

Sequential closedness of Boolean algebras of projections in Banach spaces

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Abstract. Complete and σ -complete Boolean algebras of projections acting in a Banach space were introduced by W. Bade in the 1950's. A basic fact is that every complete Boolean algebra of projections is necessarily a closed set for the strong operator topology. Here we address the analogous question for σ -complete Boolean algebras: are they always a sequentially closed set for the strong operator topology? For the atomic case the answer is shown to be affirmative. For the general case, we develop criteria which characterize when a σ -complete Boolean algebra of projections is sequentially closed. These criteria are used to show that both possibilities occur: there exist examples which are sequentially closed and others which are not (even in Hilbert space).

1. Introduction. In the Banach space setting, σ -complete and complete Boolean algebras (briefly, B.a.'s) of projection operators were intensively studied by W. Bade ([1], [2]). Such objects are a natural extension of the fundamental notion of the resolution of the identity of a normal operator in Hilbert space. The definition of such (σ -)complete B.a.'s of projections (see Section 2) is directly connected to the (*non-metrizable*) strong operator topology. Accordingly, the theory of such B.a.'s is somewhat different to the theory of abstract B.a.'s.

It was already noted from the outset of the theory that there are subtle topological and order-theoretic distinctions between the notions of σ -completeness and completeness. For instance, let X denote the Hilbert space $\ell^2([0, 1])$ and Σ the family of all Borel subsets of $[0, 1]$. For each $E \in \Sigma$ define a self-adjoint projection $P(E)$ by $P(E)x = \chi_E x$ for each $x \in X$. Then $P(E) \in \mathcal{L}(X)$, where $\mathcal{L}(X)$ denotes the space of all bounded linear operators on (any Banach space) X , and $\mathcal{M} = P(\Sigma) \subseteq \mathcal{L}(X)$ is a σ -complete B.a. of projections which fails to be complete. For a general σ -complete B.a. of pro-

2000 *Mathematics Subject Classification*: Primary 46E27, 47B40; Secondary 46B26, 46B42.

Key words and phrases: Boolean algebras of projections, sequential closedness, Bade functionals, band projections, spaces of measures.

jections \mathcal{M} in a Banach space X , the distinction is somewhat delicate. For, if Y is the closed subspace of X generated by $\{Bx_n : B \in \mathcal{M}, n \in \mathbb{N}\}$, then the restrictions $\mathcal{M}|_Y = \{B|_Y : B \in \mathcal{M}\}$ form a *complete* B.a. of projections in Y ([6, XVII, Lemma 3.21]). In particular, whenever X is separable the notions of σ -complete and complete coincide.

It is known that every complete B.a. of projections $\mathcal{M} \subseteq \mathcal{L}(X)$ is necessarily a τ_s -closed set for the strong operator topology τ_s in $\mathcal{L}(X)$ ([6, XVII, Corollary 3.7]). This raises the following

QUESTION. Is every σ -complete B.a. of projections in a Banach space X necessarily a *sequentially* τ_s -closed subset of $\mathcal{L}(X)$?

An equivalent formulation of this question appears in 323Z (Problem) of [8]. The aim of this paper is to give a complete answer to this question.

The above example given in $X = \ell^2([0, 1])$ is immediately relevant. The B.a. \mathcal{M} given there fails to be τ_s -closed in $\mathcal{L}(X)$, but it is sequentially τ_s -closed. That is, if $\{P_n\}_{n=1}^\infty \subseteq \mathcal{M}$ is any τ_s -convergent sequence, with limit $Q \in \mathcal{L}(X)$ say, then actually $Q \in \mathcal{M}$. This is no accident. It is shown in [14, p. 367] that whenever $\mathcal{M} \subseteq \mathcal{L}(X)$ is *Bade atomic* and σ -complete, then it is always sequentially τ_s -closed in $\mathcal{L}(X)$. Here *Bade atomic* means that there is a family of atoms $\{P_\alpha\}_{\alpha \in \mathcal{A}}$ in \mathcal{M} which generate \mathcal{M} in the sense that, for every $0 \neq P \in \mathcal{M}$, there exists a non-empty subset $\mathcal{B} \subseteq \mathcal{A}$ with $P = \sum_{\alpha \in \mathcal{B}} P_\alpha$; summability of the series is meant as the τ_s -limit of the net of partial sums over all finite subsets of \mathcal{B} . The above-mentioned example in $\ell^2([0, 1])$ is clearly *Bade atomic* in this sense. In Section 2 we will show that the above-mentioned result concerning *Bade atomic* B.a.'s of projections can actually be further extended to include arbitrary atomic, σ -complete B.a.'s of projections, where *atomic* is meant purely in the sense of abstract Boolean algebras. By way of examples it is shown that this extension is genuine. So, the atomic case is now completely understood.

Section 3 is concerned with presenting criteria which characterize precisely whether or not a given σ -complete B.a. \mathcal{M} is sequentially τ_s -closed in $\mathcal{L}(X)$. The idea is to represent \mathcal{M} as the range of an $\mathcal{L}(X)$ -valued *spectral measure* defined on some σ -algebra of sets Σ , and then to consider certain families Φ of non-negative measures on Σ induced by so-called *Bade functionals* ([6, XVII, Lemma 3.12]). The criteria presented are based on the notion of *sequential Φ -completeness* of the σ -algebra Σ (see Section 3 for the definition).

The most interesting part of the paper, perhaps, is the final Section 4, which is concerned with examples. It turns out that to exhibit measurable spaces (Ω, Σ) and families of non-negative measures Φ for which Σ is not sequentially Φ -complete and then, based on the properties of Φ , to construct non-atomic, σ -complete B.a.'s of projections in some appropriate

non-separable Banach space, is rather non-trivial. Briefly, the answer to the above Question turns out to be no, in general. There exist examples of non-atomic, σ -complete B.a.'s of projections $\mathcal{M} \subseteq \mathcal{L}(X)$ which are sequentially τ_s -closed but not τ_s -closed and others which fail to be sequentially τ_s -closed (even in a *Hilbert space!*). In the latter case, we point out that the *sequential closure* of \mathcal{M} , formed in the non-metrizable space $\mathcal{L}_s(X) = (\mathcal{L}(X), \tau_s)$, is always a σ -complete B.a. of projections ([13, Corollary 2.1]). Finally, we remark that there seems to be a distinct lack of concrete examples of non-atomic, σ -complete but not complete B.a.'s of projections in Banach spaces available. In particular, such B.a.'s should *not* admit any *separating vector* ([11, Lemma 1]). Hopefully, the examples of Section 4 go some way towards rectifying this situation.

2. Atomic Boolean algebras. Let X be a Banach space. All B.a.'s of projections $\mathcal{M} \subseteq \mathcal{L}(X)$ are assumed to have the identity operator I on X as their unit. We say that \mathcal{M} is *complete* (resp. *σ -complete*) *in the sense of Bade* if it is complete (resp. σ -complete) as an abstract B.a. and if, for every family (resp. countable family) $\mathcal{D} \subseteq \mathcal{M}$, we have

$$\left(\bigwedge_{D \in \mathcal{D}} D \right)(X) = \bigcap_{D \in \mathcal{D}} D(X)$$

and

$$\left(\bigvee_{D \in \mathcal{D}} D \right)(X) = \overline{\text{sp}} \left\{ \bigcup_{D \in \mathcal{D}} D(X) \right\},$$

the closed subspace of X generated by $\bigcup_{D \in \mathcal{D}} D(X)$; see [1] or [6, Chapter XVII], for example.

Let $\mathcal{M} \subseteq \mathcal{L}(X)$ be a B.a. of projections. A non-zero element $P \in \mathcal{M}$ is called an *atom* if, whenever $Q \in \mathcal{M}$ satisfies $Q \leq P$, then either $Q = 0$ or $Q = P$. We say that \mathcal{M} is *atomic* if for every $0 \neq Q \in \mathcal{M}$ there exists an atom $P \leq Q$ (see e.g. [16, Section 9]). Before proving the main result of this section, we first clarify the relation between atomic and Bade atomic B.a.'s.

