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Commutators on $(\sum \ell_q)_p$

by

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Abstract. Let T be a bounded linear operator on $X = (\sum \ell_q)_p$ with $1 \le q < \infty$ and $1 < p < \infty$. Then T is a commutator if and only if for all non-zero $\lambda \in \mathbb{C}$, the operator $T - \lambda I$ is not X-strictly singular.

1. Introduction. When studying derivations on a general Banach algebra $\mathcal{A}$, a problem that arises is to classify the commutators in the algebra, i.e., elements of the form $AB - BA$. A natural class of algebras to consider are spaces $\mathcal{L}(X)$ of all (always bounded, linear) operators on the Banach space X. After the breakthrough by Brown and Pearcy [BP] who gave a classification of the commutators in $\mathcal{L}(X)$ when X is a Hilbert space, Apostol [A1] initiated the study of commutators in $\mathcal{L}(X)$ for X a general Banach space and gave a complete classification when $X = \ell_p$, $1 < p < \infty$ [A1] and $X = c_0$ [A2], and he proved partial results for other Banach spaces. This topic was resuscitated 30+ years later by Dosev [D], who classified the commutators in $\mathcal{L}(\ell_1)$ and other spaces, and this line of investigation was continued in [DJ] and [DJS].

It seems to the authors that there are two reasons for this 30+ year gap. First, Apostol's papers, while containing the germs of many general facts, were focused on special spaces, and it is quite difficult to discern from his proofs what is needed in more general spaces X to understand the structure of commutators in $\mathcal{L}(X)$. Secondly, the geometry of most Banach spaces is much more complicated than that of ℓ_p, $1 < p < \infty$, and c_0, and this makes it much more difficult to determine which operators on them are commutators. Although the papers [D], [DJ], and [DJS], as well as this paper, are focused on classifying the commutators in $\mathcal{L}(X)$ for special spaces X, part of their value consists in building a machine that tells one for certain classes of Banach

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spaces what geometrical facts about a space are needed in order to classify the commutators on the space.

For a general Banach algebra $\mathcal{A}$, the only known obstruction for an element to be a commutator was proved in 1947 by Wintner [W]. He showed that the identity in a unital Banach algebra is not a commutator, which immediately implies that no element of the form $\lambda I + K$, where K belongs to a norm closed (proper) ideal of $\mathcal{A}$ and $\lambda \neq 0$, is a commutator in the Banach algebra $\mathcal{A}$. While in some Banach algebras there are other obstructions (such as the existence of traces), Wintner's obstruction is the only one known for $\mathcal{L}(X)$ for *any* infinite-dimensional Banach space X.

We say that a Banach space X is a *Wintner space* provided that every non-commutator in $\mathcal{L}(X)$ is of the form $\lambda I + K$, where $\lambda \neq 0$ and K lies in a proper ideal. (In [DJ] the property of being a Wintner space was called property **P**.)

Wild Conjecture. *Every infinite-dimensional Banach space is a Wintner space.*

We do not believe that this Wild Conjecture is true. In fact, there may be an infinite-dimensional Banach space such that every finite rank commutator on X has zero trace! Nevertheless, every infinite-dimensional Banach space on which the commutators are classified is a Wintner space, and the conjecture that every Banach space that admits a Pełczyński decomposition (defined below) is a Wintner space is much tamer. In this paper we verify that the spaces $Z_{p,q} := (\sum \ell_q)_p$ with $1 \leq q < \infty$ and $1 < p < \infty$ are Wintner spaces. Each of these spaces does admit a Pełczyński decomposition; in fact, it is clear that $Z_{p,q}$ is isometrically isomorphic to $(\sum Z_{p,q})_p$. Recall that, given a sequence (X_n) of Banach spaces and $p \in [1,\infty] \cup \{0\}$, $(\sum X_n)_p$ is the space of all sequences (x_n) with $x_n \in X_n$ and $\|(x_n)\| := \|(\|x_n\|)\|_p < \infty$, and the formula $\|(x_n)\| := \|(\|x_n\|)\|_p < \infty$ for $p = 0$ is used in the sense $(\|x_n\|) \in c_0$. The space X is said to have a *Pełczyński decomposition* provided X is isomorphic to $(\sum X)_p$ for some p.

2. The Main Theorem. For a Banach space X denote by S_X the unit sphere of X. We say that a linear operator between two Banach spaces $T : X \to Y$ is an *isomorphism* if T is an injective bounded linear map with closed range. If in addition T is surjective then we will say that T is an *onto isomorphism*. Let X, Y and Z be Banach spaces. An operator from X to Y is said to be Z-*strictly singular* provided that there is no subspace Z_0 of X which is isomorphic to Z for which $T|_{Z_0}$ is an isomorphism. Thus an operator is strictly singular in the usual sense if and only if it is Z-strictly singular for every infinite-dimensional space Z. For any two subspaces (possibly not closed) M and N of a Banach space X let $d(M,N) = \inf\{\|m-n\| : m \in S_M,\, n \in N\}$, so that when $M \cap N = \{0\}$, the projection from $M+N$ onto M with kernel N has norm $d(M,N)^{-1}$.

