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Some new inhomogeneous Triebel–Lizorkin spaces on metric measure spaces and their various characterizations

by

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Abstract. Let $(X,\varrho,\mu)_{d,\theta}$ be a space of homogeneous type, i.e. X is a set, ϱ is a quasi-metric on X with the property that there are constants $\theta\in(0,1]$ and $C_0>0$ such that for all $x,x',y\in X$,

$$|\varrho(x,y)-\varrho(x',y)|\le C_0\varrho(x,x')^\theta[\varrho(x,y)+\varrho(x',y)]^{1-\theta},$$

and μ is a nonnegative Borel regular measure on X such that for some $d>0$ and all $x\in X$,

$$\mu(\{y\in X:\varrho(x,y)<r\})\sim r^d.$$

Let $\varepsilon\in(0,\theta]$, $|s|<\varepsilon$ and $\max\{d/(d+\varepsilon),d/(d+s+\varepsilon)\}<q\le\infty$. The author introduces new inhomogeneous Triebel–Lizorkin spaces $F^s_{\infty q}(X)$ and establishes their frame characterizations by first establishing a Plancherel–Pólya-type inequality related to the norm $\|\cdot\|_{F^s_{\infty q}(X)}$, which completes the theory of function spaces on spaces of homogeneous type. Moreover, the author establishes the connection between the space $F^s_{\infty q}(X)$ and the homogeneous Triebel–Lizorkin space $\dot F^s_{\infty q}(X)$. In particular, he proves that $\mathrm{bmo}(X)$ coincides with $F^0_{\infty 2}(X)$.

1. Introduction. Analysis on metric spaces has recently aroused an increasing interest; see [25, 20, 9, 22]. Especially, the theory of function spaces on metric measure spaces, or more generally, spaces of homogeneous type in the sense of Coifman and Weiss [2, 3] has been well developed; see [23, 24, 13–18, 31, 34]. The homogeneous Besov and Triebel–Lizorkin spaces on spaces of homogeneous type have been studied in [16, 11]. In [13], the inhomogeneous Besov and Triebel–Lizorkin spaces on spaces of homogeneous type were introduced by use of the generalized Littlewood–Paley g-functions when $p,q\ge1$. In [14], the inhomogeneous Triebel–Lizorkin spaces were generalized to the cases where $0<p_0<p\le1\le q<\infty$ via

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the generalized Littlewood–Paley S-functions. Using the discrete Calderón reproducing formulae of [15], Han and the present author [18] further developed the theory of the inhomogeneous Besov and Triebel–Lizorkin spaces when $p \le 1$ or $q \le 1$. Some applications are given in [17, 18, 31, 33].

The main purpose of this paper is to generalize inhomogeneous Triebel–Lizorkin spaces on spaces of homogeneous type to the case $p = \infty$. The theory of the corresponding homogeneous spaces has been established in [34]. However, due to the inhomogeneity, some new ideas and techniques are necessary.

We begin by recalling some necessary definitions and notation for spaces of homogeneous type.

A *quasi-metric* ϱ on a set X is a function $\varrho : X \times X \to [0, \infty)$ satisfying:

(i) $\varrho(x, y) = 0$ if and only if $x = y$;
(ii) $\varrho(x, y) = \varrho(y, x)$ for all $x, y \in X$;
(iii) there exists a constant $A \in [1, \infty)$ such that for all $x, y, z \in X$,
$$\varrho(x, y) \le A[\varrho(x, z) + \varrho(z, y)].$$

Any quasi-metric defines a topology for which the balls
$$B(x, r) = \{y \in X : \varrho(y, x) < r\}$$
for all $x \in X$ and all $r > 0$ form a basis.

We set $\operatorname{diam} X = \sup\{\varrho(x, y) : x, y \in X\}$ and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. We also make the following conventions. We write $f \sim g$ when there is a constant $C > 0$ independent of the main parameters such that $C^{-1}g < f < Cg$. Throughout the paper, we denote by C a positive constant which is independent of the main parameters, but it may vary from line to line. Constants with subscripts, such as C_1, do not change in different occurrences. For any $q \in [1, \infty]$, we denote by q' its conjugate index, namely, $1/q + 1/q' = 1$. If A is a set then χ_A denotes the characteristic function of A.

Definition 1. Let $d > 0$ and $\theta \in (0, 1]$. A *space of homogeneous type*, $(X, \varrho, \mu)_{d,\theta}$, is a set X together with a quasi-metric ϱ and a nonnegative Borel regular measure μ on X, such that there exists a constant $C_0 > 0$ such that for all $0 < r < \operatorname{diam} X$ and all $x, x', y \in X$,
$$\mu(B(x, r)) \sim r^d, \tag{1.1}$$
$$|\varrho(x, y) - \varrho(x', y)| \le C_0 \varrho(x, x')^\theta [\varrho(x, y) + \varrho(x', y)]^{1-\theta}. \tag{1.2}$$

The above notion was introduced in [17]; it is a variant of the space of homogeneous type introduced by Coifman and Weiss [2]. In [23], Macías and Segovia have proved that one can replace the quasi-metric ϱ of the space of homogeneous type in the sense of Coifman and Weiss by another quasi-metric $\bar{\varrho}$ which yields the same topology and $(X, \bar{\varrho}, \mu)$ is as in Definition 1 with $d = 1$.

In the most part of this paper, $\mu(X)$ can be infinite or finite. This means that the spaces of homogeneous type considered by us include various fractals. It is well known that spaces of homogeneous type in the sense of Definition 1 include metric measure spaces, the Euclidean space, the C^∞-compact Riemannian manifolds, the boundaries of Lipschitz domains and, in particular, the Lipschitz manifolds introduced recently by Triebel [30], as well as the isotropic and anisotropic d-sets in $\mathbb{R}^n$. It has been proved by Triebel [28, 29] that the isotropic and anisotropic d-sets in $\mathbb{R}^n$ include various kinds of self-affine fractals, for example, the Cantor set, the generalized Sierpiński carpet and so forth. We particularly point out that the spaces of homogeneous type in the sense of Definition 1 also include the postcritically finite self-similar fractals studied by Kigami [21] and by Strichartz [26], and the metric spaces with heat kernel studied by Grigor'yan, Hu and Lau [8]. More examples of spaces of homogeneous type can be found in [2, 3, 25].

Let us now recall the definition of the space of test functions.

Definition 2 ([10]). Fix $\gamma > 0$ and $\beta > 0$. A function f defined on X is said to be a *test function of type* (x_0, r, β, γ) with $x_0 \in X$ and $r > 0$ if f satisfies the following conditions:

(i) $|f(x)| \le C \dfrac{r^\gamma}{(r + \varrho(x, x_0))^{d+\gamma}}$;

(ii) $|f(x) - f(y)| \le C\left(\dfrac{\varrho(x,y)}{r + \varrho(x, x_0)}\right)^\beta \dfrac{r^\gamma}{(r + \varrho(x, x_0))^{d+\gamma}}$

$$\text{for } \varrho(x,y) \le \frac{1}{2A}\,[r + \varrho(x, x_0)].$$

If f is a test function of type (x_0, r, β, γ), we write $f \in \mathcal{G}(x_0, r, \beta, \gamma)$, and the norm of f in $\mathcal{G}(x_0, r, \beta, \gamma)$ is defined by

$$\|f\|_{\mathcal{G}(x_0,r,\beta,\gamma)} = \inf\{C : \text{(i) and (ii) hold}\}.$$

Now fix $x_0 \in X$ and let $\mathcal{G}(\beta, \gamma) = \mathcal{G}(x_0, 1, \beta, \gamma)$. It is easy to see that

$$\mathcal{G}(x_1, r, \beta, \gamma) = \mathcal{G}(\beta, \gamma)$$

with equivalent norms for all $x_1 \in X$ and $r > 0$. Furthermore, it is easy to check that $\mathcal{G}(\beta, \gamma)$ is a Banach space with respect to the norm in $\mathcal{G}(\beta, \gamma)$. Also, let the dual space $(\mathcal{G}(\beta, \gamma))'$ be all linear functionals $\mathcal{L}$ from $\mathcal{G}(\beta, \gamma)$ to $\mathbb{C}$ with the property that there exists $C \ge 0$ such that for all $f \in \mathcal{G}(\beta, \gamma)$,

$$|\mathcal{L}(f)| \le C\|f\|_{\mathcal{G}(\beta,\gamma)}.$$

We denote by $\langle h, f\rangle$ the natural pairing between $h \in (\mathcal{G}(\beta, \gamma))'$ and $f \in \mathcal{G}(\beta, \gamma)$. Clearly, for all $h \in (\mathcal{G}(\beta, \gamma))'$, $\langle h, f\rangle$ is well defined for all $f \in \mathcal{G}(x_0, r, \beta, \gamma)$ with $x_0 \in X$ and $r > 0$.

It is well known that even when $X = \mathbb{R}^n$, $\mathcal{G}(\beta_1, \gamma)$ is not dense in $\mathcal{G}(\beta_2, \gamma)$ if $\beta_1 > \beta_2$, which will bring us some inconvenience. To overcome this defect, in what follows, for a given $\varepsilon \in (0, \theta]$, we let $\mathring{\mathcal{G}}(\beta, \gamma)$ be the completion of the space $\mathcal{G}(\varepsilon, \varepsilon)$ in $\mathcal{G}(\beta, \gamma)$ when $0 < \beta, \gamma < \varepsilon$.

Definition 3 ([11]). A sequence $\{S_k\}_{k \in \mathbb{Z}_+}$ of linear operators is said to be an *approximation to the identity* of order $\varepsilon \in (0, \theta]$ if there exists $C_1 > 0$ such that for all $k \in \mathbb{Z}_+$ and all $x, x', y, y' \in X$, $S_k(x, y)$, the kernel of S_k, is a function from $X \times X$ into $\mathbb{C}$ satisfying

(i) $|S_k(x, y)| \le C_1 \dfrac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x, y))^{d+\varepsilon}}$;

(ii) $|S_k(x, y) - S_k(x', y)| \le C_1 \left(\dfrac{\varrho(x, x')}{2^{-k} + \varrho(x, y)} \right)^{\varepsilon} \dfrac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x, y))^{d+\varepsilon}}$ for $\varrho(x, x') \le \dfrac{1}{2A} (2^{-k} + \varrho(x, y))$;

(iii) $|S_k(x, y) - S_k(x, y')| \le C_1 \left(\dfrac{\varrho(y, y')}{2^{-k} + \varrho(x, y)} \right)^{\varepsilon} \dfrac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x, y))^{d+\varepsilon}}$ for $\varrho(y, y') \le \dfrac{1}{2A} (2^{-k} + \varrho(x, y))$;

(iv) $|[S_k(x, y) - S_k(x, y')] - [S_k(x', y) - S_k(x', y')]|$
$$\le C_1 \left(\frac{\varrho(x, x')}{2^{-k} + \varrho(x, y)} \right)^{\varepsilon} \left(\frac{\varrho(y, y')}{2^{-k} + \varrho(x, y)} \right)^{\varepsilon} \frac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x, y))^{d+\varepsilon}}$$
for $\varrho(x, x') \le \dfrac{1}{2A} (2^{-k} + \varrho(x, y))$ and $\varrho(y, y') \le \dfrac{1}{2A} (2^{-k} + \varrho(x, y))$;

(v) $\int_X S_k(x, y)\, d\mu(y) = 1$;

(vi) $\int_X S_k(x, y)\, d\mu(x) = 1$.

Remark 1. By Coifman's construction [4], one can construct an approximation to the identity of order θ such that $S_k(x, y)$ has a compact support when one variable is fixed, namely, there is a constant $C_2 > 0$ such that for all $k \in \mathbb{Z}_+$, $S_k(x, y) = 0$ if $\varrho(x, y) \ge C_2 2^{-k}$.

We also need the following construction given by Christ [1], which provides an analogue of the grid of Euclidean dyadic cubes on spaces of homogeneous type, and the discrete Calderón reproducing formulae in [15].

Lemma 1. *Let X be a space of homogeneous type. Then there exists a collection*
$$\{Q_\alpha^k \subset X : k \in \mathbb{Z},\, \alpha \in I_k\}$$

of open subsets, where I_k *is some index set, and constants* $\delta \in (0,1)$ *and* $C_3, C_4 > 0$ *such that*

(i) $\mu(X \setminus \bigcup_\alpha Q_\alpha^k) = 0$ *for each fixed* k *and* $Q_\alpha^k \cap Q_\beta^k = \emptyset$ *if* $\alpha \neq \beta$;

(ii) *for any* α, β, k, l *with* $l \geq k$, *either* $Q_\beta^l \subset Q_\alpha^k$ *or* $Q_\beta^l \cap Q_\alpha^k = \emptyset$;

(iii) *for each* (k, α) *and each* $l < k$ *there is a unique* β *such that* $Q_\alpha^k \subset Q_\beta^l$;

(iv) $\operatorname{diam}(Q_\alpha^k) \leq C_3 \delta^k$;

(v) *each* Q_α^k *contains some ball* $B(z_\alpha^k, C_4\delta^k)$, *where* $z_\alpha^k \in X$.

In fact, we can think of Q_α^k as being a dyadic cube with diameter roughly δ^k and centered at z_α^k. In what follows, we always suppose $\delta = 1/2$. See [16] for how to remove this restriction. Also, in the following, for $k \in \mathbb{Z}_+$ and $\tau \in I_k$, we will denote by $Q_\tau^{k,\nu}$, $\nu = 1, \ldots, N(k, \tau)$, the set of all cubes $Q_{\tau'}^{k+j} \subset Q_\tau^k$, where j is a fixed large positive integer. Denote by $y_\tau^{k,\nu}$ a point in $Q_\tau^{k,\nu}$. For any dyadic cube Q and any $f \in L^1_{\mathrm{loc}}(X)$, we set

$$m_Q(f) = \frac{1}{\mu(Q)} \int_Q f(x)\, d\mu(x).$$

The plan of this paper is as follows. In the next section, we first introduce the norm $\|\cdot\|_{F^s_{\infty q}(X)}$. By using the Calderón reproducing formulae of [15], we then establish an inequality of Plancherel–Pólya type related to this norm; see Theorem 1 below. Applying this inequality, we show that $\|\cdot\|_{F^s_{\infty q}(X)}$ is independent of the choice of approximations to the identity (Proposition 1) and the choice of the spaces of distributions (Theorem 2). We then introduce the inhomogeneous Triebel–Lizorkin spaces $F^s_{\infty q}(X)$ in Definition 5 and prove in Theorem 3 that if $1 \leq q \leq \infty$, these spaces can be characterized quite similarly to the Euclidean case in [6].

