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Minimal ideals of group algebras

by

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Abstract. We first study the behavior of weights on a simply connected nilpotent Lie group G. Then for a subalgebra A of $L^1(G)$ containing the Schwartz algebra $\mathcal{S}(G)$ as a dense subspace, we characterize all closed two-sided ideals of A whose hull reduces to one point which is a character.

Introduction. Let G be a simply connected nilpotent Lie group, $\mathfrak{g}$ its Lie algebra, and A a subalgebra of $L^1(G)$. To every character χ_l of A we will associate a finite-dimensional translation invariant subspace $\mathcal{P}_l$ of the vector space $\mathcal{P}(G)$ of complex polynomials on G and we will show that the set of closed two-sided ideals of A with hull $\{\operatorname{Ker}\chi_l\}$ is in bijection with the set of nonzero translation-invariant subspaces of $\mathcal{P}_l$. As an example of A we can take the weighted algebra $L^1_w(G)$ where w is a weight with polynomial growth. Such weights appear in a natural way in the following manner: let π be a unitary continuous irreducible representation of G in a Hilbert space $\mathcal{H}_\pi$. We denote by $\mathcal{U}(\mathfrak{g})$ the enveloping algebra of $\mathfrak{g}$. Fix a nonzero integer k and denote by $\mathcal{U}(\mathfrak{g})_k$ the vector space generated by the elements of $\mathcal{U}(\mathfrak{g})$ with degree less than k. Let $\mathcal{H}_\pi^{(k)}$ be the space of k times differentiable vectors in $\mathcal{H}_\pi$, i.e.

$$\mathcal{H}_\pi^{(k)} = \{\xi \in \mathcal{H}_\pi \mid \forall z \in \mathcal{U}(\mathfrak{g})_k : d\pi(z)\xi \in \mathcal{H}_\pi\}.$$

Fix a basis $(z^i)_{|i|\le k}$ of $\mathcal{U}(\mathfrak{g})_k$. We equip $\mathcal{H}_\pi^{(k)}$ with the norm

$$\|\xi\|_k = \Big(\sum_{|i|\le k} \|d\pi(z^i)\xi\|^2\Big)^{1/2}.$$

The space $\mathcal{H}_\pi^{(k)}$ with this norm is complete. Denoting by $\|\pi(x)\|_{\mathrm{op}}$ the norm of the operator $\pi(x) : \mathcal{H}_\pi^{(k)} \to \mathcal{H}_\pi^{(k)}$, we then have

$$\|\pi(x)\|_{\mathrm{op}} \le \|\mathrm{Ad}(x)|_{\mathcal{U}(\mathfrak{g})_k}\|_{\mathrm{HS}}$$

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where $\| \|_{\mathrm{HS}}$ denotes the Hilbert–Schmidt norm. Denote by $w^{(k)}(x)$ the right side of this inequality. The function $x \mapsto w^{(k)}(x)$ is a natural example of a weight on G attached to π. By a *weight* on a topological group G, we mean a measurable function w on G with values in $[1, +\infty[$ such that for all s and t in G,

$$w(st) \leq w(s)w(t).$$

The preceding result leads naturally to the study of weights on nilpotent Lie groups. The first section will give another example of a natural weight. Other examples of weights come from Banach space representations of topological groups. Let X be a Banach space and let (T, X) be a Banach space representation of G on X. That means that for every s in G, we have a bounded invertible operator $T(s)$ on X such that the mapping $s \mapsto T(s)$ is a homomorphism of groups and the mappings $s \mapsto T(s)x$ are continuous for every x in X. Then the operator norm $\|T(s)\|_{\mathrm{op}}$ is a measurable function on G and defines a symmetric weight $w_T : s \mapsto \max(\|T(s)\|_{\mathrm{op}}, \|T(s^{-1})\|_{\mathrm{op}})$.

Take for example the 3-dimensional Heisenberg group $G = H_1$. For x in H_1 write $x = (a, b, t)$ and let (X, Y, Z) be a basis of the Lie algebra $\mathfrak{h}_1$ of H_1 with $[X, Y] = Z$. We have

$$\mathrm{Ad}(x)X = X - bZ, \quad \mathrm{Ad}(x)Y = Y + aZ, \quad \mathrm{Ad}(x)Z = Z.$$

After an easy computation, we find

$$w^{(2)}(x) = (9 + 7a^2 + 7b^2 + a^2b^2 + a^4 + b^4)^{1/2}$$

and

$$\frac{1}{\sqrt{2}}(1 + a^2 + b^2) \leq w^{(2)}(x) \leq 3(1 + a^2 + b^2).$$

1. Weights on topological (in particular nilpotent Lie) groups. Weights allow us to define Banach subalgebras of $L^1(G)$, the so-called Beurling algebras. This section studies the growth of the "most natural" weight attached to a connected locally compact group. This weight is of importance because it dominates all common weights. We end this section with a restriction property of this weight.

DEFINITION. Let G be a topological group and S a subset of G. We write $S^0 = \{e\}$ and for all n in $\mathbb{N}^*$,

$$S^n = \{s_1 \ldots s_n \mid s_i \in S\}.$$

When G is locally compact, for s in G, we denote by $\mathfrak{V}_G(s)$ the set of compact neighborhoods of s in G.

In the following proposition we recall the "most natural" weight attached to a connected locally compact group as in [9].

1.1. Proposition. *Let G be a connected locally compact group and U an element of $\mathfrak{V}_G(e)$. Then $G = \bigcup_{n\in\mathbb{N}} U^n$ and the map $\tau_U : G \to \mathbb{N}$ defined by*

$$\tau_U(s) = \min\{n \in \mathbb{N} \mid s \in U^n\}$$

is measurable and satisfies

$$\tau_U(s) = 0 \;\Leftrightarrow\; s = e, \qquad \tau_U(st) \le \tau_U(s) + \tau_U(t).$$

If in addition U is symmetric, then

$$\tau_U(s^{-1}) = \tau_U(s).$$

It seems difficult to define canonically the notion of a "polynomial function" on any group G. In the absence of such a notion, the following definition tries to define in a natural way a function "of polynomial growth" on a class of groups as large as possible.

1.2. Definition. Let G be a connected locally compact group. A function $f : G \to \mathbb{C}$ is said to be *of polynomial growth* if for all U in $\mathfrak{V}_G(e)$, there exists a polynomial P_U in one variable, with real coefficients, such that for all s in G,

$$|f(s)| \le P_U(\tau_U(s)).$$

For example for a connected compact group G, the functions with polynomial growth on G are bounded functions. More generally, it is easy to check that under the conditions of 1.2, a function with polynomial growth is bounded on all compact subsets. Since for any two elements U and V of $\mathfrak{V}_G(e)$, there exist strictly positive numbers k and k' such that

$$\tau_V \le k\tau_U \le k'\tau_V,$$

it follows that if $f : G \to \mathbb{C}$ satisfies $|f| \le P_U \circ \tau_U$ for *one* compact neighborhood U of e in G, then such a relation is true for all compact neighborhoods of e in G, i.e. f is of polynomial growth.

Notation. Let G be a group. For $f : G \to \mathbb{C}$, we denote by $\check{f}$ the function $s \mapsto f(s^{-1})$.

It is clear that the set of weights on G is stable under pointwise multiplication, involution $w \mapsto \check{w}$, finite simple limit, finite upper hull, and left composition by functions of the form $\exp\circ f \circ \ln$, where f is an increasing and subadditive function $\mathbb{R}_+ \to \mathbb{R}_+$. Such functions are studied in [8].

1.3. Example. For a connected locally compact group G and U in $\mathfrak{V}_G(e)$, the map $1 + \tau_U$, denoted by w_U, is clearly a weight on G, satisfying in addition

$$w_U(st) \le w_U(s) + w_U(t).$$

This weight will be studied in detail in the following when G will be assumed to be a nilpotent Lie group. By [6], we have:

1.4. Proposition. *Let G be a connected Lie group with Lie algebra $\mathfrak{g}$. A norm $\| \ \|$ on the vector space $\mathfrak{g}$ being fixed, for all U in $\mathfrak{V}_G(e)$, there exists a strictly positive number c_U such that for all X in $\mathfrak{g}$,*

$$w_U(\exp X) < 2 + c_U\|X\|.$$

Until the end of this section, G denotes a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Starting with $\mathfrak{g}_0 = \mathfrak{g}$, we define $\mathfrak{g}_m$ for m in $\mathbb{N}^*$ as the real vector space generated by the set of $[X, Y]$ where X runs through $\mathfrak{g}$ and Y runs through $\mathfrak{g}_{m-1}$. The step of nilpotency of $\mathfrak{g}$ is denoted by n; this means that $\mathfrak{g}_n$ reduces to $\{0\}$ and $\mathfrak{g}_{n-1}$ is nonzero. Hence, an element X of $\mathfrak{g}$ belongs to $\mathfrak{g}_i$ if and only if X is a linear combination of terms requiring at least i brackets in all. For all i in $\{1, \ldots, n\}$, choose a complementary subspace V_i of $\mathfrak{g}_i$ in $\mathfrak{g}_{i-1}$. Then

$$\mathfrak{g} = \bigoplus_{i=1}^{n} V_i.$$

For all k in $\{0, \ldots, n\}$, let G_k be $\exp \mathfrak{g}_k$. Then G_k is the closure in G of the subgroup generated by the elements $xyx^{-1}y^{-1}$ where x runs through G and y runs through G_{k-1}. The exponential map exp is a C^∞ diffeomorphism of $\mathfrak{g}$ onto G, which allows us to identify G with the real vector space $\mathfrak{g}$ as manifolds. If $\mathfrak{g}$ is endowed with the Baker–Campbell–Hausdorff product

$$X \cdot Y = X + Y + \frac{1}{2}[X, Y] + \frac{1}{12}([X, [X, Y]] + [Y, [Y, X]]) + (\text{commutators of order 3 at least})$$

then exp is an isomorphism of topological groups from $\mathfrak{g}$ onto G, which allows us to identify the groups G and $(\mathfrak{g}, \cdot)$. For this group law, $-X$ is the inverse of X. Finally, for X and Y in $\mathfrak{g}$, we set

$$\{X, Y\} = X \cdot Y \cdot (-X) \cdot (-Y).$$

By [21], we have:

1.5. Lemma. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of step n. For all $X_1, \ldots, X_n$ in $\mathfrak{g}$, we have*

$$[X_1, [X_2, [\ldots, X_{n-1}] \ldots]] \equiv \{X_1, \{X_2, \{\ldots, X_{n-1}\} \ldots\}\} \bmod \mathfrak{g}_{n-1},$$
$$[X_1, [X_2, [\ldots, X_n] \ldots]] = \{X_1, \{X_2, \{\ldots, X_n\} \ldots\}\}.$$

In the following proposition, the bracket of two elements X and Y will be written as product in the group $\mathfrak{g}$ of a_iX and b_iY where a_i and b_i are real numbers. We give a bound for the number of factors in the product, which improves a result of [21].

