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Descriptive properties of elements of biduals of Banach spaces

by

PAVEL LUDVÍK and JIŘÍ SPURNÝ (Praha)

Abstract. If E is a Banach space, any element x^{**} in its bidual E^{**} is an affine function on the dual unit ball B_{E^*} that might possess a variety of descriptive properties with respect to the weak* topology. We prove several results showing that descriptive properties of x^{**} are quite often determined by the behaviour of x^{**} on the set of extreme points of B_{E^*}, generalizing thus results of J. Saint Raymond and F. Jellett. We also prove a result on the relation between Baire classes and intrinsic Baire classes of L_1-preduals which were introduced by S. A. Argyros, G. Godefroy and H. P. Rosenthal (2003). Also, several examples witnessing natural limits of our positive results are presented.

1. Introduction and main results. If E is a (real or complex) Banach space, an element x^{**} of its bidual may possess interesting descriptive properties if x^{**} is understood as a function on the dual space endowed with the weak* topology. Since the dual unit ball B_{E^*} is weak* compact, the set $\operatorname{ext} B_{E^*}$ of its extreme points is nonempty and its weak* closed convex hull is the whole unit ball. Hence one might expect that the behavior of x^{**} on the set $\operatorname{ext} B_{E^*}$ in some sense determines the behaviour of x^{**} on B_{E^*}. The aim of our paper is to substantiate this general idea by presenting several results on transferring descriptive properties of $x^{**}|_{\operatorname{ext} B_{E^*}}$ to $x^{**}|_{B_{E^*}}$. To formulate our results precisely, we need to recall several notions.

Since the main results are mostly formulated for Banach spaces over the real or complex field, we need to work with vector spaces over both real and complex numbers. So all the notions are considered, if not stated otherwise, with respect to the field of complex numbers. All topological spaces considered are assumed to be Tychonoff (i.e., completely regular, see [6, p. 39]), in particular they are Hausdorff.

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If K is a compact topological space, a *positive Radon measure* on K is a finite complete measure with values in $[0,\infty)$ defined at least on the σ-algebra of all Borel sets that is inner regular with respect to compact sets (see [8, Definition 411H]). A signed or complex measure μ on K is a Radon measure if its total variation $|\mu|$ is Radon. We often write $\mu(f)$ instead of $\int f\,d\mu$. We denote by $\mathcal{M}(K)$, $\mathcal{M}^+(K)$ and $\mathcal{M}^1(K)$ the sets of all Radon measures, positive Radon measures and probability Radon measures, respectively. Using the Riesz representation theorem we view $\mathcal{M}(K)$ as the dual space to the space $\mathcal{C}(K)$ of all continuous functions on K. Unless stated otherwise, we consider the space $\mathcal{M}(K)$ endowed with the weak* topology. A function $f\colon K \to \mathbb{C}$ is *universally measurable* if f is μ-measurable for every $\mu \in \mathcal{M}(K)$. If $\mathcal{F}$ is a family of functions, we write $\mathcal{F}^b$ for the set of all bounded elements of $\mathcal{F}$.

Let X be a compact convex subset of a locally convex space. Then any measure $\mu \in \mathcal{M}^1(X)$ has its unique *barycenter* $x \in X$, i.e., the point $x \in X$ satisfying $\mu(f) = f(x)$ for each $f \in \mathfrak{A}^c(X)$ (here $\mathfrak{A}^c(X)$ stands for the space of all continuous affine functions on X). We write $\mathcal{M}_x(X)$ for the set of all probability measures with x as the barycenter. The mapping $r\colon \mathcal{M}^1(X) \to X$ assigning to every probability measure on X its barycenter is a continuous affine surjection (see [1, Proposition I.2.1] or [22, Proposition 2.38]). A function $f\colon X \to \mathbb{C}$ is said to be *strongly affine* (or to satisfy the *barycentric formula*) if f is universally measurable and $\mu(f) = f(r(\mu))$ for every $\mu \in \mathcal{M}^1(X)$. It is easy to deduce that any strongly affine function is bounded (see e.g. [22, Lemma 4.5]).

If E is Banach space, B_{E^*} with the weak* topology is a compact convex set. We call an element $f \in E^{**}$ *strongly affine* if its restriction to B_{E^*} is a strongly affine function. We also mention that a continuous affine function f on B_{E^*}, which satisfies $f(0) = 0$ and $f(ix^*) = if(x^*)$ for $x^* \in B_{E^*}$, is in fact an element of E, i.e., there exists $x \in E$ with $f(x^*) = x^*(x)$ for $x^* \in B_{E^*}$.

Further we need to recall descriptive classes of functions in topological spaces. We follow the notation of [33]. If X is a Tychonoff topological space, a *zero set* in X is the inverse image of a closed set in $\mathbb{R}$ under a continuous function $f\colon X \to \mathbb{R}$. The complement of a zero set is a *cozero set*. A countable union of closed sets is called an F_σ *set*, the complement of an F_σ set is a G_δ *set*. If X is normal, it follows from Tietze's theorem that a closed set is a zero set if and only if it is also a G_δ set. We recall that *Borel sets* are members of the σ-algebra generated by the family of all open subsets of X, and *Baire sets* are members of the σ-algebra generated by the family of all cozero sets in X. We write $\mathrm{Bos}(X)$ and $\mathrm{Bas}(X)$ for the algebras generated by open or cozero sets in X, respectively.

A set $A \subset X$ is *resolvable* (or an H-*set*) if for any nonempty $B \subset X$ (equivalently, for any nonempty closed $B \subset X$) there exists a relatively open $U \subset B$ such that either $U \subset A$ or $U \cap A = \emptyset$. It is easy to see that the family $\mathrm{Hs}(X)$ of all resolvable sets is an algebra (see e.g. [20, §12, VI]). Let $\Sigma_2(\mathrm{Bas}(X))$, $\Sigma_2(\mathrm{Bos}(X))$ and $\Sigma_2(\mathrm{Hs}(X))$ denote countable unions of sets from the respective algebras.

Let $\mathrm{Baf}_1(X)$ denote the family of all $\Sigma_2(\mathrm{Bas}(X))$-measurable functions on X, i.e., the functions $f\colon X \to \mathbb{C}$ satisfying $f^{-1}(U) \in \Sigma_2(\mathrm{Bas}(X))$ for all $U \subset \mathbb{C}$ open. Analogously we define the families $\mathrm{Bof}_1(X)$ and $\mathrm{Hf}_1(X)$.

Now we use pointwise limits to create higher hierarchies of functions. More precisely, if Φ is a family of functions on X, we define $\Phi_0 = \Phi$ and, for each countable ordinal α, Φ_α consists of all pointwise limits of sequences from $\bigcup_{\beta<\alpha} \Phi_\beta$. Starting from $\mathrm{Baf}_1(X)$ and creating higher families $\mathrm{Baf}_\alpha(X)$ as pointwise limits of sequences contained in $\bigcup_{1\le\beta<\alpha} \mathrm{Baf}_\beta(X)$, we obtain the hierarchy of *Baire measurable* functions. Analogously we define, for $\alpha \in [1,\omega_1)$, the families $\mathrm{Bof}_\alpha(X)$ and $\mathrm{Hf}_\alpha(X)$ of *Borel measurable* functions and *resolvably measurable* functions. (Theorem 5.2 in [33] explains the term "measurability" in these definitions.)

If we start the inductive process from the family $\Phi_0 = \Phi = \mathcal{C}(X)$, we obtain the families $\mathcal{C}_\alpha(X)$ of Baire-α functions on X, $\alpha < \omega_1$. Then the union $\bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha(X)$ is the family of all *Baire* functions. It is easy to see that $\mathcal{C}_1(X) = \mathrm{Baf}_1(X)$ (see Proposition 2.3) and thus $\mathcal{C}_\alpha(X) = \mathrm{Baf}_\alpha(X)$ for any $\alpha \in [1,\omega_1)$.

Now we can state our first result concerning preservation of descriptive properties. For separable Banach spaces and Baire functions, the results can be obtained from [29, Corollaire 8].

THEOREM 1.1. *Let E be a (real or complex) Banach space and $f \in E^{**}$ be strongly affine. Then*

- *for $\alpha \in [1,\omega_1)$, $f|_{\overline{\mathrm{ext}\, B_{E^*}}} \in \mathrm{Hf}_\alpha(\overline{\mathrm{ext}\, B_{E^*}})$ if and only if $f \in \mathrm{Hf}_\alpha(B_{E^*})$,*
- *for $\alpha \in [1,\omega_1)$, $f|_{\overline{\mathrm{ext}\, B_{E^*}}} \in \mathrm{Bof}_\alpha(\overline{\mathrm{ext}\, B_{E^*}})$ if and only if $f \in \mathrm{Bof}_\alpha(B_{E^*})$,*
- *for $\alpha \in [0,\omega_1)$, $f|_{\overline{\mathrm{ext}\, B_{E^*}}} \in \mathcal{C}_\alpha(\overline{\mathrm{ext}\, B_{E^*}})$ if and only if $f \in \mathcal{C}_\alpha(B_{E^*})$.*

We remark that the assumption of strong affinity is necessary because otherwise the transfer of properties fails spectacularly. An example witnessing this phenomenon can be constructed as follows. Consider the real Banach space $E = \mathcal{C}([0,1])$ and the function $f\colon \mathcal{M}([0,1]) \to \mathbb{R}$ assigning to each $\mu \in \mathcal{M}([0,1])$ its continuous part evaluated at the function 1. Then f is a weak* discontinuous element of E^{**} contained in $\mathcal{C}_2(B_{\mathcal{M}([0,1])})$ that vanishes on $\mathrm{ext}\, B_{\mathcal{M}([0,1])}$. (Details can be found e.g. in [25, Chapter 14], [2, p. 1048] or [22, Proposition 2.63].)

The next theorem in a way extends the result of F. Jellett in [14, Theorem].

THEOREM 1.2. *Let E be a (real or complex) Banach space such that $\operatorname{ext} B_{E^*}$ is a Lindelöf set. Let $f \in E^{**}$ be a strongly affine element satisfying $f|_{\operatorname{ext} B_{E^*}} \in \mathcal{C}_\alpha(\operatorname{ext} B_{E^*})$ for some $\alpha \in [0, \omega_1)$. Then*

$$f \in \begin{cases} \mathcal{C}_{\alpha+1}(B_{E^*}), & \alpha \in [0, \omega_0), \\ \mathcal{C}_\alpha(B_{E^*}), & \alpha \in [\omega_0, \omega_1). \end{cases}$$

Under a stronger assumption on $\operatorname{ext} B_{E^*}$ we may ensure the preservation of all classes, including the finite ones.

THEOREM 1.3. *Let E be a (real or complex) Banach space such that $\operatorname{ext} B_{E^*}$ is a resolvable Lindelöf set. Let $f \in E^{**}$ be a strongly affine element satisfying $f|_{\operatorname{ext} B_{E^*}} \in \mathcal{C}_\alpha(\operatorname{ext} B_{E^*})$ for some $\alpha \in [1, \omega_1)$. Then $f \in \mathcal{C}_\alpha(B_{E^*})$.*

We remark that the shift of classes may really occur without the assumption of resolvability, as is witnessed by Example 8.1. One may also ask whether results analogous to the ones of Theorems 1.2 and 1.3 remain true for functions from Bof_α and Hf_α. Examples 8.2 and 8.3 show that this is not the case.

Further we observe that, for a separable space E, the topological condition imposed on $\operatorname{ext} B_{E^*}$ in Theorem 1.3 is equivalent to the requirement that $\operatorname{ext} B_{E^*}$ is an F_σ set. This can be seen from the following two facts: a subset of a compact metrizable space is a resolvable set if and only if it is both an F_σ and a G_δ (use [20, §26, X] and the Baire category theorem); the set of extreme points in a metrizable compact convex set is a G_δ (see [1, Corollary I.4.4] or [22, Proposition 3.43]).

We also point out that the topological assumption in Theorem 1.3 is satisfied provided $\operatorname{ext} B_{E^*}$ is an F_σ set. To see this, we first notice that $\operatorname{ext} B_{E^*}$ is then a Lindelöf space. Second, we need to check that $\operatorname{ext} B_{E^*}$ is a resolvable set in B_{E^*}. To this end, assume that $F \subset B_{E^*}$ is a nonempty closed set such that both $F \cap \operatorname{ext} B_{E^*}$ and $F \setminus \operatorname{ext} B_{E^*}$ are dense in F. By [37, Théorème 2], we can write

$$\operatorname{ext} B_{E^*} = \bigcap_{n=1}^{\infty} (H_n \cup V_n),$$

where $H_n \subset B_{E^*}$ is closed and $V_n \subset B_{E^*}$ is open, for all $n \in \mathbb{N}$. Thus both $F \setminus \operatorname{ext} B_{E^*}$ and $F \cap \operatorname{ext} B_{E^*}$ are comeager disjoint sets in F, in contradiction with the Baire category theorem. Hence $\operatorname{ext} B_{E^*}$ is a resolvable set.