LEMMA 2.1. *Let X be a Banach space and $\mathcal{M} \subseteq \mathcal{L}(X)$ be a B.a. of projections.*

- (i) *If \mathcal{M} is Bade atomic, then it is atomic.*
- (ii) *If \mathcal{M} is Bade complete, then \mathcal{M} is Bade atomic if and only if it is atomic.*

Proof. Statement (i) follows immediately from the definition of Bade atomic (see Section 1). For the proof of (ii) it suffices to show that if \mathcal{M} is atomic and Bade complete, then \mathcal{M} is Bade atomic. To this end, let $\{P_\alpha : \alpha \in \mathcal{A}\}$ be the collection of all atoms in \mathcal{M} . Given $0 \neq Q \in \mathcal{M}$, let $\mathcal{B} = \{\alpha \in \mathcal{A} : P_\alpha \leq Q\}$. It is clear that $Q = \bigvee_{\alpha \in \mathcal{B}} P_\alpha$. Denoting by

\mathcal{F} the collection of all finite subsets of \mathcal{B} and defining $P_F = \sum_{\alpha \in F} P_\alpha$ for all $F \in \mathcal{F}$, it is clear that $P_F \uparrow_{\mathcal{F}} Q$ in \mathcal{M} . Since \mathcal{M} is Bade complete, we may conclude by Lemma XVII.3.4 in [6] that $Q = \lim_{F \in \mathcal{F}} P_F$ in $\mathcal{L}_s(X)$, i.e., $Q = \sum_{\alpha \in \mathcal{B}} P_\alpha$ as a τ_s -summable series. ■

THEOREM 2.2. *Let X be a Banach space. Then every atomic, Bade σ -complete B.a. of projections in X is sequentially closed in $\mathcal{L}_s(X)$.*

Proof. Let $\mathcal{M} \subseteq \mathcal{L}(X)$ be a Bade σ -complete, atomic B.a. and suppose that $\{H_n\}_{n=1}^\infty \subseteq \mathcal{M}$ is a sequence satisfying $H_n \rightarrow H$ in $\mathcal{L}_s(X)$ for some $H \in \mathcal{L}(X)$. It is clear that H is a projection and commutes with all members of \mathcal{M} . Since \mathcal{M} is σ -complete, the projections

$$Q = \liminf H_n = \bigvee_{n=1}^\infty \bigwedge_{k=n}^\infty H_k, \quad R = \limsup H_n = \bigwedge_{n=1}^\infty \bigvee_{k=n}^\infty H_k$$

both belong to \mathcal{M} . We claim that $Q \leq H \leq R$. Indeed, the sequence $\{\tilde{H}_n\}_{n=1}^\infty$, defined by $\tilde{H}_n = \bigwedge_{k=n}^\infty H_k$, increases to Q in the order of \mathcal{M} and so, by Lemma XVII.3.4 in [6], $\tilde{H}_n \rightarrow Q$ in $\mathcal{L}_s(X)$. Since $\tilde{H}_n \leq H_n$ (i.e., $\tilde{H}_n H_n = \tilde{H}_n$) for each $n = 1, 2, \dots$ and both $\tilde{H}_n \rightarrow Q$ and $H_n \rightarrow H$ in $\mathcal{L}_s(X)$, we deduce that $QH = Q$, that is, $Q \leq H$. A similar argument yields $H \leq R$, which proves our claim. Therefore, it is sufficient to show that $Q = R$. Since \mathcal{M} is atomic, it suffices to prove that $PQ = PR$ for all atoms P in \mathcal{M} .

Given a fixed atom $P \in \mathcal{M}$, we have $PH_n \rightarrow PH$ in $\mathcal{L}_s(X)$ as $n \rightarrow \infty$. Since $PH_n \in \{0, P\}$ for all n , it follows that there exists $N \in \mathbb{N}$ such that either $PH_n = 0$ or $PH_n = P$ for all $n \geq N$. In the first case,

$$\begin{aligned} PQ &= P(\liminf H_n) = \liminf(PH_n) = 0 \\ &= \limsup(PH_n) = P(\limsup H_n) = PR, \end{aligned}$$

and similarly, in the second case we find that $PQ = P = PR$. Hence, in either case $PQ = PR$. ■

REMARK 2.3. Let X be a Banach space and $\mathcal{M} \subseteq \mathcal{L}(X)$ be a Bade σ -complete B.a. of projections. Suppose that $\{P_n\}_{n=1}^\infty$ is any sequence in \mathcal{M} such that $P_n \rightarrow P$ in $\mathcal{L}_s(X)$ as $n \rightarrow \infty$. Then P is a projection and it commutes with all elements of \mathcal{M} . It follows from the proof of Theorem 2.2 that always

$$\liminf P_n \leq P \leq \limsup P_n,$$

where $\liminf P_n$ and $\limsup P_n$ exist in \mathcal{M} . The proof of Theorem 2.2 is based on the fact that if, in addition, \mathcal{M} is atomic, then necessarily $\liminf P_n = \limsup P_n$ always holds. This is not true in general for non-atomic \mathcal{M} , even if \mathcal{M} is Bade complete; see Remark 4.5.

In the remainder of this section we exhibit examples of Bade σ -complete B.a.'s of projections which are atomic but not Bade atomic. In view of Lemma 2.1(ii) such B.a.'s cannot be Bade complete. We discuss these examples in some detail, since we will use such types of B.a.'s again in other sections of the paper.

First we recall some terminology and facts from the theory of Banach lattices. For details we refer the reader to any of the books on this subject (e.g. [9]). Let $(Y, \|\cdot\|)$ be a *Dedekind complete* Banach lattice with positive cone Y^+ . Then

$$\mathcal{B}_Y = \{P \in \mathcal{L}(Y) : P^2 = P \text{ and } 0 \leq Py \leq y \text{ for all } y \in Y^+\}$$

is the B.a. of all *band projections* in Y . Note that $\|P\| \leq 1$ for any $P \in \mathcal{B}_Y$. Since Y is assumed to be Dedekind complete, \mathcal{B}_Y is an abstractly complete B.a. of projections and has the property that an upwards directed system $\{P_\alpha\} \subseteq \mathcal{B}_Y$ increases to $P \in \mathcal{B}_Y$ in the order of \mathcal{B}_Y if, and only if, $P_\alpha y \uparrow_\alpha Py$ in the order of Y for every $y \in Y^+$. Moreover, each $P \in \mathcal{B}_Y$ is *disjointness preserving* (i.e., $|Py_1| \wedge |Py_2| = 0$ whenever $|y_1| \wedge |y_2| = 0$ in Y) and satisfies $|Py| = P|y|$ for every $y \in Y$. If, in addition, Y has *order continuous norm*, then it follows that \mathcal{B}_Y is Bade complete.

Let (Ω, Σ) be a measurable space and let $X = \text{ca}(\Sigma)$ denote the (real) Banach space of all σ -additive measures $\mu : \Sigma \rightarrow \mathbb{R}$ with norm $\|\mu\| = |\mu|(\Omega)$, where $|\mu|$ denotes the *variation measure* of μ . If $\mu, \nu \in X$, then we write $\mu \leq \nu$ whenever $\mu(E) \leq \nu(E)$ for all $E \in \Sigma$. Equipped with this order, X is a Dedekind complete Banach lattice with order continuous norm. In fact, X is an *abstract L -space*, meaning that $\|\mu + \nu\| = \|\mu\| + \|\nu\|$ whenever $\mu, \nu \in X^+$. We point out that two elements $\mu, \nu \in X$ are *disjoint* (i.e., $|\mu| \wedge |\nu| = 0$) if, and only if, $\mu \perp \nu$, that is, μ and ν are *mutually singular* as measures ([9, p. 11]). Moreover, we note that the principal band generated by an element $\mu \in \text{ca}(\Sigma)$ consists precisely of those $\nu \in \text{ca}(\Sigma)$ which are absolutely continuous with respect to μ . This principal band will be denoted by $\text{AC}(\mu)$. The positive cone of $\text{ca}(\Sigma)$ is denoted by $\text{ca}^+(\Sigma)$.

Let $Y \subseteq \text{ca}(\Sigma)$ be any band. Then Y is itself a Dedekind complete Banach lattice with order continuous norm. In particular, the B.a. \mathcal{B}_Y of all band projections in Y is a Bade complete B.a. Given $\mu \in \text{ca}(\Sigma)$ and $E \in \Sigma$, let $\mu_E \in \text{ca}(\Sigma)$ be defined by $\mu_E(F) = \mu(E \cap F)$ for all $F \in \Sigma$. Since $|\mu_E| = |\mu|_E \leq |\mu|$ and Y is solid in $\text{ca}(\Sigma)$, it follows that $\mu_E \in Y$ whenever $\mu \in Y$ and $E \in \Sigma$. Given $E \in \Sigma$, the mapping $P_Y(E) : Y \rightarrow Y$ defined by

$$(1) \quad P_Y(E)\mu = \mu_E, \quad \mu \in Y,$$

is a band projection in Y . The map $P_Y : \Sigma \rightarrow \mathcal{L}(Y)$ given by $E \mapsto P_Y(E)$ for $E \in \Sigma$ is clearly finitely additive, multiplicative (i.e., $P_Y(E \cap F) = P_Y(E)P_Y(F)$ for all $E, F \in \Sigma$) and $P_Y(\Omega) = I$. Actually, P_Y is σ -additive in $\mathcal{L}_s(Y)$ because $\|P_Y(E)\mu\| = |\mu_E|(\Omega) \leq |\mu|(E)$ for $E \in \Sigma$, and each $\mu \in Y$

is σ -additive on Σ . Hence, P_Y is a *spectral measure* in Y . It follows that the range $\mathcal{M}_Y = P_Y(\Sigma)$ of P_Y is a Bade σ -complete B.a. in $\mathcal{L}(Y)$ (see Corollary XVII.3.10 in [6]). In particular, \mathcal{M}_Y is a σ -subalgebra (in the sense of [16, Section 23]) of the Bade complete B.a. \mathcal{B}_Y of all band projections in Y . For the case $Y = \text{ca}(\Sigma)$ we will omit the subscript Y and simply write \mathcal{M} and P .