In Section 3, we first give the frame characterization of the spaces $F^s_{\infty q}(X)$ (Theorems 4 and 5). We then establish the connection between $F^s_{\infty q}(X)$ and the homogeneous Triebel–Lizorkin space $\dot{F}^s_{\infty q}(X)$ in Theorem 6, which is new even when $X = \mathbb{R}^n$. The relation between $F^s_{pq}(X)$ and $\dot{F}^s_{\infty q}(X)$ is also stated in Theorem 7. Finally, we verify that $\mathrm{bmo}(X) = F^0_{\infty 2}(X)$ in Proposition 4.

Applications of our results to duality, interpolation and boundedness of Calderón–Zygmund operators will be discussed in another paper; cf. [6, 32].

2. Triebel–Lizorkin space $F^s_{\infty q}(X)$. In this section, we first introduce the norm $\|\cdot\|_{F^s_{\infty q}(X)}$ in spaces of distributions by using approximations to the identity. Via an inequality of Plancherel–Pólya type related to this norm, we then prove that $\|\cdot\|_{F^s_{\infty q}(X)}$ is independent of the choice of the approximations to the identity. Under some restrictions, we also verify that the definition of $\|\cdot\|_{F^s_{\infty q}(X)}$ is independent of the choice of spaces of distributions. Finally, we introduce the inhomogeneous Triebel–Lizorkin space $F^s_{\infty q}(X)$ and give some of their basic properties.

Definition 4. Let $\varepsilon \in (0,\theta]$, $|s| < \varepsilon$ and $\{S_k\}_{k\in\mathbb{Z}_+}$ be an approximation to the identity of order ε as in Definition 3, $D_0 = S_0$ and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{N}$. If

$$\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < q \le \infty$$

and $0 < \beta, \gamma < \varepsilon$, for any $f \in (\mathring{\mathcal{G}}(\beta,\gamma))'$, we define the norm $\|f\|_{F^s_{\infty q}(X)}$ by

$$\|f\|_{F^s_{\infty q}(X)} = \max\Bigg\{ \sup_{\substack{\tau\in I_0\\ \nu=1,\ldots,N(0,\tau)}} m_{Q^{0,\nu}_\tau}(|D_0(f)|), \\ \sup_{l\in\mathbb{N}} \sup_{\alpha\in I_l} \bigg[\frac{1}{\mu(Q^l_\alpha)} \int_{Q^l_\alpha} \sum_{k=l}^\infty 2^{ksq} |D_k(f)(x)|^q \, d\mu(x)\bigg]^{1/q}\Bigg\},$$

where $Q^{0,\nu}_\tau$ with $\tau \in I_0$ and $\nu = 1,\ldots,N(0,\tau)$ are as in the preceding section and $\{Q^l_\alpha : l \in \mathbb{N},\ \alpha \in I_l\}$ is as in Lemma 1.

To verify that Definition 4 is independent of the choice of approximations to the identity, we only need to establish the following inequality of Plancherel–Pólya type by using Lemma 1.

Theorem 1. *Let $\varepsilon \in (0,\theta]$, $|s| < \varepsilon$, $\{S_k\}_{k\in\mathbb{Z}_+}$ and $\{G_k\}_{k\in\mathbb{Z}_+}$ be two approximations to the identity of order ε as in Definition 3, $D_0 = S_0$, $E_0 = G_0$, $D_k = S_k - S_{k-1}$ and $E_k = G_k - G_{k-1}$ for $k \in \mathbb{N}$. Let $\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < q \le \infty$, and $0 < \beta, \gamma < \varepsilon$. Then there is a constant $C > 0$ such that for all $f \in (\mathring{\mathcal{G}}(\beta,\gamma))'$,*

$$\max\Bigg\{ \sup_{\substack{\tau\in I_0\\ \nu=1,\ldots,N(0,\tau)}} m_{Q^{0,\nu}_\tau}(|E_0(f)|), \\ \sup_{l\in\mathbb{N}} \sup_{\alpha\in I_l} \bigg[\frac{1}{\mu(Q^l_\alpha)} \sum_{k=l}^\infty \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q^{k,\nu}_\tau) \sup_{x\in Q^{k,\nu}_\tau \subset Q^l_\alpha} |E_k(f)(x)|^q\bigg]^{1/q}\Bigg\} \\ \le C \max\Bigg\{ \sup_{\substack{\tau\in I_0\\ \nu=1,\ldots,N(0,\tau)}} m_{Q^{0,\nu}_\tau}(|D_0(f)|), \\ \sup_{l\in\mathbb{N}} \sup_{\alpha\in I_l} \bigg[\frac{1}{\mu(Q^l_\alpha)} \sum_{k=l}^\infty \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q^{k,\nu}_\tau) \inf_{x\in Q^{k,\nu}_\tau \subset Q^l_\alpha} |D_k(f)(x)|^q\bigg]^{1/q}\Bigg\},$$

where $Q^{0,\nu}_\tau$ and $\{Q^l_\alpha\}$ are as above.

To prove Theorem 1, we first recall the following discrete Calderón reproducing formula of [15].

LEMMA 2. *Let S_k an D_k be as in Theorem 1. Then there exists a family of functions $\widetilde{S}_\tau^{0,\nu}(x)$ for $\tau \in I_0$ and $\nu = 1, \cdots, N(0,\tau)$ such that for any fixed $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ with $k \in \mathbb{N}$, $\tau \in I_k$ and $\nu \in \{1, \ldots, N(k,\tau)\}$ and all $f \in (\mathring{\mathcal{G}}(\beta_1, \gamma_1))'$ with $0 < \beta_1 < \varepsilon$ and $0 < \gamma_1 < \varepsilon$,*

$$f(x) = \sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu}) D_{\tau,1}^{0,\nu}(f) \widetilde{S}_\tau^{0,\nu}(x) + \sum_{k=1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(f)(y_\tau^{k,\nu}) \widetilde{D}_k(x, y_\tau^{k,\nu}), \tag{2.1}$$

where the series converges in $(\mathring{\mathcal{G}}(\beta_1', \gamma_1'))'$ for $\beta_1 < \beta_1' < \varepsilon$ and $\gamma_1 < \gamma_1' < \varepsilon$. The function $\widetilde{S}_\tau^{0,\nu}(x)$ for $\tau \in I_0$ and $\nu = 1, \ldots, N(0,\tau)$ satisfies

(i) $\int_X \widetilde{S}_\tau^{0,\nu}(x)\, d\mu(x) = 1$;

(ii) *there is a constant $C > 0$ such that*

$$|\widetilde{S}_\tau^{0,\nu}(x)| \le C \frac{1}{(1 + \varrho(x,y))^{d+\varepsilon}}$$

for all $x \in X$ and $y \in Q_\tau^{0,\nu}$;

(iii) *for any given $\varepsilon' \in (0, \varepsilon)$,*

$$|\widetilde{S}_\tau^{0,\nu}(x) - \widetilde{S}_\tau^{0,\nu}(z)| \le C \varrho(x,z)^{\varepsilon'} \frac{1}{(1 + \varrho(x,y))^{d+\varepsilon}}$$

for all $x, z \in X$ and all $y \in Q_\tau^{0,\nu}$ satisfying

$$\varrho(x,z) \le \frac{1}{2A} (1 + \varrho(x,y)).$$

Furthermore, for $\tau \in I_0$ and $\nu = 1, \ldots, N(0,\tau)$,

$$D_{\tau,1}^{0,\nu}(f) = \int_X D_{\tau,1}^{0,\nu}(y) f(y)\, d\mu(y),$$

where

$$D_{\tau,1}^{0,\nu}(y) = \frac{1}{\mu(Q_\tau^{0,\nu})} \int_{Q_\tau^{0,\nu}} D_0(z,y)\, d\mu(z).$$

Moreover, $\widetilde{D}_k(x,y)$ for $k \in \mathbb{N}$ satisfies conditions (i) *and* (ii) *of Definition* 3 *with ε replaced by $\varepsilon' \in (0,\varepsilon)$, and*

$$\int_X \widetilde{D}_k(x,y)\, d\mu(y) = \int_X \widetilde{D}_k(x,y)\, d\mu(x) = 0, \qquad k \in \mathbb{N}.$$

REMARK 2. Property (iii) of $\widetilde{S}_\tau^{0,\nu}(x)$ for $\tau \in I_0$ and $\nu = 1, \ldots, N(0,\tau)$ in Lemma 2 is not exactly the same as in [15]. However, by a careful check on the proof there, one can find that $\widetilde{S}_\tau^{0,\nu}(x)$ for $\tau \in I_0$ and $\nu = 1, \ldots, N(0,\tau)$ does satisfy (iii) of Lemma 2.

Proof of Theorem 1. With the notation of Theorem 1 and Lemma 2, we first recall that for all $y \in Q_{\tau'}^{0,\nu'}$ with $\tau' \in I_0$ and $\nu' = 1, \dots, N(0, \tau')$ and all $x \in X$,

$$|E_0(\widetilde{S}_{\tau'}^{0,\nu'})(x)| \le C \frac{1}{(1+\varrho(x,y))^{d+\varepsilon}}, \tag{2.2}$$

and for all $k' \in \mathbb{N}$, all $x, y \in X$ and any $\varepsilon' \in (0, \varepsilon)$,

$$|E_0(\widetilde{D}_{k'})(x,y)| \le C 2^{-k'\varepsilon'} \frac{1}{(1+\varrho(x,y))^{d+\varepsilon'}}; \tag{2.3}$$

see [10, 13, 17] for the proof.

From (2.1), it follows that for $\tau \in I_0$ and $\nu = 1, \dots, N(0, \tau)$,

$$\begin{aligned} m_{Q_\tau^{0,\nu}}(|E_0(f)|) &\le \sup_{x \in Q_\tau^{0,\nu}} |E_0(f)(x)| \\ &\le \sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0,\tau')} \mu(Q_{\tau'}^{0,\nu'}) m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|) \sup_{x \in Q_\tau^{0,\nu}} |E_0(\widetilde{S}_{\tau'}^{0,\nu'})(x)| \\ &\quad + \sum_{k'=1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) |D_{k'}(f)(y_{\tau'}^{k',\nu'})| \sup_{x \in Q_\tau^{0,\nu}} |E_0(\widetilde{D}_{k'})(x, y_{\tau'}^{k',\nu'})| \\ &= G_1 + G_2. \end{aligned}$$

For G_1, the estimate (2.2) and the fact that

$$\inf_{x \in Q_\tau^{0,\nu}} \{1 + \varrho(x,y)\} \sim \sup_{z \in Q_{\tau'}^{0,\nu'}} \{1 + \varrho(y_\tau^{0,\nu}, z)\}$$

for all $y \in Q_{\tau'}^{0,\nu'}$ yield

$$\begin{aligned} G_1 &\le C \sup_{\substack{\tau' \in I_0 \\ \nu'=1,\dots,N(0,\tau')}} m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|) \sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0,\tau')} \mu(Q_{\tau'}^{0,\nu'}) \\ &\qquad \times \inf_{y \in Q_{\tau'}^{0,\nu'}} \frac{1}{(1+\varrho(y_\tau^{0,\nu}, y))^{d+\varepsilon}} \\ &\le C \sup_{\substack{\tau' \in I_0 \\ \nu'=1,\dots,N(0,\tau')}} m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|) \int_X \frac{1}{(1+\varrho(y_\tau^{0,\nu}, y))^{d+\varepsilon}} \, d\mu(y) \\ &\le C \sup_{\substack{\tau' \in I_0 \\ \nu'=1,\dots,N(0,\tau')}} m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|), \end{aligned}$$

which is the desired estimate.

To estimate G_2, we first recall the well known inequality

$$\Big(\sum_j |a_j|\Big)^q \le \sum_j |a_j|^q \tag{2.4}$$

for all $a_j \in \mathbb{C}$ and $q \in (0,1]$. From the estimate (2.3), and the inequality (2.4) if $q \le 1$ or the Hölder inequality if $q > 1$, it follows that

$$
\begin{aligned}
G_2 &\le C \sum_{k'=1}^{\infty} 2^{-k'\varepsilon'} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) |D_{k'}(f)(y_{\tau'}^{k',\nu'})| \\
&\quad \times \frac{1}{(1+\varrho(y_\tau^{0,\nu}, y_{\tau'}^{k',\nu'}))^{d+\varepsilon'}} \\
&\le \begin{cases} C \displaystyle\sum_{k'=1}^{\infty} 2^{-k'(\varepsilon'+s+d)} \Big\{ \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{k'sq} |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \Big\}^{1/q}, & q \le 1, \\ C \displaystyle\sum_{k'=1}^{\infty} 2^{-k'\varepsilon'} \Big[\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \Big]^{1/q} & \\ \qquad \times \Big[\displaystyle\int_X \frac{1}{(1+\varrho(y_\tau^{0,\nu}, y))^{d+\varepsilon'}} \, d\mu(y) \Big]^{1/q'}, & q > 1, \end{cases} \\
&\le \begin{cases} C \displaystyle\sum_{k'=1}^{\infty} 2^{-k'(\varepsilon'+s+d)} \Big\{ \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{k'sq} |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \Big\}^{1/q}, & q \le 1, \\ C \displaystyle\sum_{k'=1}^{\infty} 2^{-k'(\varepsilon'+s+d/q)} \Big[\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{k'sq} |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \Big]^{1/q}, & q > 1 \end{cases} \\
&\le C \sup_{l \in \mathbb{N}} \sup_{\alpha \in I_l} \Big[\frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) \inf_{x \in Q_\tau^{k,\nu} \subset Q_\alpha^l} |D_k(f)(x)|^q \Big]^{1/q},
\end{aligned}
$$

where we chose $\varepsilon' \in (0, \varepsilon)$ such that $\varepsilon' + s > 0$ and we used the arbitrariness of $y_{\tau'}^{k',\nu'}$ and the trivial estimate

$$
\begin{aligned}
(2.5) \qquad & \Big[\sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{k'sq} |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \Big]^{1/q} \\
&\le C \sup_{l \in \mathbb{N}} \sup_{\alpha \in I_l} \Big[\frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) \inf_{x \in Q_\tau^{k,\nu} \subset Q_\alpha^l} |D_k(f)(x)|^q \Big]^{1/q}.
\end{aligned}
$$

Thus, we have obtained the desired estimate for G_2.