For a particular class of Banach spaces, namely the L_1-preduals, one can obtain some information on the affine class of a function from its descriptive class (we recall that a Banach space E is an L_1-*predual* if E^* is isometric to some space $L_1(\mu)$; see [15, p. 59], [21, Chapter 7] or [10, Section II.5]). The affine classes $\mathfrak{A}_\alpha(X)$, $\alpha < \omega_1$, of functions on a compact convex set X are created inductively from $\mathfrak{A}_0(X) = \mathfrak{A}^c(X)$ (see [5] or [22, Definition 5.37]).

We also remark that a pointwise convergent sequence of affine functions on X is uniformly bounded, which easily follows from the uniform boundedness principle (see e.g. [22, Lemma 5.36]), and thus any function in $\bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(X)$ is strongly affine. If $X = B_{E^*}$ is the dual unit ball of a Banach space E, the affine classes are termed *intrinsic Baire classes* of E in [2, p. 1047] whereas strongly affine Baire functions on X form the hierarchy of *Baire classes* of E. Theorem 1.4 relates these classes for real L_1-preduals.

We recall that, given a compact convex set X in a real locally convex space, the real Banach space $\mathfrak{A}^c(X)$ is an L_1-predual if and only if X is a *simplex*, i.e., if for any $x \in X$ there exists a unique maximal measure $\delta_x \in \mathcal{M}^1(X)$ with $r(\delta_x) = x$ (see [7, Theorem 3.2 and Proposition 3.23]).

(A measure $\mu \in \mathcal{M}^+(X)$ is *maximal* if μ is maximal with respect to the Choquet ordering, i.e., whenever $\nu \in \mathcal{M}^+(X)$ satisfies $\mu(k) \leq \nu(k)$ for any convex continuous function k on X, then $\mu = \nu$. We refer the reader to [1, Chapter I, §3] or [22, Section 3.6] for information on maximal measures.)

THEOREM 1.4. *Let E be a real L_1-predual and $f \in E^{**}$ be a strongly affine function such that $f \in \mathcal{C}_\alpha(B_{E^*})$ for some $\alpha \in [2, \omega_1)$. Then*

$$f \in \begin{cases} \mathfrak{A}_{\alpha+1}(B_{E^*}), & \alpha \in [2, \omega_0), \\ \mathfrak{A}_\alpha(B_{E^*}), & \alpha \in [\omega_0, \omega_1). \end{cases}$$

If, moreover, $\operatorname{ext} B_{E^*}$ *is a Lindelöf resolvable set, then $f \in \mathfrak{A}_\alpha(B_{E^*})$.*

Let us point out that, for any Banach space E and a strongly affine function $f \in E^{**}$ satisfying $f \in \mathcal{C}_1(B_{E^*})$, we have $f \in \mathfrak{A}_1(B_{E^*})$. This follows from [27, Théorème 80] (see also [2, Theorem II.1.2] or [22, Theorem 4.24]). For higher Baire classes, there is a large gap between affine and Baire classes, as substantiated by M. Talagrand's example [38, Theorem] of a separable Banach space E and a strongly affine function $f \in E^{**}$ that is in $\mathcal{C}_2(B_{E^*})$ but not in $\bigcup_{\alpha<\omega_1} \mathfrak{A}_\alpha(B_{E^*})$. Further, [32, Theorem 1.1] shows that the shift of classes in Theorem 1.4 for finite ordinals may occur even for separable L_1-preduals.

The strategy of the proofs of our main results is to first reduce the problem to the case of real Banach spaces and then to consider the dual unit ball with the weak* topology as a compact convex subset of a real locally convex space. Elements of the bidual are then bounded affine functions on the dual unit ball. The key results of Sections 3–6 are thus formulated for this setting. The proof of Theorem 1.4 is moreover based upon a result of W. Lusky stating that any real L_1-predual is complemented in a simplex space (i.e., a space of type $\mathfrak{A}^c(X)$ for a simplex X) and thus the above mentioned technique can be used only for real L_1-preduals. Since it is not clear whether Lusky's result remains true for complex L_1-preduals, *the validity of Theorem 1.4 for complex spaces remains open.*

The content of our paper is the following. The second section provides a more detailed information on descriptive classes of sets and functions. Then we prepare the proof of Theorem 1.1 in Section 3. Results necessary for dealing with Lindelöf sets of extreme points are collected in Section 4. They are used in Sections 5 and 6, which prepare ground for the proof of Theorems 1.2 and 1.3. In Sections 3–6 we work with real spaces. Section 7 proves, by means of the preparatory results, the theorems stated in the introduction. Section 8 constructs spaces witnessing some natural bounds of our positive results.

When citing references, we try to include several sources to help the reader find relevant results.

2. Descriptive classes of sets and functions. We recall that, for a Tychonoff space X, $\mathrm{Bas}(X)$, $\mathrm{Bos}(X)$ and $\mathrm{Hs}(X)$ denote the algebras generated by cozero sets, open sets and resolvable sets in X, respectively. These algebras serve as a starting point of an inductive definition of descriptive classes of sets as indicated in Section 1. More precisely, if $\mathcal{F}$ is any of the families above, $\Sigma_2(\mathcal{F})$ consists of all countable unions of sets from $\mathcal{F}$, and $\Pi_2(\mathcal{F})$ consists of all countable intersections of sets from $\mathcal{F}$. Proceeding inductively, for any $\alpha \in (2, \omega_1)$ we let $\Sigma_\alpha(\mathcal{F})$ consist of all countable unions of sets from $\bigcup_{1\le\beta<\alpha} \Pi_\beta(\mathcal{F})$, and $\Pi_\alpha(\mathcal{F})$ consist of all countable intersections of sets from $\bigcup_{1\le\beta<\alpha} \Sigma_\beta(\mathcal{F})$. The family $\Pi_\alpha(\mathcal{F}) \cap \Sigma_\alpha(\mathcal{F})$ is denoted by $\Delta_\alpha(\mathcal{F})$. The union of all the additive (or multiplicative) classes defined above is then the σ-algebra generated by $\mathcal{F}$.

(These classes and their analogues were studied by several authors; see e.g. [9], [26], [12] or [11]. We describe in [33, Remark 3.5] their relations to our descriptive classes. We refer the reader to [11] for a recent survey on descriptive set theory in nonseparable and nonmetrizable spaces.)

In case X is metrizable, all the resulting classes coincide (see [33, Proposition 3.4]). These classes characterize in terms of measurability the classes $\mathrm{Baf}_\alpha(X)$, $\mathrm{Bof}_\alpha(X)$ and $\mathrm{Hf}_\alpha(X)$ defined in the introduction. (We recall that a mapping $f\colon X \to \mathbb{C}$ is called *$\mathcal{F}$-measurable* if $f^{-1}(U) \in \mathcal{F}$ for every $U \subset \mathbb{C}$ open.) Precisely, it is proved in [33, Theorem 5.2] that *given a function $f\colon X \to \mathbb{C}$ on a Tychonoff space X and $\alpha \in [1, \omega_1)$, we have*

- *$f \in \mathrm{Baf}_\alpha(X)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Bas}(X))$-measurable.*
- *$f \in \mathrm{Bof}_\alpha(X)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Bos}(X))$-measurable.*
- *$f \in \mathrm{Hf}_\alpha(X)$ if and only if f is $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable.*

It follows easily from this characterization that all the classes $\mathrm{Baf}_\alpha(X)$, $\mathrm{Bof}_\alpha(X)$ and $\mathrm{Hf}_\alpha(X)$ are stable with respect to algebraic operations and uniform convergence (see [22, Theorem 5.10]). Also, a function f is measurable with respect to the σ-algebra generated by Hs if and only if f belongs to some Hf_α. Analogous assertions hold true for Bos and Bas. Thus $\bigcup_{\alpha<\omega_1} \mathcal{C}_\alpha(X) = \bigcup_{\alpha<\omega_1} \mathrm{Baf}_\alpha(X)$ is the family of all functions measurable with respect to the σ-algebra of Baire sets.

The following characterization of functions from Hf_1 follows from the definition and the result of G. Koumoullis in [19, Theorem 2.3].

PROPOSITION 2.1. *For a function $f\colon K \to \mathbb{C}$ on a compact space K, the following assertions are equivalent:*

(i) $f \in \mathrm{Hf}_1(K)$,
(ii) $f|_F$ *has a point of continuity for every nonempty closed* $F \subset K$ (*i.e., f has the* point of continuity property),
(iii) *for each $\varepsilon > 0$ and nonempty $F \subset K$ there exists a relatively open nonempty set $U \subset F$ such that* $\operatorname{diam} f(U) < \varepsilon$ (f *is* fragmented).

Next we need to recall a characterization of resolvable sets: *a subset H of a topological space X is resolvable if and only if there exist an ordinal κ and an increasing sequence of open sets $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_\kappa = X$ and $I \subset [0, \kappa)$ such that, for a limit ordinal $\gamma \in [0, \kappa]$, we have $\bigcup\{U_\lambda \colon \lambda < \gamma\} = U_\gamma$ and $H = \bigcup\{U_{\gamma+1} \setminus U_\gamma \colon \gamma \in I\}$* (see [13, Section 2] and references therein). We call such a transfinite sequence of open sets *regular* and such a description of resolvable sets a *regular representation* (this notion is slightly more useful for us than the one used in [13, Section 2]).

A family $\mathcal{U}$ of subsets of a topological space X is *scattered* if it is disjoint and for each nonempty $\mathcal{V} \subset \mathcal{U}$ there is some $V \in \mathcal{V}$ relatively open in $\bigcup \mathcal{V}$. If $(U_\gamma)_{\gamma\le\kappa}$ is a regular sequence, then $\{U_{\gamma+1} \setminus U_\gamma \colon \gamma < \kappa\}$ is a scattered partition of X.

It is not difficult to deduce that a scattered union of resolvable sets is again a resolvable set. (Indeed, let $\{H_i \colon i \in I\}$ be a scattered family of resolvable sets. By [12, Fact 4], each H_i is a union of a scattered family $\mathcal{H}_i$ of sets in $\mathrm{Bos}(X)$. By [9, Lemma 2.2(c)], the family $\bigcup_{i\in I} \mathcal{H}_i$ is scattered, and thus again by [12, Fact 4], the set $\bigcup_{i\in I} H_i$ is resolvable.)

We will also need the fact that any resolvable subset of a compact space is universally measurable (see [19, Lemma 4.4]).

The following fact will be used in the proof of Theorem 6.4.

PROPOSITION 2.2. *Let $\alpha \in [2, \omega_1)$ and $(U_\gamma)_{\gamma\le\kappa}$ be a regular sequence in a Tychonoff space X. Let $A \subset X$ with $A \cap (U_{\gamma+1} \setminus U_\gamma) \in \Sigma_\alpha(\mathrm{Hs}(U_{\gamma+1} \setminus U_\gamma))$ for each $\gamma < \kappa$ (resp. $A \cap (U_{\gamma+1} \setminus U_\gamma) \in \Pi_\alpha(\mathrm{Hs}(U_{\gamma+1} \setminus U_\gamma))$, $\gamma < \kappa$). Then $A \in \Sigma_\alpha(\mathrm{Hs}(X))$ (resp. $A \in \Pi_\alpha(\mathrm{Hs}(X))$).*

Proof. If $\alpha = 2$, the assertion for the additive class follows from the fact mentioned above that a scattered union of resolvable sets is again resolvable. By taking complements we obtain the assertion for $\Pi_2(\mathrm{Hs})$. A straightforward transfinite induction then concludes the proof. ∎

For completeness, we include the proof of an easy observation mentioned in the introduction.

PROPOSITION 2.3. *If X is a Tychonoff space, then $\mathcal{C}_1(X) = \mathrm{Baf}_1(X)$.*

Proof. If $f \in \mathcal{C}_1(X)$, a straightforward reasoning gives $f \in \mathrm{Baf}_1(X)$. On the other hand, if $f \in \mathrm{Baf}_1(X)$, it is enough to assume that f is real-valued. If f is moreover bounded, a standard procedure (see e.g. [22, Lemma 5.7]) provides a uniform approximation of f by a sequence of simple functions, i.e., functions of the form $\sum_{i=1}^n c_i\chi_{A_i}$, where $c_1,\dots,c_n \in \mathbb{R}$ and $\{A_1,\dots,A_n\}$ is a disjoint cover of X such that each A_i is a countable union of zero sets. A moment's reflection reveals that any such function is in $\mathcal{C}_1(X)$. Hence $f \in \mathcal{C}_1(X)$ as well.