Main Theorem. *Let T be an operator on $X := Z_{p,q}$, $1 \le q < \infty$, $1 < p < \infty$. Then T is a commutator if and only if for all non-zero $\lambda \in \mathbb{C}$ the identity on X factors through $T - \lambda I$. Consequently, X is a Wintner space.*

Remark 2.1. For $p = q = 2$, the Main Theorem is of course a restatement of the classical Brown–Pearcy theorem [BP]. The case when $p = q$ was proved by C. Apostol [A1]. So, by duality (see the proof of Corollary 2.17), it is enough to look at $Z_{p,q}$ when $1 \le q < p < \infty$.

The strategy for proving the Main Theorem is the same as that in [D], [DJ], and [DJS]. The main problem is to prove structural results for $Z_{p,q}$, $1 \le q < p < \infty$, so that [DJ] can be applied. To get started, we show in Proposition 2.9 that the $Z_{p,q}$-strictly singular operators coincide with the set $\mathcal{M}_{Z_{p,q}}$ of those operators T on $Z_{p,q}$ such that the identity on $Z_{p,q}$ does not factor through T. We also need that $\mathcal{M}_{Z_{p,q}}$ is closed under addition, so that $\mathcal{M}_{Z_{p,q}}$ is the largest (proper) ideal in $\mathcal{L}(\mathcal{M}_{Z_{p,q}})$. This is part of Proposition 2.9.

We begin with a discussion of how isomorphic copies of ℓ_q in $Z_{p,q}$, $1 \le q < p < \infty$, are situated in $Z_{p,q}$. We are primarily interested in passing to a subspace which is situated in a canonical fashion. Much of what we need is known and for $Z_{p,2}$ is partly contained in [O]. We do not assume familiarity with arguments involving $Z_{p,q}$, but we do assume a basic knowledge of techniques using block basic sequences, gliding hump arguments, small perturbations of operators, and how they are applied in the study of ℓ_q. This material can be found in standard texts, including [LT1, Chapter 1]. This allows us in many places to avoid writing long strings of inequalities when an argument is standard.

First, if X is a subspace of $Z_{p,q}$ that is isomorphic to ℓ_q, then for all $\varepsilon > 0$ there is a subspace Y of X that is $(1+\varepsilon)$-isomorphic to ℓ_q. This follows from a general result of Krivine and Maurey [KM] about stable Banach spaces, but can be proved in an elementary way using James' [J] proof of the non-distortability of the norm on ℓ_1. Indeed, by passing to a subspace and making a small perturbation, we can assume that $X = \overline{\operatorname{span}}\, x_n$ with (x_n) a normalized block basis of the usual basis for $Z_{p,q}$ and where (x_n) is equivalent to the usual basis for ℓ_q. James' argument shows that there is a normalized block basis (y_n) of (x_n) such that for all scalar sequences (a_n),

$$\Big\| \sum_n a_n y_n \Big\| \ge (1+\varepsilon)^{-1} \Big(\sum_n |a_n|^q \Big)^{1/q}. \tag{2.1}$$

Let us recall the argument for (2.1): Suppose that (x_n) is a normalized basic sequence in some Banach space satisfying (2.1) with $1+\varepsilon$ replaced by $K>1$. Partition $\mathbb{N}$ into infinitely many disjoint infinite sets $(\mathbb{N}_k)$. It may be that for some k and all scalars (a_n),

$$\Big\|\sum_{n\in\mathbb{N}_k} a_n x_n\Big\| \ge K^{-1/2}\Big(\sum_{n\in\mathbb{N}_k}|a_n|^q\Big)^{1/q}.$$

If not, choose for each k a finitely non-zero sequence $(b_n)_{n\in\mathbb{N}_k}$ so that $y_k := \sum_{n\in\mathbb{N}_k} b_n x_n$ has norm one and $(\sum_{n\in\mathbb{N}_k}|b_n|^q)^{1/q} > K^{1/2}$. It is easy to check that the sequence (y_k) satisfies (2.1) with $1+\varepsilon$ replaced by $K^{1/2}>1$. Iterating, we get a normalized sequence (y_n) in $\operatorname{span} x_n$ which is disjointly supported with respect to (x_n) and which satisfies (2.1). Finally, pass to any subsequence of (y_n) that is a true block basis of (x_n). This does it, because if (z_k) is a disjoint sequence in $Z_{p,q}$, then $\|\sum_k z_k\| \le (\sum_k \|z_k\|^q)^{1/q}$.

