STUDIA MATHEMATICA 167 (1) (2005)

On the weak decomposition property (δ_w)

by

EL HASSAN ZEROUALI and HASSANE ZGUITTI (Rabat)

Abstract. We study a new class of bounded linear operators which strictly contains the class of bounded linear operators with the decomposition property (δ) or the weak spectral decomposition property (weak-SDP). We treat general local spectral properties for operators in this class and compare them with the case of operators with (δ).

1. Introduction

1.1. *Definitions.* Throughout this paper, X is a Banach space and $\mathcal{L}(X)$ denotes the space of all bounded linear operators on X. For $T \in \mathcal{L}(X)$, let $N(T)$, $\sigma(T)$, $\sigma_{\mathrm{ap}}(T)$, $\sigma_{\mathrm{com}}(T)$ and $\sigma_{\mathrm{s}}(T)$ denote the null space, the spectrum, the approximate point spectrum, the compression spectrum and the surjectivity spectrum of T, respectively.

Let $D(\lambda, r)$ be the open disc centred at $\lambda \in \mathbb{C}$ and with radius $r \geq 0$; the corresponding closed disc will be denoted by $\overline{D}(\lambda, r)$. For a closed subset F in $\mathbb{C}$, the associated *glocal spectral analytic space* $\mathcal{X}_T(F)$ is the vector space of elements $x \in X$ for which there exists an analytic function $f : \mathbb{C} \setminus F \to X$ such that $(T-\mu)f(\mu) = x$ for $\mu \in \mathbb{C} \setminus F$. The *local resolvent set* $\varrho_T(x)$ of T at $x \in X$ is defined as the set of all $\lambda \in \mathbb{C}$ for which there exists an analytic X-valued function f on some open neighbourhood U of λ such that $(T-\mu)f(\mu) = x$ for all $\mu \in U$. The *local spectrum* of T at x is $\sigma_T(x) = \mathbb{C} \setminus \varrho_T(x)$ (see [13]). Note that $\sigma_T(x)$ is a closed subspace of $\sigma(T)$ and it may be empty.

We say that T has the *single-valued extension property* (SVEP) at $\lambda \in \mathbb{C}$ if there exists $r > 0$ such that for every open subset $U \subset D(\lambda, r)$, the only analytic solution of the equation $(T-\mu)f(\mu) = 0$ is the constant function $f \equiv 0$. In this case, $\sigma_T(x) = \emptyset$ if and only if $x = 0$, and we have $\mathcal{X}_T(F) = X_T(F)$, where $X_T(F) = \{x \in X : \sigma_T(x) \subseteq F\}$. The operator T is said to satisfy the *Dunford condition* (C) if $X_T(F)$ is closed for all closed subsets F in $\mathbb{C}$; and T has the *Bishop property* (β) if for every open subset U and for any sequence $(f_n)_n$ of analytic X-valued functions on U

2000 *Mathematics Subject Classification*: Primary 47A10, 47A11; Secondary 47B40.
Key words and phrases: local spectral theory, spectral decomposition, multipliers.

with $(T-\lambda)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on compact subsets of U, we have $f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on compact subsets of U. For more details, see [3, 13].

An operator $T \in \mathcal{L}(X)$ is said to be *decomposable* provided that for every finite open cover $\{U_1, \dots, U_n\}$ of $\mathbb{C}$, there exist closed T-invariant subspaces $X_1, \dots, X_n$ of X such that

$$\sigma(T|X_i) \subseteq U_i \quad \text{for } i = 1, \dots, n, \qquad X_1 + \dots + X_n = X. \tag{1}$$

The class of decomposable operators contains all normal operators and more generally all spectral operators. Operators with totally disconnected spectrum are decomposable by the Riesz functional calculus. In particular, compact and algebraic operators are decomposable.

A weaker version of decomposable operators is given by operators that have the weak spectral decomposition property. Namely, T is said to have the *weak spectral decomposition property* (weak-SDP) if, for every finite open cover $\{U_1, \dots, U_n\}$ of $\mathbb{C}$, there exist closed T-invariant subspaces $X_1, \dots, X_n$ of X such that

$$\sigma(T|X_i) \subseteq U_i \quad \text{for } i = 1, \dots, n, \qquad \overline{X_1 + \dots + X_n} = X. \tag{2}$$

E. Albrecht [1] gives an example that shows that the class of bounded linear operators with weak-SDP contains strictly the class of decomposable operators.

An operator T has the *decomposition property* (δ) if for every finite open cover $\{U_1, \dots, U_n\}$ of $\mathbb{C}$, we have

$$\mathcal{X}_T(\overline{U}_1) + \dots + \mathcal{X}_T(\overline{U}_n) = X. \tag{3}$$

Properties (β) and (δ) are known to be dual to each other in the sense that T has (δ) if and only if T^* satisfies (β). It is also known that (β) characterizes operators with decomposable extensions, and in particular isometries and subnormal operators have (β) (see [2]). Property (β) is hence conserved by restrictions while (δ) is inherited by quotient operators. See also [13] for more details.

Definition 1.1. Let $T \in \mathcal{L}(X)$. We say that T has the *weak decomposition property* (δ_w) *at* $\lambda \in \mathbb{C}$ if there exists $r(\lambda) > 0$ such that for every $0 \le r \le r(\lambda)$ and every finite open cover $\{U_1, \dots, U_n\}$ of $\mathbb{C}$ with $\sigma(T) \setminus D(\lambda, r) \subseteq U_1$,

$$\mathcal{X}_T(\overline{U}_1) + \dots + \mathcal{X}_T(\overline{U}_n) \text{ is dense in } X. \tag{4}$$

We will say that T has (δ_w) if it has (δ_w) at every $\lambda \in \mathbb{C}$.