If f is unbounded, we take a homeomorphism $\varphi\colon \mathbb{R} \to (0,1)$ and apply the procedure above to $\varphi\circ f \in \mathrm{Baf}_1(X)$ to infer $\varphi\circ f \in \mathcal{C}_1(X)$. We can then find an approximating sequence (f_n) of continuous functions on X such that $0 < f_n < 1$, $n \in \mathbb{N}$. Then $\varphi^{-1}\circ f_n \to f$, and $f \in \mathcal{C}_1(X)$. ∎

3. Transfer of descriptive properties from $\overline{\mathrm{ext}\,X}$ to X. Throughout this section we work with real spaces. The main result is Theorem 3.5 on transferring descriptive properties of strongly affine functions from the closure of the set of extreme points.

LEMMA 3.1. *Let K be a compact space and H a universally measurable subset of K. Let $\widetilde{H}\colon \mathcal{M}^1(K) \to \mathbb{R}$ be defined by $\widetilde{H}(\mu) = \mu(H)$ for $\mu \in \mathcal{M}^1(K)$. Then*

- $\widetilde{H} \in \mathrm{Hf}_1(\mathcal{M}^1(K))$ *if* $H \in \mathrm{Hs}(K)$,
- $\widetilde{H} \in \mathrm{Bof}_1(\mathcal{M}^1(K))$ *if* $H \in \mathrm{Bos}(K)$.

Proof. We first assume that H is a resolvable set. We select a regular sequence $(U_\gamma)_{\gamma\le\kappa}$ which provides a regular representation of H as mentioned in Section 2. We prove by transfinite induction that, *for every $\gamma \le \kappa$, the function $\mu \mapsto \mu(H\cap U_\gamma)$ is in* $\mathrm{Hf}_1(\mathcal{M}^1(K))$.

The statement holds trivially for $\gamma = 0$.

Suppose that $\gamma \le \kappa$ is of the form $\gamma = \delta + 1$ and the claim is valid for δ. Then, for every $\mu \in \mathcal{M}^1(K)$, we have

$$\mu(H\cap U_\gamma) = \mu(H\cap U_\delta) + \mu(H\cap (U_{\delta+1}\setminus U_\delta)).$$

The second summand is either 0 or $\mu(U_{\delta+1}) - \mu(U_\delta)$. Since $\mu \mapsto \mu(U)$ is lower semicontinuous on $\mathcal{M}^1(K)$ for every open set $U \subset K$, it follows e.g. from [19, Theorem 2.3] that $\mu \mapsto \mu(U_{\delta+1}) - \mu(U_\delta)$ is in $\mathrm{Hf}_1(\mathcal{M}^1(K))$.

The function $\mu \mapsto \mu(H\cap U_\delta)$ is in $\mathrm{Hf}_1(\mathcal{M}^1(K))$ by the induction hypothesis. Thus $\mu \mapsto \mu(H)$, as a sum of two functions in $\mathrm{Hf}_1(\mathcal{M}^1(K))$, is in $\mathrm{Hf}_1(\mathcal{M}^1(K))$ as well.

Assume now that $\gamma \le \kappa$ is a limit ordinal and the statement holds for each ordinal smaller than γ. Let $\widetilde{f}(\mu) = \mu(H \cap U_\gamma)$ for $\mu \in \mathcal{M}^1(K)$. By Proposition 2.1, it is sufficient to show that $\widetilde{f}$ is fragmented. Let $M \subset \mathcal{M}^1(K)$ be nonempty and $\varepsilon > 0$. Let

$$s = \sup\{\mu(U_\gamma) \colon \mu \in M\}$$

and choose $\mu_0 \in M$ with $\mu_0(U_\gamma) > s - \varepsilon/4$. By the regularity of μ_0, there exists $\delta < \gamma$ with $\mu_0(U_\delta) > s - \varepsilon/4$. Then the set

$$V = \{\mu \in \mathcal{M}^1(K) \colon \mu(U_\delta) > s - \varepsilon/4\}$$

is an open neighborhood of μ_0.

Let $\widetilde{h} \colon \mathcal{M}^1(K) \to \mathbb{R}$ be defined by $\widetilde{h}(\mu) = \mu(H \cap U_\delta)$. Then for $\mu \in M \cap V$ we have

$$\begin{aligned} |\widetilde{h}(\mu) - \widetilde{f}(\mu)| &= |\mu(H \cap U_\delta) - \mu(H \cap U_\gamma)| \\ &\le |\mu(U_\gamma \setminus U_\delta)| \le s - (s - \varepsilon/4) = \varepsilon/4, \end{aligned}$$

and, by the induction hypothesis, $\widetilde{h}$ is in $\mathrm{Hf}_1(\mathcal{M}^1(K))$, which means that $\widetilde{h}$ is fragmented.

Thus there exists an open set $W \subset \mathcal{M}^1(K)$ intersecting $M \cap V$ such that $\operatorname{diam} \widetilde{h}(M \cap V \cap W) < \varepsilon/4$. Then for $\mu_1, \mu_2 \in M \cap V \cap W$ we have

$$|\widetilde{f}(\mu_1) - \widetilde{f}(\mu_2)| \le |\widetilde{f}(\mu_1) - \widetilde{h}(\mu_1)| + |\widetilde{h}(\mu_1) - \widetilde{h}(\mu_2)| + |\widetilde{h}(\mu_2) - \widetilde{f}(\mu_2)| \le \frac{3}{4}\varepsilon.$$

Hence $\operatorname{diam} \widetilde{f}(M \cap V \cap W) < \varepsilon$ and $\widetilde{f}$ is fragmented. This proves the claim as well as the first assertion (taking $\gamma = \kappa$).

Assume now that $H \in \mathrm{Bos}(K)$. Then H can be written as a finite disjoint union of differences of closed sets (see e.g. [22, Lemma 5.12]), i.e., $H = \bigcup_{i=1}^{n} (E_i \setminus F_i)$, where $F_i \subset E_i$ are closed and the family $\{E_1 \setminus F_1, \ldots, E_n \setminus F_n\}$ is disjoint. Then the function $\mu \mapsto \mu(E_i \setminus F_i)$, as a difference of upper semicontinuous functions on $\mathcal{M}^1(K)$, is in $\mathrm{Bof}_1(\mathcal{M}^1(K))$ for each i.

Hence $\mu \mapsto \mu(H)$, $\mu \in \mathcal{M}^1(K)$, is a finite sum of functions contained in $\mathrm{Bof}_1(\mathcal{M}^1(K))$, and thus is in $\mathrm{Bof}_1(\mathcal{M}^1(K))$. ■

LEMMA 3.2. *Let K be a compact space, $f \colon K \to \mathbb{R}$ a bounded universally measurable function and let $\widetilde{f} \colon \mathcal{M}^1(K) \to \mathbb{R}$ be defined by $\widetilde{f}(\mu) = \mu(f)$ for $\mu \in \mathcal{M}^1(K)$. Then*

- $\widetilde{f} \in \mathrm{Hf}_1(\mathcal{M}^1(K))$ *if* $f \in \mathrm{Hf}_1(K)$,
- $\widetilde{f} \in \mathrm{Bof}_1(\mathcal{M}^1(K))$ *if* $f \in \mathrm{Bof}_1(K)$.

Proof. Let $f \in \mathrm{Hf}_1(K)$. First, if $f = \chi_A$ is the characteristic function of $A \in \Delta_2(\mathrm{Hs}(K))$, we write $A = \bigcup_n A_n$, where $A_1 \subset A_2 \subset \cdots$ are in $\mathrm{Hs}(K)$.

For any $c \in \mathbb{R}$, Lemma 3.1 shows that

$$\{\mu \in \mathcal{M}^1(K)\colon \widetilde{f}(\mu) > c\} = \bigcup_{n=1}^{\infty} \{\mu \in \mathcal{M}^1(K)\colon \mu(A_n) > c\} \in \Sigma_2(\mathrm{Hs}(K)).$$

On the other hand, $K \setminus A \in \Sigma_2(\mathrm{Hs}(K))$ and hence the above reasoning yields

$$\{\mu \in \mathcal{M}^1(K)\colon \widetilde{f}(\mu) < c\} = \{\mu \in \mathcal{M}^1(K)\colon \mu(K \setminus A) > 1 - c\} \in \Sigma_2(\mathrm{Hs}(K)).$$

We conclude that $\widetilde{f}$ is $\Sigma_2(\mathrm{Hs}(\mathcal{M}^1(K)))$-measurable and so $\widetilde{f} \in \mathrm{Hf}_1(\mathcal{M}^1(K))$.

If $f \in \mathrm{Hf}_1(K)$ is bounded, it can be uniformly approximated by simple functions in $\mathrm{Hf}_1(K)$, i.e., functions of the form $\sum_{i=1}^{n} c_i \chi_{A_i}$, where $A_1, \dots, A_n \in \Delta_2(\mathrm{Hs}(K))$ are pairwise disjoint and $c_1, \dots, c_n \in \mathbb{R}$ (this standard procedure can be found e.g. in [22, Lemma 5.7]). Hence $\widetilde{f}$ can be uniformly approximated by functions in $\mathrm{Hf}_1(\mathcal{M}^1(K))$, and thus $\widetilde{f} \in \mathrm{Hf}_1(\mathcal{M}^1(K))$.

The proof for $f \in \mathrm{Bof}_1(K)$ is similar. ∎

Lemma 3.3. *Let K be a compact space and $f\colon K \to \mathbb{R}$ be a bounded universally measurable function. Let $\widetilde{f}\colon \mathcal{M}^1(K) \to \mathbb{R}$ be defined by $\widetilde{f}(\mu) = \mu(f)$ for $\mu \in \mathcal{M}^1(K)$. Then:*

(a) *for $\alpha \in [1, \omega_1)$, $f \in \mathrm{Hf}_\alpha(K)$ if and only if $\widetilde{f} \in \mathrm{Hf}_\alpha(\mathcal{M}^1(K))$,*
(b) *for $\alpha \in [1, \omega_1)$, $f \in \mathrm{Bof}_\alpha(K)$ if and only if $\widetilde{f} \in \mathrm{Bof}_\alpha(\mathcal{M}^1(K))$,*
(c) *for $\alpha \in [0, \omega_1)$, $f \in \mathcal{C}_\alpha(K)$ if and only if $\widetilde{f} \in \mathcal{C}_\alpha(\mathcal{M}^1(K))$.*

Proof. The "if" parts easily follow from the fact that $f = \widetilde{f} \circ \phi$ where $\phi : K \to \mathcal{M}^1(K)$ sending $x \in K$ to the Dirac measure ε_x at x is a homeomorphic embedding.

The proof of the "only if" parts is by transfinite induction. If $\alpha = 1$ in (a) and (b), the assertion follows from Lemma 3.2; the case $\alpha = 0$ in (c) is obvious.

The induction step is straightforward. ∎

Remark 3.4. It is worth noting that part (c) of the previous lemma holds for a general completely regular topological space K if we consider the space $\mathcal{M}^1(K)$ endowed with the weak topology introduced in [40, Chapter 8].

As mentioned in the introduction, the following theorem is a generalization of [29, Corollaire 8].

Theorem 3.5. *Let X be a compact convex set and $f\colon X \to \mathbb{R}$ be a strongly affine function. Then:*

- *for $\alpha \in [1, \omega_1)$, $f|_{\overline{\mathrm{ext}\,X}} \in \mathrm{Hf}_\alpha(\overline{\mathrm{ext}\,X})$ if and only if $f \in \mathrm{Hf}_\alpha(X)$,*
- *for $\alpha \in [1, \omega_1)$, $f|_{\overline{\mathrm{ext}\,X}} \in \mathrm{Bof}_\alpha(\overline{\mathrm{ext}\,X})$ if and only if $f \in \mathrm{Bof}_\alpha(X)$,*
- *for $\alpha \in [0, \omega_1)$, $f|_{\overline{\mathrm{ext}\,X}} \in \mathcal{C}_\alpha(\overline{\mathrm{ext}\,X})$ if and only if $f \in \mathcal{C}_\alpha(X)$.*

Proof. It is easy to realize that all the families Hf_α, Bof_α and $\mathcal{C}_\alpha$ are preserved by taking restrictions to subspaces of X. This gives the "if" parts.

For the "only if" parts, let $f\colon X\to\mathbb{R}$ be a strongly affine function with $f|_{\overline{\operatorname{ext} X}}\in\mathcal{F}(\overline{\operatorname{ext} X})$ where $\mathcal{F}$ is any of the classes Hf_α, Bof_α or $\mathcal{C}_\alpha$. Then the function $\widetilde{g}\colon\mathcal{M}^1(\overline{\operatorname{ext} X})\to\mathbb{R}$ defined by

$$\widetilde{g}(\mu)=\mu(f),\quad \mu\in\mathcal{M}^1(\overline{\operatorname{ext} X}),$$

is in $\mathcal{F}(\mathcal{M}^1(\overline{\operatorname{ext} X}))$ by Lemma 3.3.