If $T: X\to Y$ is an operator between Banach spaces and Z is a subspace of X, define

$$f(T,Z) = \inf\{\|Tz\| : z\in Z,\ \|z\|=1\} \quad (= \|T_{|Z}^{-1}\|^{-1}). \tag{2.2}$$

Then $f(T,Z)>0$ iff $T_{|Z}$ is an isomorphism; $f(T,Z)=\|T\|>0$ iff $T_{|Z}$ is a multiple of an isometry; and $\|T\|\ge f(T,Z_1)\ge f(T,Z_2)$ if $Z_1\subset Z_2\subset X$.

LEMMA 2.2. *Let T be an operator from ℓ_q into $Z_{p,q}$, $1\le q<p<\infty$. Then, for all $\varepsilon>0$, there exists a block subspace Z of ℓ_q (i.e. Z is the closed linear span of a block basis of the unit vector basis for ℓ_q) which is isometric to ℓ_q and such that $\|T|_Z\| \le f(T,Z)+\varepsilon$.*

Proof. If T is strictly singular, which is to say that $f(T,Z)=0$ for all infinite-dimensional subspaces Z of ℓ_q, then this is a standard textbook exercise. So we can assume, by passing to a suitable block subspace spanned by a block basis of the unit vector basis (δ_n) of ℓ_q, that T is an isomorphism. Using the fact that subspaces of $Z_{p,q}$ which are isomorphic to ℓ_q contain smaller subspaces almost isometric to ℓ_q, and keeping in mind that the $\varepsilon>0$ gives wiggle room, Lemma 2.2 reduces to the case where the operator maps ℓ_q into an isometric copy of ℓ_q, which of course is easy and is contained e.g. in [AK, Section 2.1]. ∎

LEMMA 2.3. *Let $T: Z_{p,q}\to Z_{p,q}$ $(1\le q<p<\infty)$ be an operator. Then, for every positive integer m, $\lim_{k\to\infty}\|(P_{[1,m]}T)|_{P_{[k,\infty)}Z_{p,q}}\|=0$, where $P_{[m,n]}$ is the projection from $Z_{p,q}$ onto the direct sum from the mth ℓ_q to the nth ℓ_q.*

Proof. Suppose not. Then there exist a positive integer m, a positive number δ and a normalized block basis (x_n) of the natural basis for $Z_{p,q}$ which is equivalent to the unit vector basis of ℓ_p and such that $\|P_{[1,m]}Tx_n\| \ge \delta$. By passing to a subsequence of (x_n), we may assume that $(P_{[1,m]}Tx_n)$ is equivalent to the unit vector basis of ℓ_q. This yields an obvious contradiction since $q < p$ and T is bounded. ∎

LEMMA 2.4. *Let T be an operator from ℓ_q into $Z_{p,q}$ $(1 \le q < p < \infty)$. Then, for all $\varepsilon > 0$, there exist a positive integer N and a block subspace X of ℓ_q which is isometric to ℓ_q and such that $\|(P_{[N,\infty)}T)|_X\| < \varepsilon$.*

Proof. If T is strictly singular then there is a normalized block basis (x_n) of the unit vector basis (δ_n) of ℓ_q such that $\|T_{|\operatorname{span} x_n}\| < \varepsilon$, and we are done. Otherwise, by passing to a suitable block subspace of (δ_n), we can assume that T is an isomorphism and hence $f(T, \ell_q) > 0$. By Lemma 2.2, for a value of $\delta = \delta(\varepsilon)$ to be specified momentarily, we can pass to another block subspace, say Z, such that

$$\|T_{|Z}\| < f(T,Z) + \delta f(T,\ell_q) \le (1+\delta) f(T,Z),$$

and, by replacing T with $\|T_{|Z}\|^{-1}T$, also $\|T_{|Z}\| = 1$. Moreover, just as in Lemma 2.2, we can assume that $T\delta_n$ are disjointly supported in $Z_{p,q}$. This reduces to the case where $\|T\| = 1$ and $f(T,\ell_q) > (1+\delta)^{-1}$.

Now if $\|P_{[N,\infty)}T_{|\operatorname{span}(\delta_k)_{k=n}^{\infty}}\| > \varepsilon$ for all N and n, we get $N_1 < N_2 < \cdots$ and a normalized block basis (x_n) of (δ_n) such that for all k,

$$\|P_{[N_k,N_{k+1})}Tx_k\| > \varepsilon.$$

Keeping in mind that (Tx_n) is disjointly supported and thus has an upper q estimate and a lower p estimate, we see that for all m,