LEMMA 2.4. *Let (Ω, Σ) be a measurable space, Y a band in $\text{ca}(\Sigma)$ and $\mathcal{M}_Y = P_Y(\Sigma)$ be the range of the spectral measure $P_Y : \Sigma \rightarrow \mathcal{L}_s(Y)$ given by (1). Then $\mathcal{B}_Y = \overline{\mathcal{M}_Y}$, the τ_s -closure of \mathcal{M}_Y in $\mathcal{L}(Y)$.*

Proof. The closedness of \mathcal{B}_Y in $\mathcal{L}_s(Y)$ can easily be verified directly in this situation; it also follows from the Bade completeness of \mathcal{B}_Y by Corollary XVII.3.7 in [6]. Hence, $\overline{\mathcal{M}_Y} \subseteq \mathcal{B}_Y$.

An application of the Hahn decomposition theorem yields the following:

FACT 1. *If $Q \in \mathcal{B}_Y$ and $\mu \in Y$, then there exists $A \in \Sigma$ with $P_Y(A)\mu = Q\mu$.*

As a consequence we also have:

FACT 2. *If $Q \in \mathcal{B}_Y$ and $\mu_1, \dots, \mu_n \in Y$, then there exists $A \in \Sigma$ such that*

$$P_Y(A)\mu_j = Q\mu_j, \quad j = 1, \dots, n.$$

Indeed, define $\mu = |\mu_1| \vee \dots \vee |\mu_n|$. By Fact 1, there exists $A \in \Sigma$ with $P_Y(A)\mu = Q\mu$. Observe that, since both $P_Y(A)$ and Q are band projections, the collection of all $\nu \in Y$ satisfying $P_Y(A)\nu = Q\nu$ is a band in Y . In particular, $P_Y(A)\mu = Q\mu$ implies that $P_Y(A)\nu = Q\nu$ for all $\nu \in Y$ satisfying $|\nu| \leq \mu$. This establishes Fact 2.

To complete the proof of the lemma, fix $Q \in \mathcal{B}_Y$. For any finite subset $F \subseteq Y$, Fact 2 guarantees the existence of a set $A_F \in \Sigma$ such that $P_Y(A_F)\mu = Q\mu$ for all $\mu \in F$. Directing the collection \mathcal{F} of all finite subsets of Y by inclusion, it is now clear that the net $\{P_Y(A_F) : F \in \mathcal{F}\}$ is τ_s -convergent to Q . Accordingly, $Q \in \overline{\mathcal{M}_Y}$ and so $\mathcal{B}_Y \subseteq \overline{\mathcal{M}_Y}$. ■

EXAMPLE 2.5. *A Bade σ -complete B.a. of projections which is atomic, but not Bade atomic.*

Let $\Omega = [0, 1]$ and Σ be the σ -algebra of all Borel subsets of Ω . Let $Y = \text{ca}(\Sigma)$ and $\mathcal{M} = P(\Sigma)$ with $P = P_Y$ given by (1). The atoms in \mathcal{M} are all projections of the form $P(\{w\})$ with $w \in \Omega$. If $\emptyset \neq E \in \Sigma$, then $P(\{w\}) \leq P(E)$ for all $w \in E$, so it is clear that \mathcal{M} is an atomic B.a.

Let $\lambda \in \text{ca}(\Sigma)$ denote Lebesgue measure (or any other non-zero, continuous measure). Since $P(\{w\})\lambda = 0$ for every $w \in \Omega$, it is clear that I cannot be the τ_s -limit of sums of the form $\sum_{w \in F} P(\{w\}) = P(F)$ with $F \subseteq \Omega$ a finite set. Hence, \mathcal{M} is not Bade atomic.

In view of Lemma 2.1(ii) we see that \mathcal{M} cannot be Bade complete. This also follows immediately from Lemma 2.4. Indeed, it is easily verified that the band projection $P_\lambda \in \mathcal{B}_Y = \overline{\mathcal{M}}$ onto the principal band $\text{AC}(\lambda)$ generated by λ does not belong to \mathcal{M} .

We conclude this section with some comments about the connection between \mathcal{M} and its τ_s -closure $\overline{\mathcal{M}}$ being Bade atomic. The following observation is routine to verify.

LEMMA 2.6. *Let X be a Banach space and $\mathcal{M} \subseteq \mathcal{L}(X)$ be a Bade σ -complete B.a. of projections. If $P \in \mathcal{M}$ is an atom, then P is also an atom in the Bade complete B.a. $\overline{\mathcal{M}}$.*

The next result is a consequence of Lemma 2.6 and of Lemma XVII.3.4 in [6], together with the fact that if $\{P_\alpha\}_{\alpha \in \mathcal{A}}$ is the collection of all the atoms in a Bade σ -complete B.a. \mathcal{M} , then \mathcal{M} is Bade atomic if, and only if, $I = \sum_{\alpha \in \mathcal{A}} P_\alpha$ as a τ_s -summable series in $\mathcal{L}_s(X)$. We leave the details to the reader.

PROPOSITION 2.7. *Let X be a Banach space and $\mathcal{M} \subseteq \mathcal{L}(X)$ be a Bade σ -complete B.a. of projections. Then \mathcal{M} is Bade atomic if, and only if, $\overline{\mathcal{M}}$ is atomic and every atom of $\overline{\mathcal{M}}$ belongs to \mathcal{M} .*

We remark that *both* requirements on $\overline{\mathcal{M}}$ are necessary. Indeed, the B.a. \mathcal{M} of Example 2.5 is not Bade atomic, all atoms of $\overline{\mathcal{M}} = \mathcal{B}_Y$ belong to \mathcal{M} and $\overline{\mathcal{M}}$ is not atomic. On the other hand, let $\Omega = [0, 1]$ and Σ denote the σ -algebra of all subsets of Ω which are either countable or co-countable. Again let $Y = \text{ca}(\Sigma)$ and $\mathcal{M} = P(\Sigma)$, where $P = P_Y$ is the spectral measure given by (1). Denote by λ the measure on Σ which takes the value 0 on all countable sets and the value 1 on all co-countable sets. It turns out that $\text{ca}(\Sigma) \cong \ell^1(\Omega) \oplus \langle \lambda \rangle$, where $\langle \lambda \rangle$ denotes the 1-dimensional subspace generated by λ . Let P_c denote the (band) projection in $\text{ca}(\Sigma)$ onto $\langle \lambda \rangle$. It can be verified that \mathcal{M} is an atomic Bade σ -complete B.a. (the atoms in \mathcal{M} are all projections of the form $P(\{w\})$ with $w \in \Omega$), but is not Bade atomic. In this case, $\overline{\mathcal{M}}$ is atomic as well (the atoms in $\overline{\mathcal{M}}$ are the atoms in \mathcal{M} together with P_c), but an additional atom P_c exists in $\overline{\mathcal{M}}$ which does not occur in \mathcal{M} .

3. Criteria for sequential closedness. Theorem 2.2 raises the question of whether or not there is a similar result for Bade σ -complete B.a.'s which are not atomic. In the final section this will be answered using the criteria of this section.

Let (Ω, Σ) be a measurable space and $\Phi \subseteq \text{ca}^+(\Sigma)$. The σ -algebra Σ is said to be *sequentially Φ -complete* if, whenever $\{A_n\}_{n=1}^\infty \subseteq \Sigma$ satisfies $\nu(A_n \Delta A_m) \rightarrow 0$ as $n, m \rightarrow \infty$ for every $\nu \in \Phi$, then there exists $A \in \Sigma$

with $\nu(A \Delta A_n) \rightarrow 0$ as $n \rightarrow \infty$ for every $\nu \in \Phi$. Of course, $B \Delta C$ denotes the symmetric difference of any pair of subsets B, C of Ω .

Given any $\Phi \subseteq \text{ca}^+(\Sigma)$, let X_Φ denote the band generated by Φ in $\text{ca}(\Sigma)$ and let X'_Φ denote the dual space of X_Φ , which is itself a Banach lattice. The *absolute weak-star topology* $|\sigma|(X'_\Phi, X_\Phi)$ on X'_Φ is that determined by the family of seminorms $\{q_\nu : \nu \in X_\Phi\}$, where

$$q_\nu(\xi) = \langle |\nu|, |\xi| \rangle, \quad \xi \in X'_\Phi.$$

Each bounded Σ -measurable function g on Ω defines an element of X'_Φ by

$$\langle \nu, g \rangle = \int_{\Omega} g d\nu, \quad \nu \in X_\Phi,$$

and

$$q_\nu(g) = \langle |\nu|, |g| \rangle = \int_{\Omega} |g| d|\nu|$$

for each $\nu \in X_\Phi$. For $Y = X_\Phi$, the spectral measure defined by (1) is denoted simply by $P_\Phi : \Sigma \rightarrow \mathcal{L}(X_\Phi)$, in which case $\mathcal{M}_\Phi = P_\Phi(\Sigma)$ is a Bade σ -complete B.a. of projections in X_Φ .