1.6. Proposition. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of step n greater than 2.*

1) *There exists an integer* m, *depending only on* n *and* $2m$ *real numbers* $a_1, \dots, a_m$, $b_1, \dots, b_m$, *such that for all* X *and* Y *in* $\mathfrak{g}$, *we have*

$$X + Y = \prod_{i=1}^{m} (a_i X) \cdot (b_i Y).$$

2) *There exists an integer* p, *depending only on* n *and* $2p$ *real numbers* $c_1, \dots, c_p$, $d_1, \dots, d_p$, *such that for all* X *and* Y *in* $\mathfrak{g}$, *we have*

$$[X, Y] = \prod_{i=1}^{p} (c_i X) \cdot (d_i Y).$$

In addition m *and* p *are less than* $2^n(2^n - 5) + 2n + 2$.

Proof. 1) If $n = 2$, then for all X and Y in $\mathfrak{g}$, we have

$$X + Y = \frac{X}{2} \cdot Y \cdot \frac{X}{2}.$$

Assume the result is true for a nilpotent Lie algebra of step $n - 1 \geq 2$ and let $\mathfrak{g}$ be a nilpotent Lie algebra of step n. Since $\mathfrak{g}/\mathfrak{g}_{n-1}$ is nilpotent of step $n - 1$, for all X_1 and X_2 in $\mathfrak{g}$, we have by the induction hypothesis

$$X_1 + X_2 = \prod_{i=1}^{m} (c_i X_1) \cdot (d_i X_2) + u(X_1, X_2)$$

where $u(X_1, X_2)$ belongs to $\mathfrak{g}_{n-1}$, hence to the center of $\mathfrak{g}$. There exist real numbers $c_{i_1 \dots i_n}$, where $(i_1, \dots, i_n)$ runs through $\{1, 2\}^n$, such that

$$\begin{aligned} u(X_1, X_2) &= \sum_{(i_1, \dots, i_n) \in \{1,2\}^n} c_{i_1 \dots i_n} [X_{i_1}, [X_{i_2}, [\dots, X_{i_n}] \dots]] \\ &= \prod_{(i_1, \dots, i_n) \in \{1,2\}^n} [c_{i_1 \dots i_n} X_{i_1}, [X_{i_2}, [\dots, X_{i_n}] \dots]]. \end{aligned}$$

By Lemma 1.5, we have

$$u(X_1, X_2) = \prod_{(i_1, \dots, i_n) \in \{1,2\}^n} \{c_{i_1 \dots i_n} X_{i_1}, \{X_{i_2}, \{\dots, X_{i_n}\} \dots\}\}$$

and then

$$X_1 + X_2 = \prod_{i=1}^{m} (c_i X_1) \cdot (d_i X_2) \cdot \prod_{(i_1, \dots, i_n) \in \{1,2\}^n} \{c_{i_1 \dots i_n} X_{i_1}, \{X_{i_2}, \{\dots, X_{i_n}\} \dots\}\}$$

where $c_{1 \dots 1}$ and $c_{2 \dots 2}$ are zero. Denoting by m_n the number of factors sufficient to write $X_1 + X_2$ as a product when $\mathfrak{g}$ is nilpotent of step n, we have shown that $m_2 = 3$; we can check that

$$m_n = m_{n-1} + (2^n - 2)(3 \cdot 2^{n-1} - 2)$$

and consequently

$$m_n = 2^{n+1}(2^n - 5) + 4n + 3.$$

2) Let us prove the second assertion. If $n = 2$, then for all X and Y in $\mathfrak{g}$, we have

$$[X, Y] = X \cdot Y \cdot (-X) \cdot (-Y).$$

The proof of the rest of the assertion is similar and we find that

$$p_n = 2^{n+1}(2^n - 5) + 4n + 4$$

where p_n indicates the number of factors sufficient to write $[X_1, X_2]$ as a product when $\mathfrak{g}$ is nilpotent of step n. ■

1.7. Corollary. *Let $\mathfrak{g}$ be a nilpotent Lie algebra of step n greater than 2. Let $X_1, \ldots, X_p$ be elements of $\mathfrak{g}$ of the form*

$$X_i = [X_i^1, [X_i^2, [\ldots, X_i^{k_i}] \ldots]].$$

Then there exists an integer q, depending only on p and n, such that

$$\sum_{i=1}^{p} X_i = \prod_{j=1}^{q} \prod_{\substack{1 \le i_j \le p \\ 1 \le l_{i_j} \le k_{i_j}}} c_{i_j l_{i_j}} X_{i_j}^{l_{i_j}}.$$

Proof. It suffices to apply the previous proposition as many times as necessary. ■

We recall that G denotes a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. In the rest of this section fix a euclidean norm $\| \ \|$ on $\mathfrak{g}$, and denote by U the unit ball B of $\mathfrak{g}$.

1.8. Corollary. *There exists a real number c_1 such that for all j in $\{1, \ldots, n-1\}$, and X in $\mathfrak{g}_j$,*

$$w_U(X) \le c_1(1 + \|X\|)^{1/j+1}.$$

Proof. Let j be in $\{1, \ldots, n-1\}$ and fix a basis $(X_1, \ldots, X_p)$ of $\mathfrak{g}_j$. Each X_i can be chosen such that

$$X_i = [X_i^1, [X_i^2, [\ldots, X_i^{j+1}] \ldots]]$$

for certain vectors X_i^k. Let X be in $\mathfrak{g}_j$. We can write

$$(1 + \|X\|)^{-1} X = \sum_{i=1}^{p} c_i X_i$$

where $|c_i| < 1$, and then

$$X = \sum_{i=1}^{p} c_i(1 + \|X\|)[X_i^1, [X_i^2, [\ldots, X_i^{j+1}] \ldots]]$$
$$= \sum_{i=1}^{p} [c_i(1 + \|X\|)^{1/j+1} X_i^1,$$
$$[(1 + \|X\|)^{1/j+1} X_i^2, [\ldots, (1 + \|X\|)^{1/j+1} X_i^{j+1}] \ldots]].$$

By Corollary 1.7, it follows that

$$X = \prod_{m=1}^{q} \prod_{\substack{1 \le i_m \le p \\ 1 \le r_m \le j+1}} c_{i_m r_m} (1 + \|X\|)^{1/j+1} X_{i_m}^{r_m}$$

for a certain integer q and some real numbers $c_{i_m r_m}$, depending only on j, n, p. Let s be the number of factors in the above product. Put

$$c = \max\{|c_{i_m r_m}| \mid 1 \le i_m \le p,\ 1 \le r_m \le j+1,\ 1 \le m \le q\},$$
$$t = \max\{\|X_i^k\| \mid 1 \le i \le p,\ 1 \le k \le j+1\}.$$

Hence

$$X \in U^{s(1+E(ct(1+\|X\|)^{1/j+1}))}$$

where E indicates the integer part function, from which, by definition of w_U,

$$w_U(X) \le 1 + s(1 + E(ct(1 + \|X\|)^{1/j+1}))$$
$$\le 1 + s + sct(1 + \|X\|)^{1/j+1}$$
$$\le (1 + s + sct)\,(1 + \|X\|)^{1/j+1}. \ \blacksquare$$

1.9. Proposition. *There exists a real number c_2 such that for all X in $\mathfrak{g}$ and all j in $\{1, \ldots, n\}$ we have*

$$(1 + \|X_j\|)^{1/j} \le c_2\, w_U(X)$$

where X_j indicates the component of X belonging to V_j.

Proof. 1) Let ε be a strictly positive number. Let us show by induction on m that there exists a real number $a_\varepsilon = O(\varepsilon)$ such that if $X \in (\varepsilon B)^m$, then $\|X_j\| < a_\varepsilon(1 + m)^j$.

If $m = 1$, then $\|X_j\| < \varepsilon$, hence we take $a_\varepsilon = \varepsilon\, 2^{-j}$.