$$\begin{aligned}
&(1+\delta)^{-1}m^{1/q} \\
&\quad\le \Big\|\sum_{k=1}^{m} Tx_k\Big\| \le \Big\|\sum_{k=1}^{m}(I - P_{[N_k,N_{k+1})})Tx_k\Big\| + \Big\|\sum_{k=1}^{m} P_{[N_k,N_{k+1})}Tx_k\Big\| \\
&\quad\le \Big(\sum_{k=1}^{m}\|(I-P_{[N_k,N_{k+1})})Tx_k\|^q\Big)^{1/q} + \Big(\sum_{k=1}^{m}\|P_{[N_k,N_{k+1})}Tx_k\|^p\Big)^{1/p} \\
&\quad\le \Big(\sum_{k=1}^{m}(\|Tx_k\|^p - \|P_{[N_k,N_{k+1}}Tx_k\|^p)^{q/p}\Big)^{1/q} + m^{1/q} \\
&\quad\le \Big(\sum_{k=1}^{m}(1-\varepsilon^p)^{q/p}\Big)^{1/q} + m^{1/p} = m^{1/q}(1-\varepsilon^p)^{1/p} + m^{1/p},
\end{aligned}$$

which gives a contradiction if $(1+\delta)(1-\varepsilon^p)^{1/p} < 1$. ∎

We also need that copies of ℓ_q in $Z_{p,q}$ contain almost isometric copies of ℓ_q which are almost norm one complemented in $Z_{p,q}$. This can be done using the special structure of $Z_{p,q}$, but in fact it follows from the general results of Lemmas 2.5 and 2.6, which were proved by G. Schechtman and the second author recently when they discussed a preliminary version of this paper (and probably also thirty years ago), and the lemmas may well be somewhere in the literature. We state the lemmas for spaces with an unconditional basis, but the same proofs (modulo incorporating some standard theory of Banach lattices into the proof) yield the same result for general Banach lattices. In the proofs we assume the reader is familiar with the notions of p-convex and p-concave function lattices and the related notions of p-convexification and p-concavification of spaces with a monotone unconditional basis; see, e.g., [LT2, 40–58].

LEMMA 2.5 (W. Johnson and G. Schechtman). *Suppose that X has an unconditionally monotone basis with p-convexity constant one, and $(x_k)_{k=1}^n$ ($n \in \mathbb{N} \cup \{\infty\}$) is a disjoint sequence in X such that for some $0 < \theta < 1$ and all scalars (α_k),*

$$\theta\Big(\sum_k |\alpha_k|^p\Big)^{1/p} \le \Big\|\sum_k \alpha_k x_k\Big\| \le \Big(\sum_k |\alpha_k|^p\Big)^{1/p}. \tag{2.3}$$

Then there is an unconditionally monotone norm $!\cdot!$ on X with p-convexity constant one such that for all scalars (α_k),

(1) $\theta !x! \le \|x\| \le !x!$ *for all* $x \in X$*;*
(2) $(\sum_k |\alpha_k|^p)^{1/p} = !\sum_k \alpha_k x_k!$.

Proof. Without loss of generality we assume that $x_k \ge 0$ for all k so that the closed span of (x_k) is a sublattice of X. Assume first that $p = 1$. By the lattice version of the Hahn–Banach theorem and the hypothesis on (x_k) there is a linear functional $x^* \ge 0$ on X with $\|x^*\| \le \theta^{-1}$ so that $\langle x^*, x_k\rangle = 1$ for all k. Define $!\cdot!$ on X by $!x! := \|x\| \vee \langle x^*, |x|\rangle$. This clearly does the job. In the general case, apply the case $p = 1$ to the p-concavification of X and take the p-convexification of the resulting norm. ■

LEMMA 2.6 (W. Johnson and G. Schechtman). *Suppose that X has an unconditionally monotone basis with p-convexity constant one* ($1 \le p < \infty$), *and $(x_k)_{k=1}^n$ ($n \in \mathbb{N} \cup \{\infty\}$) is a disjoint sequence of unit vectors in X which is isometrically equivalent to the unit vector basis for ℓ_p. Then $\overline{\operatorname{span}}\, x_k$ is norm one complemented in X.*

Proof. Since the unit ball of ℓ_p is weak* compact, the case $n = \infty$ follows from the case $n < \infty$, so we assume $n < \infty$. We can also assume that $x_k \ge 0$ for all k and that the union of the supports of the x_k is the entire unconditional basis for X.

First proof. The idea is to situate X between $L_1(\mu)$ and $L_\infty(\mu)$ with μ a probability measure so that both inclusions have norm one. Since X has p-convexity constant one, it then follows from an argument in [JMST, p. 14] that in fact $L_p(\mu) \supset X$ with the inclusion having norm one. We set this up so that $\sum_k x_k$ is the constant $n^{1/p}$ function and the norm of each x_k in

$L_1(\mu)$ is $n^{1/p}/n = n^{-1/p'}$; this forces the $L_p(\mu)$ norm of each x_k to be one. Since in X the sequence (x_k) is 1-equivalent to the unit vector basis for ℓ_p^n, the injection $I_{X,p}$ from X into $L_p(\mu)$ is an isometry on $\operatorname{span}(x_k)_{k=1}^n$. But of course $\operatorname{span}(x_k)_{k=1}^n$ is norm one complemented in $L_p(\mu)$ and hence also in X.