To finish the proof of Theorem 1, we need the following estimates: for all $k \in \mathbb{N}$, $x \in X$ and $\varepsilon' \in (0, \varepsilon)$,

$$
(2.6) \qquad |E_k(\widetilde{S}_{\tau'}^{0,\nu'})(x)| \le C 2^{-k\varepsilon'} \frac{1}{(1+\varrho(x, y_{\tau'}^{0,\nu'}))^{d+\varepsilon}},
$$

and for all $k, k' \in \mathbb{N}$, $x, y \in X$ and $\varepsilon' \in (0, \varepsilon)$,

$$|E_k \widetilde{D}_{k'}(x,y)| \le C2^{-|k-k'|\varepsilon'} \frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')\varepsilon'} + \varrho(x,y))^{d+\varepsilon'}}; \tag{2.7}$$

here and in what follows, $k \wedge k' = \min\{k, k'\}$.

The proof of (2.7) can be found in [10, 13, 17]. For the reader's convenience, we now give some details for the proof of (2.6). By the vanishing moment of E_k and the regularity of $\widetilde{D}_{k'}$, we obtain

$$\begin{aligned}
|E_k(\widetilde{S}^{0,\nu'}_{\tau'})(x)| &= \Big| \int_X E_k(x,y) \widetilde{S}^{0,\nu'}_{\tau'}(y)\, d\mu(y) \Big| \\
&= \Big| \int_X E_k(x,y) [\widetilde{S}^{0,\nu'}_{\tau'}(y) - \widetilde{S}^{0,\nu'}_{\tau'}(x)]\, d\mu(y) \Big| \\
&\le \int_{\varrho(x,y) \le \frac{1}{2A}(1+\varrho(x, y^{0,\nu'}_{\tau'}))} |E_k(x,y)|\, |\widetilde{S}^{0,\nu'}_{\tau'}(y) - \widetilde{S}^{0,\nu'}_{\tau'}(x)|\, d\mu(y) \\
&\quad + \int_{\varrho(x,y) > \frac{1}{2A}(1+\varrho(x, y^{0,\nu'}_{\tau'}))} |E_k(x,y)| [|\widetilde{S}^{0,\nu'}_{\tau'}(y)| + |\widetilde{S}^{0,\nu'}_{\tau'}(x)|]\, d\mu(y) \\
&\le \int_{\varrho(x,y) \le \frac{1}{2A}(1+\varrho(x, y^{0,\nu'}_{\tau'}))} \frac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x,y))^{d+\varepsilon}} \frac{\varrho(x,y)^{\varepsilon'}}{(1+\varrho(x, y^{0,\nu'}_{\tau'}))^{d+\varepsilon}}\, d\mu(y) \\
&\quad + \int_{\varrho(x,y) > \frac{1}{2A}(1+\varrho(x, y^{0,\nu'}_{\tau'}))} \frac{2^{-k\varepsilon}}{(2^{-k} + \varrho(x,y))^{d+\varepsilon}} \\
&\quad \times \Big[\frac{1}{(1+\varrho(y, y^{0,\nu'}_{\tau'}))^{d+\varepsilon}} + \frac{1}{(1+\varrho(x, y^{0,\nu'}_{\tau'}))^{d+\varepsilon}} \Big]\, d\mu(y) \\
&\le C2^{-k\varepsilon'} \frac{1}{(1+\varrho(x, y^{0,\nu'}_{\tau'}))^{d+\varepsilon}},
\end{aligned}$$

which is just (2.6).

For any $l \in \mathbb{N}$ and $\alpha \in I_l$, from (2.1), it follows that

$$\begin{aligned}
&\Big[\frac{1}{\mu(Q^l_\alpha)} \sum_{k=l}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q^{k,\nu}_\tau) \sup_{x \in Q^{k,\nu}_\tau \subset Q^l_\alpha} |E_k(f)(x)|^q \Big]^{1/q} \\
&\quad \le C \Big\{ \frac{1}{\mu(Q^l_\alpha)} \sum_{k=l}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q^{k,\nu}_\tau) \chi_{\{(k,\tau,\nu)\,:\, Q^{k,\nu}_\tau \subset Q^l_\alpha\}}(k,\tau,\nu) \\
&\qquad \times \Big[\sum_{\tau' \in I_0} \sum_{\nu'=1}^{N(0,\tau')} \mu(Q^{0,\nu'}_{\tau'}) m_{Q^{0,\nu'}_{\tau'}}(|D_0(f)|) \sup_{x \in Q^{k,\nu}_\tau} |E_k(\widetilde{S}^{0,\nu'}_{\tau'})(x)| \Big]^q \Big\}^{1/q}
\end{aligned}$$

$$+ C\Big\{\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}\mu(Q_\tau^{k,\nu})\chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu)$$

$$\times\Big[\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|\sup_{x\in Q_\tau^{k,\nu}}|E_k\widetilde{D}_{k'}(x,y_{\tau'}^{k',\nu'})|\Big]^q\Big\}^{1/q}$$

$= H_1 + H_2$.

The estimate (2.6) and Lemma 1 lead us to

$$H_1 \le C\sup_{\substack{\tau'\in I_0\\ \nu'=1,\dots,N(0,\tau')}} m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\Big\{\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}2^{k(s-\varepsilon')q}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})$$

$$\times\chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu)\Big[\int_X\frac{1}{(1+\varrho(y_\tau^{k,\nu},y))^{d+\varepsilon}}\,d\mu(y)\Big]^q\Big\}^{1/q}$$

$$\le C\sup_{\substack{\tau'\in I_0\\ \nu'=1,\dots,N(0,\tau')}} m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|)\Big\{\sum_{k=l}^{\infty}2^{k(s-\varepsilon')q}\Big\}^{1/q}$$

$$\le C\sup_{\substack{\tau'\in I_0\\ \nu'=1,\dots,N(0,\tau')}} m_{Q_{\tau'}^{0,\nu'}}(|D_0(f)|),$$

where we chose $\varepsilon'\in(0,\varepsilon)$ such that $\varepsilon' > s$.

To estimate H_2, by (2.7), we further decompose H_2 into

$$H_2 \le C\Big\{\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}\mu(Q_\tau^{k,\nu})\chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu)$$

$$\times\Big[\sum_{k'=1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|2^{-|k-k'|\varepsilon'}$$

$$\times\frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')}+\varrho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{d+\varepsilon'}}\Big]^q\Big\}^{1/q}$$

$$\le C\Big\{\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}\mu(Q_\tau^{k,\nu})\chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu)$$

$$\times\Big[\sum_{k'=1}^{l}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|2^{-(k-k')\varepsilon'}$$

$$\times\frac{2^{-k'\varepsilon'}}{(2^{-k'}+\varrho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{d+\varepsilon'}}\Big]^q\Big\}^{1/q}$$

$$
\begin{aligned}
&+ C\bigg\{\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}\mu(Q_\tau^{k,\nu})\chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu)\\
&\quad\times\bigg[\sum_{k'=l+1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|2^{-|k-k'|\varepsilon'}\\
&\quad\times\frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')}+\varrho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{d+\varepsilon'}}\bigg]^q\bigg\}^{1/q}\\
&= J_1+J_2.
\end{aligned}
$$

For J_1, by (2.5), (2.4) if $q\le 1$ or the Hölder inequality if $q>1$, we then have

$$
J_1\le\begin{cases}
C\bigg\{\dfrac{1}{\mu(Q_\alpha^l)}\displaystyle\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}\mu(Q_\tau^{k,\nu})\chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu)\\
\quad\times\bigg[\displaystyle\sum_{k'=1}^{l}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{-(k-k')\varepsilon' q}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q\bigg]^q\bigg\}^{1/q}, & q\le 1,\\
C\bigg\{\dfrac{1}{\mu(Q_\alpha^l)}\displaystyle\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}\mu(Q_\tau^{k,\nu})\chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu)\\
\quad\times\bigg(\displaystyle\sum_{k'=1}^{l}2^{-(k-k')\varepsilon'}\bigg[\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}|D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q\bigg]^{1/q}\\
\quad\times\bigg[\displaystyle\int_X\frac{2^{-k'\varepsilon'}}{(2^{-k'}+\varrho(y_\tau^{k,\nu},y))^{d+\varepsilon'}}\,d\mu(y)\bigg]^{1/q'}\bigg)^q\bigg\}^{1/q}, & q>1,
\end{cases}
$$

$$
\begin{aligned}
&\le C\sup_{l\in\mathbb{N}}\sup_{\alpha\in I_l}\bigg[\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}\mu(Q_\tau^{k,\nu})\inf_{x\in Q_\tau^{k,\nu}\subset Q_\alpha^l}|D_k(f)(x)|^q\bigg]^{1/q}\\
&\quad\times\begin{cases}
\bigg\{\displaystyle\sum_{k=l}^{\infty}2^{k(s-\varepsilon')q}\bigg\}^{1/q}\bigg\{\sum_{k'=1}^{l}2^{k'(\varepsilon'-s)q}\bigg\}^{1/q}, & q\le 1,\\
\bigg\{\displaystyle\sum_{k=l}^{\infty}2^{k(s-\varepsilon')q}\bigg\}^{1/q}\bigg\{\sum_{k'=1}^{l}2^{k'(\varepsilon'-s)}\bigg\}, & q>1,
\end{cases}\\
&\le C\sup_{l\in\mathbb{N}}\sup_{\alpha\in I_l}\bigg[\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}\mu(Q_\tau^{k,\nu})\inf_{x\in Q_\tau^{k,\nu}\subset Q_\alpha^l}|D_k(f)(x)|^q\bigg]^{1/q},
\end{aligned}
$$

where we chose $\varepsilon'\in(0,\varepsilon)$ such that $\varepsilon'>s$.

To estimate J_2, let z_α^l be the center of Q_α^l, as in Lemma 1. Choose $m_1 \in \mathbb{N}$ such that $B(z_\alpha^l, A^2C_3 2^{m_1-l}) \supset Q_\alpha^l$, and for all $y \in Q_\alpha^l$ and $x \notin B(z_\alpha^l, A^2C_3 2^{m_1-l})$, $\varrho(x,y) \geq 2^{-l}$, where m_1 is independent of P. By Lemma 1 again, there is $m \in \mathbb{N}$ independent of Q_α^l such that $B(z_\alpha^l, A^2C_3 2^{m_1-l}) \subset \bigcup_{i=1}^m Q_{\tau^i}^{l+1}$, where $\tau^i \in I_{l+1}$ for $i = 1, \ldots, m$. With this choice, we now further decompose J_2 into

$$\begin{aligned} J_2 \leq C\Bigg\{ &\frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) \chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu} \subset Q_\alpha^l\}}(k,\tau,\nu) \\ &\times \bigg[\sum_{k'=l+1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) |D_{k'}(f)(y_{\tau'}^{k',\nu'})| 2^{-|k-k'|\varepsilon'} \\ &\times \chi_{\{(k',\tau',\nu')\,:\,Q_{\tau'}^{k',\nu'} \subset \bigcup_{i=1}^m Q_{\tau^i}^{l+1}\}}(k',\tau',\nu') \\ &\times \frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')} + \varrho(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'}))^{d+\varepsilon'}} \bigg]^q \Bigg\}^{1/q} \\ &+ C\Bigg\{ \frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) \chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu} \subset Q_\alpha^l\}}(k,\tau,\nu) \\ &\times \bigg[\sum_{k'=l+1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) |D_{k'}(f)(y_{\tau'}^{k',\nu'})| 2^{-|k-k'|\varepsilon'} \\ &\times \chi_{\{(k',\tau',\nu')\,:\,Q_{\tau'}^{k',\nu'} \cap \bigcup_{i=1}^m Q_{\tau^i}^{l+1} = \emptyset\}}(k',\tau',\nu') \\ &\times \frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')} + \varrho(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'}))^{d+\varepsilon'}} \bigg]^q \Bigg\}^{1/q} \\ = J_2^1 + J_2^2. \end{aligned}$$

For J_2^1, the inequality (2.4) if $q \leq 1$ or the Hölder inequality if $q > 1$ yields

$$\begin{aligned} J_2^1 \leq C \sum_{i=1}^m \Bigg\{ &\frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) \chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu} \subset Q_\alpha^l\}}(k,\tau,\nu) \\ &\times \bigg[\sum_{k'=l+1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) |D_{k'}(f)(y_{\tau'}^{k',\nu'})| 2^{-|k-k'|\varepsilon'} \\ &\times \chi_{\{(k',\tau',\nu')\,:\,Q_{\tau'}^{k',\nu'} \subset Q_{\tau^i}^{l+1}\}}(k',\tau',\nu') \frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')} + \varrho(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'}))^{d+\varepsilon'}} \bigg]^q \Bigg\}^{1/q} \end{aligned}$$

$$\leq \begin{cases} C\sum_{i=1}^{m}\Big\{\frac{1}{\mu(Q_{\tau^i}^{l+1})}\sum_{k'=l+1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sq}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \\ \quad\times\chi_{\{(k',\tau',\nu'):\,Q_{\tau'}^{k',\nu'}\subset Q_{\tau^i}^{l+1}\}}(k',\tau',\nu')\Big[\sum_{k=l}^{\infty}2^{(k-k')sq-|k-k'|\varepsilon' q+k'd(1-q)} \\ \quad\times\int_X\frac{2^{-(k\wedge k')\varepsilon' q}}{(2^{-(k\wedge k')}+\varrho(x,y_{\tau'}^{k',\nu'}))^{(d+\varepsilon')q}}\,d\mu(x)\Big]\Big\}^{1/q}, & q\leq 1, \\ C\sum_{i=1}^{m}\Big\{\frac{1}{\mu(Q_{\tau^i}^{l+1})}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\chi_{\{(k,\tau,\nu):\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu) \\ \quad\times\Big[\sum_{k'=l+1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sq}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q 2^{(k-k')s-|k-k'|\varepsilon'} \\ \quad\times\chi_{\{(k',\tau',\nu'):\,Q_{\tau'}^{k',\nu'}\subset Q_{\tau^i}^{l+1}\}}(k',\tau',\nu')\frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')}+\varrho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{d+\varepsilon'}}\Big] \\ \quad\times\Big[\sum_{k'=l+1}^{\infty}2^{(k-k')s-|k-k'|\varepsilon'} \\ \quad\times\int_X\frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')}+\varrho(y_\tau^{k,\nu},y))^{d+\varepsilon'}}\,d\mu(y)\Big]^{q/q'}\Big\}^{1/q}, & q>1, \end{cases}$$