The decomposition property (δ) and the Dunford condition (C) characterize the decomposability of bounded linear operators. Indeed, $T \in \mathcal{L}(X)$ is decomposable if and only if it has both (C) and (δ).

Operators with (δ_w) and (C) are called *quasi-decomposable* and have been treated in the literature (see [1, 9]). It is clear that quasi-decomposable operators satisfy weak-SDP.

In this paper we investigate the properties of operators with (δ_w). Our main objective is to compare them systematically with operators having (δ). Examples showing that the class of operators with (δ_w) is different from known classes are given at the end of this section. In Section 2 we show that the adjoints of operators with (δ_w) have the single-valued extension property but may fail to satisfy the Dunford condition (C).

We also link the (δ_w) property with generalized derivations to obtain results on the stability of this property under some transformations.

The localizable spectrum and the support points set are introduced in Section 3. A description of these spectral sets is given for operators with (δ_w).

Section 4 is devoted to property (δ_w) for multipliers.

1.2. *Examples.* From the definition, it follows that (δ) and weak-SDP each imply (δ_w). The question of whether (δ_w) implies (δ) or weak-SDP arises naturally.

We now exhibit operators satisfying (δ_w) without satisfying (δ) or weak-SDP. This shows that the class of operators considered here is strictly larger than the class of operators with (δ) or with weak-SDP.

A first example uses the classical shift on the Hardy space.

Example 1.1. Let B be the backward unilateral shift on $X_1 = l^2(\mathbb{N})$. The operator B has the decomposition property (δ) and does not have weak-SDP. Indeed, since B^* is an isometry and isometries have the Bishop property (β), it follows that B has (δ). According to Beurling's characterization of the subspaces invariant under B^* (see [8]), it is easy to see that B does not have weak-SDP. In particular, B satisfies (δ_w) but not weak-SDP.

An example with (δ_w) but without (δ) is provided by the operator $R_0 \in \mathcal{L}(X_2)$, for some Banach space, of Albrecht [1], since weak-SDP implies (δ_w).

Now set $R = 3I + R_0$ and take $T = B \oplus R \in \mathcal{L}(X_1 \oplus X_2)$. It is not difficult to see that for every closed subset F in $\mathbb{C}$,

$$(\mathcal{X}_1 \oplus \mathcal{X}_2)_T(F) = \mathcal{X}_{1B}(F) \oplus \mathcal{X}_{2R}(F).$$

Thus T has (δ_w). If T had weak-SDP, then for an open cover $\{U, V\}$ of $\mathbb{C}$ there would exist closed T-invariant subspaces Y and Z of $X_1 \oplus X_2$ such that $\sigma(T|Y) \subseteq U$, $\sigma(T|Z) \subseteq V$ and $Y + Z$ is dense in $X_1 \oplus X_2$. If Y and Z are trivial then $Y = Y_1 \oplus Y_2$ and $Z = Z_1 \oplus Z_2$, where Y_1 and Z_1 (resp. Y_2 and Z_2) are closed B-invariant subspaces (resp. R-invariant subspaces) of X_1 (resp. X_2). Then $\sigma(B|Y_1) \subseteq U$, $\sigma(B|Z_1) \subseteq V$ and $Y_1 + Z_1$ is dense in X_1. Now suppose that Y or Z is not trivial. Since $\sigma(R_0) = \overline{D}(0, 1)$ we have $\eta(\sigma(B)) \cap \eta(\sigma(R)) = \emptyset$, where $\eta(\cdot)$ is the polynomially convex hull. Then it follows from [19] that $Y = Y_1 \oplus Y_2$ and $Z = Z_1 \oplus Z_2$. Hence B has weak-SDP, which is not the case. Thus T has (δ_w) but neither (δ) nor weak-SDP.

EXAMPLE 1.2. Let T be the unilateral weighted shift on $X = l^2(\mathbb{N})$ defined by

$$Te_n = \begin{cases} 0 & \text{if } n = p! \text{ for some } p \in \mathbb{N}, \\ e_{n+1} & \text{otherwise.} \end{cases}$$

The adjoint operator of T is given by

$$T^*e_n = \begin{cases} 0 & \text{if } n = 0 \text{ or } n = p! + 1 \text{ for some } p \in \mathbb{N}, \\ e_{n-1} & \text{otherwise.} \end{cases}$$

Since $\sigma_{\mathrm{p}}(T) = \sigma_{\mathrm{p}}(T^*) = \{0\}$, (SVEP) holds for T and T^*. It is also clear that $T^{n!}e_n = 0$ for all $n \geq 1$ and hence $\sigma_T(e_n) = \{0\}$. It follows that $\mathcal{X}_T(\{0\})$ $(\subseteq X_T(\{0\}))$ is dense in X. Thus T has (δ_w). Since $\sigma(T) = \overline{D}(0,1) = \sigma(T^*)$, T does not have the Dunford property (C). Indeed, otherwise

$$\{0\} = \sigma(T|X_T(\{0\})) = \sigma(T|X) = \overline{D}(0,1).$$

The same argument shows that T^* has (δ_w) and fails (C). So T and T^* do not have (δ). To see that T does not have weak-SDP, suppose that there exist nontrivial closed T-invariant subspaces X_1, X_2 of X such that

$$\sigma(T|X_1) \subseteq D(0,1/2),$$
$$\sigma(T|X_2) \subseteq \mathbb{C} \setminus \overline{D}(0,1/4) \quad \text{and} \quad X_1 + X_2 \text{ is dense in } X.$$

In particular $0 \notin \sigma_T(x)$ for all $x \in X_2 \setminus \{0\}$. But $\bigcap_{n\geq 0} T^n(X) = \{0\}$ implies that $0 \in \sigma_T(x)$ for every nonzero $x \in X$. Contradiction.