Assume the result is true for $m - 1$ and let X be in $(\varepsilon B)^m$. Then X can be written as $Y \cdot W$ where $Y \in (\varepsilon B)^{m-1}$ and $W \in \varepsilon B$. By the induction hypothesis,

(1) $$\|Y_j\| \le a_\varepsilon m^j$$

where $a_\varepsilon = O(\varepsilon)$. By the Baker–Campbell–Hausdorff formula, we have

$$(Y \cdot W)_j = Y_j + W_j + Q_j(Y, W) \tag{2}$$

where

$$Q_j(Y, W) = \sum_{\substack{i_1,\dots,i_p \geq 1 \\ i_1+\dots+i_p \leq j}} c^j_{i_1\dots i_p} [T_{i_1}, [\dots, T_{i_p}]\dots]_j \tag{3}$$

and where each T_{i_k} is Y_{i_k} or W_{i_k}, i.e. an element of V_{i_k}. Since each W_{i_k} appears at least once in each bracket, it follows that for ε small enough

$$\begin{aligned} \| [T_{i_1}, [\dots, T_{i_p}]\dots]_j \| \leq \| [T_{i_1}, [\dots, T_{i_p}]\dots] \| &\leq \varepsilon \, \|T_{i_1}\| \dots \|\widehat{T}_{i_k}\| \dots \|T_{i_p}\| \\ &\leq \varepsilon \, a_\varepsilon m^{i_1} \dots \widehat{a_\varepsilon m^{i_k}} \dots a_\varepsilon m^{i_p} \leq \varepsilon m^{j-1} \end{aligned}$$

and hence, by (3),

$$\|Q_j(Y, W)\| \leq \varepsilon \, m^{j-1} \sum_{\substack{i_1,\dots,i_p \geq 1 \\ i_1+\dots+i_p \leq j}} |c^j_{i_1\dots i_p}| \leq \varepsilon c N m^{j-1}$$

where

$$c = \max \{ |c^j_{i_1\dots i_p}| \mid i_1, \dots, i_p \geq 1 \text{ and } i_1 + \dots + i_p \leq j \}$$

and N is the number of terms in the preceding sum. We then deduce, by (1) and (2), that

$$\begin{aligned} \|(Y \cdot W)_j\| &\leq \|Y_j\| + \|W_j\| + \|Q_j(Y, W)\| \\ &\leq a_\varepsilon m^j + \varepsilon + \varepsilon c N m^{j-1} \leq c_\varepsilon \, (1+m)^j \end{aligned}$$

where $c_\varepsilon = a_\varepsilon + \varepsilon + \varepsilon c N$. Finally $\|X_j\| \leq c_\varepsilon \, (1+m)^j$ where $c_\varepsilon = O(\varepsilon)$. We now choose our new a_ε as c_ε.

2) Let X be in U, ε be a strictly positive number and M_ε the integer such that

$$M_\varepsilon - 1 < \varepsilon^{-1} \leq M_\varepsilon.$$

Then

$$\|M_\varepsilon^{-1} X\| \leq \varepsilon \|X\| \leq \varepsilon,$$

therefore $M_\varepsilon^{-1} X$ belongs to εB and consequently $X \in (\varepsilon B)^{M_\varepsilon}$.

3) Let X be a nonzero element of $\mathfrak{g}$. Fix ε small enough so that $a_\varepsilon < 1$ in 1). By definition, X belongs to $U^{w_U(X)-1}$, then by 2) to $(\varepsilon B)^{M_\varepsilon(w_U(X)-1)}$, and by 1),

$$(1 + \|X_j\|)^{1/j} \leq (1 + M_\varepsilon^j \,)^{1/j} \, w_U(X). \ \blacksquare$$

1.10. Proposition. *There exists a real number c_3 such that for all $Y_1, \dots, Y_n$ where each Y_j belongs to $\mathfrak{g}_{j-1}$, we have*

$$\|X_j\|^{1/j} \leq c_3 \max_{1 \leq i \leq j} (1 + \|Y_i\|)^{1/i}, \qquad \|Y_j\|^{1/j} \leq c_3 \max_{1 \leq i \leq j} (1 + \|X_i\|)^{1/i},$$

where X_j indicates the component of $Y_1 \dots Y_n$ belonging to V_j.

Proof. Fix j in $\{1,\ldots,n\}$. By the Baker–Campbell–Hausdorff formula,

$$X_j = Y_j + \sum_{\substack{i_1,\ldots,i_p\ge 1\\ i_1+\ldots+i_p\le j}} c^j_{i_1\ldots i_p}\,[Y_{i_1},[\ldots,Y_{i_p}],\ldots]_j, \tag{1}$$

hence

$$\begin{aligned}\|X_j\| &\le \|Y_j\| + \sum_{\substack{i_1,\ldots,i_p\ge 1\\ i_1+\ldots+i_p\le j}} |c^j_{i_1\ldots i_p}|\,\|Y_{i_1}\|\ldots\|Y_{i_p}\|\\ &\le \|Y_j\| + c\sum_{\substack{i_1,\ldots,i_p\ge 1\\ i_1+\ldots+i_p\le j}} (\|Y_{i_1}\|^{1/i_1})^{i_1}\ldots(\|Y_{i_p}\|^{1/i_p})^{i_p}\\ &\le (1+cN)(\max_{1\le i\le j}(1+\|Y_i\|)^{1/i})^j\end{aligned}$$

where

$$c=\max\{|c^j_{i_1\ldots i_p}| \mid i_1,\ldots,i_p\ge 1 \text{ and } i_1+\ldots+i_p\le j\}$$

and N is the number of terms in the previous sum. Finally,

$$\|X_j\|^{1/j} \le (1+cN)^{1/j}\max_{1\le i\le j}(1+\|Y_i\|)^{1/i}.$$

The second relation follows similarly. ■

By 1.2 all the weights w_U are equivalent on a connected group G. Hence we fix a compact neighborhood U of e in G and we write w_G instead of w_U. We can then summarize the previous results in the following theorem:

1.11. THEOREM. *Let G be a simply connected nilpotent Lie group with Lie algebra $\mathfrak{g}$. Let $\mathfrak{g}=\bigoplus_{i=1}^n V_i$ where $V_i\oplus\mathfrak{g}_i=\mathfrak{g}_{i-1}$ and where $(\mathfrak{g}_i)_{0\le i\le n-1}$ is the central decreasing sequence of $\mathfrak{g}$. Then there exist real numbers c and c' such that for all X in $\mathfrak{g}$, we have*

$$c\max_{1\le i\le n}(1+\|X_i\|)^{1/i}\le w_G(\exp X)\le c'\max_{1\le i\le n}(1+\|X_i\|)^{1/i}$$

where X_i indicates the component of X belonging to V_i.

Proof. Proposition 1.9 shows the existence of c. Let now X be in $\mathfrak{g}$. We can find $Y_1,\ldots,Y_n$, where each Y_j belongs to $\mathfrak{g}_{j-1}$, such that $\exp X=\exp Y_1\ldots\exp Y_n$. Hence, by 1.3, we have

$$w_G(\exp X)\le\sum_{j=1}^n w_G(\exp Y_j)$$

and by Corollary 1.8,

$$w_G(\exp X)\le c_1\sum_{j=1}^n(1+\|Y_j\|)^{1/j}.$$

It now follows from Proposition 1.10 that

$$\begin{aligned} w_G(\exp X) &\le c_1 \sum_{j=1}^{n} [1 + c_3^j \max_{1\le i\le j} (1+\|X_i\|)^{j/i}]^{1/j} \\ &\le c_1 \sum_{j=1}^{n} 1 + c_3 \max_{1\le i\le j} (1+\|X_i\|)^{1/i} \\ &\le c_1 n(1+c_3) \max_{1\le i\le n} (1+\|X_i\|)^{1/i}. \ \blacksquare \end{aligned}$$

NOTATION. For all k in $\{1,\dots,n\}$, the set $U \cap G_k$ denoted by V_k is a symmetric compact neighborhood of e in G_k, and the weight w_{V_k} on G_k defined in 1.3 will be denoted by w_{G_k}.

1.12. THEOREM. *There exists a strictly positive number c such that for all k in $\{1,\dots,n\}$, we have*

$$w_G|_{G_k} \le c w_{G_k}^{1/k+1}.$$

Proof. We easily show by induction on i that for all i in $\mathbb{N}$,

(1) $$(\mathfrak{g}_k)_i \subset \mathfrak{g}_{(k+1)(i+1)-1}.$$

Denote by p_k the step of nilpotency of $\mathfrak{g}_k$ and let Y be an element of $\mathfrak{g}_k$. Then

$$Y = Y_1 \dots Y_{p_k} = X_1 + \dots + X_{p_k}$$

where each Y_i belongs to $(\mathfrak{g}_k)_{i-1}$ and X_i to $(V_k)_i$ where, as at the beginning of this section, $(\mathfrak{g}_k)_{i-1}$ is the direct sum of $(\mathfrak{g}_k)_i$ and $(V_k)_i$ for all i in $\{1,\dots,p_k\}$. As noticed in 1.3, we have

$$w_G(\exp Y) = w_G(\exp Y_1 \dots \exp Y_{p_k}) \le w_G(\exp Y_1) + \dots + w_G(\exp Y_{p_k}).$$