To effect this situation, use the lattice version of the Hahn–Banach theorem to get $x^* \geq 0$ in X^* such that for each k, $\langle x^*, x_k\rangle = n^{-1/p'}$. Define a seminorm on X by $\|x\|_1 := \langle x^*, |x|\rangle$. This is an L_1 (semi)norm on X and the inclusion from X into this L_1 space has norm one. The L_∞ structure on X is defined by specifying $n^{-1/p}\sum_k x_k$ to be the constant one function; i.e., by taking the unit ball to be those vectors x in X such that $|x| \leq n^{-1/p}\sum_k x_k$.

Second proof. As in the proof of Lemma 2.5, we use p-concavification to reduce to the case $p = 1$, but in a different way. In the p-concavification $X^{(1/p)}$ of X, the sequence (x_k^p) is a disjoint sequence that is 1-equivalent to the unit vector basis of ℓ_1^n, so there is a norm one functional $x^* \geq 0$ in $(X^{(1/p)})^*$ such that $\langle x^*, x_k^p\rangle = 1$ for all k. The (semi)norm $\|x\|_p := \langle x^*, |x|^p\rangle^{1/p}$ turns X into an abstract L_p space and (x_k) are disjoint unit vectors in this abstract L_p space, hence in it (x_k) is 1-equivalent to the unit vector basis for ℓ_p^n and $\operatorname{span}(x_k)$ is norm one complemented (either do a direct argument or use the deeper fact [LT2, Theorem 1.b.2] that an abstract L_p space is isometrically lattice isomorphic to $L_p(\mu)$ for some measure μ). Since $\|\cdot\|_p \leq \|\cdot\|_X$ and in X the sequence (x_k) is 1-equivalent to the unit vector basis for ℓ_p^n, we conclude that $\operatorname{span}(x_k)$ is also norm one complemented in X. ∎

LEMMA 2.7. *Let X be a subspace of $Z_{p,q}$, $1 \leq q < p < \infty$, which is isomorphic to $Z_{p,q}$. Then for all $\varepsilon > 0$, there is a subspace Y of X that is $(1+\varepsilon)$-isomorphic to $Z_{p,q}$ and $(1+\varepsilon)$-complemented in $Z_{p,q}$.*

Proof. Write $X = \sum_k X_k$ where each X_k is isomorphic to ℓ_q and the sum is (isomorphically) an ℓ_p-sum. By the remarks at the beginning, we can assume by passing to subspaces of each X_k that X_k has a normalized basis $(x_{n,k})_{n=1}^\infty$ that is $(1+\varepsilon_k)$-equivalent to the unit vector basis of ℓ_q with $\varepsilon_k \downarrow 0$ as fast as we like. Also, by doing a small perturbation we can assume that $(x_{n,k})_{n,k}$ are disjointly supported with respect to the canonical basis for $Z_{p,q}$. Finally, using Lemmas 2.3 and 2.4 we can assume, by passing to a subsequence of subspaces of (X_k), that there are $N_1 < N_2 < \cdots$ such that for all k, $\|P_{[N_k,N_{k+1})}x - x\| \leq \varepsilon_k\|x\|$ for all x in X_k. Doing one more perturbation, we might as well assume in fact that $P_{[N_k,N_{k+1})}$ is the identity on X_k. Using Lemmas 2.5 and 2.6, we get a projection Q_k from $P_{[N_k,N_{k+1})}Z_{p,q}$ onto X_k with $\|Q_k\| \leq 1+\varepsilon_k$. Then $\sum_k Q_k P_{[N_k,N_{k+1})}$ is a projection from $Z_{p,q}$ onto X of norm at most $1+\varepsilon_1$. ∎

Remark. Note that the argument for Lemma 2.7 also shows that if T is a $Z_{p,q}$-strictly singular operator on $Z_{p,q}=(\sum Y_k)_p$ with each Y_k isometrically isomorphic to ℓ_q, then $f(T,Y_k)\to 0$ as $k\to\infty$.

Proposition 2.8. *Let T be a $Z_{p,q}$-strictly singular operator on $Z_{p,q}$, $1\le q<p<\infty$. Then for all $\varepsilon>0$, there is a subspace X of $Z_{p,q}$ which is isometrically isomorphic to $Z_{p,q}$ and such that $\|T|_X\|<\varepsilon$. Consequently, the set of $Z_{p,q}$-strictly singular operators on $Z_{p,q}$ is a linear subspace of $L(Z_{p,q})$.*

Proof. As usual, write $Z_{p,q}=(\sum Y_k)_p$ with each Y_k isometrically isomorphic to ℓ_q. By passing to subspaces of each Y_k, we can by Lemma 2.2 assume that for each k, $\|T_{|Y_k}\|<f(T,Y_k)+2^{-k}$, so that $\|T_{|Y_k}\|\to 0$ by the Remark after Lemma 2.7. Let $X=(\sum_n Y_{k_n})_p$, where $k_n\uparrow$ is chosen so that $\sum_n \|T_{|Y_{k_n}}\|<\varepsilon$. ■

Proposition 2.9 below is an immediate consequence of Lemma 2.7 and Proposition 2.8. Actually Proposition 2.8 says that the set of all $Z_{p,q}$-strictly singular operators on $Z_{p,q}$ is an ideal and Lemma 2.7 tells us that it is maximal.