We require the following technical result. Recall if P, Q are two commuting projections in a Banach space X , then $P \Delta Q$ is defined to be the projection $P(I - Q) + (I - P)Q$ which, in turn, equals $(P \vee Q) - (P \wedge Q)$ whenever P, Q are elements of a B.a. of projections on X (since $P \vee Q = P + Q - PQ$ and $P \wedge Q = PQ$).

LEMMA 3.1. *Let X be a Banach space and $\mathcal{M} \subseteq \mathcal{L}(X)$ be a B.a. such that $M = \sup\{\|P\| : P \in \mathcal{M}\} < \infty$. Let $P, Q \in \mathcal{M}$. Then, for each $x \in X$, we have*

$$(2M)^{-1}\|Px - Qx\| \leq \|(P \Delta Q)x\| \leq 2M\|Px - Qx\|.$$

Proof. Suppose first that $PQ = 0$. Then

$$\|(P + Q)x\| = \|(P - Q)(Px - Qx)\| \leq 2M\|Px - Qx\|.$$

Also, since $P - Q = (P - Q)(P + Q)$, it follows that

$$\|Px - Qx\| \leq \|P - Q\| \|Px + Qx\| \leq 2M\|Px + Qx\|.$$

Now suppose that P, Q are arbitrary. Since $(P - PQ)(Q - PQ) = 0$ we see from the above that

$$\begin{aligned} \|Px - Qx\| &= \|(P - PQ)x - (Q - PQ)x\| \\ &\leq 2M\|(P - PQ)x + (Q - PQ)x\| = 2M\|(P \Delta Q)x\|. \end{aligned}$$

Similarly,

$$\begin{aligned} \|(P \Delta Q)x\| &= \|(P - PQ)x + (Q - PQ)x\| \\ &\leq 2M\|(P - PQ)x - (Q - PQ)x\| = 2M\|Px - Qx\|. \quad \blacksquare \end{aligned}$$

In the following theorem some characterizations of sequential Φ -completeness of Σ are presented.

THEOREM 3.2. *Let (Ω, Σ) be a measurable space and $\Phi \subseteq \text{ca}^+(\Sigma)$. The following statements are equivalent.*

- (i) Σ is sequentially Φ -complete.
- (ii) $\{\chi_A : A \in \Sigma\} \subseteq X'_\Phi$ is sequentially complete for the $|\sigma|(X'_\Phi, X_\Phi)$ topology.
- (iii) The Bade σ -complete B.a. \mathcal{M}_Φ is sequentially closed in $\mathcal{L}_s(X_\Phi)$.

Proof. (i) \Rightarrow (iii). Let $\{P_\Phi(A_n)\}_{n=1}^\infty \subseteq \mathcal{M}_\Phi$ converge to P in $\mathcal{L}_s(X_\Phi)$, in which case $\{P_\Phi(A_n)\}_{n=1}^\infty$ is Cauchy in $\mathcal{L}_s(X_\Phi)$. Lemma 3.1 implies that $\|P_\Phi(A_n \Delta A_m)\mu\| \rightarrow 0$ as $m, n \rightarrow \infty$ for every $\mu \in X_\Phi$. In particular, for all $\nu \in \Phi \subseteq X_\Phi$, we see that

$$0 \leq \nu(A_n \Delta A_m) = \|\nu_{A_n \Delta A_m}\| = \|P_\Phi(A_n \Delta A_m)\nu\| \rightarrow 0$$

as $m, n \rightarrow \infty$. By hypothesis there is $A \in \Sigma$ such that

$$\|P_\Phi(A_n \Delta A)\nu\| = \nu(A_n \Delta A) \rightarrow 0, \quad \nu \in \Phi,$$

as $n \rightarrow \infty$. The same conclusion holds for any $\mu \in \text{ca}(\Sigma)$ satisfying $|\mu| \leq \sum_{j=1}^\infty c_j \nu_j$ for some $0 \leq c_j \in \mathbb{R}$ and $\nu_j \in \Phi$ and, by a standard approximation argument and using Lemma 3.1, it follows that $P_\Phi(A_n) \rightarrow P_\Phi(A)$ in $\mathcal{L}_s(X_\Phi)$. Accordingly, $P = P_\Phi(A) \in \mathcal{M}_\Phi$.

(iii) \Rightarrow (ii). Let $\{\chi_{A_n}\}_{n=1}^\infty \subseteq X'_\Phi$ be $|\sigma|(X'_\Phi, X_\Phi)$ -Cauchy. The identities $q_\mu(\chi_{A_n} - \chi_{A_m}) = |\mu|(A_n \Delta A_m)$, for each $\mu \in X_\Phi$, together with Lemma 3.1, imply that $\{P_\Phi(A_n)\}_{n=1}^\infty$ is Cauchy in $\mathcal{L}_s(X_\Phi)$. Hence, by sequential completeness of $\mathcal{L}_s(X_\Phi)$ and sequential closedness of \mathcal{M}_Φ , there is $A \in \Sigma$ such that $P_\Phi(A_n) \rightarrow P_\Phi(A)$ in $\mathcal{L}_s(X_\Phi)$. Arguing as above, it follows that $\chi_{A_n} \rightarrow \chi_A$ in X'_Φ with respect to $|\sigma|(X'_\Phi, X_\Phi)$.

(ii) \Rightarrow (i). Let $\{A_n\}_{n=1}^\infty \subseteq \Sigma$ satisfy $\nu(A_n \Delta A_m) \rightarrow 0$ as $m, n \rightarrow \infty$ for all $\nu \in \Phi$. Via a routine argument (similar to the one used above), it follows that actually $|\mu|(A_n \Delta A_m) \rightarrow 0$ as $m, n \rightarrow \infty$ for every $\mu \in X_\Phi$. Hence, $\{\chi_{A_n}\}_{n=1}^\infty$ is a $|\sigma|(X'_\Phi, X_\Phi)$ -Cauchy sequence in X'_Φ and so, by hypothesis, there exists $A \in \Sigma$ such that $\chi_{A_n} \rightarrow \chi_A$ with respect to $|\sigma|(X'_\Phi, X_\Phi)$. In particular, for each $\nu \in \Phi$ we conclude that

$$\nu(A \Delta A_n) = q_\nu(\chi_A - \chi_{A_n}) \rightarrow 0, \quad n \rightarrow \infty,$$

that is, Σ is sequentially Φ -complete. ■

Let $\mathcal{M} \subseteq \mathcal{L}(X)$ be a Bade σ -complete B.a. of projections in a Banach space X . Any spectral measure $P : \Sigma \rightarrow \mathcal{L}_s(X)$, defined on some measurable space (Ω, Σ) and satisfying $P(\Sigma) = \mathcal{M}$, is said to *represent* \mathcal{M} . Such spectral measures always exist (see e.g. [6, Corollary XVII.3.10]), but are surely not unique. Given $x \in X$, any $x' \in X'$ with the properties:

- (i) $\langle Bx, x' \rangle \geq 0$ for all $B \in \mathcal{M}$,
- (ii) $\langle Bx, x' \rangle = 0$ for some $B \in \mathcal{M}$ if and only if $Bx = 0$,

is called a *Bade functional* for x (relative to \mathcal{M}); see [6, Lemma XVII.3.12]. In this case, the set function $\langle P(\cdot)x, x' \rangle : E \mapsto \langle P(E)x, x' \rangle$ for $E \in \Sigma$ belongs to $\text{ca}^+(\Sigma)$ and is mutually absolutely continuous to the X -valued vector measure $P(\cdot)x : E \mapsto P(E)x$ for $E \in \Sigma$. In view of property (ii), for each $x \in X$ the measures $\{\langle P(\cdot)x, x' \rangle : x' \in B(P; x)\}$, where

$$B(P; x) = \{x' \in X' : x' \text{ is a Bade functional for } x\}$$

are all mutually absolutely continuous. Any family of measures of the form

$$(2) \quad B(P) = \{\langle P(\cdot)x, \varphi_x \rangle : x \in X, \varphi_x \in B(P; x)\} \subseteq \text{ca}^+(\Sigma)$$

is called a *Bade family* relative to P .