Now each Y_i belongs to $\mathfrak{g}_{(k+1)i-1}$ by (1), and so by Corollary 1.8,

$$w_G(\exp Y_i) \le c_1(1+\|Y_i\|)^{1/(k+1)i};$$

hence

$$\begin{aligned} w_G(\exp Y) &\le c_1 \sum_{i=1}^{p_k} (1+\|Y_i\|)^{1/(k+1)i} \\ &\le c_1 p_k \max_{1\le i\le p_k} (1+\|Y_i\|)^{1/(k+1)i} \\ &\le c_1 p_k + c_1 p_k (\max_{1\le i\le p_k} \|Y_i\|^{1/i})^{1/k+1} \\ &\le c_1 p_k + c_1 p_k (c_3 \max_{1\le i\le p_k} (1+\|X_i\|)^{1/i})^{1/k+1} \qquad (2) \\ &\le c_1 p_k + c_1 p_k (c_3\, c_2\, w_{G_k}(\exp Y))^{1/k+1} \qquad (3) \\ &\le (c_1 p_k + c_1 p_k (c_2 c_3)^{1/k+1})\, (w_{G_k}(\exp Y))^{1/k+1} \end{aligned}$$

where (2) and (3) result from Propositions 1.10 and 1.9 respectively. ■

1.13. Corollary. *Let N be a subgroup of G_1 and let (π, X) be a Banach space representation of G on X. If $\pi|_N$ is given by $\chi 1_X$ for some character χ of N, then χ must be unitary.*

Proof. Assume that $\pi|_N$ is a (continuous) nonunitary character of N. Denote by $\mathfrak{n}$ the Lie algebra of N. Let U be in $\mathfrak{V}_G(e)$. First, for all s in G distinct from e we have

$$s = s_1 \ldots s_{\tau_U(s)},$$

hence

$$|\pi(s)| = |\pi(s_1)| \ldots |\pi(s_{\tau_U(s)})| \le e^{k_U \tau_U(s)} \le e^{k_U w_G(s)} \tag{1}$$

where

$$e^{k_U} = \sup_{s\in U} |\pi(s)|.$$

By hypothesis, there exist two real linear forms α and β on $\mathfrak{n}$, with $\alpha \neq 0$, such that

$$\pi(\exp X) = e^{\langle \alpha + i\beta, X\rangle}, \qquad X \in \mathfrak{n}.$$

Fix X in $\mathfrak{n}$ such that $\langle \alpha, X\rangle = 1$. Then for all t in $\mathbb{R}$, from (1) we have

$$e^t = |\pi(\exp(tX))| \le e^{k_U w_G(\exp(tX))}.$$

Let $V = U \cap G_1$. By Theorem 1.12

$$w_G(\exp(tX)) = w_G|_{G_1}(\exp(tX)) \le c(w_{G_1}(\exp(tX)))^{1/2}$$

and by Proposition 1.4,

$$w_{G_1}(\exp(tX)) < 2 + c_V \|tX\|,$$

hence

$$e^t \le e^{k_U c \sqrt{2 + c_V |t| \|X\|}}.$$

This last inequality is false for t large enough. ∎

2. Spectral synthesis for nilpotent Lie groups. Let G be a connected Lie group, $\mathfrak{g}$ its Lie algebra, and $\mathfrak{g}^*$ the dual vector space of $\mathfrak{g}$. The set of equivalence classes of irreducible continuous unitary representations of G is denoted by $\widehat{G}$. When G is abelian, by Schur's lemma, $\widehat{G}$ is in bijection with the group of continuous characters of G into the multiplicative group U of complex numbers of norm 1. When G is not abelian, $\widehat{G}$ is not known in general. In 1962, A. Kirillov managed to determine $\widehat{G}$ when G is nilpotent and simply connected [11]: the unitary dual $\widehat{G}$ of G is described by the orbits of the elements of $\mathfrak{g}^*$ under the coadjoint action of G; this action is defined by the relation

$$x \cdot l = l \circ \operatorname{Ad}(x^{-1}), \qquad l \in \mathfrak{g}^*,\ x \in G.$$

From now on, G denotes a simply connected real nilpotent Lie group with Lie algebra $\mathfrak{g}$. For l in $\mathfrak{g}^*$, there exists a *polarization* $\mathfrak{m}$ at l, i.e. a subalgebra $\mathfrak{m}$ of $\mathfrak{g}$ which is maximal isotropic for the skew-symmetric bilinear form

$$B_l(X,Y) = l[X,Y], \qquad X, Y \in \mathfrak{g}.$$

Denote by M the connected subgroup $\exp \mathfrak{m}$ of G associated to $\mathfrak{m}$. The map

$$\chi_{l,M} : M \to U, \qquad \exp X \mapsto e^{i\langle l,X\rangle},$$

is a character of M. We write

$$\pi_{l,M} = \operatorname{ind}_M^G \chi_{l,M}.$$

Then $\pi_{l,M}$ is irreducible and the correspondence

$$\mathfrak{g}^*/\mathrm{Ad}^*(G) \to \widehat{G}, \qquad [l] \mapsto [\pi_{l,M}],$$

is a bijective mapping, called *Kirillov's bijection*, where

$$l \sim l' \Leftrightarrow \exists x \in G : l' = \mathrm{Ad}^*(x)l.$$

The set $\widehat{G}$ is also in bijection with $\mathrm{Prim}(G)$, the space of primitive ideals of the C^*-algebra of G by [6], and by [3] in bijection with

$$\mathrm{Prim}^* L^1(G) = \{\mathrm{Ker}\,\pi \mid |\pi \text{ a } *\text{-topologically irreducible representation of } L^1(G)\}.$$

We equip these two sets with the Jacobson topology: for a subset S of $L^1(G)$, we define its *hull* by

$$h(S) = \{J \in \mathrm{Prim}^* L^1(G) \mid |S \subset J\},$$

and for a subset C of $\mathrm{Prim}^* L^1(G)$ or $\mathrm{Prim}(G)$, we define its *kernel* by

$$k(C) = \bigcap_{J\in C} J.$$

Then, by definition, C is closed in $\mathrm{Prim}^* L^1(G)$, respectively in $\mathrm{Prim}(G)$, if and only if $C = h(k(C))$. By Brown's theorem [4], Kirillov's bijection is a homeomorphism.

The Jacobson topology is in general not Hausdorff, but always accessible, i.e. each point is closed, which means that every element in $\mathrm{Prim}^* L^1(G)$, respectively in $\mathrm{Prim}(G)$, is maximal. This follows from the fact that the coadjoint orbits of nilpotent Lie groups are closed [18].

PROBLEM. Given a closed subset C of $\mathrm{Prim}^* L^1(G)$, can we determine the set $\mathcal{J}(C)$ of closed two-sided ideals of $L^1(G)$ with hull C?

When $\mathcal{J}(C) = \{k(C)\}$, the subset C is said to be *of synthesis* or *spectral*. The first result of spectral synthesis is the famous theorem of N. Wiener stating that $\emptyset$ is of synthesis in $\mathrm{Prim}^* L^1(\mathbb{R})$, i.e. each proper closed ideal of $L^1(\mathbb{R})$ is contained in the kernel of a $*$-topologically irreducible representation of $L^1(\mathbb{R})$. I. Segal [20] next showed that each point of $\mathrm{Prim}^* L^1(\mathbb{R})$ is of synthesis; then I. Kaplansky [10] generalized this result to $\mathrm{Prim}^* L^1(G)$ where G is abelian. The first result when G is not abelian was obtained by H. Leptin [12] who showed that if G is nilpotent of step 2, then each point in $\mathrm{Prim}^* L^1(G)$ is of synthesis. If G is nilpotent of step 3, J. Ludwig [14] showed that $\mathcal{J}(\{\mathrm{Ker}\,\pi\})$ is in bijection with $\mathcal{J}(\{\mathrm{Ker}\,\chi\})$ where χ is a character of $L^1_w(\mathbb{R}^n)$, and w is a weight of polynomial growth on $\mathbb{R}^n$. J. Ludwig shows that $\mathcal{J}(\{\mathrm{Ker}\,\pi\})$ then contains in general an infinity of elements, and consequently $\{\mathrm{Ker}\,\pi\}$ is not of synthesis in these cases. If G is nilpotent of step 4, the computations become much more difficult and no general result is known. We have however the following theorem due to J. Ludwig [13], which gives the existence of a smallest element in $\mathcal{J}(C)$:

THEOREM. *Let G be a locally compact group with polynomial growth such that $L^1(G)$ is symmetric*, *and C a closed subset of* $\mathrm{Prim}^* L^1(G)$. *Then there exists a single closed two-sided ideal $j(C)$ of $L^1(G)$ such that*

$$h(j(C)) = C$$

and

$$(J \triangleleft L^1(G),\, h(J) \subset C) \;\Rightarrow\; j(C) \subset J.$$

This theorem applies in particular when G is a simply connected nilpotent Lie group [6]. For example, if G is abelian, then $j(C)$ is the closure in $L^1(G)$ of the ideal of $L^1(G)$ of functions for which the support of the Fourier transform is compact and disjoint from C [19].

Notice that for a closed subset C of $\mathrm{Prim}^* L^1(G)$, each element of $\mathcal{J}(C)$ is contained in $k(C)$. Hence there exists a "minimal" ideal and a "maximal" ideal with hull C. The subset C is then of synthesis if and only if these two ideals are equal.

Let π be an element of $\widehat{G}$. In order to determine $\mathcal{J}(\{\mathrm{Ker}\,\pi\})$ when the step of G is larger than 3, it is natural to begin with the determination of $j(\{\mathrm{Ker}\,\pi\})$, since the latter is contained in each element of $\mathcal{J}(\{\mathrm{Ker}\,\pi\})$. The result obtained by J. Ludwig when G is of step 3 forces us to look for this ideal not in $L^1(G)$ but in a weighted L^1-algebra on $\mathbb{R}^n$.

By Kirillov's bijection, π is associated to the orbit $O(l)$ of a certain linear form l on $\mathfrak{g}$, and the easiest case is when the orbit $O(l)$ is a single point. The rest of this paper is devoted to the determination of $j(\{\mathrm{Ker}\,\pi\})$ in this case. This will be done in a quite general class of algebras which contain weighted algebras, and for nilpotent Lie groups of any step. The principal result of this paper is based in fact on a general property of $C^\infty(G)$-modules of finite dimension, where G is solvable. This property is dealt with in [2].