The mapping $r\colon\mathcal{M}^1(\overline{\operatorname{ext} X})\to X$ which assigns to $\mu\in\mathcal{M}^1(\overline{\operatorname{ext} X})$ its barycenter $r(\mu)\in X$ is a continuous surjection of the compact space $\mathcal{M}^1(\overline{\operatorname{ext} X})$ onto X (see [1, Proposition I.4.6 and Theorem I.4.8] or [22, Theorem 3.65 and Proposition 3.64]).

From the strong affinity of f we have $\widetilde{g}=f\circ r$. Now we use the fact that $\widetilde{g}\in\mathcal{F}(\mathcal{M}^1(\overline{\operatorname{ext} X}))$ if and only if $f\in\mathcal{F}(X)$. This fact can be found in [28, Theorem 5.9.13] and [22, Theorem 5.26] for $\mathcal{C}_\alpha$, and in [13, Theorems 4 and 10] for Bof_α and Hf_α (see also [22, Theorem 5.26]). Thus f is in $\mathcal{F}(X)$. ∎

4. Auxiliary result on compact convex sets with $\operatorname{ext} X$ Lindelöf. Throughout this section we work with spaces over the field of real numbers. We aim for the proof of Proposition 4.8, a fact to be used both in Sections 5 and 6. We recall that a topological space X is *K-analytic* if it is the image of a Polish space under an upper semicontinuous compact-valued map (see [28, Section 2.1]).

LEMMA 4.1. *Let $\varphi\colon X\to Y$ be a continuous surjection of a K-analytic space X onto a K-analytic space Y and let $g\colon Y\to\mathbb{R}$. Hence g is a Baire function on Y if and only if $g\circ\varphi$ is a Baire function on X.*

Proof. If g is a Baire function Y, then $g\circ\varphi$ is clearly a Baire function on X. Conversely, if $f=g\circ\varphi$ is a Baire function on X and $U\subset\mathbb{R}$ is an open set, then both $f^{-1}(U)$ and $f^{-1}(\mathbb{R}\setminus U)$ are Baire sets in X. Hence they are K-analytic sets in X (see [28, Section 2]), and thus

$$g^{-1}(U)=\varphi(f^{-1}(U)),\quad g^{-1}(\mathbb{R}\setminus U)=\varphi(f^{-1}(\mathbb{R}\setminus U))$$

are K-analytic as well. It follows from the proof of the standard separation theorem (see [28, Theorem 3.3.1]) that they are Baire sets. Hence g is measurable with respect to the σ-algebra of Baire sets, and thus it is a Baire function. ∎

LEMMA 4.2. *Let B be a Lindelöf subset of a compact space X and f be a bounded continuous function on B. Then there exists a bounded Baire function on X extending f.*

Proof. Without loss of generality, let $0\le f\le 1$. If

$$h(x)=\begin{cases} f(x), & x\in B,\\ \limsup\limits_{y\to x,\, y\in B} f(y), & x\in\overline{B}\setminus B,\\ 0, & x\in X\setminus\overline{B},\end{cases}$$

then h is an upper semicontinuous function on X. Hence

$$h = \inf\{a \in \mathcal{C}(X) \colon h \le a \le 1\}.$$

By the Lindelöf property of B and the continuity of f (see [22, Lemma A.54]), there exists a countable family $\{a_n \colon n \in \mathbb{N}\}$ of continuous functions on X with $h \le a_n \le 1$, $n \in \mathbb{N}$, such that $f = \inf\{a_n \colon n \in \mathbb{N}\}$ on B. Then $g = \inf\{a_n \colon n \in \mathbb{N}\}$ is a Baire function on X with values in $[0,1]$ extending f. ∎

Lemma 4.3. *Let $f\colon X \to \mathbb{R}$ be a strongly affine function on a compact convex set X for which there exists a Baire set $B \supset \operatorname{ext} X$ such that $f|_B$ is a Baire function. Then f is a Baire function on X.*

Proof. Let

$$M = \{\mu \in \mathcal{M}^1(X) \colon \mu(B) = 1\}.$$

Since the characteristic function of B is a Baire function, the function $\widetilde{B}(\mu) = \mu(B)$, $\mu \in \mathcal{M}^1(X)$, is a Baire function on $\mathcal{M}^1(X)$ as well, by Lemma 3.3(c), and thus $M = \{\mu \in \mathcal{M}^1(X) \colon \widetilde{B}(\mu) = 1\}$ is a Baire and consequently K-analytic set in $\mathcal{M}^1(X)$.

Since $f|_B$ is a Baire function on B, it extends to a bounded Baire function g on X by Lemma 4.2 and transfinite induction (we remark that a Baire subset of a compact space is Lindelöf by [28, Theorem 2.7.1]). Then

$$\widetilde{g}(\mu) = \mu(g), \quad \mu \in \mathcal{M}^1(X),$$

is a Baire function on $\mathcal{M}^1(X)$ by Lemma 3.3(c).

Further, the function $\widetilde{f}\colon M \to \mathbb{R}$ defined by

$$\widetilde{f}(\mu) = \mu(f), \quad \mu \in M,$$

coincides on M with $\widetilde{g}$. Hence $\widetilde{f}$ is a Baire function on M.

Then $r\colon M \to X$ is a continuous surjective mapping satisfying $\widetilde{f} = f \circ r$ (see [1, Corollary I.4.12 and the subsequent remark] or [22, Theorem 3.79]). By Lemma 4.1, f is a Baire function. ∎

Lemma 4.4. *Let X be a compact convex set with $\operatorname{ext} X$ Lindelöf, $\mu \in \mathcal{M}^1(X)$ be maximal and $B \supset \operatorname{ext} X$ be μ-measurable. Then $\mu(B) = 1$.*

Proof. By the regularity of μ it is enough to show that $\mu(K) = 0$ for every $K \subset X \setminus B$ compact. Given such a K, for every $x \in \operatorname{ext} X$ we select a closed neighborhood U_x of x disjoint from K. By the Lindelöf property we choose a countable set $\{x_n \colon n \in \mathbb{N}\} \subset \operatorname{ext} X$ with $\operatorname{ext} X \subset \bigcup U_{x_n}$. By Corollary I.4.12 and the subsequent remark in [1] (see also [22, Theorem 3.79]), $\mu(\bigcup U_{x_n}) = 1$. Hence $\mu(K) = 0$. ∎

Lemma 4.5. *Let X be a compact convex set with $\operatorname{ext} X$ Lindelöf and $f \in \mathcal{C}^b(\operatorname{ext} X)$. Then there exist a decreasing sequence (u_n) of continuous* *concave functions on X and an increasing sequence (l_n) of continuous convex functions on X such that*

$$\inf f(\operatorname{ext} X) \le \inf l_1(X), \quad \sup u_1(X) \le \sup f(\operatorname{ext} X),$$

and

$$u_n \searrow f, \quad l_n \nearrow f \quad \text{on } \operatorname{ext} X.$$

Proof. Without loss of generality we may assume that

$$0 \le i = \inf f(\operatorname{ext} X) \le \sup f(\operatorname{ext} X) = s \le 1.$$

We construct a decreasing sequence (u_n) of continuous concave functions on X with values in $[0,1]$ such that $u_n \searrow f$ on $\operatorname{ext} X$. To do this, we define $h\colon \overline{\operatorname{ext} X} \to [0,1]$ by

$$h(x) = \begin{cases} f(x), & x \in \operatorname{ext} X, \\ \limsup\limits_{y \to x,\, y \in \operatorname{ext} X} f(y), & x \in \overline{\operatorname{ext} X} \setminus \operatorname{ext} X. \end{cases}$$

Then h is upper semicontinuous on $\overline{\operatorname{ext} X}$ and the function

$$h^* = \inf\{a \in \mathfrak{A}^c(X) \colon a \ge f \text{ on } \operatorname{ext} X\}$$

satisfies $h = h^* = f$ on $\operatorname{ext} X$ by [1, Proposition I.4.1] (see also [22, Theorem 3.24]). Hence

$$f = \inf\{a \in \mathfrak{A}^c(X) \colon a \ge f \text{ on} \operatorname{ext} X\} \quad \text{on} \operatorname{ext} X.$$

Since $\operatorname{ext} X$ is a Lindelöf space, there exists a countable family $\mathcal{H} = \{h_n : n \in \mathbb{N}\}$ of functions in $\mathfrak{A}^c(X)$ majorizing f on $\operatorname{ext} X$ such that $f = \inf \mathcal{H}$ on $\operatorname{ext} X$ (see [14, Lemma] or [22, Lemma A.54]). Then we obtain the desired sequence by setting

$$u_1 = s \wedge h_1, \quad u_n = s \wedge h_1 \wedge \cdots \wedge h_n, \quad n \in \mathbb{N}.$$

Analogously we obtain an increasing sequence (l_n) of convex continuous functions converging to f on $\operatorname{ext} X$. ∎

LEMMA 4.6. *Let X be a compact convex set with* $\operatorname{ext} X$ *Lindelöf and let $f \in \mathcal{C}_\alpha(\operatorname{ext} X)$ have values in $[0,1]$. Then there exist a Baire set $B \supset \operatorname{ext} X$ and a function $g \in \mathcal{C}_\alpha(B)$ such that*

- *$g = f$ on $\operatorname{ext} X$,*
- *$0 \le g \le 1$ on B,*
- *$g(r(\mu)) = \mu(g)$ for any $\mu \in \mathcal{M}^1(X)$ with $\mu(B) = 1$ and $r(\mu) \in B$.*

Proof. We proceed by transfinite induction on the class of f.

Assume first that f is continuous on $\operatorname{ext} X$. Using Lemma 4.5 we find relevant sequences (u_n) and (l_n), and define $u = \inf_{n\in\mathbb{N}} u_n$, $l = \sup_{n\in\mathbb{N}} l_n$. Then we observe that $l \le u$ by the minimum principle (see [1, Theorem I.5.3] or [22, Theorem 3.16]), both functions are Baire, u is upper semicontinuous concave and l is lower semicontinuous convex. Let

$$B = \{x \in X \colon u(x) = l(x)\} \quad \text{and} \quad g(x) = u(x), \quad x \in B.$$

Then B is a Baire set containing $\operatorname{ext} X$ and, for $x \in B$ and $\mu \in \mathcal{M}_x(X)$ with $\mu(B) = 1$, we have, by [22, Proposition 4.7],

$$g(x) = u(x) \geq \mu(u) = \mu(l) \geq l(x) = g(x).$$

Since g is continuous on B, the proof is finished for the case $\alpha = 0$.

Assume now that the claim holds true for all β smaller than some countable ordinal α. Given $f \in \mathcal{C}_\alpha(\operatorname{ext} X)$ with values in $[0,1]$, let (f_n) be a sequence of functions with $f_n \in \mathcal{C}_{\alpha_n}(\operatorname{ext} X)$ for some $\alpha_n < \alpha$, $n \in \mathbb{N}$, such that $f_n \to f$. Without loss of generality we may assume that all functions f_n have values in $[0,1]$. For each $n \in \mathbb{N}$, we use the induction hypothesis and find a Baire set $B_n \supset \operatorname{ext} X$ along with a function $g_n \in \mathcal{C}_{\alpha_n}(B_n)$ with values in $[0,1]$ that coincides with f_n on $\operatorname{ext} X$ and satisfies $g_n(r(\mu)) = \mu(g_n)$ for any $\mu \in \mathcal{M}^1(X)$ satisfying $\mu(B_n) = 1$ and $r(\mu) \in B_n$.

We set

$$B = \Big\{x \in \bigcap_{n=1}^{\infty} B_n \colon (g_n(x)) \text{ converges}\Big\} \quad \text{and} \quad g(x) = \lim_{n\to\infty} g_n(x),\ x \in B.$$

Then B is a Baire set containing $\operatorname{ext} X$, $g \in \mathcal{C}_\alpha(B)$ with values in $[0,1]$,

$$g_n(x) = f_n(x) \to f(x) \quad \text{for every } x \in \operatorname{ext} X,$$

and, for $x \in B$ and $\mu \in \mathcal{M}_x(X)$ with $\mu(B) = 1$,

$$g(x) = \lim_{n\to\infty} g_n(x) = \lim_{n\to\infty} \mu(g_n) = \mu(g). \ \blacksquare$$

Lemma 4.7. *Let X be a compact convex set with $\operatorname{ext} X$ Lindelöf and let $f \colon X \to \mathbb{R}$ be a strongly affine function such that $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X)$. Then there exists a Baire set $B \supset \operatorname{ext} X$ such that $f \in \mathcal{C}_\alpha(B)$.*

Proof. We can assume that $0 \leq f \leq 1$. Using Lemma 4.6 we find a Baire set $B \supset \operatorname{ext} X$ together with a function $g \in \mathcal{C}_\alpha(B)$ with values in $[0,1]$ such that $g = f$ on $\operatorname{ext} X$ and $g(x) = \mu(g)$ for each $x \in B$ and $\mu \in \mathcal{M}_x(X)$ with $\mu(B) = 1$.