$$\leq \begin{cases} C\sum_{i=1}^{m}\Big\{\frac{1}{\mu(Q_{\tau^i}^{l+1})}\sum_{k'=l+1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sq}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \\ \quad\times\chi_{\{(k',\tau',\nu'):\,Q_{\tau'}^{k',\nu'}\subset Q_{\tau^i}^{l+1}\}}(k',\tau',\nu')\Big\}^{1/q}, & q\leq 1, \\ C\sum_{i=1}^{m}\Big\{\frac{1}{\mu(Q_{\tau^i}^{l+1})}\sum_{k'=l+1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sq}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \\ \quad\times\chi_{\{(k',\tau',\nu'):\,Q_{\tau'}^{k',\nu'}\subset Q_{\tau^i}^{l+1}\}}(k',\tau',\nu')\Big[\sum_{k=l}^{\infty}2^{(k-k')s-|k-k'|\varepsilon'} \\ \quad\times\int_X\frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')}+\varrho(x,y_{\tau'}^{k',\nu'}))^{d+\varepsilon'}}\,d\mu(x)\Big]\Big\}^{1/q}, & q>1, \end{cases}$$

$$\leq C\sum_{i=1}^{m}\Big\{\frac{1}{\mu(Q_{\tau^i}^{l+1})}\sum_{k'=l+1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sq}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q$$

$$\times\chi_{\{(k',\tau',\nu'):\,Q_{\tau'}^{k',\nu'}\subset Q_{\tau^i}^{l+1}\}}(k',\tau',\nu')\Big\}^{1/q}$$

$$\leq C \sup_{l\in\mathbb{N}} \sup_{\alpha\in I_l} \left[\frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) \inf_{x\in Q_\tau^{k,\nu}\subset Q_\alpha^l} |D_k(f)(x)|^q \right]^{1/q},$$

where we used the arbitrariness of $y_{\tau'}^{k',\nu'}$ and we chose $\varepsilon' \in (0,\varepsilon)$ such that $\varepsilon' > |s|$ and $q > \max\{d/(d+\varepsilon'), d/(d+s+\varepsilon')\}$.

Finally, to finish the proof of Theorem 1, we estimate J_2^2 by first considering the case $q \leq 1$. In this case, the inequality (2.4) and Lemma 1 tell us that

$$\begin{aligned}
J_2^2 &\leq C\Bigg\{ \frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) \chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu) \\
&\quad \times \Bigg[\sum_{k'=l+1}^{\infty} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'})^q |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q 2^{-|k-k'|\varepsilon' q} \\
&\quad \times \chi_{\{(k',\tau',\nu')\,:\,Q_{\tau'}^{k',\nu'}\cap \bigcup_{i=1}^m Q_{\tau^i}^{l+1}=\emptyset\}}(k',\tau',\nu') \\
&\quad \times \frac{2^{-(k\wedge k')\varepsilon' q}}{(2^{-(k\wedge k')} + \varrho(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'}))^{(d+\varepsilon')q}} \Bigg] \Bigg\}^{1/q} \\
&\leq C\Bigg\{ \frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) \chi_{\{(k,\tau,\nu)\,:\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu) \\
&\quad \times \Bigg[\sum_{\substack{\alpha'\in I_{l+1}\\ \alpha'\neq\tau^1,\dots,\tau^m}} \sum_{k'=l+1}^{\infty} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'})^q |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q 2^{-|k-k'|\varepsilon' q} \\
&\quad \times \chi_{\{(k',\tau',\nu')\,:\,Q_{\tau'}^{k',\nu'}\subset Q_{\alpha'}^{l+1}\}}(k',\tau',\nu') \frac{2^{-(k\wedge k')\varepsilon' q}}{(2^{-(k\wedge k')} + \varrho(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'}))^{(d+\varepsilon')q}} \Bigg] \Bigg\}^{1/q} \\
&\leq C\Bigg\{ \frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \Bigg[\sum_{j=0}^{\infty} \sum_{\substack{\alpha'\in I_{l+1}\\ \alpha'\neq\tau^1,\dots,\tau^m}} \chi_{\{\alpha'\,:\,\varrho(z_{\alpha'}^{l+1}, z_\alpha^l)\sim 2^{j-l}\}}(\alpha') \\
&\quad \times \sum_{k'=l+1}^{\infty} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{(k-k')sq - |k-k'|\varepsilon' q - (k\wedge k')\varepsilon' q} 2^{k'sq} \mu(Q_{\tau'}^{k',\nu'})^q \\
&\quad \times |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \chi_{\{(k',\tau',\nu')\,:\,Q_{\tau'}^{k',\nu'}\subset Q_{\alpha'}^{l+1}\}}(k',\tau',\nu') \\
&\quad \times \int_{\{y\in Q_\alpha^l\,:\,\varrho(y, y_{\tau'}^{k',\nu'})\sim C2^{j-l}\}} \frac{1}{\varrho(y, y_{\tau'}^{k',\nu'})^{(d+\varepsilon')q}} \, d\mu(y) \Bigg] \Bigg\}^{1/q}
\end{aligned}$$

$$\leq C\Big\{\sum_{j=0}^{\infty}2^{-j(d+\varepsilon')q}\sum_{\substack{\alpha'\in I_{l+1}\\ \alpha'\neq\tau^1,\dots,\tau^m}}\chi_{\{\alpha':\,\varrho(z_{\alpha'}^{l+1},z_\alpha^l)\sim 2^{j-l}\}}(\alpha')\frac{1}{\mu(Q_{\alpha'}^{l+1})}$$

$$\times\sum_{k'=l+1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sq}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q$$

$$\times\chi_{\{(k',\tau',\nu'):\,Q_{\tau'}^{k',\nu'}\subset Q_{\alpha'}^{l+1}\}}(k',\tau',\nu')$$

$$\times\Big[\sum_{k=l}^{\infty}2^{(k-k')sq-|k-k'|\varepsilon'q+k'd(1-q)-(k\wedge k')\varepsilon'q+l[(d+\varepsilon')q-d]}\Big]\Big\}^{1/q}$$

$$\leq C\Big\{\sum_{j=0}^{\infty}2^{j[d-(d+\varepsilon')q]}\Big\}^{1/q}$$

$$\times\sup_{l\in\mathbb{N}}\sup_{\alpha\in I_l}\left[\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}\mu(Q_\tau^{k,\nu})\inf_{x\in Q_\tau^{k,\nu}\subset Q_\alpha^l}|D_k(f)(x)|^q\right]^{1/q}$$

$$\leq C\sup_{l\in\mathbb{N}}\sup_{\alpha\in I_l}\left[\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}\mu(Q_\tau^{k,\nu})\inf_{x\in Q_\tau^{k,\nu}\subset Q_\alpha^l}|D_k(f)(x)|^q\right]^{1/q},$$

where we used the arbitrariness of $y_{\tau'}^{k',\nu'}$ and we chose $\varepsilon'\in(0,\varepsilon)$ such that $\varepsilon'>s$ and $q>\max\{d/(d+\varepsilon'),d/(d+s+\varepsilon')\}$.

We now finish the estimate for J_2^2 by considering the case $q>1$. In this case, the Hölder inequality and Lemma 1 yield

$$J_2^2\leq C\Big\{\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\chi_{\{(k,\tau,\nu):\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu)$$

$$\times\Big[\sum_{k'=l+1}^{\infty}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sq}\mu(Q_{\tau'}^{k',\nu'})|D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q2^{(k-k')s-|k-k'|\varepsilon'}$$

$$\times\chi_{\{(k',\tau',\nu'):\,Q_{\tau'}^{k',\nu'}\cap\bigcup_{i=1}^m Q_{\tau^i}^{l+1}=\emptyset\}}(k',\tau',\nu')\frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')}+\varrho(y_\tau^{k,\nu},y_{\tau'}^{k',\nu'}))^{d+\varepsilon'}}\Big]$$

$$\times\Big[\sum_{k'=l+1}^{\infty}2^{(k-k')s-|k-k'|\varepsilon'}\int_X\frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')}+\varrho(y_\tau^{k,\nu},y))^{d+\varepsilon'}}\,d\mu(y)\Big]^{q/q'}\Big\}^{1/q}$$

$$\leq C\Big\{\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)}\mu(Q_\tau^{k,\nu})\chi_{\{(k,\tau,\nu):\,Q_\tau^{k,\nu}\subset Q_\alpha^l\}}(k,\tau,\nu)$$

$$
\begin{aligned}
&\times \Big[\sum_{\substack{\alpha' \in I_{l+1} \\ \alpha' \neq \tau^1, \dots, \tau^m}} \sum_{k'=l+1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{k'sq} \mu(Q_{\tau'}^{k',\nu'}) |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \\
&\times 2^{(k-k')s-|k-k'|\varepsilon'} \chi_{\{(k',\tau',\nu') : Q_{\tau'}^{k',\nu'} \subset Q_{\alpha'}^{l+1}\}}(k',\tau',\nu') \\
&\times \frac{2^{-(k\wedge k')\varepsilon'}}{(2^{-(k\wedge k')} + \varrho(y_\tau^{k,\nu}, y_{\tau'}^{k',\nu'}))^{d+\varepsilon'}} \Big]\Big\}^{1/q} \\
\le{}& C\Big\{ \frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \Big[\sum_{j=0}^{\infty} \sum_{\substack{\alpha' \in I_{l+1} \\ \alpha' \neq \tau^1, \dots, \tau^m}} \chi_{\{\alpha' : \varrho(z_{\alpha'}^{l+1}, z_\alpha^l) \sim 2^{j-l}\}}(\alpha') \\
&\times \sum_{k'=l+1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{(k-k')s-|k-k'|\varepsilon'-(k\wedge k')\varepsilon'} 2^{k'sq} \mu(Q_{\tau'}^{k',\nu'}) |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \\
&\times \chi_{\{(k',\tau',\nu') : Q_{\tau'}^{k',\nu'} \subset Q_{\alpha'}^{l+1}\}}(k',\tau',\nu') \\
&\times \int_{\{y \in Q_\alpha^l : \varrho(y, y_{\tau'}^{k',\nu'}) \sim C2^{j-l}\}} \frac{1}{\varrho(y, y_{\tau'}^{k',\nu'})^{d+\varepsilon'}} \, d\mu(y) \Big]\Big\}^{1/q} \\
\le{}& C\Big\{ \sum_{j=0}^{\infty} 2^{-j(d+\varepsilon')} \sum_{\substack{\alpha' \in I_{l+1} \\ \alpha' \neq \tau^1, \dots, \tau^m}} \chi_{\{\alpha' : \varrho(z_{\alpha'}^{l+1}, z_\alpha^l) \sim 2^{j-l}\}}(\alpha') \frac{1}{\mu(Q_{\alpha'}^{l+1})} \\
&\times \sum_{k'=l+1}^{\infty} \sum_{\tau' \in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} 2^{k'sq} \mu(Q_{\tau'}^{k',\nu'}) |D_{k'}(f)(y_{\tau'}^{k',\nu'})|^q \\
&\times \chi_{\{(k',\tau',\nu') : Q_{\tau'}^{k',\nu'} \subset Q_{\alpha'}^{l+1}\}}(k',\tau',\nu') \\
&\times \Big[\sum_{k=l}^{\infty} 2^{(k-k')s-|k-k'|\varepsilon'-(k\wedge k')\varepsilon'+l\varepsilon'} \Big]\Big\}^{1/q} \\
\le{}& C\Big\{ \sum_{j=0}^{\infty} 2^{-j\varepsilon'} \Big\}^{1/q} \\
&\times \sup_{l \in \mathbb{N}} \sup_{\alpha \in I_l} \Big[\frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) \inf_{x \in Q_\tau^{k,\nu} \subset Q_\alpha^l} |D_k(f)(x)|^q \Big]^{1/q} \\
\le{}& C \sup_{l \in \mathbb{N}} \sup_{\alpha \in I_l} \Big[\frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) \inf_{x \in Q_\tau^{k,\nu} \subset Q_\alpha^l} |D_k(f)(x)|^q \Big]^{1/q},
\end{aligned}
$$

where we used the arbitrariness of $y_{\tau'}^{k',\nu'}$ and we chose $\varepsilon' \in (0,\varepsilon)$ such that $\varepsilon' > |s|$.

This finishes the proof of Theorem 1.

REMARK 3. From the proof of Theorem 1, it is easy to see that the key role played by $\{E_k\}_{k\in\mathbb{Z}}$ is in the estimates (2.2), (2.3), (2.6) and (2.7). However, to establish these estimates, we only need to use the regularity (iii) as in Definition 3 of E_k for $k \in \mathbb{Z}_+$; see also [10, 13, 17]. This means that if we replace the operators E_k by some other operators $\overline{D}_k$ for $k \in \mathbb{Z}_+$ whose kernels have the same properties as the kernels of E_k except for the regularity (ii) of Definition 3, then the conclusion Theorem 1 still holds. This observation is useful in some applications.

From Theorem 1, Lemma 1 and the construction of the cubes $\{Q_\tau^{k,\nu} : k \in \mathbb{Z}_+, \tau \in I_k, \nu = 1,\dots,N(k,\tau)\}$, it is easy to deduce the following proposition.