2. Properties of operators with (δ_w)**.** In this section we generalize some known results for operators with (δ) to the class of operators with (δ_w).

THEOREM 2.1. *If $T \in \mathcal{L}(X)$ has (δ_w) at λ, then T^* has* (SVEP) *at λ.*

Proof. Let $U \subset D(\lambda,\varepsilon)$ be an open set and $\varphi(\cdot)$ be an analytic X^*-valued function on U such that $(T^* - \mu)\varphi(\mu) = 0$ for all $\mu \in U$. Choose $\varepsilon < r(\lambda)$ such that $U \cap D(\lambda,\varepsilon/4)) \neq \emptyset$ and $U \cap (\mathbb{C} \setminus D(\lambda,\varepsilon/2)) \neq \emptyset$. As T has (δ_w) at λ, it follows that

$$\mathcal{X}_T(\overline{D}(\lambda,\varepsilon/2)) + \mathcal{X}_T(\mathbb{C} \setminus D(\lambda,\varepsilon/4)) \text{ is dense in } X. \tag{5}$$

For each $x \in \mathcal{X}_T(\overline{D}(\lambda,\varepsilon/2)) + \mathcal{X}_T(\mathbb{C} \setminus D(\lambda,\varepsilon/4))$ there are $x_1 \in \mathcal{X}_T(\overline{D}(\lambda,\varepsilon/2))$ and $x_2 \in \mathcal{X}_T(\mathbb{C} \setminus D(\lambda,\varepsilon/4))$ such that $x = x_1 + x_2$. Let $f_1(\mu)$ and $f_2(\mu)$ be analytic X-valued functions on $\mathbb{C} \setminus \overline{D}(\lambda,\varepsilon/2)$ and $D(\lambda,\varepsilon/4)$ respectively such that $x_i = (T - \mu)f_i(\mu)$ for $i = 1, 2$.

Now for $\mu \in U \cap D(\lambda,\varepsilon/4)$, we have

$$\langle \varphi(\mu), x_2 \rangle = \langle \varphi(\mu), (T - \mu)f_2(\mu) \rangle = \langle (T^* - \mu)\varphi(\mu), f_2(\mu) \rangle = 0.$$

Since $\mu \to \langle \varphi(\mu), x_2 \rangle$ is an analytic function on U which vanishes on $U \cap D(\lambda, \varepsilon/4)$, it is identically null, and hence $\langle \varphi(\mu), x_2 \rangle = 0$ for all $\mu \in U$. Similarly, $\langle \varphi(\mu), x_1 \rangle = 0$ on U. Then $\langle \varphi(\mu), x \rangle = 0$ for all $x \in \mathcal{X}_T(\overline{D}(\lambda, \varepsilon/2)) + \mathcal{X}_T(\mathbb{C} \setminus D(\lambda, \varepsilon/4))$. It follows from (5) that $\varphi \equiv 0$. ∎

The following corollaries are immediate.

COROLLARY 2.1. *If* $T \in \mathcal{L}(X)$ *has* (δ_w), *then* T^* *has* (SVEP), *in particular*

$$\sigma(T) = \sigma_{\mathrm{ap}}(T).$$

Recall that $T \in \mathcal{L}(X)$ is said to be *semi-Fredholm* (resp. *Fredholm*) if $T(X)$ is closed and either $N(T)$ or $X/T(X)$ is finite-dimensional (resp. both are). The *semi-Fredholm spectrum* is defined by $\sigma_{\mathrm{sF}}(T) := \{\lambda \in \mathbb{C} : T - \lambda$ is not semi-Fredholm on $X\}$.

COROLLARY 2.2. *If* $T \in \mathcal{L}(X)$ *has* (δ_w) *and* (SVEP), *then every point of* $\sigma(T) \setminus \sigma_{\mathrm{sF}}(T)$ *is an isolated point.*

Proof. It follows from Corollary 2.1 that T^* has (SVEP); then the result follows by [16, Corollary 1.8]. ∎

REMARK 2.1. 1) Corollary 2.1 generalizes [13, Proposition 1.4] concerning the case when T has (δ) or weak-SDP.

2) It is well known that T having (δ) implies that T^* satisfies the Dunford condition (C) and hence has (SVEP). As shown by Example 1.2, if T has (δ_w) then T^* does not necessarily have (C).

3) Corollary 2.2 together with [11, Lemma 1] yields the following result: If T has (δ_w), then $\sigma_{\mathrm{sF}}(T)$ consists of all cluster points of $\sigma(T)$ and the isolated points $\lambda \in \sigma(T)$ for which $\mathcal{X}_T(\{\lambda\})$ is infinite-dimensional. This extends Corollary 1 of K. B. Laursen [11].

Just as for operators with (δ), property (δ_w) is inherited by quotients and some limits, and is preserved by functional calculus as shown by the next propositions.