We claim that $f = g$ on B. To verify this, pick $x \in B$ and a maximal measure $\mu \in \mathcal{M}_x(X)$. Then μ is supported by B and $f = g$ μ-almost everywhere. (Indeed, the set $\{y \in X \colon f(y) = g(y)\}$ is μ-measurable and contains $\operatorname{ext} X$, so we can apply Lemma 4.4.) Hence

$$g(x) = \mu(g) = \mu(f) = f(x),$$

where the last equality follows from the strong affinity of f. $\blacksquare$

Proposition 4.8. *Let X be a compact convex set with* $\operatorname{ext} X$ *Lindelöf and let $f\colon X \to \mathbb{R}$ be a strongly affine function such that $f|_{\operatorname{ext} X}$ is Baire. Then f is a Baire function on X.*

Proof. The assertion follows from Lemmas 4.7 and 4.3. ∎

5. Transfer of descriptive properties on compact convex sets with $\operatorname{ext} X$ **Lindelöf.** The notions considered in this section are over the real numbers. The following key factorization result uses a method of metrizable reduction for Baire functions, which can be found e.g. in [5], [28, Theorem 5.9.13], [39, Theorem 1], [3] or [22, Theorem 9.12]. The main result in this section, Theorem 5.2, is then a consequence of a selection theorem by M. Talagrand (see [36]).

Lemma 5.1. *Let X be a compact convex set with* $\operatorname{ext} X$ *Lindelöf and let $f\colon X \to \mathbb{R}$ be strongly affine such that $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X)$ for some $\alpha \in [1, \omega_1)$. Then there exist a metrizable compact convex set Y, an affine surjection $\varphi\colon X \to Y$, a strongly affine Baire function $\widetilde{f}\colon Y \to \mathbb{R}$ and $\widetilde{g} \in \mathcal{C}^b_\alpha(\operatorname{ext} Y)$ such that*

$$\widetilde{g}(\varphi(x)) = f(x), \quad x \in \operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y),$$

and

$$f(x) = \widetilde{f}(\varphi(x)), \quad x \in X.$$

Proof. We may assume that $0 \le f \le 1$. Let $\mathcal{F} = \{g_n \colon n \in \mathbb{N}\} \subset \mathcal{C}(\operatorname{ext} X)$ be a countable family of functions with values in $[0,1]$ satisfying $f \in \mathcal{F}_\alpha$.

For a fixed $n \in \mathbb{N}$, using Lemma 4.5 we select finite families $\mathcal{U}^k_n$ and $\mathcal{L}^k_n$, $k \in \mathbb{N}$, of functions in $\mathfrak{A}^c(X)$ with values in $[0,1]$ such that, for

$$u^k_n = \inf \mathcal{U}^k_n, \quad l^k_n = \sup \mathcal{L}^k_n,$$

we have

- $\lim_{k\to\infty} l^k_n(x) = \lim_{k\to\infty} u^k_n = g_n(x)$ for each $x \in \operatorname{ext} X$,
- $(l^k_n)_{k=1}^\infty$ is increasing and $(u^k_n)_{k=1}^\infty$ is decreasing.

Further, by Proposition 4.8, f is a Baire function on X, say of class β. Let $\mathcal{F}' = \{h_n \colon n \in \mathbb{N}\} \subset \mathcal{C}(X)$ be a countable family satisfying $f \in (\mathcal{F}')_\beta$. For any $n, k \in \mathbb{N}$, by [1, Proposition I.1.1] (or [22, Proposition 3.11]) there exist finite families $\mathcal{V}^k_n, \mathcal{W}^k_n \subset \mathfrak{A}^c(X)$ such that, for $v^k_n = \inf \mathcal{V}^k_n$, $w^k_n = \sup \mathcal{W}^k_n$, we have

$$\|h_n - (v^k_n + w^k_n)\| < 1/k.$$

By setting $\mathcal{G} = \{v^k_n, w^k_n \colon n, k \in \mathbb{N}\}$, we obtain a family satisfying $f \in \mathcal{G}_\beta$.

We set

$$\Phi = \bigcup_{n,k\in\mathbb{N}} (\mathcal{U}^k_n \cup \mathcal{L}^k_n \cup \mathcal{V}^k_n \cup \mathcal{W}^k_n)$$

and define $\varphi\colon X \to \mathbb{R}^{\mathbb{N}}$ by

$$\varphi(x) = (\phi(x))_{\phi\in\Phi}, \quad x \in X.$$

Then $Y = \varphi(X)$ is a metrizable compact convex set and, for each $\phi \in \Phi$, there exists $\widetilde{\phi} \in \mathfrak{A}^c(Y)$ with $\widetilde{\phi} \circ \varphi = \phi$.

For fixed $n, k \in \mathbb{N}$, let $\widetilde{\mathcal{U}}_n^k \subset \mathfrak{A}^c(Y)$ be such that

$$\mathcal{U}_n^k = \{\widetilde{u} \circ \varphi\colon \widetilde{u} \in \widetilde{\mathcal{U}}_n^k\}.$$

Analogously we pick $\widetilde{\mathcal{L}}_n^k$, $\widetilde{\mathcal{V}}_n^k$ and $\widetilde{\mathcal{W}}_n^k$ in $\mathfrak{A}^c(Y)$. Then

$$\widetilde{u}_n^k = \inf \widetilde{\mathcal{U}}_n^k,\ \widetilde{l}_n^k = \sup \widetilde{\mathcal{L}}_n^k, \quad \widetilde{v}_n^k = \inf \widetilde{\mathcal{V}}_n^k \quad \text{and} \quad \widetilde{w}_n^k = \sup \widetilde{\mathcal{W}}_n^k$$

satisfy

$$\widetilde{u}_n^k \circ \varphi = u_n^k, \quad \widetilde{l}_n^k \circ \varphi = l_n^k, \quad \widetilde{v}_n^k \circ \varphi = v_n^k \quad \text{and} \quad \widetilde{w}_n^k \circ \varphi = w_n^k.$$

Given $y \in \operatorname{ext} Y$, we select $x \in \operatorname{ext} X \cap \varphi^{-1}(y)$. Then

$$\lim_{k\to\infty} \widetilde{u}_n^k(y) = \lim_{k\to\infty} \widetilde{u}_n^k(\varphi(x)) = \lim_{k\to\infty} u_n^k(x) = g_n(x),$$
$$\lim_{k\to\infty} \widetilde{l}_n^k(y) = \lim_{k\to\infty} \widetilde{l}_n^k(\varphi(x)) = \lim_{k\to\infty} l_n^k(x) = g_n(x).$$

Thus $(\widetilde{u}_n^k)_{k=1}^\infty$ is a decreasing sequence on $\operatorname{ext} Y$, $(\widetilde{l}_n^k)_{k=1}^\infty$ is increasing on $\operatorname{ext} Y$and both converge to a common limit $\widetilde{g}_n\colon \operatorname{ext} Y \to \mathbb{R}$ given by

$$\widetilde{g}_n(y) = \lim_{k\to\infty} \widetilde{u}_n^k(y), \quad y \in \operatorname{ext} Y,$$

which is a continuous function on $\operatorname{ext} Y$ with values in $[0, 1]$.

Thus, for every $n \in \mathbb{N}$, there exists a function $\widetilde{g}_n \in \mathcal{C}^b(\operatorname{ext} Y)$ satisfying $\widetilde{g}_n \circ \varphi = g_n$ on $\operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y)$. Let $\widetilde{\mathcal{F}} = \{\widetilde{g}_n\colon n \in \mathbb{N}\}$.

Now we claim that, for each $\gamma \in [0, \alpha]$ and $h \in \mathcal{F}_\gamma$, there exists $\widetilde{h} \in \widetilde{\mathcal{F}}_\gamma$ such that $h = \widetilde{h} \circ \varphi$ on $\operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y)$. To verify this, we proceed by transfinite induction. The claim is obvious for $\gamma = 0$. Assume that it holds for all $\gamma' < \gamma$ for some $\gamma \le \alpha$ and that we are given $h \in \mathcal{F}_\gamma$. Let $\gamma_n < \gamma$ and $h_n \in \mathcal{F}_{\gamma_n}$, $n \in \mathbb{N}$, be such that $h = \lim h_n$. By the inductive assumption, there exist $\widetilde{h}_n \in \widetilde{\mathcal{F}}_{\gamma_n}$ satisfying $h_n = \widetilde{h}_n \circ \varphi$ on $\operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y)$. Then the sequence $(\widetilde{h}_n(y))$ converges for every $y \in \operatorname{ext} Y$. Hence we may define $\widetilde{h} \in \widetilde{\mathcal{F}}_\gamma$ by

$$\widetilde{h}(y) = \lim_{n\to\infty} \widetilde{h}_n(y), \quad y \in \operatorname{ext} Y,$$

and then, for every $y \in \operatorname{ext} Y$ and $x \in \varphi^{-1}(y) \cap \operatorname{ext} X$,

$$\widetilde{h}(y) = \lim_{n\to\infty} \widetilde{h}_n(y) = \lim_{n\to\infty} h_n(x) = h(x).$$

This proves the claim.

It follows from the claim that there exists $\widetilde{g} \in \mathcal{C}_\alpha(\operatorname{ext} Y)$ such that

$$\widetilde{g}(\varphi(x)) = f(x), \quad x \in \operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y).$$

Analogously, let $\widetilde{\mathcal{G}}$ be the family satisfying

$$\mathcal{G} = \{\widetilde{z} \circ \varphi \colon \widetilde{z} \in \widetilde{\mathcal{G}}\}.$$

Then, for each $\gamma \in [0, \beta]$ and $h \in \mathcal{G}_\gamma$, there exists $\widetilde{h} \in \widetilde{\mathcal{G}}_\gamma$ satisfying $h = \widetilde{h} \circ \varphi$. Hence there exists $\widetilde{f} \in \widetilde{\mathcal{G}}_\beta$ satisfying $f = \widetilde{f} \circ \varphi$. Obviously, $\widetilde{f}$ is a Baire function, and moreover it is strongly affine by [30, Proposition 3.2] (see also [22, Proposition 5.29]). ∎

THEOREM 5.2. *Let X be a compact convex set with* $\operatorname{ext} X$ *Lindelöf and $f \colon X \to \mathbb{R}$ be a strongly affine function. If $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X)$, then*

$$f \in \begin{cases} \mathcal{C}_{\alpha+1}(X), & \alpha \in [0, \omega_0), \\ \mathcal{C}_\alpha(X), & \alpha \in [\omega_0, \omega_1). \end{cases}$$

Proof. Let f be a strongly affine function f whose restriction to $\operatorname{ext} X$ is of Baire class α. If $\alpha = 0$, i.e., f is continuous and bounded on $\operatorname{ext} X$, Lemma 4.5 provides the relevant sequences (u_n) and (l_n). For $n \in \mathbb{N}$, $x \in X$ and $\mu_1, \mu_2 \in \mathcal{M}_x(X)$, we have

$$\mu_1(l_n) \le \mu_1(f) = f(x) = \mu_2(f) \le \mu_2(u_n).$$

If we denote

$$(l_n)^* = \inf\{h \in \mathfrak{A}^c(X) \colon h \ge l_n \text{ on } X\},$$
$$(u_n)_* = \sup\{h \in \mathfrak{A}^c(X) \colon h \le u_n \text{ on } X\},$$

then by [1, Corollary I.3.6] (see also [22, Lemma 3.21]),

$$(l_n)^* \le f \le (u_n)_*.$$

Using an argument based upon the Hahn–Banach theorem (see e.g. [22, Lemma 4.11]), there exists a sequence (h_n) of functions in $\mathfrak{A}^c(X)$ such that

$$(l_n)^* - 1/n < h_n < (u_n)_* + 1/n, \quad n \in \mathbb{N}.$$

Then $f \in \mathcal{C}_1(X)$ because $h_n \to f$ on $\operatorname{ext} X$, and thus on X. (Indeed, given $x \in X$, let $\mu \in \mathcal{M}_x(X)$ be maximal. Then the set

$$B = \{y \in X \colon h_n(y) \to f(y)\}$$

is μ-measurable and contains $\operatorname{ext} X$. By Lemma 4.4, $\mu(B) = 1$. Hence $f(x) = \mu(f) = \lim \mu(h_n) = h_n(x)$.)