PROPOSITION 1. *With the notation of Theorem* 1, *for all* $f \in (\mathring{\mathcal{G}}(\beta,\gamma))'$,

$$\max\Bigg\{\sup_{\substack{\tau\in I_0\\ \nu=1,\dots,N(0,\tau)}} m_{Q_\tau^{0,\nu}}(|D_0(f)|),\\ \sup_{l\in\mathbb{N}}\sup_{\alpha\in I_l}\bigg[\frac{1}{\mu(Q_\alpha^l)}\int_{Q_\alpha^l}\sum_{k=l}^{\infty} 2^{ksq}|D_k(f)(x)|^q\,d\mu(x)\bigg]^{1/q}\Bigg\}$$

$$\sim \max\Bigg\{\sup_{\substack{\tau\in I_0\\ \nu=1,\dots,N(0,\tau)}} m_{Q_\tau^{0,\nu}}(|E_0(f)|),\\ \sup_{l\in\mathbb{N}}\sup_{\alpha\in I_l}\bigg[\frac{1}{\mu(Q_\alpha^l)}\int_{Q_\alpha^l}\sum_{k=l}^{\infty} 2^{ksq}|E_k(f)(x)|^q\,d\mu(x)\bigg]^{1/q}\Bigg\}.$$

Proof. By Theorem 1, Lemma 1 and the construction of the cubes $\{Q_\tau^{k,\nu} : k \in \mathbb{Z}_+, \tau \in I_k, \nu = 1,\dots,N(k,\tau)\}$, we have

$$\max\Bigg\{\sup_{\substack{\tau\in I_0\\ \nu=1,\dots,N(0,\tau)}} m_{Q_\tau^{0,\nu}}(|D_0(f)|),\\ \sup_{l\in\mathbb{N}}\sup_{\alpha\in I_l}\bigg[\frac{1}{\mu(Q_\alpha^l)}\int_{Q_\alpha^l}\sum_{k=l}^{\infty} 2^{ksq}|D_k(f)(x)|^q\,d\mu(x)\bigg]^{1/q}\Bigg\}$$

$$\le \max\Bigg\{\sup_{\substack{\tau\in I_0\\ \nu=1,\dots,N(0,\tau)}} m_{Q_\tau^{0,\nu}}(|D_0(f)|),\\ \sup_{l\in\mathbb{N}}\sup_{\alpha\in I_l}\bigg[\frac{1}{\mu(Q_\alpha^l)}\sum_{k=l}^{\infty}\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} 2^{ksq}\mu(Q_\tau^{k,\nu})\sup_{x\in Q_\tau^{k,\nu}\subset Q_\alpha^l}|D_k(f)(x)|^q\bigg]^{1/q}\Bigg\}$$

$$\leq C \max\Big\{ \sup_{\substack{\tau\in I_0\\ \nu=1,\ldots,N(0,\tau)}} m_{Q_\tau^{0,\nu}}(|E_0(f)|),$$
$$\sup_{l\in\mathbb{N}} \sup_{\alpha\in I_l} \bigg[\frac{1}{\mu(Q_\alpha^l)} \sum_{k=l}^\infty \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) \inf_{x\in Q_\tau^{k,\nu}\subset Q_\alpha^l} |E_k(f)(x)|^q\bigg]^{1/q}\Big\}$$
$$\leq C \max\Big\{ \sup_{\substack{\tau\in I_0\\ \nu=1,\ldots,N(0,\tau)}} m_{Q_\tau^{0,\nu}}(|E_0(f)|),$$
$$\sup_{l\in\mathbb{N}} \sup_{\alpha\in I_l} \bigg[\frac{1}{\mu(Q_\alpha^l)} \int_{Q_\alpha^l} \sum_{k=l}^\infty 2^{ksq} |E_k(f)(x)|^q \, d\mu(x)\bigg]^{1/q}\Big\}.$$

By symmetry the proof of Proposition 1 is complete.

From Proposition 1, we deduce that the definition of the norm $\|\cdot\|_{F^s_{\infty q}(X)}$ with $|s| < \varepsilon$ and $\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < q \leq \infty$ is independent of the choice of approximations to the identity. We now verify that under some restrictions on β and γ, it is also independent of the choice of spaces of distributions.

THEOREM 2. *Let* $\varepsilon \in (0,\theta]$, $|s| < \varepsilon$ *and* $\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < q \leq \infty$. *If* $f \in (\mathring{\mathcal{G}}(\beta_1,\gamma_1))'$ *with* $\max\{0, d(1-1/q)_+ - s - d\} < \beta_1 < \varepsilon$, $0 < \gamma_1 < \varepsilon$ *and* $\|f\|_{F^s_{\infty q}(X)} < \infty$, *then* $f \in (\mathring{\mathcal{G}}(\beta_2,\gamma_2))'$ *with* $\max\{0, d(1-1/q)_+ - s - d\} < \beta_2 < \varepsilon$, $0 < \gamma_2 < \varepsilon$.

Proof. Let $h \in \mathcal{G}(\varepsilon,\varepsilon)$. With the notation of Lemma 2, we first claim that for $\tau \in I_0$ and $\nu = 1,\ldots,N(0,\tau)$,

$$|\langle \widetilde{S}_\tau^{0,\nu}, h\rangle| \leq C\|h\|_{\mathcal{G}(\beta_2,\gamma_2)} \frac{1}{(1+\varrho(y_\tau^{0,\nu},x_0))^{d+\gamma_2}}, \tag{2.8}$$

and for all $k \in \mathbb{N}$ and all $x, y \in X$,

$$|\langle \widetilde{D}_k(\cdot,y), h\rangle| \leq C2^{-k\beta_2}\|h\|_{\mathcal{G}(\beta_2,\gamma_2)} \frac{1}{(1+\varrho(y,x_0))^{d+\gamma_2}}; \tag{2.9}$$

see [10, 13, 17, 18] for the proofs.

By (2.8), (2.9) and Lemma 2, we obtain

$$|\langle f, h\rangle|$$
$$\leq C\|h\|_{\mathcal{G}(\beta_2,\gamma_2)} \Big\{ \sum_{\tau\in I_0} \sum_{\nu=1}^{N(0,\tau)} \mu(Q_\tau^{0,\nu}) m_{Q_\tau^{0,\nu}}(|D_0(f)|) \frac{1}{(1+\varrho(y_\tau^{0,\nu},x_0))^{d+\gamma_2}}$$
$$+ \sum_{k=1}^\infty 2^{-k\beta_2} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) |D_k(f)(y_\tau^{k,\nu})| \frac{1}{(1+\varrho(y_\tau^{k,\nu},x_0))^{d+\gamma_2}} \Big\}.$$

If $q \le 1$, by (2.4), (2.5) and Proposition 1, we have

$$(2.10)\quad |\langle f,h\rangle| \le C\|h\|_{\mathcal{G}(\beta_2,\gamma_2)}\|f\|_{F^s_{\infty q}(X)} \times \Big\{ \int\limits_X \frac{1}{(1+\varrho(y,x_0))^{d+\gamma_2}}\,d\mu(y) + \sum_{k=1}^{\infty} 2^{-k(\beta_2+s+d)}\Big\} \le C\|h\|_{\mathcal{G}(\beta_2,\gamma_2)}\|f\|_{F^s_{\infty q}(X)},$$

where we used the fact that $\beta_2 > -s-d$. If $q>1$, the Hölder inequality, (2.5) and Proposition 1 then tell us that

$$(2.11)\quad |\langle f,h\rangle| \le C\|h\|_{\mathcal{G}(\beta_2,\gamma_2)}\Big\{\|f\|_{F^s_{\infty q}(X)} \int\limits_X \frac{1}{(1+\varrho(y,x_0))^{d+\gamma_2}}\,d\mu(y) + \sum_{k=1}^{\infty} 2^{-k\beta_2}\Big[\sum_{\tau\in I_k}\sum_{\nu=1}^{N(k,\tau)} \mu(Q^{k,\nu}_\tau)|D_k(f)(y^{k,\nu}_\tau)|^q\Big]^{1/q} \times \Big[\int\limits_X \frac{1}{(1+\varrho(y,x_0))^{d+\gamma_2}}\,d\mu(y)\Big]^{1/q'}\Big\} \le C\|h\|_{\mathcal{G}(\beta_2,\gamma_2)}\|f\|_{F^s_{\infty q}(X)}\Big\{1+\sum_{k=1}^{\infty}2^{-k(\beta_2+s+d/q)}\Big\} \le C\|h\|_{\mathcal{G}(\beta_2,\gamma_2)}\|f\|_{F^s_{\infty q}(X)},$$

where we used the fact that $\beta_2 > -s-d/q$ in this case.

Suppose now $h \in \mathring{\mathcal{G}}(\beta_2,\gamma_2)$. We choose $h_n \in \mathcal{G}(\varepsilon,\varepsilon)$ for any $n\in\mathbb{N}$ such that

$$\|h_n - h\|_{\mathcal{G}(\beta_2,\gamma_2)} \to 0$$

as $n\to\infty$. The estimates of (2.10) and (2.11) show that for all $n,m\in\mathbb{N}$,

$$|\langle f, h_n - h_m\rangle| \le C\|f\|_{F^s_{\infty q}(X)}\|h_n-h_m\|_{\mathcal{G}(\beta_2,\gamma_2)},$$

which shows that $\lim_{n\to\infty}\langle f,h_n\rangle$ exists and is independent of the choice of h_n. Therefore, we define

$$\langle f,h\rangle = \lim_{n\to\infty}\langle f,h_n\rangle.$$

By (2.10) and (2.11), for all $h\in\mathring{\mathcal{G}}(\beta_2,\gamma_2)$,

$$|\langle f,h\rangle| \le C\|f\|_{F^s_{\infty q}(X)}\|h\|_{\mathcal{G}(\beta_2,\gamma_2)}.$$

Thus, $f\in(\mathring{\mathcal{G}}(\beta_2,\gamma_2))'$. This finishes the proof of Theorem 2.

We now introduce the space $F^s_{\infty q}(X)$.

Definition 5. Let $\varepsilon\in(0,\theta]$, $\{S_k\}_{k\in\mathbb{Z}_+}$ be an approximation to the identity of order ε as in Definition 3, $D_0 = S_0$ and $D_k = S_k - S_{k-1}$ for

$k \in \mathbb{N}$. Let $|s| < \varepsilon$,

$$\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < q \le \infty,$$

$\max\{s_+, d(1-1/q)_+ - s - d\} < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$. We define the *inhomogeneous Triebel–Lizorkin space* $F^s_{\infty q}(X)$ to be the set of all $f \in (\mathring{\mathcal{G}}(\beta,\gamma))'$ such that

$$\|f\|_{F^s_{\infty q}(X)} = \max\Big\{ \sup_{\substack{\tau\in I_0 \\ \nu=1,\dots,N(0,\tau)}} m_{Q^{0,\nu}_\tau}(|D_0(f)|), \\ \sup_{l\in\mathbb{N}} \sup_{\alpha\in I_l} \Big[\frac{1}{\mu(Q^l_\alpha)} \int_{Q^l_\alpha} \sum_{k=l}^{\infty} 2^{ksq}|D_k(f)(x)|^q\, d\mu(x)\Big]^{1/q}\Big\} < \infty,$$

where $Q^{0,\nu}_\tau$ with $\tau \in I_0$ and $\nu = 1,\dots,N(0,\tau)$ are as in the preceding section and $\{Q^l_\alpha : l \in \mathbb{N},\ \alpha \in I_l\}$ is as in Lemma 1.

Proposition 1 and Theorem 2 tell us that the definition of $F^s_{\infty q}(X)$ is independent of the choice of approximations to the identity and spaces of distributions.

REMARK 4. To guarantee that the definition of $F^s_{\infty q}(X)$ is independent of the choice of the distribution space $(\mathring{\mathcal{G}}(\beta,\gamma))'$, we only need the restriction

$$\max\{0, d(1-1/q)_+ - s - d\} < \beta < \varepsilon$$

and $0 < \gamma < \varepsilon$; see Theorem 3. However, if $s_+ < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$, we prove below that the space of test functions, $\mathcal{G}(\beta,\gamma)$, is contained in $F^s_{\infty q}(X)$. Thus, $F^s_{\infty q}(X)$ is non-empty for β and γ as in Definition 5.

PROPOSITION 2. *Let $\varepsilon \in (0,\theta]$ and $|s| < \varepsilon$.*

(i) *If* $\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < p, q \le \infty$, *then*

$$B^s_{p,\min(p,q)}(X) \subset F^s_{pq}(X) \subset B^s_{p,\max(p,q)}(X).$$

(ii) *If $f \in \mathcal{G}(\beta,\gamma)$ with $\max\{0,s\} < \beta < \varepsilon$ and $0 < \gamma < \varepsilon$, then $f \in F^s_{\infty q}(X)$ with $\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < q \le \infty$.*

Proof. The proof of (i) is trivial; see [27, Proposition 2.3.2/2, p. 47] and [29, Proposition 2.3].

Let $f \in \mathcal{G}(\beta,\gamma)$ and $\{D_k\}_{k\in\mathbb{Z}_+}$ be as in Definition 5. To verify (ii), we first claim that for all $k \in \mathbb{Z}_+$ and all $x \in X$,

$$|D_k(f)(x)| \le C2^{-k\beta}\|f\|_{\mathcal{G}(\beta,\gamma)} \frac{1}{(1+\varrho(x,x_0))^{d+\gamma}}; \tag{2.12}$$

see (2.8) and (2.9) and also [10, 13, 17, 18] for the proof.

From (2.12) and Definition 5, it follows that

$$\|f\|_{F^s_{\infty q}(X)} \le C\|f\|_{\mathcal{G}(\beta,\gamma)} \max\Big\{1, \sum_{k=1}^{\infty} 2^{k(s-\beta)q}\Big\} \le C\|f\|_{\mathcal{G}(\beta,\gamma)},$$

which finishes the proof of Proposition 2.

The following theorem gives a new characterization of the spaces $F^s_{\infty q}(X)$ when $|s| < \varepsilon$ and $1 \le q \le \infty$.