Notation. By [3], the set $\mathrm{Prim}^* L^1(G)$ is in bijection with $\mathfrak{g}^*/\mathrm{Ad}^*(G)$. In order to make the reading easier, closed subsets C of $\mathrm{Prim}^* L^1(G)$ and closed subsets of $\widehat{G}$ will be identified with closed $\mathrm{Ad}^*(G)$-invariant subsets of $\mathfrak{g}^*$. So, for π_l in $\widehat{G}$, associated to the orbit $O(l)$ of a linear form l on $\mathfrak{g}$, the minimal ideal $j(\{\mathrm{Ker}\,\pi_l\})$ of $L^1(G)$ and the set $\mathcal{J}(\{\mathrm{Ker}\,\pi_l\})$ of closed two-sided ideals of $L^1(G)$ with hull $\{\mathrm{Ker}\,\pi_l\}$ will be denoted $j(l)$ and $\mathcal{J}(l)$ respectively.

Conventions. Unless otherwise stated, a function will always be complex-valued. For any group, e will indicate the identity element. For a normed algebra A the relation $I \lhd A$ means that I is a closed two-sided ideal of A.

3. Polynomials and group algebras. In the following, λ will indicate a Haar measure on a simply connected nilpotent Lie group G and $d\lambda(x)$ will be denoted by dx.

3.1. Notation. Let G be a locally compact group, λ a left Haar measure on G, and w a weight on G. We denote by $L^1_w(G)$ the subalgebra of $L^1(G)$ of measurable functions f such that $\int_G |f| w\, d\lambda$ is finite, and we define a norm $\| \, \|_w$ on $L^1_w(G)$ by

$$\|f\|_w = \int_G |f| w\, d\lambda.$$

We thus obtain the *Beurling algebra* $L^1_w(G)$. The algebra of polynomials on G is denoted by $\mathcal{P}(G)$. For X in $\mathfrak{g}$ and for a C^∞ function f on G, we let $X * f$ be the left derivative of f in direction X, and $f * X$ the right derivative of f in direction X:

$$X * f(y) = \frac{d}{dt} f(\exp(-tX)y)\Big|_{t=0}, \qquad y \in G,$$
$$f * X(y) = \frac{d}{dt} f(y \exp(tX))\Big|_{t=0}, \qquad y \in G.$$

A basis $(X_1, \ldots, X_d)$ of $\mathfrak{g}$ being fixed, for a multi-index $(\alpha_1, \ldots, \alpha_d)$ of $\mathbb{N}^d$, denoted by α, and a C^∞ function f on G, we write

$$X^\alpha * f = X_1^{\alpha_1} * \ldots * X_d^{\alpha_d} * f, \qquad f * X^\alpha = f * X_1^{\alpha_1} * \ldots * X_d^{\alpha_d},$$
$$|\alpha| = \alpha_1 + \ldots + \alpha_d.$$

We denote by $\mathcal{S}(G)$ the Schwartz space of C^∞ functions f on G such that for all positive integers N,

$$p_N(f) = \sum_{|\alpha| \le N} \int_G |X^\alpha * f|\, w^N\, d\lambda$$

is finite, where w is the weight w_U defined in 1.3. One can check that the definition of $\mathcal{S}(G)$ is independent of the choice of the basis of $\mathfrak{g}$ and of U. We have (see [17])

$$p_N(g * f) \leq p_N(g)\|f\|_{w^N}.$$

We denote by $\mathcal{D}(G)$ the subspace of $\mathcal{S}(G)$ of functions with compact support. The space $\mathcal{S}(G)$ equipped with the convolution multiplication and with the family of seminorms $(p_N)_{N\in\mathbb{N}}$ is then a Fréchet algebra and $\mathcal{S}(G)$ is dense in $(L^1(G), \| \|_1)$.

3.2. The determination of the "minimal ideal" in Section 5 will be given for a quite general class of algebras. Indeed, in this paper we consider a Banach subalgebra $(A, \| \|)$ of $L^1(G)$ containing $\mathcal{S}(G)$ as a dense subspace and satisfying

$$\begin{cases} \exists N \in \mathbb{N}, \forall f \in \mathcal{S}(G) : \|f\| \leq p_N(f), \\ \forall f \in A : \|f\|_1 \leq \|f\|, \end{cases}$$

which means that the norm $\| \|$ of A makes the injections of $\mathcal{S}(G)$ into A and of A into $L^1(G)$ continuous.

3.3. Recall that the characters of G, i.e. the continuous homomorphisms of the group G into $\mathbb{C}^\times$, are of the form $\exp X \mapsto \chi_l(\exp X) = e^{il(X)}$ where l is an $\mathbb{R}$-linear form on $\mathfrak{g}$ with complex values such that $l[X, Y]$ is zero for all X and Y in $\mathfrak{g}$. For real-valued l we obtain the unitary characters of G.

For l in $\mathfrak{g}^*$ such that l is zero on $\mathfrak{g}_1$, we denote by $\mathcal{P}_l$ the vector space of polynomials P, with complex coefficients, such that the continuous linear form $P\chi_l$ on $\mathcal{S}(G)$ mapping f to $\int_G fP\chi_l\, d\lambda$ extends to a continuous linear form on A, meaning that there exists a positive number c such that for all f in $\mathcal{S}(G)$, we have

$$\Big|\int_G fP\chi_l\, d\lambda\Big| \leq c\|f\|.$$

Let G be a group and s be an element of G. For a function $f : G \to \mathbb{C}$, we denote by $L_s f$ or ${}_s f$ the left translate of f by s, mapping t to $f(s^{-1}t)$, and by $R_s f$ or f_s the right translate of f by s, mapping t to $f(ts)$.

Let P be in $\mathcal{P}_l$ and f, g be elements of A. Then $P\chi_l$ defines a continuous linear form on A by definition, and consequently $\langle P\chi_l, g * f\rangle$ exists. For g in A, we write $\check{g} * (P\chi_l)$ for the continuous linear form on A defined by

$$\langle \check{g} * (P\chi_l), f\rangle = \langle P\chi_l, g * f\rangle.$$

In the same way, $P\chi_l * \check{g}$ denotes the continuous linear form on A defined by

$$\langle P\chi_l * \check{g}, f\rangle = \langle P\chi_l, f * g\rangle.$$

3.4. THEOREM. *The vector space $\mathcal{P}_l$ is finite-dimensional.*

Proof. 1) Let f be in $\mathcal{S}(G)$, Q a polynomial and χ_q a unitary character of G. After an easy computation, for all x in G we have

$$(f * (Q\chi_q))(x) = P(x)\chi_q(x)$$

where P is another polynomial.

2) Let Q be in $\mathcal{P}_l$ and g in $\mathcal{S}(G)$. By 1),

$$g * (Q\chi_l) = Q_g\chi_l$$

where Q_g is a polynomial, and for all f in $\mathcal{S}(G)$,

$$\begin{aligned}|\langle g * (Q\chi_l), f\rangle| &= |\langle Q\chi_l, \check{g} * f\rangle| \leq \|Q\chi_l\|_{\mathrm{op}}\, \|\check{g} * f\| \\ &\leq \|Q\chi_l\|_{\mathrm{op}}\, p_N(\check{g} * f) \leq \|Q\chi_l\|_{\mathrm{op}}\, p_N(\check{g})\, \|f\|_{w^N}\end{aligned}$$

where N is an integer depending on Q and l. Hence $g * (Q\chi_l)$ is in the dual space of $L^1_{w^N}(G)$, and so

$$\|Q_g/w^N\|_\infty < \infty.$$

Denote by $\mathcal{P}_N$ the vector space of polynomials P such that $\|P/w^N\|_\infty$ is finite. Since the weight w^N has a polynomial growth, the space $\mathcal{P}_N$ is finite-dimensional and we have shown that for all Q in $\mathcal{P}_l$ and all g in $\mathcal{S}(G)$, $g * (Q\chi_l)$ belongs to $\mathcal{P}_N\chi_l \cap \mathcal{P}_l\chi_l$.

3) Let Q be in $\mathcal{P}_l$. Since the weak star topology on $\mathcal{P}_N$ with respect to $L^1_{w^N}(G)$ coincides with the norm topology, and since for any approximate identity (g_n) in $\mathcal{S}(G)$, $(g_n * Q\chi_l)$ converges in the weak star topology to $Q\chi_l$, it follows that $(g_n * Q\chi_l)$ inside $\mathcal{P}_N$ converges to $Q\chi_l$ in the operator norm, and so $Q\chi_l \in \mathcal{P}_N$. Hence $\mathcal{P}_l \subset \mathcal{P}_N$. ∎

3.5. Notation. Until the end of this paper, W indicates a nonzero subspace of $\mathcal{P}_l$ which is invariant under left and right translations, and $W\chi_l$ is denoted by W_l. We also write

$$I(W) = \{f \in A \mid \forall P \in W : \langle P\chi_l, f\rangle = 0\} = (W\chi_l)^\circ.$$

We then have the following proposition.