Proposition 2.9. *Let $1\le q<p<\infty$. The set of all $Z_{p,q}$-strictly singular operators on $Z_{p,q}$ is equal to $\mathcal{M}_{Z_{p,q}}$ and forms the unique maximal ideal in $\mathcal{L}(Z_{p,q})$.*

Let $(X_i)_{i=0}^\infty$ be a sequence of Banach spaces. In our case, all the (X_i)'s are uniformly isomorphic to $Z_{p,q}$ so that their ℓ_p direct sum $(\sum_{i=0}^\infty X_i)_p$ is isomorphic to $(\sum_{i=0}^\infty Z_{p,q})_p$, which in turn is isometrically isomorphic to $Z_{p,q}$. We are interested in the case when (X_i) is a sequence of subspaces of $Z_{p,q}$ which are uniformly isomorphic to $Z_{p,q}$, $\operatorname{span}\bigcup_{i=0}^\infty X_i$ is dense in $Z_{p,q}$ and the mapping that identifies $X_i, 0\le i<\infty$, in $Z_{p,q}$ with X_i in $(\sum_{j=0}^\infty X_j)_p$ extends to an isomorphism from $Z_{p,q}$ onto $(\sum_{j=0}^\infty X_j)_p$. Since commutators are preserved under similarity transformations, without confusion we will identify an operator on $(\sum_{i=0}^\infty X_i)_p$ with the corresponding operator on $\overline{\operatorname{span}}\,(X_i)_{i=0}^\infty=Z_{p,q}$. For $x=(x_i)\in(\sum_{i=0}^\infty X_i)_p$ with $x_i\in X_i$, we define the right and left shifts as follows:

$$R(x)=(0,x_0,x_1,\ldots),\qquad L(x)=(x_1,x_2,\ldots).$$

Let $\mathcal{A}=\{T\in\mathcal{L}((\sum_{i=0}^\infty X_i)_p):\sum_{n=0}^\infty R^nTL^n \text{ is strongly convergent}\}$. By Lemma 3 in [D], if T is in $\mathcal{A}$, then T is a commutator. The proof of the next theorem shows that if T is a $Z_{p,q}$-strictly singular operator on $Z_{p,q}$ then there is an ℓ_p-decomposition of $Z_{p,q}$ such that T is in $\mathcal{A}$ and hence is a commutator.

Theorem 2.10. *Let $1\le q<p<\infty$. If $T:Z_{p,q}\to Z_{p,q}$ is $Z_{p,q}$-strictly singular, then T is a commutator.*

Proof. We first make a partition of the natural numbers to infinitely many infinite subsets, denoted by $\mathbb{N}=\bigcup_{n=0}^\infty I_n$. For each $n\ge 1$, $(\sum_{i\in I_n} Z_{p,q})_p$

is isometric to $Z_{p,q}$ and the restriction of T to $\sum_{i\in I_n} Z_{p,q}$ is $Z_{p,q}$-strictly singular. By Proposition 2.8, for each n there is a subspace X_n of $\sum_{i\in I_n} Z_{p,q}$ such that X_n is isometric to $Z_{p,q}$ and $\|T|_{X_n}\| < \varepsilon_n$. Moreover, by Lemma 2.6 X_n can be chosen 1-complemented in $\sum_{i\in I_n} Z_{p,q}$. By passing to appropriate subspaces of X_n, we can assume that the complement Z_n to X_n in $\sum_{i\in I_n} Z_{p,q}$ contains a 1-complemented subspace that is uniformly isomorphic (actually even isometric) to $Z_{p,q}$ and hence Z_n is uniformly isomorphic to $Z_{p,q}$ by the Pełczyński decomposition method [LT1, p. 54] (the decomposition method applies because $Z_{p,q}$ is isometric to $(\sum_{i=0}^\infty Z_{p,q})_p$). So we get a sequence $(X_n)_{n=1}^\infty$ of subspaces of $(\sum_{n=0}^\infty Z_{p,q})_{\ell_p}$ such that

(1) X_n is isometric to $Z_{p,q}$ and 1-complemented in $Z_{p,q}$;
(2) $\|T|_{X_n}\| < \varepsilon_n$;
(3) $\|\sum_{n=1}^\infty x_n\| = (\sum_{n=1}^\infty \|x_n\|^p)^{1/p}$ for all $x_n \in X_n$;
(4) $Z_{p,q} = (\sum_{n=1}^\infty X_n)_p \oplus X_0$ and X_0 is isomorphic to $Z_{p,q}$.