Theorem 3. *Let* $\varepsilon \in (0,\theta]$, $|s| < \varepsilon$ *and* $1 \le q \le \infty$. *Let* $\{D_k\}_{k\in\mathbb{Z}_+}$ *be as in Definition* 5. *Then* $f \in F^s_{\infty q}(X)$ *if and only if* $f \in (\mathring{\mathcal{G}}(\beta,\gamma))'$ *with* $\max\{s_+, -s-d/q\} < \beta < \varepsilon$ *and* $0 < \gamma < \varepsilon$, *and*

$$\sup_{l\in\mathbb{Z}_+} \sup_{\alpha\in I_l} \bigg[\frac{1}{\mu(Q^l_\alpha)} \int_{Q^l_\alpha} \sum_{k=l}^{\infty} 2^{ksq} |D_k(f)(x)|^q \, d\mu(x)\bigg]^{1/q} < \infty,$$

where we used the notation of Lemma 1. *Moreover*, *in this case*,

$$\|f\|_{F^s_{\infty q}(X)} \sim \sup_{l\in\mathbb{Z}_+} \sup_{\alpha\in I_l} \bigg[\frac{1}{\mu(Q^l_\alpha)} \int_{Q^l_\alpha} \sum_{k=l}^{\infty} 2^{ksq} |D_k(f)(x)|^q \, d\mu(x)\bigg]^{1/q}.$$

Proof. By the Hölder inequality and Lemma 1, it is easy to see that there is a constant $C > 0$ such that for all $f \in F^s_{\infty q}(X)$,

$$\|f\|_{F^s_{\infty q}(X)} \le C \sup_{l\in\mathbb{Z}_+} \sup_{\alpha\in I_l} \bigg[\frac{1}{\mu(Q^l_\alpha)} \int_{Q^l_\alpha} \sum_{k=l}^{\infty} 2^{ksq} |D_k(f)(x)|^q \, d\mu(x)\bigg]^{1/q}. \tag{2.13}$$

Lemma 1 again tells us that to establish the reverse inequality, it suffices to verify that there is a constant $C > 0$ such that for all $f \in F^s_{\infty q}(X)$ and $\tau \in I_0$,

$$\bigg[\frac{1}{\mu(Q^0_\tau)} \int_{Q^0_\tau} |D_0(f)(x)|^q \, d\mu(x)\bigg]^{1/q} \le C\|f\|_{F^s_{\infty q}(X)}. \tag{2.14}$$

By the construction of $\{Q^{0,\nu}_\tau : \tau \in I_0,\ \nu = 1,\dots,N(0,\tau)\}$, we can further control the left-hand side of (2.14) by

$$\begin{aligned}
&\bigg[\frac{1}{\mu(Q^0_\tau)} \int_{Q^0_\tau} |D_0(f)(x)|^q \, d\mu(x)\bigg]^{1/q} \\
&\qquad \le \bigg[\frac{1}{\mu(Q^0_\tau)} \sum_{\nu=1}^{N(0,\tau)} \mu(Q^{0,\nu}_\tau) \sup_{x\in Q^{0,\nu}_\tau} |D_0(f)(x)|^q\bigg]^{1/q} \\
&\qquad \le \sup_{\nu=1,\dots,N(0,\tau)} \sup_{x\in Q^{0,\nu}_\tau} |D_0(f)(x)|.
\end{aligned}$$

Using Lemma 2 and repeating the estimates for G_1 and G_2 as in the proof of Theorem 1, we can verify that for all $\tau \in I_0$ and $\nu = 1, \ldots, N(0,\tau)$,

$$\sup_{x \in Q_\tau^{0,\nu}} |D_0(f)(x)| \le C\|f\|_{F^s_{\infty q}(X)},$$

where $C > 0$ is independent of τ, ν and f. This shows (2.4) and completes the proof of Theorem 3.

3. Some characterizations. We first establish the frame characterization of the spaces $F^s_{\infty q}(X)$. The frame characterizations of the spaces $B^s_{pq}(X)$ and $F^s_{pq}(X)$ with $p \neq \infty$ can be found in [32, 17]. To this end, we first introduce a space of sequences, $f^s_{\infty q}(X)$. Let

$$\lambda = \{\lambda_\tau^{k,\nu} : k \in \mathbb{Z}_+,\ \tau \in I_k,\ \nu = 1, \ldots, N(k,\tau)\} \tag{3.1}$$

be a sequence of complex numbers. The space $f^s_{\infty q}(X)$ with $s \in \mathbb{R}$ and $0 < q \le \infty$ is the set of all λ as in (3.1) such that

$$\|\lambda\|_{f^s_{\infty q}(X)} = \max\Bigg\{ \sup_{\substack{\tau \in I_0 \\ \nu=1,\ldots,N(0,\tau)}} |\lambda_\tau^{0,\nu}|, \sup_{l \in \mathbb{N}} \sup_{\alpha \in I_l} \Bigg[\frac{1}{\mu(Q_\alpha^l)} \times \sum_{k=l}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} \mu(Q_\tau^{k,\nu}) |\lambda_\tau^{k,\nu}|^q \chi_{\{(k,\tau,\nu)\,:\, Q_\tau^{k,\nu} \subset Q_\alpha^l\}}(k,\tau,\nu)\Bigg]^{1/q}\Bigg\} < \infty.$$

THEOREM 4. *Let $\varepsilon \in (0,\theta]$ and $|s| < \varepsilon$. Let λ be a sequence as in* (3.1). *With the notation of Lemma* 2, *if* $\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < q \le \infty$ *and* $\|\lambda\|_{f^s_{\infty q}(X)} < \infty$, *then the series*

$$\sum_{\tau \in I_0} \sum_{\nu=1}^{N(0,\tau)} \lambda_\tau^{0,\nu} \mu(Q_\tau^{0,\nu}) \widetilde{S}_\tau^{0,\nu}(x) + \sum_{k=1}^{\infty} \sum_{\tau \in I_k} \sum_{\nu=1}^{N(k,\tau)} \lambda_\tau^{k,\nu} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k(x, y_\tau^{k,\nu}) \tag{3.2}$$

converges in $(\mathring{\mathcal{G}}(\beta,\gamma))'$ *with*

$$\max\{0, d(1-1/q)_+ - s - d\} < \beta < \varepsilon, \qquad 0 < \gamma < \varepsilon. \tag{3.3}$$

Moreover,

$$\|f\|_{F^s_{\infty q}(X)} \le C\|\lambda\|_{f^s_{\infty q}(X)}. \tag{3.4}$$

Proof. The proof is similar to that of Proposition 4.1 in [17] and Theorem 2.1 in [32]. We only give an outline. First, we need to verify that if $\|\lambda\|_{f^s_{\infty q}(X)} < \infty$, then the series in (3.2) converges in $(\mathring{\mathcal{G}}(\beta,\gamma))'$ with β and γ as in (3.3). Without loss of generality, we may assume that $I_k = \mathbb{N}$ for all

$k \in \mathbb{Z}_+$. For $L \in \mathbb{N}$, we define

$$f_L(x) = \sum_{\tau=1}^{L} \sum_{\nu=1}^{N(0,\tau)} \lambda_\tau^{0,\nu} \mu(Q_\tau^{0,\nu}) \widetilde{S}_\tau^{0,\nu}(x) + \sum_{k=1}^{L} \sum_{\tau=1}^{L} \sum_{\nu=1}^{N(k,\tau)} \lambda_\tau^{k,\nu} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k(x, y_\tau^{k,\nu}).$$

We note that by construction, $N(k,\tau)$ is always finite for all $k \in \mathbb{Z}_+$ and $\tau \in I_k$.

Let $\psi \in \mathcal{G}(\beta,\gamma)$ with $\beta,\ \gamma$ as in (3.3). For any $L_1, L_2 \in \mathbb{N}$ with $L_1 < L_2$, we write

$$\begin{aligned}
|\langle f_{L_2} - f_{L_1}, \psi\rangle| &\le \Big|\Big\langle \sum_{\tau=L_1+1}^{L_2} \sum_{\nu=1}^{N(0,\tau)} \lambda_\tau^{0,\nu} \mu(Q_\tau^{0,\nu}) \widetilde{S}_\tau^{0,\nu}(\cdot), \psi\Big\rangle\Big| \\
&\quad + \Big|\Big\langle \sum_{k=L_1+1}^{L_2} \sum_{\tau=1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} \lambda_\tau^{k,\nu} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k(\cdot, y_\tau^{k,\nu}), \psi\Big\rangle\Big| \\
&\quad + \Big|\Big\langle \sum_{k=1}^{L_1} \sum_{\tau=L_1+1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} \lambda_\tau^{k,\nu} \mu(Q_\tau^{k,\nu}) \widetilde{D}_k(\cdot, y_\tau^{k,\nu}), \psi\Big\rangle\Big| \\
&= M_1 + M_2 + M_3.
\end{aligned}$$

The estimate (2.8) and the fact that $1 + \varrho(y_\tau^{0,\nu}, x_0) \sim 1 + \varrho(y, x_0)$ for all $y \in Q_\tau^{0,\nu}$ tell us that

$$\begin{aligned}
M_1 &\le \sum_{\tau=L_1+1}^{L_2} \sum_{\nu=1}^{N(0,\tau)} |\lambda_\tau^{0,\nu}| \mu(Q_\tau^{0,\nu}) |\langle \widetilde{S}_\tau^{0,\nu}(\cdot), \psi\rangle| \\
&\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)} \sum_{\tau=L_1+1}^{L_2} \sum_{\nu=1}^{N(0,\tau)} |\lambda_\tau^{0,\nu}| \mu(Q_\tau^{0,\nu}) \frac{1}{(1 + \varrho(y_\tau^{0,\nu}, x_0))^{d+\gamma}} \\
&\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)} \|\lambda\|_{f^s_{\infty q}(X)} \int_{\bigcup_{\tau=L_1+1}^{L_2} \bigcup_{\nu=1}^{N(0,\tau)} Q_\tau^{0,\nu}} \frac{1}{(1 + \varrho(y, x_0))^{d+\gamma}}\, d\mu(y) \\
&\to 0 \quad \text{as } L_1, L_2 \to \infty,
\end{aligned}$$

since

$$\int_X \frac{1}{(1 + \varrho(y, x_0))^{d+\gamma}}\, d\mu(y) < \infty. \tag{3.5}$$

Using (2.9), (2.4) if $q \le 1$ and the Hölder inequality if $q > 1$, and the fact that

$$1 + \varrho(y_\tau^{k,\nu}, x_0) \sim 1 + \varrho(y, x_0) \tag{3.6}$$

for all $y \in Q_\tau^{k,\nu}$, we obtain

$$M_2 \le C\|\psi\|_{\mathcal{G}(\beta,\gamma)} \sum_{k=L_1+1}^{L_2} \sum_{\tau=1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} |\lambda_\tau^{k,\nu}| \mu(Q_\tau^{k,\nu}) 2^{-k\beta} \frac{1}{(1+\varrho(y_\tau^{k,\nu}, x_0))^{d+\gamma}}$$

$$\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)} \begin{cases} \displaystyle\sum_{k=L_1+1}^{L_2} 2^{-k(\beta+s+d)} \Big[\sum_{\tau=1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} |\lambda_\tau^{k,\nu}|^q\Big]^{1/q}, & q \le 1, \\ \displaystyle\sum_{k=L_1+1}^{L_2} 2^{-k(\beta+s-d/q)} \Big[\sum_{\tau=1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} |\lambda_\tau^{k,\nu}|^q\Big]^{1/q} & \\ \displaystyle\qquad \times \Big[\int_X \frac{1}{(1+\varrho(y,x_0))^{d+\gamma}}\, d\mu(y)\Big]^{1/q'}, & q > 1, \end{cases}$$

$$\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)} \|\lambda\|_{f^s_{\infty q}(X)} \begin{cases} \displaystyle\sum_{k=L_1+1}^{L_2} 2^{-k(\beta+s+d)}, & q \le 1, \\ \displaystyle\sum_{k=L_1+1}^{L_2} 2^{-k(\beta+s-d/q)}, & q > 1, \end{cases}$$

$$\to 0 \quad \text{as } L_1, L_2 \to \infty,$$

where we used (3.3) and the trivial estimate

$$\Big[\sum_{\tau=1}^{\infty} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} |\lambda_\tau^{k,\nu}|^q\Big]^{1/q} \le C\|\lambda\|_{f^s_{\infty q}(X)}. \tag{3.7}$$

From (2.9), (3.6), (2.4) if $q \le 1$ and the Hölder inequality if $q > 1$ again, it follows that

$$M_3 \le C\|\psi\|_{\mathcal{G}(\beta,\gamma)} \sum_{k=1}^{L_1} \sum_{\tau=L_1+1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} |\lambda_\tau^{k,\nu}| \mu(Q_\tau^{k,\nu}) 2^{-k\beta} \frac{1}{(1+\varrho(y_\tau^{k,\nu}, x_0))^{d+\gamma}}$$

$$\le C\|\psi\|_{\mathcal{G}(\beta,\gamma)}$$

$$\times \begin{cases} \displaystyle\sum_{k=1}^{L_1} 2^{-k(\beta+s+d)} \Big[\sum_{\tau=L_1+1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} |\lambda_\tau^{k,\nu}|^q\Big]^{1/q}, & q \le 1, \\ \displaystyle\sum_{k=1}^{L_1} 2^{-k(\beta+s-d/q)} \Big[\sum_{\tau=L_1+1}^{L_2} \sum_{\nu=1}^{N(k,\tau)} 2^{ksq} |\lambda_\tau^{k,\nu}|^q\Big]^{1/q} & \\ \displaystyle\qquad \times \Big[\int_{\bigcup_{\tau=L_1+1}^{L_2} \bigcup_{\nu=1}^{N(k,\tau)} Q_\tau^{k,\nu}} \frac{1}{(1+\varrho(y,x_0))^{d+\gamma}}\, d\mu(y)\Big]^{1/q'}, & q > 1, \end{cases}$$

$$\to 0 \quad \text{as } L_1, L_2 \to \infty,$$

where we used (3.3), the fact that if $q<\infty$, then

$$\Big[\sum_{\tau=L_1+1}^{L_2}\sum_{\nu=1}^{N(k,\tau)}2^{ksq}|\lambda_\tau^{k,\nu}|^q\Big]^{1/q}\to 0 \quad \text{as } L_1,L_2\to\infty$$

by (3.7), and the fact that if $q=\infty$, then

$$\int_{\bigcup_{\tau=L_1+1}^{L_2}\bigcup_{\nu=1}^{N(k,\tau)}Q_\tau^{k,\nu}}\frac{1}{(1+\varrho(y,x_0))^{d+\gamma}}\,d\mu(y)\to 0 \quad \text{as } L_1,L_2\to\infty$$

by (3.5).

Thus, for any given $\psi\in\mathcal{G}(\beta,\gamma)$, $\{\langle f_L,\psi\rangle\}_{L\in\mathbb{N}}$ is a Cauchy sequence, which means that the series in (3.2) converges to some $f\in(\mathring{\mathcal{G}}(\beta,\gamma))'$ with β, γ as in (3.3) if $\lambda\in f_{\infty q}^s(X)$. Moreover, by repeating the argument of Theorem 1, we can verify (3.4). This completes the proof of Theorem 4.

Combining Theorems 4 and 1, we obtain the frame characterization of the space $F_{\infty q}^s(X)$.