PROPOSITION 2.1. *Let* $T \in \mathcal{L}(X)$ *have* (δ_w) *and let* $S \in \mathcal{L}(Y)$. *If* $RT = SR$ *for some* $R \in \mathcal{L}(X, Y)$ *with dense range*, *then* S *has* (δ_w).

Proof. Note that $R(\mathcal{X}_T(F)) \subseteq \mathcal{Y}_S(F)$ for every closed subset F of $\mathbb{C}$. Let $\{U_1, \ldots, U_n\}$ be an open cover of $\mathbb{C}$. Since T has (δ_w), we have

$$\begin{aligned} Y = \overline{R(X)} &\subseteq \overline{R(\mathcal{X}_T(\overline{U}_1) + \cdots + \mathcal{X}_T(\overline{U}_n))} \\ &\subseteq \overline{R(\mathcal{X}_T(\overline{U}_1)) + \cdots + R(\mathcal{X}_T(\overline{U}_n))} \subseteq \overline{\mathcal{Y}_S(\overline{U}_1) + \cdots + \mathcal{Y}_S(\overline{U}_n)}. \ ∎ \end{aligned}$$

For $T \in \mathcal{L}(X)$ and $S \in \mathcal{L}(Y)$, let $\delta_{S,T}$ be the generalized derivation induced by T and S defined on $\mathcal{L}(X, Y)$ by $\delta_{S,T}(R) = SR - RT$ for all $R \in \mathcal{L}(X, Y)$.

Next we give an asymptotic version of Proposition 2.1.

THEOREM 2.2. *Let $T_k \in \mathcal{L}(X)$ be a sequence of operators with (δ_w) and let $S \in \mathcal{L}(Y)$. Suppose that there exist $R_k \in \mathcal{L}(X,Y)$ with dense range such that $r_{\delta_{S,T_k}}(R_k) \to 0$ as $k \to \infty$. Then S has (δ_w).*

Proof. Let $\{U_1, \ldots, U_n\}$ be an open cover of $\mathbb{C}$. Choose an open cover $\{V_1, \ldots, V_n\}$ of $\mathbb{C}$ such that $V_i \subseteq \overline{V}_i \subseteq U_i$ for $i = 1, \ldots, n$. Since $r_k := r_{\delta_{S,T_k}}(R_k) \to 0$ as $k \to \infty$ and the sets $(\overline{V}_i + \overline{D}(0, r_k)) \cap \sigma(S)$ are compact, there exists $k_0 \geq 0$ such that

$$(\overline{V}_i + \overline{D}(0, r_{k_0})) \cap \sigma(S) \subseteq U_i \quad \text{for } i = 1, \ldots, n.$$

Hence by [13, Proposition 3.4.2], for $i = 1, \ldots, n$,

$$R_{k_0}(\mathcal{X}_T(\overline{V}_i)) \subseteq \mathcal{Y}_S(\overline{V}_i + \overline{D}(0, r_{k_0})) = \mathcal{Y}_S((\overline{V}_i + \overline{D}(0, r_{k_0})) \cap \sigma(S)) \subseteq \mathcal{Y}_S(\overline{U}_i).$$

Now we proceed as in the proof of Proposition 2.1. ∎

The following corollary is immediate by setting $R_n = (T - \lambda)^{-1}$ for some $\lambda \notin \sigma(T)$ in Theorem 2.2.

COROLLARY 2.3. *Let T_n be operators with (δ_w) norm converging to T. If T commutes with T_n for all $n \geq 0$, then T has (δ_w).*

PROPOSITION 2.2. *Let T have (δ_w) and let $f : U \to X$ be an analytic function on an open neighbourhood U of $\sigma(T)$. Then $f(T)$ has (δ_w).*

Proof. We recall that $\mathcal{X}_{f(T)}(F) = \mathcal{X}_T(f^{-1}(F))$ for all closed subsets f of $\mathbb{C}$ (see [13, Theorem 3.3.6]). Let $\{U_1, \ldots, U_n\}$ be an open cover of $\mathbb{C}$. Since $\{f^{-1}(U_1), \ldots, f^{-1}(U_n)\}$ is an open cover of $\sigma(T)$ and T has (δ_w), it follows that

$$\begin{aligned} X &= \overline{\mathcal{X}_T(\overline{f^{-1}(U_1)}) + \cdots + \mathcal{X}_T(\overline{f^{-1}(U_n)})} \\ &\subseteq \overline{\mathcal{X}_T(f^{-1}(\overline{U}_1)) + \cdots + \mathcal{X}_T(f^{-1}(\overline{U}_n))} \\ &= \overline{\mathcal{X}_{f(T)}(\overline{U}_1) + \cdots + \mathcal{X}_{f(T)}(\overline{U}_n)}. \end{aligned}$$

Hence $f(T)$ has (δ_w). ∎

PROPOSITION 2.3. *Let $T \in \mathcal{L}(X)$ have (δ_w) and let $S \in \mathcal{L}(Y)$ satisfy the Dunford condition* (C). *Then $\delta_{S,T}$ has* (SVEP).