3.6. Proposition. *The vector space W is invariant under translations and under convolution by elements of $\mathcal{S}(G)$. So $I(W)$ is a closed two-sided ideal of A.*

4. Hull

Definition. For a Banach algebra A, we denote by $\mathrm{Prim}(A)$ the set of primitive ideals of A, i.e. the set of the kernels of algebraically irreducible representations of A in Banach spaces. The *kernel* of a subset C of $\mathrm{Prim}(A)$ is the set

$$k(C) = \bigcap_{J \in C} J,$$

and the *hull* of a subset S of A is the set

$$h(S) = \{J \in \mathrm{Prim}(A) \mid S \subset J\}.$$

NOTATION. For a Banach algebra A, the set $\mathrm{Prim}(A)$ is equipped with the Jacobson topology: by definition, a subset C of $\mathrm{Prim}(A)$ is closed in $\mathrm{Prim}(A)$ if and only if $C = h(k(C))$. We denote by $\mathcal{J}(C)$ the set of closed two-sided ideals of A with hull C:

$$\mathcal{J}(C) = \{J \triangleleft A \mid h(J) = C\}.$$

In the present case, the set $\{\mathrm{Ker}\,\chi_l\}$ is closed in $\mathrm{Prim}(A)$, and as stipulated in Section 2, the set $\mathcal{J}(\{\mathrm{Ker}\,\chi_l\})$ will be denoted $\mathcal{J}(l)$ by abuse of notation.

4.1. PROPOSITION. *With the above hypothesis on* A, *we have*

$$\mathrm{Prim}(A) = \{\mathrm{Ker}(\pi|_A) \mid \pi \in \widehat{G}\}.$$

Proof. 1) Let π be a unitary topologically irreducible representation of G; denote also by π the corresponding representation of $L^1(G)$. Since A is dense in $L^1(G)$, $\pi|_A$ is topologically irreducible on the Hilbert space $\mathcal{H}$. Let

$$\mathcal{H}_0 = \mathrm{Span}\{\pi(f)\xi \mid \xi \in \mathcal{H},\ f \in A,\ \pi(f) \text{ of finite rank}\}.$$

Since $\pi(\mathcal{S}(G))$ contains many operators of finite rank, $\mathcal{H}_0$ is an A-invariant nontrivial subspace of $\mathcal{H}$ and the restriction of π to $\mathcal{H}_0$ defines a simple module of A (see [6]). Hence $\mathrm{Ker}(\pi|_A)$ is a primitive ideal:

$$\{\mathrm{Ker}(\pi|_A) \mid \pi \in \widehat{G}\} \subset \mathrm{Prim}(A).$$

Let us prove the other inclusion. If (T, V) is a simple A-module on a Banach space V then $(T|_{\mathcal{S}(G)}, V)$ is a topologically irreducible $\mathcal{S}(G)$-module. Hence by [16] there exists a $\pi \in \widehat{G}$ such that

$$\mathrm{Ker}(T|_{\mathcal{S}(G)}) = \mathrm{Ker}(\pi|_{\mathcal{S}(G)}).$$

By [15] we know that $\mathrm{Ker}(\pi|_{\mathcal{S}(G)})$ is dense in $\mathrm{Ker}(\pi|_A)$. Hence $\mathrm{Ker}\,T$ contains $\mathrm{Ker}(\pi|_A)$.

2) Let us prove that $\mathrm{Ker}(\pi|_A)$ is a maximal two-sided ideal of A. Let M be a closed two-sided ideal of A containing $\mathrm{Ker}(\pi|_A)$. Suppose that $M \neq \mathrm{Ker}(\pi|_A)$. Then there exists g in M such that $g \notin \mathrm{Ker}(\pi|_A)$. By [15], the two-sided ideal

$$R = \{f \in \mathcal{S}(G) \mid \pi(f) \text{ of finite rank}\}$$

is dense in $\mathcal{S}(G)$ and then in A. Hence $R * g * R$ is not contained in $\mathrm{Ker}(\pi|_A)$ and so M contains an element h such that $\pi(h) = P_\lambda$ is the orthogonal projector onto a C^∞ vector λ of $\mathcal{H}_\pi$. Let f in $\mathcal{S}(G)$ be such that $\pi(f) = P_\mu$ is also a one-dimensional orthogonal projector with $\langle \lambda, \mu \rangle \neq 0$. Then

$$\pi(f) = |\langle \lambda, \mu \rangle|^{-2}\, P_\mu \circ P_\lambda \circ P_\mu = \pi(\langle \lambda, \mu \rangle^{-2}\, f * h * f).$$

Hence

$$f - \langle \lambda, \mu \rangle^{-2} f * h * f \in \operatorname{Ker} \pi \subset M$$

and consequently $f \in M$. Since R is generated as an ideal by those elements f, this shows that M contains the ideal R and finally $M = A$ since M is closed. This proves that $\operatorname{Ker} T = \operatorname{Ker}(\pi|_A)$. ∎

The aim of this section is to determine the hull of $I(W)$ where W is defined in 3.5. Since W is finite-dimensional, we have the following proposition.

4.2. Proposition. *The space W is invariant under derivations: for all X in $\mathfrak{g}$ and all P in W, $X * P$ and $P * X$ belong to W.*

By [5], we have:

4.3. Proposition. *There exists a function* deg *on the complex vector space of polynomials on G such that for all X in $\mathfrak{g}$ and all polynomials P, we have*

$$\deg(X * P) < \deg P.$$

Hence for all X in $\mathfrak{g}$, there exists a natural k such that for all P in W, $X^k * P$ is zero.

4.4. Proposition. *The hull $h(I(W))$ of $I(W)$ contains* $\operatorname{Ker} \chi_l$.

Proof. For X in $\mathfrak{g}$ and P in W, $\pi(X)(P\chi_l) = X * (P\chi_l) = (X * P)\chi_l + i\langle l, X\rangle(P\chi_l)$ defines a representation π of the Lie algebra $\mathfrak{g}$ in W_l. By Lie's theorem (see [7]), there exists a nonzero element P in W such that for all X in $\mathfrak{g}$, $\pi(X)(P\chi_l) = \lambda(X)(P\chi_l)$ where λ is a linear form on $\mathfrak{g}$. Since $\deg(X * P) < \deg P$, we have $\lambda(X) = i\langle l, X\rangle$ and so $(X * P)\chi_l = 0$. Hence $X * P = 0$ and the polynomial P is constant. Consequently, $\chi_l \in W_l$ and hence $I(W) \subset \operatorname{Ker} \chi_l$ and $\operatorname{Ker} \chi_l \subset h(I(W))$. ∎

Notation. For f in $L^1(G)$, the Fourier transform of f at l is denoted $\widehat{f}(l)$ and is defined by

$$\widehat{f}(l) = \int_G f \overline{\chi}_l \, d\lambda.$$

Let P be a polynomial in the variables $X_1, \ldots, X_d$. We define the differential operator $P(D)$ in the $D_j = i\,\partial/\partial X_j$ with

$$D^\alpha = \prod_{i=1}^{d} D_j^{\alpha_j}, \qquad \alpha = (\alpha_1, \ldots, \alpha_d).$$

We have the well known result:

4.5. Lemma. *For all f in A,*

$$f \in I(W) \;\Leftrightarrow\; \forall P \in W : (P(D)(\widehat{f}\,))(-l) = 0,$$

where $\widehat{f}$ indicates the Fourier transform of f.

4.6. Theorem. *The hull $h(I(W))$ of $I(W)$ is* $\{\operatorname{Ker} \chi_l\}$.

Proof. By Proposition 4.4, $\operatorname{Ker}\chi_l \in h(I(W))$.

Let π be a topologically irreducible $*$-representation of $L^1(G)$ in a Hilbert space whose kernel in A contains $I(W)$. By Theorem 3.4, $I(W)$ is of finite codimension in A, hence π is finite-dimensional and defines an irreducible continuous unitary representation $\widetilde{\pi}$ of the nilpotent group G. By Lie's theorem (see [7]), $\widetilde{\pi}$ is a character. Then π is a character $\chi_{l'}$ where l' is a real linear form on $\mathfrak{g}$ which is zero on $[\mathfrak{g},\mathfrak{g}]$ by 3.3. If l' is different from l, there exists f in $\mathcal{S}(\mathfrak{g})$ such that $\widehat{f}(-l') = 1$ and $\widehat{f}$ is zero on a neighborhood of $-l$. Then f does not belong to $\operatorname{Ker}\chi_{l'}$ and belongs to $I(W)$ by Lemma 4.5. Since this contradicts the hypothesis, l' is equal to l. ∎

5. Minimal ideal

5.1. Proposition. *For each closed subset C of* $\operatorname{Prim}^*(A)$, *there exists a closed two-sided ideal $j(C)$ of A with hull C such that each closed two-sided ideal of A whose hull is contained in C contains $j(C)$.*

Proof. The proof given in [13] adapts to the general case. ∎

Taking in the previous theorem $W = \mathcal{P}_l$, we have $j(\operatorname{Ker}\chi_l) \subset I(\mathcal{P}_l)$. The following theorem will show the other inclusion.

5.2. Lemma. *Let F be a finite-dimensional A-left invariant subspace of the dual A' of the algebra A. Then each element of F is a finite sum of functions of the form $P\chi_q$, where P is a polynomial, and χ_q a unitary character of G.*

Proof. Let us show that the elements of F are C^∞ functions on G. Let $(\mu_1,\ldots,\mu_n)$ be a basis of F. Then $\mathcal{D}(G)*\mu_1+\ldots+\mathcal{D}(G)*\mu_n$ is dense in the finite-dimensional vector space F, hence is equal to F. Every μ in F defines a tempered distribution on G. Let g be in $\mathcal{D}(G)$. For all f in $\mathcal{S}(G)$,

$$\langle g*\mu, f\rangle = \langle \mu, \check{g}*f\rangle = \int_G \varphi(x)\,(1-\Delta)^N(\check{g}*f)(x)\,dx$$

for a certain function φ with moderate growth, of class C^∞ on G, and a certain integer N, where Δ indicates the Laplacian of G (by [17]).