Assume now that $\alpha \ge 1$. Then we use Lemma 5.1 to find a continuous affine surjection φ of X onto a metrizable compact convex set Y, $\widetilde{g} \in \mathcal{C}^b_\alpha(\operatorname{ext} Y)$ and a Baire function $\widetilde{f} \colon X \to \mathbb{R}$ such that

$$f = \widetilde{g} \circ \varphi \text{ on } \operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y) \quad \text{and} \quad f = \widetilde{f} \circ \varphi \text{ on } X. \tag{5.1}$$

Since $\operatorname{ext} Y$ is a G_δ set and $\alpha \ge 1$, we can extend $\widetilde{g}$ to the whole Y (and denote it likewise) with preservation of class (see [20, §31, VI, Théorème]). By [36, Théorème 1] (see also [22, Theorem 11.41]), there exists a mapping $y \mapsto \nu_y$, $y \in Y$, such that

(a) ν_y is a maximal measure in $\mathcal{M}_y(Y)$,
(b) $y \mapsto \nu_y(h)$ is Baire-one on Y for every $h \in \mathcal{C}(Y)$.

Let

$$\widetilde{h}(y) = \nu_y(\widetilde{g}), \quad y \in Y.$$

Then

$$\widetilde{h} \in \begin{cases} \mathcal{C}_{\alpha+1}(Y), & \alpha \in [1, \omega_0), \\ \mathcal{C}_\alpha(Y), & \alpha \in [\omega_0, \omega_1). \end{cases}$$

Indeed, if $\alpha < \omega_0$, the claim follows from (b) by induction. If $\alpha = \omega_0$, let $(\widetilde{g}_n)$ be a bounded sequence of functions such that $\widetilde{g}_n \in \mathcal{C}_{\alpha_n}(Y)$ for some $\alpha_n < \omega_0$ and $\widetilde{g}_n \to \widetilde{g}$. Then the functions $\widetilde{h}_n(y) = \nu_y(\widetilde{g}_n)$ are in $\mathcal{C}_{\alpha_n+1}(Y)$ and converge to $\widetilde{h}$. Hence $\widetilde{h} \in \mathcal{C}_{\omega_0}(Y)$. For $\alpha > \omega_0$, the claim follows by transfinite induction.

Next we prove that $\widetilde{h} = \widetilde{f}$. Fix $y \in Y$. Using [22, Proposition 7.49] we find a maximal measure $\mu \in \mathcal{M}^1(X)$ satisfying $\varphi_\sharp \mu = \nu_y$ (here $\varphi_\sharp\colon \mathcal{M}^1(X) \to \mathcal{M}^1(Y)$ denotes the mapping induced by $\varphi\colon X \to Y$, see [8, Theorem 418I]). Then it is easy to check (see e.g. the proof of Proposition 5.29 in [22]) that

$$\varphi(r(\mu)) = r(\varphi_\sharp \mu) = r(\nu_y) = y. \tag{5.2}$$

Further, $\mu(\varphi^{-1}(\operatorname{ext} Y)) = 1$ and

$$\{x \in X\colon f(x) = \widetilde{g}(\varphi(x))\} \supset \operatorname{ext} X \cap \varphi^{-1}(\operatorname{ext} Y).$$

From these facts and Lemma 4.4 it follows that $f = \widetilde{g} \circ \varphi$ μ-almost everywhere. Thus from (5.2) and (5.1) we get

$$\begin{aligned}\widetilde{h}(y) &= \int_{\operatorname{ext} Y} \widetilde{g}\, d\nu_y = \int_{\operatorname{ext} Y} \widetilde{g}\, d(\varphi_\sharp \mu) = \int_X \widetilde{g} \circ \varphi\, d\mu = \int_X f\, d\mu \\ &= f(r(\mu)) = \widetilde{f}(\varphi(r(\mu))) = \widetilde{f}(y).\end{aligned}$$

Hence $\widetilde{f} = \widetilde{h}$ on Y.

By (5.1), f is of the same class as $\widetilde{f} = \widetilde{h}$. ∎

6. Transfer of descriptive properties on compact convex sets with $\operatorname{ext} X$ a resolvable Lindelöf set. Again in this section we work with real spaces. The first important ingredient is a result on separation of Lindelöf sets in Tychonoff spaces.

LEMMA 6.1 (see [17, Proposition 11]). *Let X_1 and X_2 be disjoint Lindelöf sets in a Tychonoff space X. Assume that there is no set $G \subset X$ satisfying $X_1 \subset G \subset X \setminus X_2$ which is a countable intersection of cozero sets. Then there exists a nonempty closed set $H \subset X$ with $\overline{H \cap X_1} = \overline{H \cap X_2} = H$.*

The following lemma is a kind of selection result.

Lemma 6.2. *Let $\varphi\colon X \to Y$ be a continuous surjective mapping of a compact space X onto a compact space Y and let $f\colon X \to \mathbb{R}$ be a bounded $\Sigma_\alpha(\mathrm{Bos}(X))$-measurable function for some $\alpha \in [2, \omega_1)$. Then there exists a mapping $\phi\colon Y \to X$ such that*

- $\varphi(\phi(y)) = y$, $y \in Y$,
- $f \circ \phi$ *is a* $\Sigma_\alpha(\mathrm{Bos}(Y))$*-measurable function.*

Proof. Using a standard approximation technique and [33, Proposition 2.3(f)] (see also [22, Lemma 5.7]) we construct a bounded sequence (f_n) of $\Sigma_\alpha(\mathrm{Bos}(X))$-measurable simple functions uniformly converging to f. More precisely, each f_n is of the form

$$f_n = \sum_{k=1}^{k_n} c_{nk}\chi_{A_{nk}}, \quad c_{nk} \in \mathbb{R}, A_{nk} \in \Delta_\alpha(\mathrm{Bos}(X)) \text{ for } k = 1, \dots, k_n,$$

where $\{A_{nk}\colon k = 1, \dots, k_n\}$ is a disjoint cover of X. For every A_{nk} we consider a countable family $\mathcal{A}_{nk} \subset \mathrm{Bos}(X)$ satisfying $A_{nk} \in \Sigma_\alpha(\mathcal{A}_{nk})$. We include all these families in a single family $\mathcal{A}$.

By [13, Lemma 8], there exists a mapping $\phi\colon Y \to X$ such that $\varphi(\phi(y)) = y$ for every $y \in Y$ and $\phi^{-1}(A) \in \mathrm{Bos}(Y)$ for every $A \in \mathcal{A}$. Then both $\phi^{-1}(A_{nk})$ and $\phi^{-1}(X \setminus A_{nk})$ are in $\Sigma_\alpha(\mathrm{Bos}(Y))$ for every set A_{nk}. Thus the functions $f_n \circ \phi$ are $\Sigma_\alpha(\mathrm{Bos}(Y))$-measurable and consequently, since they converge uniformly to $f \circ \phi$, the function $f \circ \phi$ is $\Sigma_\alpha(\mathrm{Bos}(Y))$-measurable as well. ∎

The next assertion provides an inductive step needed in the proof of Theorem 6.4.

Lemma 6.3. *Let X be a compact convex set with $\operatorname{ext} X$ a resolvable Lindelöf set and let $f\colon X \to \mathbb{R}$ be a strongly affine function such that $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X)$ for some $\alpha \in [1, \omega_0)$. Let $K \subset X$ be a nonempty compact set and $\varepsilon > 0$. Then there exists a nonempty open set U in K and a $\Sigma_{\alpha+1}(\mathrm{Hs}(U))$-measurable function g on U such that $|g - f| < \varepsilon$ on U.*

Proof. We assume that $0 \le f \le 1$. Let K be a compact set in X and $\varepsilon > 0$. By Lemma 4.7, there exists a Baire set $B \supset \operatorname{ext} X$ such that $f \in \mathcal{C}_\alpha(B)$. We claim that there exists a G_δ set G with

$$X \setminus B \subset G \subset X \setminus \operatorname{ext} X. \tag{6.1}$$

Indeed, if there were no such set, Lemma 6.1 applied to $X_1 = X \setminus B$ and $X_2 = \operatorname{ext} X$ (observe that $X \setminus B$ is Lindelöf since it is a Baire set; see [28, Theorem 2.7.1]) would provide a nonempty closed set $H \subset X$ satisfying $\overline{H \cap (X \setminus B)} = \overline{H \cap \operatorname{ext} X} = H$. But this would contradict the fact that $\operatorname{ext} X$ is a resolvable set.

We pick a G_δ set G satisfying (6.1) and write $F = X \setminus G = \bigcup F_n$, where the sets $F_1 \subset F_2 \subset \cdots$ are closed in X. Then $\operatorname{ext} X \subset \bigcup F_n \subset B$.

For each $n \in \mathbb{N}$, we set

$$M_n = \{\mu \in \mathcal{M}^1(X) \colon \mu(F_n) \geq 1 - \varepsilon/2\},$$
$$X_n = \{x \in X \colon \text{there exists } \mu \in M_n \text{ such that } r(\mu) = x\} \ (= r(M_n)).$$

Then each X_n is a closed set by the upper semicontinuity of the function $\mu \mapsto \mu(F_n)$ on $\mathcal{M}^1(X)$ and $X = \bigcup X_n$. Indeed, for any $x \in X$ there exists a maximal measure $\mu \in \mathcal{M}_x(X)$ which is carried by F (see [1, Corollary I.4.12 and the subsequent remark] or [22, Theorem 3.79]), and thus $\mu(F_n) \geq 1-\varepsilon/2$ for $n \in \mathbb{N}$ large enough.

Since $K \subset \bigcup X_n$, by the Baire category theorem there exists $m \in \mathbb{N}$ such that $X_m \cap K$ has nonempty interior in K. Let U denote this interior. Since $f|_{F_m} \in \mathcal{C}_\alpha(F_m)$, we can extend $f|_{F_m}$ to a function $h \in \mathcal{C}_\alpha(X)$ satisfying $h(X) \subset \overline{\operatorname{co}}\, f(F_m)$ (see [31, Corollary 3.5] or [22, Corollary 11.25]). Let $\widetilde{h}, \widetilde{f} \colon \mathcal{M}^1(X) \to \mathbb{R}$ be defined by

$$\widetilde{h}(\mu) = \mu(h), \qquad \widetilde{f}(\mu) = \mu(f), \qquad \mu \in \mathcal{M}^1(X).$$

Then

$$|\widetilde{f}(\mu) - \widetilde{h}(\mu)| < \varepsilon, \quad \mu \in M_m. \tag{6.2}$$

Indeed, for $\mu \in M_m$ we have

$$\begin{aligned} |\mu(f) - \mu(h)| &= \Big| \int_{F_m} (f-h)\, d\mu + \int_{X \setminus F_m} (f-h)\, d\mu \Big| \\ &\leq \int_{X \setminus F_m} |h - f|\, d\mu \leq \mu(X \setminus F_m) \leq \varepsilon/2 < \varepsilon. \end{aligned}$$

By Lemma 3.3(c), $\widetilde{h} \in \mathcal{C}_\alpha(\mathcal{M}^1(X))$, and thus it is $\Sigma_{\alpha+1}(\mathrm{Bos}(\mathcal{M}^1(X)))$-measurable on $\mathcal{M}^1(X)$.

We consider the mapping $r \colon M_m \to r(M_m)$ and use Lemma 6.2 to find a selection $\phi \colon r(M_m) \to M_m$ such that

- $r(\phi(x)) = x$, $x \in r(M_m)$,
- $\widetilde{h} \circ \phi$ is $\Sigma_{\alpha+1}(\mathrm{Bos}(r(M_m)))$-measurable on $r(M_m)$.

By setting $g = \widetilde{h} \circ \phi$ we obtain the desired function. Indeed, for a given $x \in r(M_m)$, the measure $\phi(x)$ is contained in $\mathcal{M}_x(X) \cap M_m$, and hence by (6.2) and the strong affinity of f, we have

$$|g(x) - f(x)| = |\widetilde{h}(\phi(x)) - \widetilde{f}(\phi(x))| < \varepsilon.$$

Thus $g|_U$ is as required because $\Sigma_{\alpha+1}(\mathrm{Bos})$-measurability implies $\Sigma_{\alpha+1}(\mathrm{Hs})$-measurability. ∎

THEOREM 6.4. *Let X be a compact convex set with $\operatorname{ext} X$ a resolvable Lindelöf set. Let $f\colon X \to \mathbb{R}$ be a strongly affine function such that $f|_{\operatorname{ext} X} \in \mathcal{C}_\alpha(\operatorname{ext} X)$ for some $\alpha \in [1, \omega_1)$. Then $f \in \mathcal{C}_\alpha(X)$.*

Proof. We assume that $0 \le f \le 1$. Also we may assume that $\alpha \in [1, \omega_0)$ since other cases are covered by Theorem 5.2. We claim that f is $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable.

Let $\varepsilon > 0$. We construct a regular sequence $\emptyset = U_0 \subset U_1 \subset \cdots \subset U_\kappa = X$ and functions

$$g_\gamma \in \Sigma_{\alpha+1}(\mathrm{Hs}(U_{\gamma+1} \setminus U_\gamma)), \quad \gamma < \kappa,$$

satisfying $|g - f| < \varepsilon$ on $U_{\gamma+1} \setminus U_\gamma$ as follows.