THEOREM 3.3. *Let X be a Banach space and $\mathcal{M} \subseteq \mathcal{L}(X)$ be a Bade σ -complete B.a. of projections. Then \mathcal{M} is sequentially closed in $\mathcal{L}_s(X)$ if, and only if, for every (for some) representing spectral measure $P : \Sigma \rightarrow \mathcal{L}(X)$ for \mathcal{M} and for every (for some) Bade family of measures $B(P)$ relative to P , the σ -algebra Σ is sequentially $B(P)$ -complete.*

Proof. Suppose that P and $B(P)$ exist with the stated property. Let $P_n = P(A_n)$ for $n = 1, 2, \dots$ be a sequence in $\mathcal{M} = P(\Sigma)$ which is τ_s -convergent to $Q \in \mathcal{L}(X)$. Fix $x \in X$ and, for $\varphi_x \in B(P; x)$ as in (2), let $\nu_x = \langle P(\cdot)x, \varphi_x \rangle$. Since $\{P_n x\}_{n=1}^\infty$ is Cauchy in X , it follows from Lemma 3.1 that $\|(P_n \Delta P_m)x\| \rightarrow 0$ as $m, n \rightarrow \infty$. Hence, $\nu_x(A_n \Delta A_m) \rightarrow 0$ as $m, n \rightarrow \infty$. This holds for every $x \in X$ and so, by hypothesis, there exists $A \in \Sigma$ such that $\nu_x(A_n \Delta A) \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in X$. Since the vector measure $P(\cdot)x$ is absolutely continuous with respect to ν_x , it follows from a theorem of Pettis (see [5, p. 10]), that also $\|P(A_n \Delta A)x\| \rightarrow 0$ as $n \rightarrow \infty$. In view of Lemma 3.1 we conclude that $P(A_n)x \rightarrow P(A)x$ in X as $n \rightarrow \infty$ for every $x \in X$. Hence, $P(A_n) \rightarrow P(A)$ in $\mathcal{L}_s(X)$ and so $Q = P(A) \in \mathcal{M}$. This shows that \mathcal{M} is sequentially closed in $\mathcal{L}_s(X)$.

Conversely, suppose that \mathcal{M} is sequentially closed in $\mathcal{L}_s(X)$. Let $P : \Sigma \rightarrow \mathcal{L}(X)$ be any representing spectral measure for \mathcal{M} and $B(P)$ be any Bade family of measures relative to P . We have to show that Σ is sequentially $B(P)$ -complete. So, let $\{A_n\}_{n=1}^\infty \subseteq \Sigma$ satisfy $\nu_x(A_n \Delta A_m) \rightarrow 0$ as $n, m \rightarrow \infty$ for all $x \in X$ (with notation ν_x as above). Using the Pettis theorem mentioned above and Lemma 3.1 we can conclude that $\{P(A_n)\}_{n=1}^\infty \subseteq \mathcal{M}$ is Cauchy in $\mathcal{L}_s(X)$. Since $\mathcal{L}_s(X)$ is sequentially complete and \mathcal{M} is τ_s -closed, there exists $A \in \Sigma$ such that $P(A_n) \rightarrow P(A)$ in $\mathcal{L}_s(X)$. It follows easily that $\nu_x(A_n \Delta A) \rightarrow 0$ as $n \rightarrow \infty$ for every $x \in X$. ■

4. Examples. In this section we apply the results of the previous section to various examples. The idea is to find suitable measurable spaces (Ω, Σ)

together with families of measures $\Phi \subseteq \text{ca}^+(\Sigma)$ such that Σ is (or is not) sequentially Φ -complete.

4.1. EXAMPLE A. *There exists a non-atomic, Bade σ -complete B.a. of projections which is sequentially τ_s -closed, but not τ_s -closed.*

Let $\Omega = [0, 1]$ and Σ be the σ -algebra of all Borel subsets of Ω . We now let $\Phi \subseteq \text{ca}^+(\Sigma)$ be the family of all non-negative, continuous measures on Σ . Then $X_\Phi = \Phi - \Phi$ is the band in $\text{ca}(\Sigma)$ consisting of all continuous measures on Σ (i.e., a measure $\mu \in \text{ca}(\Sigma)$ belongs to X_Φ if, and only if, $\mu(\{w\}) = 0$ for all $w \in \Omega$). Before proceeding, it will be convenient to recall the following fact, which indicates that X_Φ contains an abundance of measures.

LEMMA 4.1. *Let F be an uncountable Borel subset of $\Omega = [0, 1]$. Then there exists $0 < \mu \in \Phi$ satisfying $\mu(\Omega \setminus F) = 0$.*

Proof. Two sets $B_1, B_2 \in \Sigma$ are called *Borel isomorphic* if there is a bijection $\varphi : B_1 \rightarrow B_2$ such that both φ and φ^{-1} are Borel measurable (see e.g. [12, p. 7]). In particular, if ν is a continuous Borel measure on B_1 , then the image measure $\tilde{\nu} : E \mapsto \nu(\varphi^{-1}(E))$, for each Borel set $E \subseteq B_2$, is clearly continuous on B_2 .

Now choose $B_1 = \Omega$ and $B_2 = F$, in which case there exists a Borel isomorphism $\varphi : \Omega \rightarrow F$ (see Theorems 2.8 and 2.12 in [12], for example). Let λ be Lebesgue measure on Ω and define the image measure $\tilde{\lambda}$ on F as above. Then the measure $\mu : E \mapsto \tilde{\lambda}(E \cap F)$, for $E \in \Sigma$, has the required properties. ■

For $Y = X_\Phi$, let $P_\Phi = P_{X_\Phi} : \Sigma \rightarrow \mathcal{L}(X_\Phi)$ be the spectral measure given by (1) and define $\mathcal{M}_\Phi = P_\Phi(\Sigma)$, which is a Bade σ -complete B.a. in $\mathcal{L}(X_\Phi)$. It follows from Lemma 4.1 that the P_Φ -null sets are precisely the countable subsets of Ω . Moreover, it is easy to see that \mathcal{M}_Φ is non-atomic. From Lemma 2.4 we know that $\overline{\mathcal{M}_\Phi} = \mathcal{B}_{X_\Phi}$, the B.a. of all band projections in X_Φ . For any $\mu \in X_\Phi$ we denote by $Q_\mu \in \mathcal{B}_{X_\Phi}$ the band projection onto the principal band $\text{AC}(\mu)$ generated by μ . We will show that $Q_\mu \notin \mathcal{M}_\Phi$, for which we need the following observation.

LEMMA 4.2. *Suppose that $\mu \in \Phi$ satisfies $\mu(E) > 0$ for some set $E \in \Sigma$. Then there exists an uncountable Borel set $F \subseteq E$ such that $\mu(F) = 0$.*

Proof. We consider the case that $\mu = \lambda$ is Lebesgue measure on $\Omega = [0, 1]$. The general case then follows from [15] (Theorem 9 in Section 3 of Chapter 15). Now $\lambda(E) > 0$ and so we can choose a compact set $C \subseteq E$ with $\lambda(C) > 0$. Since C is clearly an uncountable G_δ -set, it follows from [10, Lemma 5.1] that there exists a nowhere dense, closed set $F \subseteq C$ with $\lambda(F) = 0$ such that F can be mapped continuously onto Ω . In particular, F must be uncountable. ■

LEMMA 4.3. *For any $0 < \mu \in \Phi$ we have $Q_\mu \notin \mathcal{M}_\Phi$. In particular, \mathcal{M}_Φ is not τ_s -closed in $\mathcal{L}(X)$, and hence, is also not Bade complete.*

Proof. Fix $0 < \mu \in \Phi$ and suppose that $Q_\mu \in \mathcal{M}_\Phi$, i.e., $Q_\mu = P_\Phi(E)$ for some $E \in \Sigma$. Since $\mu(E) = \mu(\Omega) > 0$, it follows from Lemma 4.2 that there exists an uncountable Borel set $F \subseteq E$ such that $\mu(F) = 0$. By Lemma 4.1 there is $0 < \nu \in \Phi$ such that $\nu(\Omega \setminus F) = 0$. Since $P_\Phi(F)\nu = \nu$ and $P_\Phi(F)\mu = \mu$, it is clear that $\nu \perp \mu$ and so $Q_\mu\nu = 0$. However, this contradicts the fact that also $Q_\mu\nu = P_\Phi(E)\nu = P_\Phi(E)P_\Phi(F)\nu = P_\Phi(F)\nu = \nu$. ■

To finish the example, it remains to establish the following:

PROPOSITION 4.4. *The B.a. \mathcal{M}_Φ is sequentially closed in $\mathcal{L}_s(X_\Phi)$.*

Proof. By Theorem 3.2 it suffices to show that Σ is sequentially Φ -complete. This will be achieved in a number of steps. So, let $\{A(n)\}_{n=1}^\infty \subseteq \Sigma$ be any sequence such that $\mu(A(n) \triangle A(m)) \rightarrow 0$ as $m, n \rightarrow \infty$ for all $\mu \in \Phi$. Accordingly, the sequence (of equivalence classes of) $\{\chi_{A(n)}\}_{n=1}^\infty$ is Cauchy in $L^1(\mu)$, and so converges in $L^1(\mu)$ for every $\mu \in \Phi$.