Proof. Let $R : U \to \mathcal{L}(X,Y)$ be an analytic function on an open connected set U of $\mathbb{C}$ such that $(\delta_{S,T} - \mu)R(\mu) = 0$ for all $\mu \in U$. If $\mu \in U$ and $x \in X$, then $(\delta_{S,T} - \mu)R(\mu)x = 0$ implies $SR(\mu)x = R(\mu)(T + \mu)x$. Hence $R(\mu)x \in \mathcal{Y}_S(\sigma_T(x) + \mu)$. Now let $D_1, D_2 \subseteq U$ be two closed discs with nonempty interiors and $\operatorname{dist}(D_1, D_2) > \varepsilon$. Then $R(\mu)x \in \mathcal{Y}_S(\sigma_T(x) + D_i)$ for all $\mu \in D_i$. Since S satisfies the Dunford condition (C), $\mathcal{Y}_S(\sigma_T(x) + D_i)$ is a closed subspace and by the identity theorem for analytic functions, $R(\mu)x \in \mathcal{Y}_S(\sigma_T(x) + D_i)$ for all $\mu \in U$. Then if $\operatorname{diam}(\sigma_T(x)) \leq \varepsilon$, we have

$R(\mu)x \in \mathcal{Y}_S(\sigma_T(x)+D_1)\cap\mathcal{Y}_S(\sigma_T(x)+D_2) = \mathcal{Y}_S(\emptyset) = \{0\}$. Hence $R(\mu)x = 0$ for all x with $\operatorname{diam}(\sigma_T(x)) \le \varepsilon$ and all $\mu \in U$.

Now since T has (δ_w), there exists an open cover $\{U_1, \dots, U_n\}$ of $\sigma(T)$ with $\operatorname{diam}(U_i) \le \varepsilon$ such that $\mathcal{X}_T(\overline{U}_1) + \dots + \mathcal{X}_T(\overline{U}_n)$ is dense in X. Hence $R(\mu)x = 0$ for all $x \in X$, which proves the proposition. ∎

3. The localizable spectrum for operators with (δ_w). For a bounded operator T, the *localizable spectrum* $\sigma_{\mathrm{loc}}(T)$ of T is the set of all points $\lambda \in \sigma(T)$ such that

$$X_T(\overline{D}(\lambda, r)) \neq \{0\} \quad \text{for every } r > 0. \tag{6}$$

We list some elementary observations related to the localizable spectrum:

(a) If T does not have (SVEP), then $\sigma_{\mathrm{loc}}(T) = \sigma(T)$. Indeed, $X_T(\emptyset)$ is nontrivial and contained in $X_T(D(\lambda, r))$ for all λ and r.

(b) $\sigma_{\mathrm{p}}(T) \subset \sigma_{\mathrm{loc}}(T)$, because $\sigma_T(x) \subseteq \{\lambda\}$ for any eigenvector associated with the eigenvalue λ.

(c) The localizable spectrum may be strictly contained in the spectrum; for example, if T is the unilateral forward unweighted shift on the Hardy space, then $\sigma_{\mathrm{loc}}(T) = \emptyset$, while $\sigma(T)$ is the closed unit disk. On the other hand, if T satisfies (δ) or weak-SDP, then $\sigma_{\mathrm{loc}}(T) = \sigma(T)$. See [7, 18].

If T is decomposable, then T has (δ) and (δ^*). Thus $\sigma_{\mathrm{loc}}(T) = \sigma_{\mathrm{loc}}(T^*)$. This is still valid if (δ) is replaced by (δ_w).

Proposition 3.1. *If $T \in \mathcal{L}(X)$ has (δ_w), then*

$$\sigma_{\mathrm{com}}(T) \subseteq \sigma_{\mathrm{loc}}(T^*) \subseteq \sigma_{\mathrm{loc}}(T).$$

In particular if both T and T^* have (δ_w), then $\sigma_{\mathrm{loc}}(T) = \sigma_{\mathrm{loc}}(T^*)$.

Proof. The first inclusion is trivial since $\sigma_{\mathrm{com}}(T) = \sigma_{\mathrm{p}}(T^*)$. Suppose $\lambda \notin \sigma_{\mathrm{loc}}(T)$ and let $r > 0$ be such that $X_T(D(\lambda, r)) = \{0\}$. It follows that $\mathcal{X}_T(D(\lambda, r)) = \{0\}$ and because T satisfies (δ_w), we see that $\mathcal{X}_T(\mathbb{C} \setminus D(\lambda, r/2))$ is dense. Now, from [13, Proposition 2.5.1] we have the inclusion

$$\mathcal{X}^*_{T^*}(D(\lambda, r/4)) \subseteq (\mathcal{X}_T(\mathbb{C} \setminus D(\lambda, r/2)))^{\perp}.$$

Thus $\mathcal{X}^*_{T^*}(D(\lambda, r/4)) = \{0\}$, and (SVEP) for T^* leads to $X^*_{T^*}(D(\lambda, r/4)) = \{0\}$. The proof is complete. ∎

In contrast with decomposable operators, Example 1.2 provides an operator T with (δ_w) and (δ^*_w) such that $\sigma_{\mathrm{loc}}(T) \neq \sigma(T)$.

In the proposition above, the inclusion $\sigma_{\mathrm{com}}(T) \subseteq \sigma_{\mathrm{loc}}(T)$ may be strict. Let T be the operator given in Example 1.2. Then $\sigma_{\mathrm{com}}(T) = \{0\}$. Indeed, $\{0\} \subseteq \sigma_{\mathrm{com}}(T) \subseteq \sigma_{\mathrm{loc}}(T^*) \subseteq \sigma_{\mathrm{loc}}(T)$. On the other hand, since $\bigcap_{n\ge0} T^n(H) = \{0\}$, we obtain $0 \in \sigma_T(x)$ for every nonzero x, and therefore we conclude that $\sigma_{\rm loc}(T) = \{0\}$. Thus

$$\sigma_{\rm com}(T) = \sigma_{\rm loc}(T^*) = \sigma_{\rm loc}(T) = \{0\}.$$

If T is any normal operator without eigenvectors, then $\sigma_{\rm com}(T) = \emptyset$ while the equality $\sigma_{\rm loc}(T) = \sigma(T)$ always holds.