Let $U_0 = \emptyset$. Using Lemma 6.3 we select a nonempty open set U of X along with a $\Sigma_{\alpha+1}(\mathrm{Hs}(U))$-measurable function g on U with $|g - f| < \varepsilon$ on U. We set $U_1 = U$ and $g_0 = g$.

Assume now that U_δ and g_δ are chosen for all δ less than some γ. If γ is limit, we set $U_\gamma = \bigcup_{\delta<\gamma} U_\delta$.

Let $\gamma = \lambda + 1$. If $U_\lambda = X$, we set $\kappa = \lambda$ and stop the procedure. Otherwise we apply Lemma 6.3 to $K = X \setminus U_\lambda$ and obtain an open set $U \subset X$ intersecting K along with a $\Sigma_{\alpha+1}(\mathrm{Hs}(U \cap K))$-measurable function g on $U \cap K$ satisfying $|g - f| < \varepsilon$ on $U \cap K$. We set $U_\gamma = U_\lambda \cup U$ and $g_\lambda = g$. This finishes the construction.

Let $g\colon X \to \mathbb{R}$ be defined as $g = g_\gamma$ on $U_{\gamma+1} \setminus U_\gamma$, $\gamma < \kappa$. By Proposition 2.2, g is a $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable function.

By using the procedure above we can approximate f uniformly by $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable functions, so f itself is $\Sigma_{\alpha+1}(\mathrm{Hs}(X))$-measurable. But f is a Baire function by Proposition 4.8. Thus Theorem 5.2 and Corollary 5.5 in [33] imply $f \in \mathcal{C}_\alpha(X)$. ∎

7. Proofs of the main results. Before proving the main results we recall a simple observation.

LEMMA 7.1. *Let E be a complex Banach space and let $f \in E^{**}$. Then f is strongly affine on B_{E^*} if and only if $\operatorname{Re} f$ is strongly affine on B_{E^*}.*

Proof. If f is strongly affine on B_{E^*} and $\mu \in \mathcal{M}^1(B_{E^*})$ has barycenter x^*, then

$$\operatorname{Re} f(x^*) + i \operatorname{Im} f(x^*) = f(x^*) = \mu(f) = \mu(\operatorname{Re} f) + i\mu(\operatorname{Im} f),$$

and thus $\mu(\operatorname{Re} f) = \operatorname{Re} f(x^*)$ and $\mu(\operatorname{Im} f) = \operatorname{Im} f(x^*)$.

Conversely, assuming that $\operatorname{Re} f$ is strongly affine on B_{E^*}, we infer that so is $\operatorname{Im} f$. To see this, consider the affine surjective homeomorphic mapping $\varphi\colon B_{E^*} \to B_{E^*}$ defined by

$$\varphi(y^*) = iy^*, \quad y^* \in B_{E^*}.$$

Since $\operatorname{Im} f(y^*) = -\operatorname{Re} f(iy^*)$ for $y^* \in E^*$, the function $\operatorname{Im} f$ is a composition of an affine homeomorphism and a strongly affine function, and hence it is strongly affine as well. Thus, for $\mu \in \mathcal{M}^1(B_{E^*})$ with barycenter x^*,

$$\mu(f) = \mu(\operatorname{Re} f) + i\mu(\operatorname{Im} f) = \operatorname{Re} f(x^*) + i \operatorname{Im} f(x^*) = f(x^*),$$

and f is strongly affine. ∎

Proofs of Theorems 1.1–1.3. Let E be a (real or complex) Banach space and f be an element of E^{**} whose restriction to B_{E^*} is strongly affine. By forgetting multiplication by complex numbers, we can regard B_{E^*} as a compact convex set in a real locally convex space. The function $\operatorname{Re} f$ is then a strongly affine function on a compact convex set B_{E^*} that inherits all descriptive properties from f. Thus if $f|_{\overline{\operatorname{ext} B_{E^*}}} \in \operatorname{Hf}_\alpha(\overline{\operatorname{ext} B_{E^*}})$, then $\operatorname{Re} f$ is a strongly affine real-valued function with $\operatorname{Re} f|_{\overline{\operatorname{ext} B_{E^*}}} \in \operatorname{Hf}_\alpha(\overline{\operatorname{ext} B_{E^*}})$. An application of Theorem 3.5 gives $\operatorname{Re} f \in \operatorname{Hf}_\alpha(B_{E^*})$. Thus both $\operatorname{Re} f$ and $\operatorname{Im} f$ are in $\operatorname{Hf}_\alpha(B_{E^*})$, and so $f = \operatorname{Re} f + i \operatorname{Im} f$ is in $\operatorname{Hf}_\alpha(B_{E^*})$. Similarly we prove the other assertions of Theorem 1.1.

Apparently, this procedure also verifies Theorems 1.2 and 1.3. ∎

Proof of Theorem 1.4. From now on we will be working with real spaces. We start with the following assertion which shows the required result for Banach spaces of continuous affine functions on simplices. The general result will then be obtained by applying a result of W. Lusky [23].

PROPOSITION 7.2. *Let $f: X \to \mathbb{R}$ be a strongly affine function on a simplex X such that $f \in \mathcal{C}_\alpha(X)$ for some $\alpha \geq 2$. Then*

$$f \in \begin{cases} \mathfrak{A}_{\alpha+1}(X), & \alpha \in [2, \omega_0), \\ \mathfrak{A}_\alpha(X), & \alpha \in [\omega_0, \omega_1). \end{cases}$$

If, moreover, $\operatorname{ext} X$ *is a Lindelöf resolvable set, then $f \in \mathfrak{A}_\alpha(X)$.*

Proof of Proposition 7.2. If X is a general simplex, the assertion for finite ordinals is proved in [5, Théorème 2], and for infinite ordinals in [16, Theorem 1.2].

Assume now that X is a simplex with $\operatorname{ext} X$ a Lindelöf resolvable set. For each $x \in X$, let δ_x denote the unique maximal measure in $\mathcal{M}_x(X)$. By [34, Theorem 1] and [22, Theorem 4.24], the function $Tg(x) = \delta_x(g)$, $x \in X$, is in $\mathfrak{A}_1(X)$ for any bounded $g \in \mathcal{C}_1(X)$. By induction, $Tg \in \mathfrak{A}_\beta(X)$ for any bounded function $g \in \mathcal{C}_\beta(X)$ and finite ordinal $\beta \in [2, \omega_0)$. Thus, for any $\alpha \in [2, \omega_0)$ and any strongly affine function $f \in \mathcal{C}_\alpha(X)$, $f = Tf \in \mathfrak{A}_\alpha(X)$. ∎

Let E be a real L_1-predual and $f \in E^{**}$ be a strongly affine function satisfying $f \in \mathcal{C}_\alpha(B_{E^*})$ for some $\alpha \in [2, \omega_1)$. By [23, Theorem], there exist a simplex X, an isometric embedding $j: E \to \mathfrak{A}^c(X)$ and a projection

$P\colon \mathfrak{A}^c(X) \to j(E)$ of norm 1. Further, it is proved in [23, Corollary III] that there exists an affine continuous surjection $\varphi\colon X \to B_{E^*}$ such that

(1) $\varphi(\operatorname{ext} X) = \operatorname{ext} B_{E^*} \cup \{0\}$ and $\varphi^{-1}(\operatorname{ext} B_{E^*}) \subset \operatorname{ext} X$,
(2) $\varphi|_{\operatorname{ext} X}$ is injective,
(3) $\operatorname{ext} X \setminus \varphi^{-1}(\operatorname{ext} B_{E^*})$ is a singleton,
(4) $j(e)(x) = (e \circ \varphi)(x)$, $e \in E$, $x \in X$.

(In the notation of [23], the embedding j is denoted by T and φ is denoted by q. Conditions (1)–(3) are explicitly stated in [23, Corollary III], condition (4) follows from the definitions of T on p. 175 and q on p. 176.)

The projection P provides for each $x \in X$ a measure $\mu_x \in B_{\mathcal{M}(X)}$ such that

$$Pg(x) = \mu_x(g), \quad g \in \mathfrak{A}^c(X). \tag{7.1}$$

Since P is the identity on $j(E)$, from (4) we obtain

$$\mu_x(e \circ \varphi) = (e \circ \varphi)(x), \quad x \in X, e \in E.$$

We use equality (7.1) to extend the domain of P to any bounded universally measurable function on X.

We claim that

$$\mu_x(f \circ \varphi) = f(\varphi(x)), \quad x \in X. \tag{7.2}$$

To verify this, let $x \in X$. We write

$$\mu_x = a_1\mu_1 - a_2\mu_2, \quad a_1, a_2 \geq 0 \text{ with } a_1 + a_2 \leq 1,\ \mu_1, \mu_2 \in \mathcal{M}^1(X),$$

and let $x_1, x_2 \in X$ be the barycenters of μ_1 and μ_2, respectively. Then

$$\varphi(x) = a_1\varphi(x_1) - a_2\varphi(x_2). \tag{7.3}$$

Indeed, let $e \in E$. Then

$$\begin{aligned} e(\varphi(x)) &= \mu_x(e \circ \varphi) = a_1\mu_1(e \circ \varphi) - a_2\mu_2(e \circ \varphi) \\ &= a_1 e(\varphi(x_1)) - a_2 e(\varphi(x_2)) = e(a_1\varphi(x_1) - a_2\varphi(x_2)). \end{aligned}$$

Hence (7.3) holds.

Since $f \circ \varphi$ is strongly affine on X by [32, Lemma 2.3] (see also [22, Proposition 5.29]), from (7.3) we get

$$\begin{aligned} \mu_x(f \circ \varphi) &= a_1\mu_1(f \circ \varphi) - a_2\mu_2(f \circ \varphi) = a_1 f(\varphi(x_1)) - a_2 f(\varphi(x_2)) \\ &= f(a_1\varphi(x_1) - a_2\varphi(x_2)) = f(\varphi(x)). \end{aligned}$$

This verifies (7.2).

Now we prove by induction that $Pg \in (j(E))_\beta$ provided $g \in \mathfrak{A}_\beta(X)$ for some $\beta \geq 1$. First consider the case $\beta = 1$, i.e., there exists a bounded sequence (g_n) in $\mathfrak{A}^c(X)$ with $g_n \to g$. Then $Pg_n \in j(E)$ and, by the Lebesgue dominated convergence theorem, $Pg_n \to Pg$.

Assuming the validity of the assertion for all ordinals $\widetilde{\beta}$ smaller than some β, we consider $g \in \mathfrak{A}_\beta(X)$. Let (g_n) be a bounded sequence converging pointwise to g, where $g_n \in \mathfrak{A}_{\beta_n}(X)$ for some $\beta_n < \beta$. Then $Pg_n \in (j(E))_{\beta_n}$ and, as above, $Pg_n \to Pg$.

Now we get back to the function f. Since $f \circ \varphi \in \mathcal{C}_\alpha(X)$, Proposition 7.2 implies that $f \circ \varphi \in \mathfrak{A}_\beta(X)$, where either $\beta = \alpha + 1$ if $\alpha < \omega_0$, or $\beta = \alpha$ otherwise. By the reasoning above and (7.2),

$$f \circ \varphi = P(f \circ \varphi) \in (j(E))_\beta.$$

Since $j(e) = e \circ \varphi$ for each $e \in E$, it follows that $f \in \mathfrak{A}_\beta(B_{E^*})$. This concludes the proof of the first part of the theorem.

If, moreover, we assume that $\operatorname{ext} B_{E^*}$ is a Lindelöf resolvable set, then $\operatorname{ext} X$ is a Lindelöf resolvable set as well. To show this, we first notice that $\operatorname{ext} X$ differs from the resolvable set $\varphi^{-1}(\operatorname{ext} B_{E^*})$ by a singleton (see (1) and (3)), and thus it is resolvable. Second, let $F \subset X \setminus \operatorname{ext} X$ be a compact set. By (1), $\varphi(F)$ is disjoint from $\operatorname{ext} B_{E^*}$. Since $\operatorname{ext} B_{E^*}$ is Lindelöf, [34, Lemma 14] provides an F_σ set A with

$$\operatorname{ext} B_{E^*} \subset A \subset B_{E^*} \setminus \varphi(F).$$

If $x_0 \in X$ denotes the singleton $\operatorname{ext} X \setminus \varphi^{-1}(\operatorname{ext} B_{E^*})$, then $\varphi^{-1}(A)$ is an F_σ set in X satisfying

$$\operatorname{ext} X \subset \varphi^{-1}(A) \cup \{x_0\} \subset X \setminus F.$$

By [34, Lemma 15], $\operatorname{ext} X$ is a Lindelöf space.

Now we can conclude the proof as in the first part; the only difference is that we use the second part of Proposition 7.2. ∎

8. Examples. Banach spaces constructed in this section are real L_1-preduals and they are created using the notion of a *simplicial function space*. In order to illuminate the construction, we need to recall several definitions and facts.