THEOREM 5. *Let* $\varepsilon\in(0,\theta]$, $|s|<\varepsilon$ *and* $\max\{d/(d+\varepsilon),d/(d+s+\varepsilon)\}<q\le\infty$. *With the notation of Lemma* 2, *let* $\lambda_\tau^{k,\nu}=D_k(f)(y_\tau^{k,\nu})$ *for* $k\in\mathbb{N}$, $\tau\in I_k$ *and* $\nu=1,\dots,N(k,\tau)$, *and* $\lambda_\tau^{0,\nu}=D_{\tau,1}^{0,\nu}(f)$ *for* $\tau\in I_0$ *and* $\nu=1,\dots,N(0,\tau)$. *Then* $f\in F_{\infty q}^s(X)$ *if and only if* $f\in(\mathring{\mathcal{G}}(\beta,\gamma))'$ *with* β, γ *as in Definition* 5, (2.1) *holds in* $(\mathring{\mathcal{G}}(\beta',\gamma'))'$ *with* $\beta<\beta'<\varepsilon$ *and* $\gamma<\gamma'<\varepsilon$, *and* $\lambda\in f_{\infty q}^s(X)$. *Moreover*, *in this case*,

$$\|f\|_{F_{\infty q}^s(X)}\sim\|\lambda\|_{f_{\infty q}^s(X)}.$$

Now we come to establish a connection between the inhomogeneous Triebel–Lizorkin space $F_{\infty q}^s(X)$ and the homogeneous Triebel–Lizorkin space $\dot F_{\infty q}^s(X)$ when $\mu(X)=\infty$. To this end, we first recall the definition of homogeneous approximations to the identity with compact support (see [16]).

DEFINITION 6. A sequence $\{S_k\}_{k=-\infty}^\infty$ of linear operators is said to be an *approximation to the identity* of order $\varepsilon\in(0,\theta]$ if there exist $C_5,C_6>0$ such that for all $k\in\mathbb{Z}$ and all $x,x',y,y'\in X$, $S_k(x,y)$, the kernel of S_k is a function from $X\times X$ into $\mathbb{C}$ satisfying

(i) $S_k(x,y)=0$ if $\varrho(x,y)\ge C_5 2^{-k}$ and $\|S_k\|_{L^\infty(X\times X)}\le C_6 2^{dk}$;
(ii) $|S_k(x,y)-S_k(x',y)|\le C_5 2^{k(d+\varepsilon)}\varrho(x,x')^\varepsilon$;
(iii) $|S_k(x,y)-S_k(x,y')|\le C_5 2^{k(d+\varepsilon)}\varrho(y,y')^\varepsilon$;
(iv) $|[S_k(x,y)-S_k(x,y')]-[S_k(x',y)-S_k(x',y')]|$
$$\le C_5 2^{k(d+2\varepsilon)}\varrho(x,x')^\varepsilon\varrho(y,y')^\varepsilon;$$
(v) $\int_X S_k(x,y)\,d\mu(y)=1$;
(vi) $\int_X S_k(x,y)\,d\mu(x)=1$.

The following homogeneous Triebel–Lizorkin space $\dot{F}^s_{\infty q}(X)$ was introduced in [34]. For $0 < \beta, \gamma < \varepsilon$, we define

$$\mathcal{G}_0(\beta,\gamma) = \Big\{f \in \mathcal{G}(\beta,\gamma) : \int\limits_X f(x)\,d\mu(x) = 0\Big\}.$$

DEFINITION 7. Let $\varepsilon \in (0,\theta]$ and $\{S_k\}_{k\in\mathbb{Z}}$ be an approximation to the identity of order ε as in Definition 6 and $D_k = S_k - S_{k-1}$ for $k \in \mathbb{Z}$. Let $|s| < \varepsilon$,

$$\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < q \le \infty,$$

$\max\{s_+, d(1-1/q)_+ - s - d\} < \beta < \varepsilon$, and $\max\{-s-d, s_+\} < \gamma < \varepsilon$. We define the Triebel–Lizorkin space $\dot{F}^s_{\infty q}(X)$ to be the set of all $f \in (\mathring{\mathcal{G}}_0(\beta,\gamma))'$ such that

$$\|f\|_{\dot{F}^s_{\infty q}(X)} = \sup_{l\in\mathbb{Z}}\sup_{\alpha\in I_l}\bigg\{\frac{1}{\mu(Q^l_\alpha)}\int\limits_{Q^l_\alpha}\sum_{k=l}^{\infty} 2^{ksq}|D_k(f)(x)|^q\,d\mu(x)\bigg\}^{1/q} < \infty,$$

where the notation is as in Lemma 1.

We remark that in [34], it was proved that the space defined in Definition 7 is independent of the choices of approximations to the identity and spaces of distributions with β, γ as in Definition 7.

On the relation between $\dot{F}^s_{\infty q}(X)$ and $F^s_{\infty q}(X)$, we have the following conclusion.

THEOREM 6. *Let $\varepsilon \in (0,\theta]$, $|s| < \varepsilon$ and $\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < q \le \infty$. For any $k_0 \in \mathbb{Z}$, let S_{k_0} be as in Definition* 6. *If $f \in \dot{F}^s_{\infty q}(X)$, then $f - S_{k_0}(f) \in F^s_{\infty q}(X)$ and*

$$\|f - S_{k_0}(f)\|_{F^s_{\infty q}(X)} \le C\|f\|_{\dot{F}^s_{\infty q}(X)},$$

where $C > 0$ is independent of f.

REMARK 5. If S_{k_0} in Theorem 6 does not have compact support, as in Definition 3, then the conclusion of Theorem 6 still holds. However, this needs a more complicated computation. In fact, it is easy to see that if $\psi(x,\cdot) \in \mathcal{G}(x,r,\varepsilon,\varepsilon)$ and $\psi(\cdot,y) \in \mathcal{G}(y,r,\varepsilon,\varepsilon)$ for all $x,y \in X$ and some $r > 0$, and

$$\int\limits_X \psi(x,y)\,d\mu(x) = 1,$$

then the statement of Theorem 6 is also true with S_{k_0} replaced by ψ. We leave the details to the reader.

To verify Theorem 6, we need the following discrete Calderón reproducing formula (see [12]).

LEMMA 3. *Let $\varepsilon \in (0,\theta]$, $|s| < \varepsilon$, and $\{D_k\}_{k=-\infty}^{\infty}$ be as in Definition 7. Then there is a family of functions $\{\widetilde{D}_k(x,y)\}_{k=-\infty}^{\infty}$ such that for* *all* $y_\tau^{k,\nu} \in Q_\tau^{k,\nu}$ *and all* $f \in (\mathring{\mathcal{G}}(\beta,\gamma))'$ *with* $0 < \beta, \gamma < \varepsilon$,

$$f(x) = \sum_{k\in\mathbb{Z}} \sum_{\tau\in I_k} \sum_{\nu=1}^{N(k,\tau)} \mu(Q_\tau^{k,\nu}) D_k(x, y_\tau^{k,\nu}) \widetilde{D}_k(f)(y_\tau^{k,\nu}), \tag{3.8}$$

where the series converges in $(\mathring{\mathcal{G}}(\beta',\gamma'))'$ *with* $\beta < \beta' < \varepsilon$ *and* $\gamma < \gamma' < \varepsilon$ *and*

$$\widetilde{D}_k(f)(x) = \int_X \widetilde{D}_k(x,y) f(y)\, d\mu(y).$$

Moreover, $\widetilde{D}_k(x,y)$ *for all* $k \in \mathbb{Z}$ *satisfies conditions* (i) *and* (iii) *of Definition* 3 *with* ε *replaced by* $\varepsilon' \in (0,\varepsilon)$ *and*

$$\int_X \widetilde{D}_k(x,y)\, d\mu(x) = 0 = \int_X \widetilde{D}_k(x,y)\, d\mu(y).$$

Proof of Theorem 6. In what follows, we let I be the identity operator. For simplicity, we assume that $k_0 = 0$. By (3.8), we can write

$$\begin{aligned} & S_0(I - S_0)(f)(x) \\ &\quad = \sum_{k'\in\mathbb{Z}} \sum_{\tau'\in I_{k'}} \sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'}) S_0(I - S_0) D_{k'}(x, y_{\tau'}^{k',\nu'}) \widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'}). \end{aligned} \tag{3.9}$$

Let us first verify that for all $x, y \in X$,

$$|S_0(I - S_0) D_{k'}(x,y)| \le C 2^{-|k'|\varepsilon} \frac{2^{-(0\wedge k')\varepsilon}}{(2^{-(0\wedge k')} + \varrho(x,y))^{d+\varepsilon}}. \tag{3.10}$$

To this end, we consider two cases.

Case 1: $k' \ge 0$. In this case,

$$\begin{aligned} & |S_0(I - S_0) D_{k'}(x,y)| \\ &\quad = \Big| \int_X \int_X [S_0(x,z) - S_0(x,u)] S_0(u,z) D_{k'}(z,y)\, d\mu(u)\, d\mu(z) \Big| \\ &\quad \le \Big\{ \Big| \int_X \int_X [S_0(x,z) - S_0(x,u)][S_0(u,z) - S_0(u,y)] D_{k'}(z,y)\, d\mu(u)\, d\mu(z) \Big| \\ &\qquad + \Big| \int_X [S_0(x,z) - S_0(x,y)] D_{k'}(z,y)\, d\mu(z) \Big| \Big\} \chi_{\{(x,y)\,:\,\varrho(x,y)\le C\}}(x,y) \\ &\quad \le C 2^{-k'\varepsilon} \chi_{\{(x,y)\,:\,\varrho(x,y)\le C\}}(x,y) \\ &\quad \le C 2^{-k'\varepsilon} \frac{1}{(1 + \varrho(x,y))^{d+\varepsilon}}, \end{aligned}$$

which is the desired estimate.

CASE 2: $k' < 0$. In this case,

$$
\begin{aligned}
&|S_0(I-S_0)D_{k'}(x,y)|\\
&\quad = \Big|\int_X\int_X [S_0(x,z)-S_0(x,u)]S_0(u,z)D_{k'}(z,y)\,d\mu(u)\,d\mu(z)\Big|\\
&\qquad \times \chi_{\{(x,y)\,:\,\varrho(x,y)\le C2^{-k'}\}}(x,y)\\
&\quad = \Big|\int_X\int_X [S_0(x,z)-S_0(x,u)]S_0(u,z)[D_{k'}(z,y)-D_{k'}(x,y)]\,d\mu(u)\,d\mu(z)\Big|\\
&\qquad \times \chi_{\{(x,y)\,:\,\varrho(x,y)\le C2^{-k'}\}}(x,y)\\
&\quad \le C2^{k'(d+\varepsilon)}\chi_{\{(x,y)\,:\,\varrho(x,y)\le C2^{-k'}\}}(x,y) \le C2^{k'\varepsilon}\frac{2^{-k\varepsilon'}}{(2^{-k'}+\varrho(x,y))^{d+\varepsilon}},
\end{aligned}
$$

which completes the proof of (3.10).

From (3.9), (3.10), the fact that $2^{-(0\wedge k')}+\varrho(x,y_\tau^{k,\nu}) \sim 2^{-(0\wedge k')}+\varrho(x,y)$ for all $y\in Q_{\tau'}^{k',\nu'}$, (2.4) when $q\le 1$ and the Hölder inequality when $q>1$, it follows that for all $x\in Q_\tau^{0,\nu}$,

$$
\begin{aligned}
&|S_0(I-S_0)(f)(x)|\\
&\quad \le C\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q_{\tau'}^{k',\nu'})|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})|2^{-|k'|\varepsilon}\\
&\qquad \times \frac{2^{-(0\wedge k')\varepsilon}}{(2^{-(0\wedge k')}+\varrho(x,y_{\tau'}^{k',\nu'}))^{d+\varepsilon}}\\
&\quad \le \begin{cases}
C\displaystyle\sum_{k'\in\mathbb{Z}}2^{-|k'|\varepsilon-k'(s+d)+(0\wedge k')d} \\
\qquad \times \Big[\displaystyle\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sq}|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})|^q\Big]^{1/q}, & q\le 1,\\
C\displaystyle\sum_{k'\in\mathbb{Z}}2^{-|k'|\varepsilon-k's}\Big[\displaystyle\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}2^{k'sq}|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})|^q\Big]^{1/q} \\
\qquad \times \Big[\displaystyle\int_X \frac{2^{-(0\wedge k')\varepsilon}}{(2^{-(0\wedge k')}+\varrho(x,y))^{d+\varepsilon}}\,d\mu(y)\Big]^{1/q'}, & q>1,
\end{cases}\\
&\quad \le C\|f\|_{\dot F^s_{\infty q}(X)}\begin{cases}
\displaystyle\sum_{k'\in\mathbb{Z}}2^{-|k'|\varepsilon-k'(s+d)+(0\wedge k')d}, & q\le 1,\\
\displaystyle\sum_{k'\in\mathbb{Z}}2^{-|k'|\varepsilon-k's}, & q>1,
\end{cases}\\
&\quad \le C\|f\|_{\dot F^s_{\infty q}(X)},
\end{aligned}
$$

since $|s| < \varepsilon$, where we used the estimate

$$\Big[\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')} 2^{k'sq}|\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'})|^q\Big]^{1/q} \le C\|f\|_{\dot{F}^s_{\infty q}(X)}, \tag{3.11}$$

which in fact was proved in the proof of Theorem 2.2 in [34]; see Remark 2.3 there.

Thus, for all $\tau \in I_0$ and $\nu = 1, \ldots, N(0,\tau)$, we have

$$m_{Q_\tau^{0,\nu}}(|S_0(I-S_0)(f)|) \le C\|f\|_{\dot{F}^s_{\infty q}(X)}. \tag{3.12}$$

Now for $k \in \mathbb{N}$, by (3.8), we write

$$\begin{aligned} D_k(I-S_0)(f)(x) \\ = \sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')} \mu(Q_{\tau'}^{k',\nu'})D_k(I-S_0)D_{k'}(x,y_{\tau'}^{k',\nu'})\widetilde{D}_{k'}(f)(y_{\tau'}^{k',\nu'}). \end{aligned} \tag{3.13}$$

We claim that for all $k \in \mathbb{N}$, $k' \in \mathbb{Z}$ and all $x, y \in X$,

$$|D_k(I-S_0)D_{k'}(x,y)| \le C2^{-|k-k'|\varepsilon}\,\frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\varrho(x,y))^{d+\varepsilon}}. \tag{3.14}$$

To verify (3.14), we consider three cases.