STEP 1. *For every $\nu \in \text{ca}^+(\Sigma)$, the sequence $\{\chi_{A(n)}\}_{n=1}^\infty$ has a ν -a.e. convergent subsequence.*

To see this, let $\{w_k\}_{k=1}^\infty = \{w \in [0, 1] : \nu(\{w\}) > 0\}$ be the (at most countable) set of atoms of ν . By the usual diagonalization process we can find a subsequence $\{\chi_{A(n_j)}\}_{j=1}^\infty$ which converges at each of the points w_k ($k = 1, 2, \dots$). Now decompose $\nu = \nu_c + \nu_a$ with $\nu_c \in \Phi$ and ν_a purely atomic. Since the sequence $\{\chi_{A(n_j)}\}_{j=1}^\infty$ is convergent in $L^1(\nu_c)$, it has a further subsequence converging ν_c -a.e. This final subsequence has the property that it converges ν -a.e. This establishes Step 1.

For any $\nu \in \text{ca}^+(\Sigma)$, let $M_\nu([0, 1])$ denote the locally convex Hausdorff space of all \mathbb{R} -valued, ν -measurable functions defined on $[0, 1]$, equipped with the topology of pointwise convergence on $[0, 1]$.

STEP 2. *Given $\nu \in \text{ca}^+(\Sigma)$, the sequence $\{\chi_{A(n)}\}_{n=1}^\infty$ has a cluster point in $M_\nu([0, 1])$.*

By Step 1 there is a subsequence $\{\chi_{A(n_k)}\}_{k=1}^\infty$, a Borel function f and a set $K \in \Sigma$ with $\nu(K^c) = 0$ such that

$$(3) \quad \chi_{A(n_k)} \rightarrow f \quad \text{pointwise on } K \text{ as } k \rightarrow \infty.$$

Since $\{\chi_{A(n_k)}\}_{k=1}^\infty$ is contained in the compact set $[0, 1]^{[0, 1]}$, this sequence has a cluster point $g \in [0, 1]^{[0, 1]}$. By (3) it follows that $f\chi_K = g\chi_K$. This implies that $g \in M_\nu([0, 1])$, and hence, g is a cluster point of $\{\chi_{A(n)}\}_{n=1}^\infty$ in $M_\nu([0, 1])$.

Since the result of Step 2 applies to every subsequence of $\{\chi_{A(n)}\}_{n=1}^\infty$, this shows that $\{\chi_{A(n)}\}_{n=1}^\infty$ is *relatively countably compact* in $M_\nu([0, 1])$ for all

$\nu \in \text{ca}^+(\Sigma)$. It follows from [3, Theorem 4D] that $\{\chi_{A(n)}\}_{n=1}^\infty$ is *relatively sequentially compact* in $\mathbb{R}^{[0,1]}$, and hence, has a subsequence $\{\chi_{A(n_j)}\}_{j=1}^\infty$ which converges everywhere on $[0, 1]$. Of course, the limit function is of the form χ_A for some $A \in \Sigma$ and, by the dominated convergence theorem, it is clear that $\mu(A(n_j) \triangle A) \rightarrow 0$ as $j \rightarrow \infty$ for all $\mu \in \Phi$. Since also $\mu(A(n) \triangle A(m)) \rightarrow 0$ as $m, n \rightarrow \infty$, we may conclude that $\mu(A(n) \triangle A) \rightarrow 0$ as $n \rightarrow \infty$, for every $\mu \in \Phi$. This completes the proof of the proposition. ■

Before leaving Example A, we return to Remark 2.3.

REMARK 4.5. Let (Ω, Σ) , Φ , P_Φ and $\mathcal{M}_\Phi = P_\Phi(\Sigma)$ still be as in Example A. Let $A_0 = [0, 1/2]$. Then define $A_1 = [0, 1/4]$ and $A_2 = [1/4, 1/2]$. Now form $A_3 = [0, 1/8)$, $A_4 = [1/8, 1/4)$, $A_5 = [1/4, 3/8)$ and $A_6 = [3/8, 1/2]$, and continue in the obvious way. For each $n \geq 0$, define the set

$$B_n = [1/2, 1] \setminus \{x + 1/2 : x \in A_n\}$$

and then the projection $P_n = P_\Phi(A_n \cup B_n) \in \mathcal{M}_\Phi$. It can be shown that $P_n \rightarrow P = P_\Phi([1/2, 1])$ in $\mathcal{L}_s(X_\Phi)$ as $n \rightarrow \infty$. Since $\liminf P_n = 0$ and $\limsup P_n = I$ (formed in the order of \mathcal{M}_Φ or in $\overline{\mathcal{M}_\Phi}$), we see that

$$\liminf P_n < P < \limsup P_n.$$

4.2. EXAMPLE B. *There exists a Bade σ -complete B.a. of projections which fails to be sequentially τ_s -closed.*

To produce Bade σ -complete B.a.'s of projections which *fail* to be sequentially τ_s -closed, we begin by exhibiting a measurable space (Ω, Σ) and a family of measures $\Phi \subseteq \text{ca}^+(\Sigma)$ such that Σ is *not* sequentially Φ -complete.

For the rest of this section let $\Omega = \{0, 1\}^\mathbb{N}$, equipped with the product topology, and $\Sigma = \mathcal{B}$ be the σ -algebra of all Borel subsets of Ω . Elements $\xi \in \Omega$ will be denoted as $\xi = (\xi_n)_{n=0}^\infty$ with $\xi_n \in \{0, 1\}$ for all $n \in \mathbb{N} = \{0, 1, 2, \dots\}$. For each $n \in \mathbb{N}$, let $D_n = \{\xi \in \Omega : \xi_n = 1\}$ and define

$$(4) \quad \Phi = \{\mu \in \text{ca}^+(\mathcal{B}) : \mu(D_n \triangle D_m) \rightarrow 0 \text{ as } m, n \rightarrow \infty\}.$$

PROPOSITION 4.6. *The Borel σ -algebra \mathcal{B} is not sequentially Φ -complete.*

To establish Proposition 4.6 it suffices to show that there is no set $E \in \mathcal{B}$ with the property that $\mu(E \triangle D_n) \rightarrow 0$ as $n \rightarrow \infty$ for all $\mu \in \Phi$ or, equivalently, that there is no set $E \in \mathcal{B}$ satisfying $\mu(D_n) \rightarrow \mu(E)$ as $n \rightarrow \infty$ for all $\mu \in \Phi$. Define

$$\begin{aligned} \mathcal{A} &= \{\mu \in \text{ca}^+(\mathcal{B}) : \mu(D_n) \rightarrow 0 \text{ as } n \rightarrow \infty\}, \\ \mathcal{J} &= \{A \in \mathcal{B} : \mu(A) = 0 \text{ for all } \mu \in \mathcal{A}\}, \end{aligned}$$

in which case $\mathcal{A} \subseteq \Phi$ and \mathcal{J} is a σ -ideal in the Boolean σ -algebra \mathcal{B} . Suppose that there *does* exist a set $E \in \mathcal{B}$ such that $\lim_{n \rightarrow \infty} \mu(D_n) = \mu(E)$ for all

$\mu \in \Phi$. Then clearly $E \in \mathcal{J}$. Define a homeomorphic involution $\pi : \Omega \rightarrow \Omega$ by $\pi(\xi) = (1 - \xi_0, 1 - \xi_1, \dots)$ for each $\xi \in \Omega$, and let $E^\sharp = \pi^{-1}(\Omega \setminus E)$. Using the observations that $\pi^{-1}(D_n) = D_n^c$ for all $n \in \mathbb{N}$ and that

$$\mu\pi^{-1}(D_n \triangle D_m) = \mu(D_n \triangle D_m) \rightarrow 0 \quad \text{as } m, n \rightarrow \infty$$

for all $\mu \in \Phi$, where $\mu\pi^{-1} : A \mapsto \mu(\pi^{-1}(A))$ for $A \in \mathcal{B}$ is the image measure, it follows that $\mu\pi^{-1} \in \Phi$ whenever $\mu \in \Phi$. Hence, $\mu\pi^{-1}(D_n) \rightarrow \mu\pi^{-1}(E)$ for all $\mu \in \Phi$, which implies in particular that $\mu\pi^{-1}(E) = \mu(\Omega)$ for all $\mu \in \mathcal{A}$ and so $E^\sharp \in \mathcal{J}$. Since $\Omega = E \cup \pi(E^\sharp)$ and Ω is a complete metric space, at least one of E or $\pi(E^\sharp)$ is a *non-meager* set. But π is a homeomorphism and so we can conclude that E or E^\sharp is non-meager and both $E, E^\sharp \in \mathcal{J}$. So, if such a set E exists, then necessarily \mathcal{J} contains at least one non-meager set. Hence, Proposition 4.6 follows from the following fact.

PROPOSITION 4.7. *Every element of \mathcal{J} is a meager subset of Ω .*

The proof of this proposition will be via some lemmas. First we require some further notation.