PROPOSITION 3.2. *Let $T \in \mathcal{L}(X)$, $S \in \mathcal{L}(Y)$, and let $R \in \mathcal{L}(X, Y)$ be injective such that $RT = SR$. If T has (δ_w), then*

$$\sigma_{\rm com}(T) \subseteq \sigma_{\rm loc}(S). \tag{7}$$

Proof. Since $RT = SR$, we obtain $R(X_T(F)) \subseteq Y_S(F)$ for every closed set F, and as R is injective, it follows that $\sigma_{\rm loc}(T) \subset \sigma_{\rm loc}(S)$. Proposition 3.1 allows us to conclude. ∎

REMARK 3.1. The inclusion (7) may be strict. To see this, let $T = S$ be a normal operator without eigenvectors and R the identity map. Then T has (δ_w) and $\sigma_{\rm com}(T) = \emptyset \subset \sigma_{\rm loc}(S) = \sigma(S)$.

A notion closely related to the localizable spectrum is provided by the *support points set* introduced in [17] by taking, in (6), the glocal spectral analytic space instead of the analytic spectral space. More precisely, according to [17], $\lambda \in \mathbb{C}$ is a *support point* for T if

$$\mathcal{X}_T(\overline{D}(\lambda, r)) \neq \{0\} \quad \text{for every } r > 0. \tag{8}$$

The set of all support points for T is denoted $\mathrm{spt}(T)$. It is trivial from (6) and (8) that $\sigma_{\rm p}(T) \subseteq \mathrm{spt}(T) \subseteq \sigma_{\rm loc}(T)$ and that the last inclusion is an equality if and only if T has (SVEP). It is also not difficult to see that $\mathrm{spt}(T) = \sigma(T)$ if T is decomposable, or more generally if T has weak-SDP or (δ). The latter equality fails to be true in general for operators with (δ_w) as shown by Example 1.2.

The *analytic core* $K(T)$ associated with T is the (not necessarily closed) invariant subspace of T that consists of elements $x \in X$ for which there exists $c > 0$ and a sequence $x_n \in X$ such that $x_0 = x$, $\|x_n\| \le c^n \|x\|$ and $Tx_{n+1} = x_n$. The *quasi-nilpotent part* of T is $X_0(T) := \{x \in X : \lim_{n\to\infty} \|T^n x\|^{1/n} = 0\}$. The analytic core and the quasi-nilpotent part of an operator have been extensively studied by M. Mbekhta in [14–16]. In particular

$$K(T) = X_T(\mathbb{C} \setminus \{0\}) \quad \text{and} \quad X_0(T) = \mathcal{X}_T(\{0\}).$$

The question when the analytic core is closed has interested some mathematicians and is central in [17], where the special case of operators with (δ) has been developed. In the rest of this section we discuss the same phenomena for operators with (δ_w).

For operators with (SVEP), it is proved in [17, Lemma 4] that $K(T)$ is not closed when 0 is a cluster point of $\operatorname{spt}(T)$. For operators with (δ_w), we have

THEOREM 3.1. *Let T be noninvertible and suppose T has (δ_w). If $K(T)$ is closed, then $0 \in \operatorname{spt}(T)$.*

Proof. Let $F_n = D(0, 1/n)$ and $G_n = \mathbb{C} \setminus F_{2n}$. Then $\mathcal{X}_T(\overline{F}_n) + \mathcal{X}_T(G_n)$ is dense and $\mathcal{X}_T(G_n) \subseteq K(T)$. In particular, if $\mathcal{X}_T(\overline{F}_n) = \{0\}$ for some n, then $\mathcal{X}_T(G_n) \subseteq K(T)$ is dense, and since $K(T)$ is closed, it follows that T is onto. Now since $0 \in \sigma(T)$, we have $0 \in \sigma_{\mathrm{p}}(T) \subseteq \operatorname{spt}(T)$. ∎

Set $\sigma_{\mathrm{fK}}(T) = \{\lambda \in \mathbb{C} : K(T - \lambda) \text{ is not closed}\}$. We have

COROLLARY 3.1. *If T has (δ_w), then $\sigma(T) \setminus \sigma_{\mathrm{fK}}(T) \subseteq \operatorname{spt}(T)$.*

In [17, Corollary 8], it is shown that an operator T is quasinilpotent if and only if T has (δ) and $K(T) = \{0\}$. We note that the assumption (δ) cannot be relaxed to (δ_w) as shown by Example 1.2.

4. Multipliers and (δ_w)**.** We devote this section to showing that for multipliers on semisimple Banach algebras, the notions of decomposability and (δ_w) coincide.

Let A be a semisimple commutative Banach algebra. $\Sigma(A)$ denotes the spectrum of A, that is, the set of nontrivial multiplicative linear functionals on A. For each $a \in A$, let $\widehat{a}$ denote the Gelfand transform given by $\widehat{a}(\chi) = \chi(a)$ for all $\chi \in \Sigma(A)$. The *Gelfand topology* is the coarsest topology on $\Sigma(A)$ for which all the Gelfand transforms $\widehat{a}$ are continuous.