Case 1: $k > k' > 0$. In this case,

$$\begin{aligned} &|D_k(I-S_0)D_{k'}(x,y)| \\ &\quad= \Big|\int_X\int_X [D_k(x,z)-D_k(x,u)]S_0(u,z)D_{k'}(z,y)\,d\mu(u)\,d\mu(z)\Big| \\ &\quad\le \int_X |D_k(x,z)|\,|D_{k'}(z,y)-D_{k'}(x,y)|\,d\mu(z)\,\chi_{\{(x,y)\,:\,\varrho(x,y)\le C2^{-k'}\}}(x,y) \\ &\qquad+ \chi_{\{(x,y)\,:\,\varrho(x,y)\le C\}}(x,y) \\ &\qquad\times \int_X\int_X |D_k(x,u)|\,|S_0(u,z)-S_0(x,z)|\,|D_{k'}(z,y)|\,d\mu(u)\,d\mu(z) \\ &\quad\le C2^{k'(d+\varepsilon)-k\varepsilon}\chi_{\{(x,y)\,:\,\varrho(x,y)\le C2^{-k'}\}}(x,y) + C2^{-k\varepsilon}\chi_{\{(x,y)\,:\,\varrho(x,y)\le C\}}(x,y) \\ &\quad\le C2^{(k'-k)\varepsilon}\frac{2^{-k'\varepsilon}}{(2^{-k'}+\varrho(x,y))^{d+\varepsilon}}, \end{aligned}$$

which is the desired estimate.

Case 2: $k > 0 > k'$. In this case, we estimate the left-hand side of (3.14) in the following way:

$$\begin{aligned}
&|D_k(I-S_0)D_{k'}(x,y)| \\
&= \Big|\int_X\int_X [D_k(x,z)-D_k(x,u)]S_0(u,z)D_{k'}(z,y)\,d\mu(u)\,d\mu(z)\Big| \\
&\le \chi_{\{(x,y)\,:\,\varrho(x,y)\le C2^{-k'}\}}(x,y)\Big\{\int_X |D_k(x,z)|\,|D_{k'}(z,y)-D_{k'}(x,y)|\,d\mu(z) \\
&\quad + \int_X\int_X |D_k(x,u)|\,|S_0(u,z)-S_0(x,z)|\,|D_{k'}(z,y)-D_{k'}(x,y)|\,d\mu(u)\,d\mu(z)\Big\} \\
&\le C2^{k'(d+\varepsilon)-k\varepsilon}\chi_{\{(x,y)\,:\,\varrho(x,y)\le C2^{-k'}\}}(x,y) \le C2^{(k'-k)\varepsilon}\,\frac{2^{-k'\varepsilon}}{(2^{-k'}+\varrho(x,y))^{d+\varepsilon}}.
\end{aligned}$$

CASE 3: $k'\ge k>0$. In this case, we have

$$\begin{aligned}
&|D_k(I-S_0)D_{k'}(x,y)| \\
&\qquad= \Big|\int_X\int_X [D_k(x,z)-D_k(x,u)]S_0(u,z)D_{k'}(z,y)\,d\mu(u)\,d\mu(z)\Big| \\
&\qquad\le \int_X |D_k(x,z)-D_k(x,y)|\,|D_{k'}(z,y)|\,d\mu(z)\,\chi_{\{(x,y)\,:\,\varrho(x,y)\le C2^{-k}\}}(x,y) \\
&\qquad\quad + \chi_{\{(x,y)\,:\,\varrho(x,y)\le C\}}(x,y) \\
&\qquad\quad \times \int_X\int_X |D_k(x,u)|\,|S_0(u,z)-S_0(u,y)|\,|D_{k'}(z,y)|\,d\mu(u)\,d\mu(z) \\
&\qquad\le C2^{k(d+\varepsilon)-k'\varepsilon}\chi_{\{(x,y)\,:\,\varrho(x,y)\le C2^{-k}\}}(x,y) + 2^{-k'\varepsilon}\chi_{\{(x,y)\,:\,\varrho(x,y)\le C\}}(x,y) \\
&\qquad\le C2^{(k-k')\varepsilon}\,\frac{2^{-k\varepsilon}}{(2^{-k}+\varrho(x,y))^{d+\varepsilon}},
\end{aligned}$$

which completes the estimate of (3.14).

For $l\in\mathbb{N}$ and $\alpha\in I_l$, from (3.13) and (3.14), it follows that

$$\begin{aligned}
&\frac{1}{\mu(Q^l_\alpha)}\int_{Q^l_\alpha}\sum_{k=l}^{\infty} 2^{ksq}|D_k(I-S_0)(f)(x)|^q\,d\mu(x) \\
&\quad\le C\frac{1}{\mu(Q^l_\alpha)}\int_{Q^l_\alpha}\sum_{k=l}^{\infty}2^{ksq}\Big[\sum_{k'\in\mathbb{Z}}\sum_{\tau'\in I_{k'}}\sum_{\nu'=1}^{N(k',\tau')}\mu(Q^{k',\nu'}_{\tau'})2^{-|k-k'|\varepsilon}|\widetilde{D}_{k'}(f)(y^{k',\nu'}_{\tau'})| \\
&\qquad\times \frac{2^{-(k\wedge k')\varepsilon}}{(2^{-(k\wedge k')}+\varrho(x,y^{k',\nu'}_{\tau'}))^{d+\varepsilon}}\Big]^q\,d\mu(x).
\end{aligned}$$

Then, by repeating the procedure for the estimate of H_2 in the proof of Theorem 1 and using Theorem 2.2 of [34] (see also Remark 2.3 of [34] and (3.11)), we can further verify that

$$\left[\frac{1}{\mu(Q_\alpha^l)}\int_{Q_\alpha^l}\sum_{k=l}^{\infty} 2^{ksq}|D_k(I-S_0)(f)(x)|^q\,d\mu(x)\right]^{1/q} \le C\|f\|_{\dot F^s_{\infty q}(X)}. \tag{3.15}$$

Combining (3.12) and (3.15) with Definition 5 tells us that

$$\|f-S_0(f)\|_{F^s_{\infty q}(X)} \le C\|f\|_{\dot F^s_{\infty q}(X)},$$

which completes the proof of Theorem 6.

The following homogeneous Triebel–Lizorkin space $\dot F^s_{pq}(X)$ and the inhomogeneous Triebel–Lizorkin spaces $F^s_{pq}(X)$ were studied in [16, 11] and [15, 13, 18], respectively.

Definition 8. Let $\varepsilon \in (0,\theta]$, $|s|<\varepsilon$, and $\{D_k\}_{k=-\infty}^{\infty}$ be as in Definition 7. Let $\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < p < \infty$ and $\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < q \le \infty$. The homogeneous Triebel–Lizorkin space $\dot F^s_{pq}(X)$ is defined to be the set of all $f \in (\mathring{\mathcal{G}}_0(\beta,\gamma))'$ with $\max\{s_+, -s+d(1/p-1)_+\} < \beta < \varepsilon$ and $\max\{s-d/p, d(1/p-1)_+, -s+d(1/p-1)\} < \gamma < \varepsilon$ such that

$$\|f\|_{\dot F^s_{pq}(X)} = \left\|\left\{\sum_{k=-\infty}^{\infty} 2^{ksq}|D_k(f)|^q\right\}^{1/q}\right\|_{L^p(X)} < \infty.$$

The inhomogeneous Triebel–Lizorkin space $F^s_{pq}(X)$ is the set of all $f \in (\mathring{\mathcal{G}}(\beta,\gamma))'$ with

$$\max\{s_+, -s+d(1/p-1)_+\} < \beta < \varepsilon$$

and $d(1/p-1)_+ < \gamma < \varepsilon$ such that

$$\|f\|_{F^s_{pq}(X)} = \left\|\left\{|S_0(f)|^q + \sum_{k=1}^{\infty} 2^{ksq}|D_k(f)|^q\right\}^{1/q}\right\|_{L^p(X)} < \infty.$$

It was proved in [16, 11, 15, 13, 18] that the spaces $F^s_{pq}(X)$ and $\dot F^s_{pq}(X)$ in Definition 8 are independent of the choices of approximations to the identity and spaces of distributions with β, γ as in Definition 8.

Using Lemma 2 and by a similar procedure to the proof of Theorem 6, we can verify the following theorem; we omit the details.

Theorem 7. *Let* $\varepsilon \in (0,\theta]$, $|s|<\varepsilon$, $\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < p < \infty$ *and* $\max\{d/(d+\varepsilon), d/(d+s+\varepsilon)\} < q \le \infty$. *For any* $k_0 \in \mathbb{Z}$, *let* S_{k_0} *be as in Definition* 6. *If* $f \in F^s_{pq}(X)$, *then* $f - S_{k_0}(f) \in \dot F^s_{pq}(X)$ *and*

$$\|f - S_{k_0}(f)\|_{\dot F^s_{pq}(X)} \le C\|f\|_{F^s_{pq}(X)},$$

where $C > 0$ *is independent of* f.

If $s = 0$, $X = \mathbb{R}^n$, $n/(n+1) < p \le 1$ and $q = 2$, then Theorem 7 was obtained by Goldberg [7] by noting that in this case, the spaces $\dot F^0_{p2}(X)$ and $F^0_{p2}(X)$ are the classical Hardy spaces $H^p(\mathbb{R}^n)$ and $h^p(\mathbb{R}^n)$, respectively.

Next, we turn to the spaces $F^0_{p2}(X)$ with $d/(d+\varepsilon) < p \le 1$; we denote them by $h^p(X)$. If $p \in (1,\infty)$, it was proved in [15] that $F^0_{p2}(X) = L^p(X)$ with equivalent norms. In [11], it was proved that $h^p(X)$ can be characterized by the generalized Littlewood–Paley S-function, and in [14], its atomic characterization was obtained.

Lemma 4. *Let* $\varepsilon \in (0,\theta]$, $|s| < \varepsilon$ *and* $d/(d+\varepsilon) < p \le 1$.

(i) *Let* $\{D_k\}_{k\in\mathbb{Z}_+}$ *be as in Definition* 5. *Then* $f \in h^p(X)$ *if and only if* $f \in (\mathring{\mathcal{G}}(\beta,\gamma))'$ *with* $d(1/p-1) < \beta,\gamma < \varepsilon$ *and for some* $a \in (0,\infty)$, $\|S_a(f)\|_{L^p(X)} < \infty$, *where*

$$S_a(f)(x) = \Big\{\sum_{k=0}^{\infty} \int_{\{y\,:\,\varrho(x,y)\le a2^{-k}\}} 2^{kd}|D_k(f)(y)|^2\,d\mu(y)\Big\}^{1/2}. \tag{3.16}$$

Moreover, in this case, $\|f\|_{h^p(X)} \sim \|S_a(f)\|_{L^p(X)}$. *Furthermore, the operators* D_k *in* (3.16) *can be replaced by any other operators* $\overline{D}_k$ *for* $k \in \mathbb{Z}_+$ *whose kernels have the same properties as the kernels of* D_k *except for the regularity* (ii) *of Definition* 3.

(ii) $f \in h^p(X)$ *if and only if* $f \in (\mathring{\mathcal{G}}(\beta,\gamma))'$ *with* $d(1/p-1) < \beta,\gamma < \varepsilon$, *there exist a sequence* $\{\lambda_k\}_{k=1}^\infty$ *of numbers and a collection* $\{a_k\}_{k=1}^\infty$ *of* $(p,2)$*-atoms or* $(p,2)$*-blocks with* $\operatorname{diam}(\operatorname{supp} a_k) \sim 2^{-l}$ *with* $l \in \mathbb{Z}_+$ *such that*

$$f = \sum_{k=1}^{\infty} \lambda_k a_k$$

in $(\mathcal{G}(\varepsilon,\varepsilon))'$; $\sum_{k=1}^\infty |\lambda_k|^p < \infty$; a_k *is a* $(p,2)$*-atom if* $\operatorname{diam}(\operatorname{supp} a_k) \sim 2^{-l}$ *with* $l \in \mathbb{N}$, *which means that*

(a) $\operatorname{supp} a_k \subset B_k = B_k(y_k, r) = \{y \in X : \varrho(y,y_k) \le r_k\}$ *for some* $y_k \in X$ *and* $r_k \sim 2^{-l}$ *with some* $l \in \mathbb{N}$,

(b) $\|a_k\|_{L^2(X)} \le \mu(B_k)^{1/2-1/p}$,

(c) $\int_X a_k(x)\,d\mu(x) = 0$;

a_k *is a* $(p,2)$*-block if* $\operatorname{diam}(\operatorname{supp} a_k) \sim 1$, *which means* a_k *satisfies* *only* (a) *and* (b) *for some* $y_k \in X$ *and some* $r_k \sim 1$. *Moreover*,

$$\|f\|_{h^p(X)} \sim \inf \Big\{ \Big(\sum_{k=1}^{\infty} |\lambda_k|^p \Big)^{1/p} \Big\},$$

where the infimum is taken over all the above decompositions.

Definition 9. Let $\varepsilon \in (0, \theta]$ and $d/(d+\varepsilon) < p \le 1$. We define the space $\Lambda_p(X)$ to be the set of all $f \in L^p_{\mathrm{loc}}(X)$ such that

$$\begin{aligned} &\|f\|_{\Lambda_p(X)} \\ &\quad = \sup_{x \in X,\, r \sim 2^{-l},\, l \in \mathbb{N}} \Big\{ \frac{1}{\mu(B(x,r))^{2/p-1}} \int_{B(x,r)} |f(y) - m_{B(x,r)}(f)|^2 \, d\mu(y) \Big\}^{1/2} \\ &\qquad + \sup_{x \in X,\, r \sim 1} \Big\{ \int_{B(x,r)} |f(y)|^2 \, d\mu(y) \Big\}^{1/2} < \infty. \end{aligned}$$

If $p = 1$, we denote the space $\Lambda_1(X)$ by $\mathrm{bmo}(X)$.

By the standard procedure, we can verify that the dual of $h^p(X)$ is just $\Lambda_p(X)$; see the proof of Theorem 5 in [7]. We omit the details.

Proposition 3. *Let* $\varepsilon \in (0, \theta]$ *and* $d/(d+\varepsilon) < p \le 1$. *The dual of* $h^p(X)$ *is the space* $\Lambda_p(X)$, *in the sense of Lemma* 1.8 *in* [17].

Using Lemma 4 and Proposition 3 and by a similar argument to that in [19], we can verify the following proposition; we also leave the details to the reader.

Proposition 4. *The spaces* $\mathrm{bmo}(X)$ *and* $F^0_{\infty,2}(X)$ *are equal with equivalent norms.*

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