerical index on some classical Banach spaces. J. Duncan, C. McGregor, J. Pryce and A. White [DMPA] proved that M-spaces, L-spaces and their isometric preduals have numerical index 1. By Proposition 2.3, we know that for any Banach space X, $m(X) \leq n(X) \leq 1$. In this section, for the real Banach spaces c_0, l^1 and l^∞, we will show that $m(l^\infty) = m(l^1) = m(c_0) = 1$, which is also another proof that $n(l^\infty) = n(l^1) = n(c_0) = 1$.

First, we give an important lemma, whose proof is obvious.

LEMMA 4.1. *For any $x_0 \in \operatorname{ext}(B(l^\infty))$ and any $x \in l^\infty$, we have*

$$\sup_{x_0^* \in D(x_0)} |x_0^*(x)| = \|x\|.$$

THEOREM 4.2. *For the real Banach space l^∞, we have $m(l^\infty) = 1$.*

Proof. Fix $T \in \operatorname{Lip}_0(l^\infty)$ with $\|T\|_L = 1$. Then for any $\varepsilon > 0$ there exist $x, y \in l^\infty$ such that

$$\|Tx - Ty\| \geq (1 - \varepsilon)\|x - y\|.$$

We can find $z \in l^\infty$ such that

$$\|z - x\| = \|z - y\| = \|x - y\|/2$$

and

$$z = x + \|x - y\|(\epsilon_1/2, \epsilon_2/2, \ldots) \in l^\infty$$

where $\epsilon_i = \pm 1$ for any $i \in \mathbb{N}$.

Because $z-x$ is an extreme point of $B_{\|x-y\|/2}(l^\infty)$, by Lemma 4.1 we have

$$\sup_{(z-x)^*\in D(z-x)}\left|\frac{(z-x)^*(Tz-Tx)}{\|z-x\|^2}\right| = \frac{\|Tz-Tx\|}{\|z-x\|}$$
$$\geq \frac{\|Tx-Ty\|-\|Tz-Ty\|}{\|z-x\|} \geq \frac{(1-\varepsilon)\|x-y\|-\|x-y\|/2}{\|z-x\|} = 1-2\varepsilon.$$

Since ε is arbitrary, it follows that $m(l^\infty)=1$. ∎

For l^1, we have the following lemma similar to Lemma 4.1:

Lemma 4.3. *For any $x_0 \in \operatorname{ext}(B(l^1))$ and any $x \in l^1$, we have*

$$\sup_{x_0^*\in D(x_0)} |x_0^*(x)| = \|x\|.$$

Theorem 4.4. *For the real Banach space l^1, we have $m(l^1)=1$.*

Proof. Fix $T \in \operatorname{Lip}_0(l^1)$ with $\|T\|_L = 1$. For any $\varepsilon>0$ there exist $x,y\in l^1$ such that

$$\|Tx-Ty\| \geq (1-\varepsilon)\|x-y\|.$$

We assume that $x=(\xi_1,\xi_2,\ldots)$ and $y=(\eta_1,\eta_2,\ldots)$. Let $x_0=x$ and

$$x_1=(\eta_1,\xi_2,\xi_3,\ldots),\qquad x_2=(\eta_1,\eta_2,\xi_3,\ldots),\qquad \ldots.$$

Then

$$\|x-y\| = \sum_{i=1}^\infty \|x_i-x_{i-1}\|.$$

We claim that there exists $n_0\in\mathbb{N}$ such that $\|x_{n_0}-x_{n_0-1}\|>0$ and

$$\|Tx_{n_0}-Tx_{n_0-1}\| \geq (1-\varepsilon)\|x_{n_0}-x_{n_0-1}\|.$$

If not, we have $\|Tx_i-Tx_{i-1}\| < (1-\varepsilon)\|x_i-x_{i-1}\|$ for any $i\in\mathbb{N}$. Since $Ty-Tx=\sum_{i=1}^\infty(Tx_i-Tx_{i-1})$, it follows that

$$(1-\varepsilon)\|x-y\| \leq \|Tx-Ty\| = \Big\|\sum_{i=1}^\infty (Tx_i-Tx_{i-1})\Big\|$$
$$\leq \sum_{i=1}^\infty \|Tx_i-Tx_{i-1}\| < \sum_{i=1}^\infty (1-\varepsilon)\|x_i-x_{i-1}\| = (1-\varepsilon)\|x-y\|,$$

a contradiction.