The collections of all subsets and of all finite subsets of \mathbb{N} are denoted by $\mathcal{P}(\mathbb{N})$ and $\mathcal{P}_f(\mathbb{N})$, respectively. For $0 \leq m \leq n \in \mathbb{N}$, define $[m, n) = \{k \in \mathbb{N} : m \leq k < n\}$. If $\xi \in \Omega$, define $\text{supp}(\xi) = \{n \in \mathbb{N} : \xi_n = 1\}$. The mapping $\xi \mapsto \text{supp}(\xi)$ is a bijection between Ω and $\mathcal{P}(\mathbb{N})$. For $n \in \mathbb{N} \setminus \{0\}$ we will identify $\{0, 1\}^n$ with a subset of Ω via the imbedding

$$(\xi_0, \dots, \xi_{n-1}) \mapsto (\xi_0, \dots, \xi_{n-1}, 0, 0, \dots).$$

In particular, if $1 \leq m \leq n \in \mathbb{N}$, we consider $\{0, 1\}^m$ as a subset of $\{0, 1\}^n$. If $M \in \mathcal{P}(\mathbb{N})$ is non-empty, then ξ_M denotes the restriction of ξ to M . For $n \in \mathbb{N}$, the characteristic function $\chi_{[0, n)}$ is denoted simply by χ_n .

By λ we denote the canonical Haar measure on Ω , i.e., the product measure

$$(5) \quad \lambda = \bigotimes_{n=0}^{\infty} \lambda_n,$$

where λ_n is the probability measure on $\{0, 1\}$ satisfying $\lambda_n(\{0\}) = \lambda_n(\{1\}) = 1/2$ for all $n \in \mathbb{N}$.

Given $1 \leq n \in \mathbb{N}$ and $\delta \in \{0, 1\}^n$ we define $G(n, \delta) = \{\xi \in \Omega : \chi_n \xi = \delta\}$. The collection

$$(6) \quad \{G(n, \delta) : n \in \mathbb{N} \setminus \{0\}, \delta \in \{0, 1\}^n\}$$

is a base for the topology in Ω consisting of sets which are simultaneously open and closed. For each $\xi \in \Omega$, the collection $\{G(n, \chi_n \xi)\}_{n=1}^{\infty}$ is a neighborhood base at ξ . If $n, m \in \mathbb{N} \setminus \{0\}$, $\delta \in \{0, 1\}^n$ and $\varepsilon \in \{0, 1\}^m$, then $G(m, \varepsilon) \subseteq G(n, \delta)$ if and only if $n \leq m$ and $\chi_n \varepsilon = \delta$. Moreover, if $G(m, \varepsilon) \cap G(n, \delta) \neq \emptyset$, then either $G(m, \varepsilon) \subseteq G(n, \delta)$ or $G(n, \delta) \subseteq G(m, \varepsilon)$.

LEMMA 4.8. *Let $M \subseteq \Omega$ be closed and nowhere dense and let $n \in \mathbb{N} \setminus \{0\}$ and $\delta \in \{0, 1\}^n$ be given. For every integer $l \geq n$, there exists an integer $m \geq l$ and $\varepsilon \in \{0, 1\}^m$ such that $\chi_l \varepsilon = \delta$ and $G(m, \varepsilon) \cap M = \emptyset$.*

Proof. Since the interior $\text{int}(M)$ is empty, there exists $\xi \in G(l, \delta) \setminus M$, and since M is closed, it follows that there exists $m \in \mathbb{N} \setminus \{0\}$ such that $G(m, \chi_m \xi) \subseteq G(l, \delta) \setminus M$. As observed above, this implies that $l \leq m$ and $\chi_l \varepsilon = \delta$, so we can take $\varepsilon = \chi_m \xi$. ■

LEMMA 4.9. *Let $M \subseteq \Omega$ be closed and nowhere dense and let $n \in \mathbb{N} \setminus \{0\}$ and $\delta_j \in \{0, 1\}^n$ for $j = 0, 1, \dots, k$ be given. Then there exists an integer $m \geq n$ and elements $\varepsilon_j \in \{0, 1\}^m$ for $j = 0, 1, \dots, k$ such that:*

- (i) $G(m, \varepsilon_j) \subseteq G(n, \delta_j)$ for all $j = 0, 1, \dots, k$;
- (ii) $G(m, \varepsilon_j) \cap M = \emptyset$ for all $j = 0, 1, \dots, k$;
- (iii) $\text{supp}(\varepsilon_i) \cap \text{supp}(\varepsilon_j) \subseteq [0, n)$ whenever $0 \leq i < j \leq k$.

Proof. The proof is by induction on k . For $k = 0$, the statement follows immediately from Lemma 4.8 (applied to $\delta = \delta_0$ and $l = n$). Assume now that the statement has been proved for some k . Let $\delta_j \in \{0, 1\}^n$ for $j = 0, 1, \dots, k+1$ be given. By the induction hypothesis there exists an integer $m_k \geq n$ and elements $\varepsilon_j \in \{0, 1\}^{m_k}$, for $j = 0, \dots, k$, such that (i), (ii) and (iii) are satisfied (with $m = m_k$). Now apply Lemma 4.8 to $\delta = \delta_{k+1}$ and $l = m_k$. Then there exists an integer $m_{k+1} \geq m_k$ and an element $\varepsilon_{k+1} \in \{0, 1\}^{m_{k+1}}$ such that $\chi_{m_k} \varepsilon_{k+1} = \delta_{k+1}$ and $G(m_{k+1}, \varepsilon_{k+1}) \cap M = \emptyset$. Note that $\chi_n \varepsilon_{k+1} = \delta_{k+1}$.

It is now clear that (i) and (ii) are satisfied (with $m = m_{k+1}$). Since $\text{supp}(\varepsilon_{k+1}) \subseteq [0, n) \cup [m_k, m_{k+1})$ and $\text{supp}(\varepsilon_j) \subseteq [0, m_k)$, it follows that (iii) is satisfied as well. ■

Before formulating the next lemma we recall that, for $k \in \mathbb{N}$ and for any $\tau \in \{0, 1\}^{k+1}$, we denote by $\tau_{[0, k)} \in \{0, 1\}^k$ the restriction of τ to $[0, k)$.

LEMMA 4.10. *Let $\{M_k\}_{k=0}^\infty$ be a sequence of closed, nowhere dense subsets of Ω and let $n_0 \in \mathbb{N} \setminus \{0\}$ and $\delta_\emptyset \in \{0, 1\}^{n_0}$ be given. Then there exists a sequence $n_0 < n_1 < \dots$ in \mathbb{N} with the property that, for each $k \in \mathbb{N}$ and every $\tau \in \{0, 1\}^{k+1}$, there exists $\delta_\tau \in \{0, 1\}^{n_{k+1}}$ such that:*

- (i) $\delta_\tau \chi_{n_k} = \delta_{\tau_{[0, k)}}$ for all $\tau \in \{0, 1\}^{k+1}$;
- (ii) $\text{supp}(\delta_{\tau_1}) \cap \text{supp}(\delta_{\tau_2}) \subseteq [0, n_k)$ whenever $\tau_1 \neq \tau_2$ in $\{0, 1\}^{k+1}$;
- (iii) $G(n_{k+1}, \delta_\tau) \cap M_k = \emptyset$ for all $\tau \in \{0, 1\}^{k+1}$.

Proof. Suppose that $n_0 < n_1 < \dots < n_k$ have been constructed with corresponding $\delta_\tau \in \{0, 1\}^{n_j}$ for every $\tau \in \{0, 1\}^j$ and every $j = 0, 1, \dots, k$ satisfying (i), (ii) and (iii). Choose any $\tau \in \{0, 1\}^{k+1}$, in which case $\tau_{[0, k)} \in \{0, 1\}^k$, and so we can define $\alpha_\tau = \delta_{\tau_{[0, k)}}$. Now apply Lemma 4.9 to $M = M_k$, $n = n_k$ and all the $\alpha_\tau \in \{0, 1\}^{n_k}$. It follows that there exists an integer

$n_{k+1} > n_k$ and elements $\delta_\tau \in \{0, 1\}^{n_{k+1}}$ (for each $\tau \in \{0, 1\}^{k+1}$) such that $\delta_\tau \chi_{n_k} = \alpha_\tau = \delta_{\tau_{[0,k]}}$ with $G(n_{k+1}, \delta_\tau) \cap M_k = \emptyset$ for all $\tau \in \{0, 1\}^{k+1}$, and $\text{supp}(\delta_{\tau_1}) \cap \text{supp}(\delta_{\tau_2}) \subseteq [0, n_k]$ whenever $\tau_1 \neq \tau_2$ in $\{0, 1\}^{k+1}$. ■

Before we prove Proposition 4.7, recall that a subset $F \subseteq \Omega$ has the *Baire property* if there exists an open set $V \subseteq \Omega$ such that $F \Delta V$ is *meager*. Since the collection of all subsets with the Baire property is a σ -algebra, it is clear that any Borel subset of Ω has the Baire property.