Putting $h = (1-\Delta)^N\check{g}$, we then have

$$\langle g*\mu, f\rangle = \int_G \psi f\,d\lambda \quad \text{where} \quad \psi(x) = \int_G h\varphi_x\,d\lambda.$$

The linear form $g*\mu$ is then given on $\mathcal{S}(G)$ by a function ψ of class C^∞ on G. Since $\mathcal{S}(G)$ is dense in A, the linear form $g*\mu$ can be identified with ψ, and with this identification, F consists of C^∞ functions. The lemma then results from Proposition 1 of [2]. ∎

5.3. Theorem. *The smallest closed two-sided ideal of A with hull $\{\operatorname{Ker}\chi_l\}$ is*

$$j(l) = I(\mathcal{P}_l).$$

Proof. 1) It has already been noticed that $j(l)$ is contained in $I(\mathcal{P}_l)$. By [15], there exists a natural integer N such that $j(l) = \overline{(\operatorname{Ker}\chi_l)^N}$.

Let us show by induction on n that if T is a continuous linear form on A which is zero on $(\operatorname{Ker}\chi_l)^n$ then T is of the form $P\chi_l$ where P belongs to $\mathcal{P}_l$.

The result is true if $n = 1$: the polynomial P is a nonzero constant.

2) Let m in $\mathbb{N}^*$ be such that T is zero on $(\operatorname{Ker}\chi_l)^m$ and nonzero on $(\operatorname{Ker}\chi_l)^{m-1}$.

(a) Let f_0 be in $\operatorname{Ker}\chi_l$. Then $\check{f}_0 * T$ is a continuous linear form on A and for all u in $(\operatorname{Ker}\chi_l)^{m-1}$,

$$\langle \check{f}_0 * T, u\rangle := \langle T, f_0 * u\rangle = 0$$

because $f_0 * u$ belongs to $(\operatorname{Ker}\chi_l)^m$. The induction hypothesis shows that $\check{f}_0 * T = P_{f_0}\chi_l$ where P_{f_0} belongs to $\mathcal{P}_l$.

(b) Let f and f_1 in A be such that $\chi_l(f_1) = 1$. Then $f - \chi_l(f)f_1 \in \operatorname{Ker}\chi_l$, and consequently

$$(f - \widehat{f}(-l)f_1)^\vee * T = P_f\chi_l$$

where $P_f \in \mathcal{P}_l$ by (a), i.e.

$$\check{f} * T = \widehat{f}(-l)\check{f}_1 * T + P_f\chi_l \in \mathbb{C}(\check{f}_1 * T) + \mathcal{P}_l\chi_l.$$

This shows that the complex vector space $\check{A} * T$, which is contained in A', is of finite dimension by Theorem 3.4.

3) Let ϕ be an element of A. By 2) and Lemma 5.2, $\check{\phi} * T$ is of the form

$$\check{\phi} * T = \sum_{j=1}^{p} P_j\chi_{q_j}$$

where the P_j are polynomials and the χ_{q_j} are unitary characters of G which we assume to be all distinct. Let us show that $p = 1$ and $q_1 = l$.

Let f_0 be in $\operatorname{Ker}\chi_l \cap \mathcal{S}(G)$. The function $f_0 * \phi$ belongs to $\operatorname{Ker}\chi_l$, so by 2)(a),

$$(f_0 * \phi)^\vee * T = P\chi_l$$

where P belongs to $\mathcal{P}_l$. On the other hand, the computation 1) in the proof of Theorem 3.4 shows that

$$(f_0 * \phi)^\vee * T = \sum_{j=1}^{p} \check{f}_0 * P_j\chi_{q_j} = \sum_{j=1}^{p} Q_j\chi_{q_j}$$

where the Q_j are polynomials which we can assume to be all nonzero. Finally

$$P\chi_l = \sum_{j=1}^{p} Q_j\chi_{q_j}.$$

In the module of linear combinations (whose coefficients are polynomials) of unitary characters of G, each finite family of distinct unitary characters of G is free. Consequently, $p = 1$, $q_1 = l$ and $\check{\phi} * T = Q\chi_l$ where Q is a polynomial. Since $\phi \in A$ and $T \in A'$, $\check{\phi} * T$ is continuous on A and Q belongs to $\mathcal{P}_l$.

Let us show that T itself is in $\mathcal{P}_l\chi_l$.

4) The space $\mathcal{P}_l$ being finite-dimensional, let $f_1, \ldots, f_M$ be Schwartz functions on G such that

$$(\langle P\chi_l, f_i\rangle = 0 \text{ for } i = 1, \ldots, M) \;\Rightarrow\; P = 0.$$

For all P in $\mathcal{P}_l$ let

$$\|P\chi_l\|_l = \max_{1\le i\le M} |\langle P\chi_l, f_i\rangle|.$$

Let $(\phi_n)_{n\in\mathbb{N}}$ be an approximate unit in $\mathcal{S}(G)$. For all f in $\mathcal{S}(G)$,

$$\langle \check{\phi}_n * T - T, f\rangle = \langle T, \phi_n * f - f\rangle. \tag{1}$$

The sequence $(\phi_n * f - f)_{n\in\mathbb{N}}$ converges to 0 in $\mathcal{S}(G)$, hence in A, and T being continuous on A, $(\langle \check{\phi}_n * T - T, f\rangle)_{n\in\mathbb{N}}$ tends to 0 by (1). We have

$$\|\check{\phi}_n * T - \check{\phi}_m * T\|_l = \max_{1\le i\le M} |\langle T, (\phi_n - \phi_m) * f_i\rangle|.$$

This tends to 0 because $(\phi_n - \phi_n * f_i)_{n\in\mathbb{N}}$ tends to 0 in $\mathcal{S}(G)$, hence also in A. This shows that the sequence $(\check{\phi}_n * T)_{n\in\mathbb{N}}$ is Cauchy for the norm $\| \;\|_l$, hence converges to an element $P\chi_l$ where P belongs to $\mathcal{P}_l$, the space $\mathcal{P}_l\chi_l$ being finite-dimensional. Let f be in $\mathcal{S}(G)$. For all Q in $\mathcal{P}_l$ write

$$\|Q\chi_l\|_f = \|Q\chi_l\|_l + |\langle Q\chi_l, f\rangle|.$$

Then $\| \;\|_f$ is a norm on $\mathcal{P}_l\chi_l$ equivalent to $\| \;\|_l$, since $\mathcal{P}_l\chi_l$ is finite-dimensional. Hence the sequence $(\check{\phi}_n * T)_{n\in\mathbb{N}}$ converges to $P\chi_l$ for $\| \;\|_f$ and the inequality

$$\begin{aligned}|\langle P\chi_l - T, f\rangle| &\le |\langle P\chi_l - \check{\phi}_n * T, f\rangle| + |\langle \check{\phi}_n * T - T, f\rangle|\\ &\le \|P\chi_l - \check{\phi}_n * T\|_f + |\langle \check{\phi}_n * T - T, f\rangle|,\end{aligned}$$

valid for all n in $\mathbb{N}$, gives, as $n \to \infty$,

$$\langle P\chi_l - T, f\rangle = 0.$$

Since $\mathcal{S}(G)$ is dense in A, this proves that $T = P\chi_l$ and so T is zero on $I(\mathcal{P}_l)$. For all T in $j(l)^\circ$, we know that T is zero on $(\operatorname{Ker}\chi_l)^N$ and by the preceding T belongs to $\mathcal{P}_l\chi_l$ and so to $I(\mathcal{P}_l)^\circ$. Since $\langle T, (\operatorname{Ker}\chi_l)^m\rangle = 0$ we see that T is zero on $I(\mathcal{P}_l)$. The Hahn–Banach theorem shows finally that $I(\mathcal{P}_l)$ is contained in $j(l)$. ∎

Notation. Let J be a closed two-sided ideal of A. We associate to it the vector subspace $V(J)$ of $\mathcal{P}_l$ defined by

$$V(J) = \{P \in \mathcal{P}_l \mid \forall f \in J : Pf \in \operatorname{Ker}\chi_l\}.$$

We show that the mapping $J \mapsto V(J)$ gives characterization of the closed two-sided ideals of A with hull $\{\operatorname{Ker}\chi_l\}$.

5.4. PROPOSITION. *Let J be a closed two-sided ideal of A. The vector subspace $V(J)$ of $\mathcal{P}_l$ is invariant under translations.*

Proof. The vector space generated by $\mathcal{S}(G) * V(J) * \mathcal{S}(G)$ is dense in the finite-dimensional vector space $V(J)$, hence is equal to $V(J)$. The result then follows from the formula

$$ {}_x(f * P * g)_y = {}_xf * P * g_y $$

valid for all f and g in $\mathcal{S}(G)$, P in $V(J)$, and x, y in G. ∎

NOTATION. Denote by $\mathcal{TP}_l$ the set of nonzero subspaces of $\mathcal{P}_l$ which are invariant under left and right translations. For a topological vector space E and a subset X of E, we denote by X° the orthogonal complement of X in E, i.e. the vector space of continuous linear forms on E which are zero on X:

$$ X^\circ = \{\varphi \in E' \mid \forall x \in X : \langle \varphi, x\rangle = 0\}. $$

The most important result of this paper is the following theorem:

5.5. THEOREM. *The map*

$$ \mathcal{TP}_l \to \mathcal{J}(l), \qquad W \mapsto I(W), $$

is a decreasing bijection, with inverse

$$ \mathcal{J}(l) \to \mathcal{TP}_l, \qquad J \mapsto V(J). $$

Proof. By Theorem 4.6, the map $W \mapsto I(W)$ is $\mathcal{J}(l)$-valued.