Since $x_{n_0}-x_{n_0-1}=(0,0,\ldots,\eta_{n_0}-\xi_{n_0},0,\ldots)$, it follows that $x_{n_0}-x_{n_0-1}$ is an extreme point of $B_{\|x_{n_0}-x_{n_0-1}\|}(l^1)$, and by Lemma 4.3 we have

$$\sup_{(x_{n_0}-x_{n_0-1})^*\in D(x_{n_0}-x_{n_0-1})}\left|\frac{(x_{n_0}-x_{n_0-1})^*(Tx_{n_0}-Tx_{n_0-1})}{\|x_{n_0}-x_{n_0-1}\|^2}\right| = \frac{\|Tx_{n_0}-Tx_{n_0-1}\|}{\|x_{n_0}-x_{n_0-1}\|}\geq 1-\varepsilon.$$

Since ε is arbitrary, it follows that $m(l^1)=1$. ∎

THEOREM 4.5. *For the real Banach space c_0, we have $m(c_0)=1$.*

Proof. Fix $T\in \mathrm{Lip}_0(c_0)$ with $\|T\|_L=1$. Then for any $\varepsilon>0$ there exist $x,y\in c_0$ such that

$$\|Tx-Ty\|\geq(1-\varepsilon)\|x-y\|.$$

Let

$$x=(\xi_1,\xi_2,\ldots),\qquad y=(\eta_1,\eta_2,\ldots),$$

and

$$Tx=(\alpha_1,\alpha_2,\ldots),\qquad Ty=(\gamma_1,\gamma_2,\ldots).$$

Since $x,y,Tx,Ty\in c_0$, there exists $n_0\in\mathbb{N}$ such that for any $n\geq n_0$, we have $|\eta_n-\xi_n|<\frac{1}{2}\|x-y\|$ and $|\gamma_n-\alpha_n|<\min(\frac{1}{2}\|Tx-Ty\|,\frac{1}{2}\|x-y\|)$.

We can find $z\in c_0$ satisfying $\|z-x\|=\|z-y\|=\|x-y\|/2$ and

$$z=x+\frac{\|x-y\|}{2}(\epsilon_1,\ldots,\epsilon_{n_0},0,\ldots)$$

where $\epsilon_i=\pm1$ for any $i\in\mathbb{N}$ and $1\leq i\leq n_0$. So

$$D(z-x)=\{\epsilon_1\|z-x\|e_1^*,\ldots,\epsilon_{n_0}\|z-x\|e_{n_0}^*\}$$

where

$$e_i^*(e_j)=\begin{cases}1 & \text{if } i=j,\\ 0 & \text{if } i\neq j,\end{cases}$$

and

$$\begin{aligned}\|Tz-Tx\|&\geq\|Tx-Ty\|-\|Tz-Ty\|\\ &\geq(1-\varepsilon)\|x-y\|-\|x-y\|/2=\|z-x\|-2\varepsilon\|z-x\|.\end{aligned}$$

Let $Tz=(\lambda_1,\lambda_2,\ldots)$. Then

$$\begin{aligned}Tz-Tx&=(\lambda_1-\alpha_1,\lambda_2-\alpha_2,\ldots),\\ Ty-Tz&=(\gamma_1-\lambda_1,\gamma_2-\lambda_2,\ldots).\end{aligned}$$

Since for any $n>n_0$, we have $|\gamma_n-\alpha_n|<\frac{1}{2}\|Tx-Ty\|$, it follows that there exists $1\leq i_0\leq n_0$ such that $|\gamma_{n_0}-\alpha_{n_0}|=\|Tx-Ty\|$. So

$$\begin{aligned}|\lambda_{i_0}-\alpha_{i_0}|&\geq|\alpha_{i_0}-\gamma_{i_0}|-|\gamma_{i_0}-\lambda_{i_0}|\geq\|Tx-Ty\|-\|Tz-Ty\|\\ &\geq(1-\varepsilon)\|x-y\|-\|x-y\|/2=\|z-x\|-2\varepsilon\|z-x\|.\end{aligned}$$

So

$$\sup_{(z-x)^*\in D(z-x)}\left|\frac{(z-x)^*(Tz-Tx)}{\|z-x\|^2}\right|=\frac{|\lambda_{i_0}-\alpha_{i_0}|}{\|z-x\|}\geq1-2\varepsilon.$$

Since ε is arbitrary, it follows that $m(c_0)=1$. ∎

5. Open problems

PROBLEM 5.1. In Section 2, we have shown that $m(X) \le n(X)$ for any Banach space X. We do not know if there is a Banach space X such that $m(X) < n(X)$.

PROBLEM 5.2. M. Martín and R. Payá [MP] proved some stability properties of the numerical index for operations like c_0-, l_1- and l_∞-sums, namely for a family $\{X_\lambda : \lambda \in \Lambda\}$ of Banach spaces, we have

$$n\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{c_0}\Big) = n\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{l^1}\Big) = n\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{l^\infty}\Big) = \inf_{\lambda\in\Lambda} n(X_\lambda).$$

We do not know whether this holds for the Lipschitz numerical index, i.e. whether

$$m\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{c_0}\Big) = m\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{l^1}\Big) = m\Big(\Big[\bigoplus_{\lambda\in\Lambda} X_\lambda\Big]_{l^\infty}\Big) = \inf_{\lambda\in\Lambda} m(X_\lambda).$$

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