Let R and L be the right shift and left shift with respect to the ℓ_p-decomposition $(X_n)_{n=0}^\infty$ of $Z_{p,q}$. Then it is easy to see [D, Lemma 3] that $\sum_{n=0}^\infty R^n T L^n$ is strongly convergent if $\sum_n \varepsilon_n < \infty$. ∎

The following lemma is a direct consequence of Lemmas 2.3 and 2.4. We omit the proof.

Lemma 2.11. *Let T be an operator on $Z_{p,q}$, $1 \le q < p < \infty$. Then for any sequence (ε_i) of positive numbers, there exist an infinite subset M of positive integers and a subspace Z_i of the ith ℓ_q which is isometric to ℓ_q for all $i \in M$ and such that for all $k \in M$, $\|\sum_{i\in M,\, i\ne k} P_i T|_{Z_k}\| < \varepsilon_k$, where P_i is the projection from $Z_{p,q}$ onto the ith ℓ_q.*

Let T be an operator on a Banach space X. The *left essential spectrum* of T is defined to be the set

$$\sigma_{\mathrm{l.e.}}(T) = \{\lambda \in \mathbb{C} : \inf_{x\in S_Y} \|(\lambda - T)x\| = 0 \text{ for all } Y \subset X \text{ with } \operatorname{codim} Y < \infty\}.$$

Apostol proved in [A1] that $\sigma_{\mathrm{l.e.}}(T)$ is non-empty for all operators T on any infinite-dimensional X.

Proposition 2.12. *Let T be an operator on $Z_{p,q}$, $1 \le q < p < \infty$. Then either there is a $\lambda \in \mathbb{C}$ and a subspace Y of $Z_{p,q}$ that is isomorphic to $Z_{p,q}$ and $(T - \lambda I)|_Y$ is $Z_{p,q}$-strictly singular, or there is a $\lambda \in \mathbb{C}$ and a subspace Y of $Z_{p,q}$ that is isomorphic to $Z_{p,q}$ and such that $(T - \lambda I)|_Y$ is an isomorphism and $d((T - \lambda I)(Y), Y) > 0$.*

Proof. Let (ε_i) be a sequence of positive reals decreasing to 0 fast. Let $(\widetilde{Z}_i)_{i\in M}$ be a sequence of subspaces satisfying the conclusion of Lemma 2.11. By Lemma 2.6, for each $i \in M$, $\widetilde{Z}_i$ is 1-complemented in the ith ℓ_q. Let

$\widetilde{R}_i$ be a contractive projection from the ith ℓ_q onto $\widetilde{Z}_i$. Let P_i be the natural projection from $Z_{p,q}$ onto the ith ℓ_q and set $\widetilde{Q}_i = \widetilde{R}_i P_i$. Since for all $k \in M$, $\|\sum_{i\in M, i\neq k} P_i T|_{\widetilde{Z}_k}\| < \varepsilon_k$ and $\widetilde{Q}_i P_i = \widetilde{Q}_i$, it follows that for all $k \in M$, $\|\sum_{i\in M, i\neq k} \widetilde{Q}_i T|_{\widetilde{Z}_k}\| < \varepsilon_k$. For each $i \in M$, consider the operator $\widetilde{Q}_i T|_{\widetilde{Z}_i} : \widetilde{Z}_i \to \widetilde{Z}_i$. Let λ_i be any number in $\sigma_{\mathrm{l.e.}}(\widetilde{Q}_i T|_{\widetilde{Z}_i})$. Then there is a subspace Z_i of $\widetilde{Z}_i$ which is isometric to ℓ_q and hence 1-complemented in the ith ℓ_q such that $\|(\widetilde{Q}_i T - \lambda_i I)|_{Z_i}\| < \varepsilon_i$. Since (λ_i) is uniformly bounded by $\|T\|$, by taking a limit of a subsequence of (λ_i) and passing to an infinite subset J of M, we get a complex number λ such that for all $i \in J$, $\|(\widetilde{Q}_i T - \lambda I)|_{Z_i}\| < \varepsilon_i$. Let R_i be a contractive projection from the ith ℓ_q onto Z_i and let $Q_i = R_i P_i$. Then for all $k \in J$, $\|\sum_{i\in J, i\neq k} Q_i T|_{Z_k}\| < \varepsilon_k$ and $\|(Q_k T - \lambda I)|_{Z_k}\| < \varepsilon_k$. This immediately implies that for all $k \in J$, $\|((\sum_{i\in J} Q_i T) - \lambda I)|_{Z_k}\| < 2\varepsilon_k$. Let $\widetilde{Y} = (\sum_{i\in J} Z_i)_p$. Then $Q = \sum_{i\in J} Q_i$ is a contractive projection from $Z_{p,q}$ onto $\widetilde{Y}$ and $\|(QT - \lambda I)|_{\widetilde{Y}}\| < 2\sum_i \varepsilon_i$. Since $\|(QT - \lambda I)|_{(\sum_{i\in J, i\geq k} Z_i)_p}\| < \sum_{i\geq k} \varepsilon_i$, it is straightforward to check that $(QT - \lambda I)|_{\widetilde{Y}}$ is $Z_{p,q}$-strictly singular if $(\varepsilon_i)_i$ is summable. Now consider the operator $((I - Q)T)|_{\widetilde{Y}}$. If it is $Z_{p,q}$-strictly singular, then $(T - \lambda I)|_{\widetilde{Y}} = (QT - \lambda I)|_{\widetilde{Y}} + ((I - Q)T)|_{\widetilde{Y}}$ is $Z_{p,q}$-strictly singular by Proposition 2.8. If it is not $Z_{p,q}$-strictly singular, then $(T - \lambda I)|_{\widetilde{Y}}$ is not $Z_{p,q}$-strictly singular and hence there is a subspace Y of $\widetilde{Y}$ that is isomorphic to $Z_{p,q}$ and such that for some $\mu > 0$ and all norm one vectors y in Y, $\|(T - \lambda I)y\| > \mu$. By Lemma 2.7 we may assume that Y is $1 + \varepsilon$-isomorphic to $Z_{p,q}$ and by Proposition 2.8 we can assume that $\|(QT - \lambda I)|_Y\| < 3^{-1}\mu$. So $\|(I - Q)T(y)\| > \mu - 3^{-1}\mu = 2\mu/3$ for all $y \in S_Y$. Hence if y_1, y_2 are in Y and $\|(T - \lambda I)y_1\| = 1$ (so that $\|y_1\| \geq \mu^{-1}$),