For $B \subset A$ and $E \subseteq \Sigma(A)$, we define $h(B) = \{\Psi \in \Sigma(A) : \Psi(b) = 0 \text{ for all } b \in B\}$ and $k(E) = \{a \in A : \varphi(a) = 0 \text{ for all } \varphi \in E\}$. We say that E is closed for the *hull-kernel topology* if $E = hk(E)$. The last topology is always coarser than the Gelfand topology, and they coincide exactly when A is a regular algebra. For further information, see [4, 19]. For $a \in A$ let T_a denote the corresponding multiplication operator given by $T_a(x) = ax$ for all $x \in A$.

A mapping $T : A \to A$ is called a *multiplier* if $T(x)y = xT(y)$ for all $x, y \in A$. By semisimplicity of A, every multiplier is a bounded linear operator on A. Moreover, $M(A)$, the set of multipliers on A, is a semisimple commutative unital subalgebra of $\mathcal{L}(A)$. The spectrum $\Sigma(M(A))$ may be represented as the disjoint union of $\Sigma(A)$ and $H(A)$, where $H(A) = \{\chi \in \Sigma(M(A)) : \chi(a) = 0, \forall a \in A\}$ and $\Sigma(A)$ is canonically embedded in $\Sigma(M(A))$. For $T \in M(A)$, $a, b \in A$ and $\chi \in \Sigma(A)$ we have $\chi(a)\chi(T(b)) = \chi(b)\chi(T(a))$. Hence $\chi(T(a))/\chi(a) = \chi(T(b))/\chi(b)$. This allows us to define $\widehat{T}(\chi) := \chi(T) = \chi(T(a))/\chi(a)$ for some $a \in A$ such that $\chi(a) \neq 0$. See for instance [10] and [13].

It is shown in [6, Proposition 1] that weak-SDP multipliers have hull-kernel continuous transform on $\Sigma(A)$. This result is extended as follows:

PROPOSITION 4.1. *Let A be a semisimple commutative Banach algebra. If $T \in M(A)$ has (δ_w), then $\widehat{T}|\Sigma(A)$ is hull-kernel continuous on $\Sigma(A)$.*

Proof. Suppose that $\widehat{T}$ is not hull-kernel continuous on $\Sigma(A)$. There exists a closed subset F of $\mathbb{C}$ such that $E = \{\chi \in \Sigma(A) : \chi(T) \in F\} \subset hk(E)$. Let $\chi \in hk(E) \setminus E$. Then $\chi(T) = \lambda \notin F$ and hence there exist open subsets U, V of $\mathbb{C}$ such that $\lambda \in U$, $F \subseteq V$ and $\overline{U} \cap \overline{V} = \emptyset$.

Since T has (δ_w), it follows that

$$\mathcal{A}_T(\mathbb{C} \setminus U) + \mathcal{A}_T(\mathbb{C} \setminus V) \text{ is dense in } A.$$

For each $x \in \mathcal{A}_T(\mathbb{C} \setminus U)$, there exists $y \in A$ such that $x = (T - \lambda)y$. Then $\chi(x) = 0$. Also for each $x \in \mathcal{A}_T(\mathbb{C} \setminus V)$, $\psi(x) = 0$ for all $\psi \in E$ and hence $\chi(x) = 0$. It follows that $\chi \equiv 0$ on A. Contradiction. ∎

We deduce the following corollary in the spirit of [13, Theorem 4.4.5]. The proof is a simple adaptation and is omitted.

COROLLARY 4.1. *Let A be a semisimple commutative Banach algebra and $a \in A$. Then the following statements are equivalent*:

(i) *T_a is decomposable.*
(ii) *T_a has (δ_w).*
(iii) *The Gelfand transform $\widehat{a}$ is hull-kernel continuous on $\Sigma(A)$.*

Let T be multiplier on A. It is not hard to see that $\widehat{T}(\Sigma(A)) \subset \sigma(T) = \widehat{T}(\Sigma(M(A)))$. We say that T has *natural spectrum* if

$$\sigma(T) = \overline{\widehat{T}(\Sigma(A))}. \tag{9}$$

Multipliers with weak-SDP or (δ) are known to have natural spectrum (see [6, Proposition 1] and [13, Proposition 4.6.3] respectively). This may fail to be true for an operator with (δ_w). We have

PROPOSITION 4.2. *If T is a multiplier with (δ_w), then*

$$\overline{\sigma_{\mathrm{com}}(T)} = \overline{\widehat{T}(\Sigma(A))}. \tag{10}$$

Proof. Let S be multiplication by $\widehat{T}$ on the Banach space Y of all continuous bounded $\mathbb{C}$-valued functions on $\Sigma(A)$ equipped with the sup-norm, and $R : A \to Y$ the Gelfand transform. It follows from Proposition 3.2 that $\sigma_{\mathrm{com}}(T) \subseteq \sigma(S)$.

Let $\lambda \notin \overline{\widehat{T}(\Sigma(A))}$. If $(S - \lambda)f = 0$ then $(\widehat{T}(\chi) - \lambda)f(\chi) = 0$ for all $\chi \in \Sigma(A)$. Thus $f(\chi) = 0$ and hence $S - \lambda$ is injective. Now let $g \in Y$. The mapping

$$f(\chi) = \frac{1}{\widehat{T}(\chi) - \lambda}\, g(\chi)$$

is continuous and bounded. Moreover, it satisfies $(S-\lambda)f = g$. Thus $S-\lambda$ is surjective and then $\sigma_{\mathrm{com}}(T) \subseteq \sigma(S) \subseteq \overline{\widehat{T}(\Sigma(A))}$.