For any finite-dimensional subspace U of A', we know that U is $*$-weakly closed and so $(U^\circ)^\circ = U$. This shows that the mapping $W \mapsto I(W)$ is injective.

Let us show the surjectivity. Let J be an element of $\mathcal{J}(l)$. Since $J \supset j(l)$, its orthogonal J° is finite-dimensional and is contained in $j(l)^\circ$, which means by Theorem 5.3 that $J^\circ \subset \mathcal{P}_l\chi_l$ and so $J^\circ = W\chi_l$ for some translation invariant subspace W of $\mathcal{P}_l$. Hence $J = (W\chi_l)^\circ = I(W)$, which shows the surjectivity of the map $W \mapsto I(W)$ and consequently, the bijectivity of $J \mapsto V(J)$. ∎

6. Examples. Let w be a symmetric weight with polynomial growth on G. Let N be an integer and define A_N as the subalgebra of $L^1(G)$ of classes of functions f such that $\sum_{|\alpha|\le N}\int_G(|X^\alpha * f|w + |f * X^\alpha|w)\,d\lambda$ is finite. We define a norm on A_N by putting

$$ \|f\| = \sum_{|\alpha|\le N}\int_G(|X^\alpha * f|w + |f * X^\alpha|w)\,d\lambda. $$

The algebra A_N with the norm $\| \ \|$ is a Banach algebra and satisfies the conditions given in 3.2. Consequently, Theorem 5.3 applies in this case. In particular, for N equal to zero, the weighted algebra $L^1_w(G)$ defined in 3.1 is an example of an algebra A satisfying the conditions of 3.2. The rest of this section states the principal results of the paper in this particular case.

NOTATION. Let G be a locally compact group and w a weight on G. We denote by $L^\infty_w(G)$ the vector space of (classes of) functions f essentially bounded by w, i.e. such that $\|f/w\|_\infty$ is finite, and we define a norm $\| \ \|$ on $L^\infty_w(G)$ by

$$\|f\| = \|f/w\|_\infty.$$

The following proposition, which describes the topological dual $L^1_w(G)'$ of $L^1_w(G)$, is known.

6.1. PROPOSITION. *Let G be a locally compact group, λ a nonzero positive left Haar measure on G, and w a weight on G. The map $\psi : L^\infty_w(G) \to L^1_w(G)'$ which takes $g \in L^\infty_w(G)$ to*

$$\psi g : L^1_w(G) \to \mathbb{C}, \qquad f \mapsto \langle g, f\rangle = \int_G fg\, d\lambda,$$

is an isometric isomorphism of Banach spaces.

In the following, the spaces $L^1_w(G)'$ and $L^\infty_w(G)$ will be identified. The topological dual of $L^1_w(G)$ being known, it is possible to give a more vivid description of the vector space $\mathcal{P}_l$ defined in 3.3:

NOTATION. Let w be a weight on G. We denote by $\mathcal{P}_w(G)$ the vector space of polynomials which are essentially bounded by w:

$$\mathcal{P}_w(G) = \mathcal{P}(G) \cap L^\infty_w(G).$$

By 6.1, it is clear that $\mathcal{P}_l = \mathcal{P}_w(G)$ and Theorem 5.3 can be written as

$$j(l) = I(\mathcal{P}_w(G)) = \Big\{ f \in L^1_w(G) \ \Big|\ \forall P \in \mathcal{P}_w(G) : \int_G P(x) f(x) \chi_l(x)\, dx = 0 \Big\}.$$

Particular cases. 1) If w is the constant weight equal to 1, then $L^1_w(G)$ coincides with $L^1(G)$, and $\mathcal{P}_w(G)$ contains only constants; then $j(l) = \{\operatorname{Ker} \chi_l\}$, which shows that $\{\operatorname{Ker} \chi_l\}$ is of synthesis. We find again in this case a result of [10].

2) If in a direction X_0, $w(\exp(tX_0))$ grows at least as $|t|$, then $\mathcal{P}_w(G)$ contains a nonconstant polynomial, hence $j(l)$ is strictly contained in $\{\operatorname{Ker} \chi_l\}$ and therefore $\{\operatorname{Ker} \chi_l\}$ is not of synthesis. So, for a nonconstant weight w, the one-point set $\{\operatorname{Ker} \chi_l\}$ is not of synthesis in $L^1_w(G)$ in general.

6.2. Let us take for G the 3-dimensional Heisenberg group H_1, for which the multiplication is given by

$$(x, y, z) \cdot (x', y', z') = \left(x + x', y + y', z + z' + \frac{1}{2}(xy' - yx') \right).$$

Denote by L the left regular representation of H_1 in $C^\infty(H_1)$. Let P be the polynomial

$$P = -x^2 + y^2 + z^2$$

and V the vector space generated by P and its left derivatives. Then the vector space V is 10-dimensional and $(1, x, y, z, x^2, y^2, xy, xz, yz, z^2)$ is a basis of V. For an element $(m_{ij})_{1 \leq i,j \leq 10}$, denoted by M, belonging to $\mathrm{End}(V)$, denote by $\|M\|_{\mathrm{HS}}$ its Hilbert–Schmidt norm

$$\|M\|_{\mathrm{HS}} = \Big(\sum_{1 \leq i,j \leq 10} |m_{ij}|^2 \Big)^{1/2}.$$

Finally, define

$$\omega(u, v, w) = \|L_{(u,v,w)}\|_{\mathrm{HS}}.$$

An explicit computation shows that

$$\omega(u, v, w) = \Big[10 + \frac{35}{4}(u^2 + v^2) + 7w^2 + \frac{7}{4}u^2v^2 + 2(u^2w^2 + v^2w^2) + \frac{21}{16}(u^4 + v^4) + w^4 \Big]^{1/2}.$$

The mapping ω is a weight on H_1. Let π be an element of the unitary dual $\widehat{H}_1$ of H_1 and let $\mathfrak{h}_1$ be the Lie algebra of H_1. Assume that the orbit of the linear form l on $\mathfrak{h}_1$ associated to π by the Kirillov bijection is one point, i.e. l is a character of $\mathfrak{h}_1$. So, π is a character of $L^1_\omega(H_1)$. By Theorem 5.5, the sets $\mathcal{J}(l)$ and $\mathcal{TP}_\omega(H_1)$ are in bijection. The set $\mathcal{TP}_\omega(H_1)$ is explicitly determined in [1].

References

[1] D. Alexander, *Idéaux minimaux d'algèbres de groupes*, thèse, Université de Metz, 2000.

[2] D. Alexander and J. Ludwig, *Undecomposable finite dimensional representations of solvable Lie groups*, preprint.

[3] J. Boidol, H. Leptin, J. Schurmann und D. Vahle, *Raüme primitiver Ideale von Gruppenalgebren*, Math. Ann. 236 (1978), 1–13.

[4] I. D. Brown, *Dual topology of a nilpotent Lie group*, Ann. Sci. École Norm. Sup. 6 (1973), 407–411.

[5] S. Dhieb et J. Ludwig, *Caractérisation des convoluteurs de Schwartz des groupes de Lie nilpotents*, J. Funct. Anal. 144 (1997), 46–53.

[6] J. Dixmier, *Opérateurs de rang fini dans les représentations unitaires*, Inst. Hautes Études Sci. Publ. Math. 6 (1960), 13–35.
[7] —, *Algèbres enveloppantes*, Cahiers Sci. 37, Gauthier-Villars, 1974.
[8] E. Hille and R. S. Phillips, *Functional Analysis and Semi-Groups*, Amer. Math. Soc., Providence, 1957.
[9] A. Hulanicki, *Subalgebra of $L_1(G)$ associated with Laplacian on a Lie group*, Colloq. Math. 31 (1974), 259–287.
[10] I. Kaplansky, *Primary ideals in group algebras*, Proc. Natl. Acad. Sci. USA 35 (1949), 133–136.
[11] A. Kirillov, *Unitary representations of nilpotent Lie groups*, Uspekhi Mat. Nauk 17 (1962), no. 4, 57–110 (in Russian); English transl.: Russian Math. Surveys 17 (1962), no. 4, 53–104.
[12] H. Leptin, *Ideal theory in group algebras of locally compact groups*, Invent. Math. 31 (1976), 259–278.
[13] J. Ludwig, *Polynomial growth and ideals in group algebras*, Manuscripta Math. 30 (1980), 215–221.
[14] —, *On the spectral synthesis problem for points in the dual of a nilpotent Lie group*, Ark. Math. 21 (1983), 127–144.
[15] —, *Minimal C^*-dense ideals and algebraically irreducible representations in the Schwartz algebra of a nilpotent Lie group*, in: Lecture Notes in Math. 1359, Springer, Berlin, 1988, 209–217.
[16] —, *Topologically irreducible representations of the Schwartz algebra of a nilpotent Lie group*, Arch. Math. (Basel) 54 (1990), 284–292.
[17] J. Ludwig and C. Molitor-Braun, *A restriction theorem for ideals in the Schwartz algebra of a nilpotent Lie group*, ibid. 67 (1996), 199–210.
[18] L. Pukanszky, *Leçons sur les représentations des groupes*, Dunod, 1967.
[19] H. Reiter, *Classical Harmonic Analysis and Locally Compact Groups*, Clarendon Press, 1968.
[20] I. Segal, *The group algebra of a locally compact group*, Trans. Amer. Math. Soc. 61 (1947), 69–105.
[21] N. Varopoulos, *Analysis and Geometry on Groups*, Cambridge Univ. Press, 1992.

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