If K is a compact topological space, then $\mathcal{H} \subset \mathcal{C}(K)$ is a *function space* if $\mathcal{H}$ is a subspace of $\mathcal{C}(K)$, contains the constant functions and separates the points of K. For simplicity, we will construct real Banach spaces, and thus we will deal in this section only with real spaces $\mathcal{C}(K)$. For $x \in K$, we write $\mathcal{M}_x(\mathcal{H})$ for the set of all measures $\mu \in \mathcal{M}^1(K)$ with $\mu(h) = h(x)$ for all $h \in \mathcal{H}$. Let $\operatorname{Ch}_{\mathcal{H}}(K)$ be the *Choquet boundary* of $\mathcal{H}$, i.e., the set of those points $x \in K$ with $\mathcal{M}_x(\mathcal{H}) = \{\varepsilon_x\}$. By defining $\mathcal{A}^c(\mathcal{H}) = \{f \in \mathcal{C}(K) \colon \mu(f) = f(x) \text{ for all } x \in K \text{ and } \mu \in \mathcal{M}_x(\mathcal{H})\}$ we obtain a closed function space satisfying $\mathcal{H} \subset \mathcal{A}^c(\mathcal{H})$ (see [22, Definition 3.8]) and $\operatorname{Ch}_{\mathcal{H}}(K) = \operatorname{Ch}_{\mathcal{A}^c(\mathcal{H})}(K)$ (this follows easily from the definitions).

Let

$$\mathbf{S}(\mathcal{H}) = \{s \in \mathcal{H}^* \colon s \geq 0,\ \|s\| = 1\}$$

denote the *state space* of $\mathcal{H}$. Then $\mathbf{S}(\mathcal{H})$, endowed with the weak* topology, is a compact convex set and K is homeomorphically embedded in $\mathbf{S}(\mathcal{H})$ via the mapping $\phi\colon K \to \mathbf{S}(\mathcal{H})$ assigning to each $x \in K$ the point evaluation at x. Moreover, $\phi(\mathrm{Ch}_{\mathcal{H}}(K)) = \operatorname{ext} \mathbf{S}(\mathcal{H})$ (see [25, Proposition 6.2] or [22, Proposition 4.26]).

The function space $\mathcal{H}$ is called *simplicial* if $\mathbf{S}(\mathcal{A}^c(\mathcal{H}))$ is a simplex (see [22, Theorem 6.54]).

Further, let $\mathcal{H}^{\perp\perp}$ denote the space of all universally measurable functions $f\colon K \to \mathbb{R}$ satisfying $\mu(f) = 0$ for every $\mu \in \mathcal{H}^{\perp} \subset \mathcal{M}(K)$. It is proved in [32, Theorem 2.5] (see also [22, Corollary 5.41]) that for any $f \in \mathcal{H}^{\perp\perp}$ there exists a strongly affine function $\widetilde{f}\colon \mathbf{S}(\mathcal{H}) \to \mathbb{R}$ with $f = \widetilde{f} \circ \phi$. Moreover, $\widetilde{f}$ inherits from f all descriptive properties considered in this paper; more precisely, for any $\alpha \in [1, \omega_1)$ we have $f \in \mathcal{C}_\alpha(K)$, $f \in \mathrm{Bof}_\alpha(K)$ and $f \in \mathrm{Hf}_\alpha(K)$ if and only if $\widetilde{f} \in \mathcal{C}_\alpha(\mathbf{S}(\mathcal{H}))$, $\widetilde{f} \in \mathrm{Bof}_\alpha(\mathbf{S}(\mathcal{H}))$ and $\widetilde{f} \in \mathrm{Hf}_\alpha(\mathbf{S}(\mathcal{H}))$, respectively (the first two assertions are proved in [22, Corollary 5.41], the last one follows from Theorem 3.5).

A standard construction from [4, Section VII] of a simplicial function space $\mathcal{H}$ satisfying $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$ goes as follows. Take a compact space L and a subset B of L and define

$$K = (L \times \{0\}) \cup (B \times \{-1, 1\})$$

with the "porcupine topology", i.e., points of $K \setminus (L \times \{0\})$ are isolated and a point $(x, 0) \in K$ has a basis of neighborhoods consisting of the sets of the form

$$K \cap (U \times \{-1, 0, 1\}) \setminus F,$$

where $U \subset L$ is a neighborhood of x and $F \subset K \setminus (L \times \{0\})$ is finite. Then K is a compact space and

$$\mathcal{H} = \{f \in \mathcal{C}(K)\colon f(x, 0) = \tfrac{1}{2}(f(x, 1) + f(x, -1)),\ x \in B\}$$

is a simplicial function space satisfying $\mathcal{H} = \mathcal{A}^c(\mathcal{H})$ and

$$\mathrm{Ch}_{\mathcal{H}}(K) = K \setminus (B \times \{0\})$$

(see [35] or [22, Definition 6.13 and Lemma 6.14]).

If $f\colon K \to \mathbb{R}$ is a bounded universally measurable function satisfying $f(x, 0) = \frac{1}{2}(f(x, 1) + f(x, -1))$ for each $x \in B$, it is easy to verify that $f \in \mathcal{H}^{\perp\perp}$ (see [22, Corollary 6.12]), and thus f induces a strongly affine function $\widetilde{f}\colon \mathbf{S}(\mathcal{H}) \to \mathbb{R}$ which satisfies $f = \widetilde{f} \circ \phi$ and shares with f all descriptive properties.

By this procedure we obtain a simplex $X = \mathbf{S}(\mathcal{H})$ and a strongly affine function on X with the desired descriptive properties. It is well known (see e.g. [22, Propositions 4.31 and 4.32]) that, given a compact convex set X, the dual space $(\mathfrak{A}^c(X))^*$ can be identified with $\operatorname{span} X$ and the dual unit ball with $\mathrm{co}(X\cup(-X))$, whereas the second dual $(\mathfrak{A}^c(X))^{**}$ equals the space of all affine bounded functions on X. Hence the construction of a simplex X along with a strongly affine function f with the prescribed descriptive properties yields the resulting L_1-predual E: we set $E=\mathfrak{A}^c(X)$ and the element $x^{**}\in E^{**}$ is the function f.

This general construction is now used in the following examples.

Example 8.1. There exist a separable L_1-predual E and a strongly affine function $f\in E^{**}$ such that $f|_{\operatorname{ext} B_{E^*}}\in\mathcal{C}_1(\operatorname{ext} B_{E^*})$ and $f\notin\mathcal{C}_1(B_{E^*})$.

Proof. Let $L=[0,1]$ and let B denote the set of all rational numbers in L. Let K, $\mathcal{H}$ and X be constructed as above. Then K is metrizable, and thus $E=\mathfrak{A}^c(X)$ is a separable space. Let $f\colon K\to\mathbb{R}$ be defined by

$$f(x,t)=\begin{cases}1, & x\in B,\\ 0, & x\notin B,\end{cases}\qquad (x,t)\in K.$$

Then $f|_{\mathrm{Ch}_{\mathcal{H}}(K)}\in\mathcal{C}_1(\mathrm{Ch}_{\mathcal{H}}(K))$ since $f|_{\mathrm{Ch}_{\mathcal{H}}(K)}$ is the characteristic function of an open set in $\mathrm{Ch}_{\mathcal{H}}(K)$. On the other hand, f has no point of continuity on $L\times\{0\}$, and thus $f\notin\mathcal{C}_1(K)$. ∎

Example 8.2. There exist an L_1-predual E and a strongly affine function $f\in E^{**}$ such that $\operatorname{ext} B_{E^*}$ is an open set in $\overline{\operatorname{ext} B_{E^*}}$ (hence $\operatorname{ext} B_{E^*}\in\mathrm{Bos}(B_{E^*})$), $f|_{\operatorname{ext} B_{E^*}}\in\mathcal{C}(\operatorname{ext} B_{E^*})$ and f is not resolvably measurable on B_{E^*}.

Proof. Let $L=B=[0,1]$ and A be an analytic non-Borel set in L (see [18, Theorem 14.2]) and let K, $\mathcal{H}$ and X be constructed as above. Then $\mathrm{Ch}_{\mathcal{H}}(K)=K\setminus(L\times\{0\})$ is an open set in $\overline{\mathrm{Ch}_{\mathcal{H}}(K)}=K$. Further, let $f\colon K\to\mathbb{R}$ be defined by

$$f(x,t)=\begin{cases}1, & x\in A,\\ 0, & x\notin A,\end{cases}\qquad (x,t)\in K.$$

Then $f|_{\mathrm{Ch}_{\mathcal{H}}(K)}\in\mathcal{C}(\mathrm{Ch}_{\mathcal{H}}(K))$ since $f|_{\mathrm{Ch}_{\mathcal{H}}(K)}$ is the characteristic function of a clopen set in $\mathrm{Ch}_{\mathcal{H}}(K)$. Since A is μ-measurable for any Radon measure μ on $[0,1]$, f is universally measurable on K (see [18, Theorem 21.10]). Obviously, $f|_{L\times\{0\}}$ is not Borel on $L\times\{0\}$. Since the σ-algebra of Borel sets in L coincides with the σ-algebra generated by the resolvable sets in L (see [33, Proposition 3.4]), f is not measurable on K with respect to the σ-algebra generated by resolvable sets. ∎

Example 8.3. Assuming (CH), there exist an L_1-predual E with $\operatorname{ext} B_{E^*}$ Lindelöf and a strongly affine function $f\in E^{**}$ such that f is not a resolvably measurable function and $f|_{\operatorname{ext} B_{E^*}}\in\mathrm{Bof}_1(\operatorname{ext} B_{E^*})$.

Proof. Let $L=[0,1]$ and let Q stand for the set of all rational numbers in L. Assuming the continuum hypothesis, by the method of proof of [24, Proposition 4.9] we construct an uncountable set B disjoint from Q that concentrates around Q (i.e., $B\setminus U$ is countable for any open $U\supset Q$). Let K,

$\mathcal{H}$ and X be as above. Then $\mathrm{Ch}_{\mathcal{H}}(K) = K \setminus (B \times \{0\})$ is Lindelöf. Indeed, if $\mathcal{U}$ is an open cover of $\mathrm{Ch}_{\mathcal{H}}(K)$, we select a countable family $\mathcal{V} \subset \mathcal{U}$ satisfying

$$(L \times \{0\}) \setminus (B \times \{0\}) \subset V = \bigcup \{U \cap (L \times \{0\}) : U \in \mathcal{V}\}.$$

Then V is an open set in $L \times \{0\}$ containing $Q \times \{0\}$, and thus $B \setminus V$ is countable. Hence we may extract a countable family $\mathcal{W} \subset \mathcal{U}$ which covers that part of $\mathrm{Ch}_{\mathcal{H}}(K)$ not already contained in V. Thus $\mathcal{V} \cup \mathcal{W}$ is a countable subcover of $\mathrm{Ch}_{\mathcal{H}}(K)$.

Define a function $f\colon K \to \mathbb{R}$ by

$$f(x,t) = \begin{cases} 1, & x \in B, \\ 0, & x \notin B, \end{cases} \qquad (x,t) \in K.$$

Then f is universally measurable on K. To see this, it is enough to verify that B is universally measurable. If $\mu \in \mathcal{M}^1([0,1])$ is a continuous measure (i.e., $\mu(\{x\}) = 0$ for each $x \in [0,1]$), let (U_n) be a sequence of open sets satisfying $\mu(U_n) < 1/n$ and $U_n \supset Q$. Then $\mu(\bigcap U_n) = 0$ and $B \setminus \bigcap U_n$ is countable, and thus μ-measurable. Hence B is μ-measurable for every continuous measure. Obviously, B is μ-measurable for any discrete probability measure μ, and hence B is universally measurable.

On the other hand, B is not Borel, because otherwise, as an uncountable set, it would contain a copy of the Cantor set (see [18, Theorem 13.6]), which would contradict its concentration around Q.

Since $f|_{\mathrm{Ch}_{\mathcal{H}}(K)}$ is the characteristic function of an open set in $\mathrm{Ch}_{\mathcal{H}}(K)$, we have $f|_{\mathrm{Ch}_{\mathcal{H}}(K)} \in \mathrm{Bof}_1(\mathrm{Ch}_{\mathcal{H}}(K))$. On the other hand, f is not resolvably measurable on K because f is not Borel on $L \times \{0\}$ and the σ-algebra of Borel sets in $L \times \{0\}$ coincides with the σ-algebra generated by the resolvable sets in $L \times \{0\}$ (see [33, Proposition 3.4]). Thus f is the required function. ∎

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Pavel Ludvík, Jiří Spurný
Department of Mathematical Analysis
Faculty of Mathematics and Physics
Charles University
Sokolovská 83
186 75 Praha 8, Czech Republic
E-mail: ludvik@karlin.mff.cuni.cz
spurny@karlin.mff.cuni.cz

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