Proof of Proposition 4.7. Let $E \in \mathcal{B}$ be non-meager in Ω . It suffices to show that there exists $\mu \in \mathcal{A}$ with $\mu(E) > 0$, as then $E \notin \mathcal{J}$. Since the set E has the Baire property and since the sets (6) form a base for the topology in Ω , it follows that there exist $n_0 \in \mathbb{N} \setminus \{0\}$ and $\delta_\emptyset \in \{0, 1\}^{n_0}$ such that $G(n_0, \delta_\emptyset) \setminus E$ is meager. Let $\{M_k\}_{k=0}^\infty$ be a sequence of closed, nowhere dense sets in Ω such that

$$(7) \quad G(n_0, \delta_\emptyset) \setminus E \subseteq \bigcup_{k=0}^\infty M_k.$$

Apply Lemma 4.10 to obtain a sequence $n_0 < n_1 < \dots$ in \mathbb{N} such that, for each $k \in \mathbb{N}$ and every $\tau \in \{0, 1\}^{k+1}$, there exists $\delta_\tau \in \{0, 1\}^{n_{k+1}}$ satisfying properties (i), (ii) and (iii) of the lemma.

For each $\xi \in \Omega$ define $d(\xi) = \bigcup_{k \in \mathbb{N}} \text{supp}(\delta_{\xi_{[0,k]}})$ and a function $f : \Omega \rightarrow \Omega$ by $f(\xi) = \chi_{d(\xi)}$. It follows from (i) of Lemma 4.10 that for each $k \in \mathbb{N}$,

$$(8) \quad f(\xi)(n) = \delta_{\xi_{[0,k]}}(n), \quad n \in [0, n_k].$$

In particular, if $\xi, \eta \in \Omega$ satisfy $\xi_{[0,k]} = \eta_{[0,k]}$ for some $k \in \mathbb{N} \setminus \{0\}$, then $f(\xi)\chi_{n_k} = f(\eta)\chi_{n_k}$. This implies that f is continuous.

Choose $\xi \in \Omega$. It follows from (8) that

$$(9) \quad f(\xi) \in G(n_k, \delta_{\xi_{[0,k]}}) \subseteq G(n_0, \delta_\emptyset), \quad k \in \mathbb{N} \setminus \{0\}.$$

Since $G(n_k, \delta_{\xi_{[0,k]}}) \cap M_{k-1} = \emptyset$ for all $k \in \mathbb{N} \setminus \{0\}$ (cf. (iii) of Lemma 4.10), this implies that $f(\xi) \notin \bigcup_{k=0}^\infty M_k$. It now follows from (7) and (9) that $f(\xi) \in G(n_0, \delta_\emptyset) \cap E \subseteq E$. Accordingly,

$$(10) \quad f(\Omega) \subseteq E.$$

Let λ be Haar measure on Ω as given by (5) and define the probability measure $\mu = \lambda f^{-1}$, that is, $\mu(A) = \lambda(f^{-1}(A))$ for each $A \in \mathcal{B}$. Then (10) implies that $\mu(E) = 1$. It follows from the definitions of D_n and f that

$$\mu(D_n) = \lambda(\{\xi \in \Omega : n \in d(\xi)\}), \quad n \in \mathbb{N}.$$

Fix $k \in \mathbb{N}$ and assume that $n \in [n_k, n_{k+1})$. It follows from (i) in Lemma 4.10 that $n \in d(\xi)$ if and only if $n \in \text{supp}(\delta_{\xi_{[0,k+1]}})$, and hence,

$$\mu(D_n) = \lambda(\{\xi \in \Omega : n \in \text{supp}(\delta_{\xi_{[0,k+1]}})\}), \quad n \in \mathbb{N}.$$

By Lemma 4.10(ii) there is at most one $\tau \in \{0, 1\}^{k+1}$ such that $n \in \text{supp}(\delta_\tau)$. If no such τ exists, then $\mu(D_n) = 0$. If there exists $\tau_0 \in \{0, 1\}^{k+1}$ such that $n \in \text{supp}(\delta_{\tau_0})$, then

$$\{\xi \in \Omega : n \in \text{supp}(\delta_{\xi_{[0, k+1]}})\} = \{\xi \in \Omega : \xi_{[0, k+1]} = \tau_0\}$$

and so $\mu(D_n) = 2^{-(k+1)}$. Hence, $\mu(D_n) \leq 2^{-(k+1)}$ for all $n \in [n_k, n_{k+1})$. Accordingly, $\lim_{n \rightarrow \infty} \mu(D_n) = 0$ and so $\mu \in \mathcal{A}$. Since $\mu(E) > 0$, the proof of Proposition 4.7 is complete. In particular, Proposition 4.6 is thereby also established. ■

Now it is clear how to construct an example of the type announced at the beginning of this subsection. Indeed, with Φ as defined by (4), let $X_\Phi \subseteq \text{ca}(\mathcal{B})$ be the band generated by Φ . By Proposition 4.6, \mathcal{B} is not sequentially Φ -complete. Then Theorem 3.2 implies that the Bade σ -complete B.a. \mathcal{M}_Φ (notation as in Theorem 3.2) cannot be sequentially closed in $\mathcal{L}_s(X_\Phi)$.

We are indebted to G. Debs for an invaluable conversation, leading in particular to the suggestion that the ideal \mathcal{J} is meager; a similar question was examined in [4].

4.3. Concluding remarks. Finally, we indicate how the previous example can be used to obtain an example of a Bade σ -complete B.a. of projections in a Hilbert space which fails to be sequentially closed.

The setting is still $\Omega = \{0, 1\}^{\mathbb{N}}$ and with Φ as given by (4). Define $\tilde{\Omega} = \Omega \times \Phi$ and, for every $G \subseteq \tilde{\Omega}$ and $\nu \in \Phi$, let $G^\nu = \{t \in \Omega : (t, \nu) \in G\}$. Then $\Gamma = \{G \subseteq \tilde{\Omega} : G^\nu \in \mathcal{B} \text{ for every } \nu \in \Phi\}$ is a σ -algebra of subsets of $\tilde{\Omega}$. Moreover, $\mu : \Gamma \rightarrow [0, \infty]$ defined by $\mu(G) = \sum_{\nu \in \Phi} \nu(G^\nu)$ for $G \in \Gamma$ is a σ -additive measure and, being decomposable, $(\tilde{\Omega}, \Gamma, \mu)$ is a Maharam measure space (cf. [7, Section 64]). Consider now the Hilbert space $\mathcal{H} = L^2(\mu)$. For each $A \in \mathcal{B}$ we define $P(A) \in \mathcal{L}(\mathcal{H})$ by $P(A)f = \chi_{A \times \Phi} f$ for all $f \in \mathcal{H}$. Then $P : \mathcal{B} \rightarrow \mathcal{L}(\mathcal{H})$ is a spectral measure and $\mathcal{M} = \{P(A) : A \in \mathcal{B}\}$ is a Bade σ -complete B.a. of (self-adjoint) projections in \mathcal{H} . Given $\nu \in \Phi$ and a function f on $\tilde{\Omega}$, define $f^\nu : t \mapsto f(t, \nu)$ for each $t \in \Omega$. Then $f \in \mathcal{H}$ if and only if there exist countably many $\{\nu_k\}_{k=1}^\infty \subseteq \Phi$ such that each $f^{\nu_k} \in L^2(\nu_k)$ with $\sum_{k=1}^\infty \|f^{\nu_k}\|_2^2 < \infty$ and $f^\nu = 0$ (ν -a.e.) for all other $\nu \in \Phi$. This observation, together with the fact that $\nu(D_n \triangle D_m) \rightarrow 0$ as $m, n \rightarrow \infty$ for each $\nu \in \Phi$, and a standard truncation argument, shows that $\|P(D_n)f - P(D_m)f\|_2 \rightarrow 0$ as $m, n \rightarrow \infty$ for each $f \in \mathcal{H}$. Hence, there exists a self-adjoint projection Q in \mathcal{H} such that $P(D_n) \rightarrow Q$ in $\mathcal{L}_s(\mathcal{H})$ as $n \rightarrow \infty$. Assuming that $Q = P(E)$ for some $E \in \mathcal{B}$, it is easy to see that this would imply $\nu(D_n \triangle E) \rightarrow 0$ as $n \rightarrow \infty$ for each $\nu \in \Phi$. However, from Example B we know that no such set $E \in \mathcal{B}$ exists. Consequently, \mathcal{M} is not sequentially closed in $\mathcal{L}_s(\mathcal{H})$.

In conclusion, we remark that in the above example the space $\mathcal{H} = L^2(\mu)$ may be replaced by the Banach space $X = L^p(\mu)$ for any $1 \leq p < \infty$, and we still maintain all the features that occur in $L^2(\mu)$.

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