On the other hand, $\widehat{T}(\Sigma(A)) \subseteq \sigma_{\mathrm{com}}(T)$. Indeed, let $\lambda \notin \sigma_{\mathrm{com}}(T)$. If $\lambda \in \widehat{T}(\Sigma(A))$, then there exists $\chi \in \Sigma(A)$ such that $\widehat{T}(\chi) = \lambda$. Hence $\chi((T-\lambda)a) = 0$ for all $a \in A$. Since $(T-\lambda)A$ is dense in A, it follows that χ vanishes on A. ■

If A is a regular semisimple commutative Tauberian Banach algebra, then it follows from the proof of [6, Proposition 5] that every multiplier T on A has (δ_w). By Corollary 2.1, $\sigma(T) = \sigma_{\mathrm{ap}}(T)$. Thus in Proposition 4.8.6 of [13] the assumption that $\Sigma(A)$ is discrete is not necessary.

Let G be a locally compact abelian group, Γ its dual group, $L^1(G)$ the space of $\mathbb{C}$-valued functions on G integrable with respect to Haar measure and $M(G)$ the Banach algebra of regular complex Borel measures on G. We recall that $L^1(G)$ is a regular semisimple commutative Tauberian Banach algebra. Then we have the following proposition.

PROPOSITION 4.3. *Let G be a locally compact abelian group, $\mu \in M(G)$ and $X = L^1(G)$. Then every convolution operator $T_\mu : X \to X$, $T_\mu(k) = \mu \star k$, has (δ_w) and*

$$\overline{\sigma_{\mathrm{com}}(T_\mu)} = \overline{\widehat{\mu}(\Gamma)}.$$

Note that when G is nondiscrete there exists a measure $\mu \in M(G)$ such that the convolution operator T_μ does not have a natural spectrum (see for instance [5, Corollary 3] or [13, pp. 370]). In that example $\overline{\widehat{\mu}(T_\mu)} \subset \sigma(T_\mu)$. Hence this also gives another example of an operator with (δ_w) but with neither weak-SDP nor (δ).

Acknowledgments. The authors are grateful to the referee for many valuable comments, which led to improvement of some results of this paper.

References

[1] E. Albrecht, *An example of a weakly decomposable operator, which is not decomposable*, Rev. Roumaine Math. Pures Appl. 20 (1975), 855–861.

[2] E. Albrecht and J. Eschmeier, *Analytic functional models and local spectral theory*, Proc. London Math. Soc. (3) 75 (1997), 323–348.

[3] E. Bishop, *A duality theorem for an arbitrary operator*, Pacific J. Math. 9 (1959), 379–397.

[4] F. F. Bonsall and J. Duncan, *Complete Normed Algebras*, Springer, Berlin, 1973.

[5] J. Eschmeier, *Operator decomposability and weakly continuous representations of locally compact abelian groups*, J. Operator Theory 7 (1982), 201–208.

[6] J. Eschmeier, K. B. Laursen and M. M. Neumann, *Multipliers with natural local spectra on commutative Banach algebras*, J. Funct. Anal. 138 (1996), 273–294.

[7] J. Eschmeier and B. Prunaru, *Invariant subspaces and localizable spectrum*, Integral Equations Operator Theory 42 (2002), 461–471.

[8] P. R. Halmos, *A Hilbert Space Problem Book*, Van Nostrand, New York, 1967.

[9] A. A. Jafarian, *Weak and quasi-decomposable operators*, Rev. Roumaine Math. Pures Appl. 22 (1977), 195–212.

[10] R. Larsen, *An Introduction to the Theory of Multipliers*, Springer, New York, 1971.

[11] K. B. Laursen, *Essential spectra through local spectral theory*, Proc. Amer. Math. Soc. 125 (1997), 1425–1434.

[12] K. B. Laursen and M. M. Neumann, *Asymptotic intertwining and spectral inclusions on Banach spaces*, Czechoslovak Math. J. 43 (118) (1993), 483–497.

[13] —, —, *An Introduction to Local Spectral Theory*, Clarendon Press, Oxford, 2000.

[14] M. Mbekhta, *Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux,* Glasgow Math. J. 29 (1987), 159–175.

[15] —, *Résolvant généralisé et théorie spectrale,* J. Operator Theory 21(1989), 69–105.

[16] —, *Local spectrum and generalized spectrum*, Proc. Amer. Math. Soc. 112 (1991), 457–463.

[17] T. L. Miller, V. G. Miller and M. M. Neumann, *On operators with closed analytic core,* Rend. Circ. Mat. Palermo (2) 51 (2002), 495–502.

[18] B. Prunaru, *Invariant subspaces for bounded operators with large localizable spectrum*, Proc. Amer. Math. Soc. 129 (2001), 2365–2372.

[19] C. E. Rickart, *General Theory of Banach Algebras*, Van Nostrand, New York, 1974.

Département de Mathématiques et Informatique
Faculté des Sciences de Rabat
BP 1014 Agdal
Rabat, Maroc
E-mail: zerouali@fsr.ac.ma
zguitti@hotmail.com

Received May 6, 2003
Revised version March 29, 2004 (5198)