$$\|(T - \lambda I)y_1 - y_2\| \geq \frac{1}{2}\|(I - Q)[(T - \lambda I)y_1 - y_2]\| = \frac{1}{2}\|(I - Q)Ty_1\|$$
$$\geq \frac{1}{2} \cdot \frac{2}{3} \cdot \mu\|y_1\| \geq \frac{1}{3}.$$

This implies that $d((T - \lambda I)(Y), Y) \geq 1/3$. ∎

To prove our last theorem, we use the following lemma which is an immediate consequence of Theorems 3.2 and 3.3 in [DJ].

LEMMA 2.13. *Let X be a Banach space such that X is isomorphic to $(\sum X)_p$, $p \in [1, \infty] \cup \{0\}$. Let T be a bounded linear operator on X such that there exists a subspace Y of X such that Y is isomorphic to X, $T|_Y$ is an isomorphism, $d(TY, Y) > 0$ and $Y + T(Y)$ is complemented in X. Then T is a commutator.*

An infinite-dimensional Banach space X is said to be *complementably homogeneous* if every subspace of X that is isomorphic to X must contain a smaller subspace that is isomorphic to X and is complemented in X. It is clear that if X is complementably homogeneous then $\mathcal{M}_X$, the set of operators T on X such that the identity does not factor through T, is equal to the set of X-strictly singular operators on X. Lemma 2.7 implies that $Z_{p,q}$ is complementably homogeneous for $1 \le q < p < \infty$. (We did not attempt to check that $Z_{p,q}$ is complementably homogeneous for other values of p and q because that is not needed to prove our Main Theorem.)

Let T be a bounded linear operator on the complementably homogeneous space X. An important fact which was used repeatedly in [D, DJ, DJS] is that in certain spaces of this type, if there exists a subspace Y of X such that Y is isomorphic to X, $T|_Y$ is an isomorphism and $d(Y, TY) > 0$, then there is a subspace Z of Y isomorphic to X and such that $Z + T(Z)$ is complemented in X. The next lemma gives a formulation of this fact for a general class of spaces for which the statement is true.

Lemma 2.14. *Let X be a complementably homogeneous Banach space. Let T be a bounded linear operator on X for which there is a subspace Y of X isomorphic to X such that $T|_Y$ is an isomorphism and $d(Y, TY) > 0$. Then there is a subspace Z of Y isomorphic to X and such that $Z + T(Z)$ is complemented in X.*

Proof. Since X is complementably homogeneous, there is a subspace W of TY that is isomorphic to X and is the range of some projection P_W. Then $(T|_Y)^{-1}W$ is also isomorphic to X and is the range of the projection $(T|_Y)^{-1}P_W T$. Consequently, without loss of generality we assume that Y and TY are complemented in X.

The main step of the proof consists in finding a subspace Y_1 of Y that is isomorphic to X and such that there is a projection P_{TY_1} onto TY_1 for which $P_{TY_1}Y_1 = \{0\}$. Having done that, we have, as mentioned above, a projection P_{Y_1} onto Y_1. Then $P_{TY_1} + P_{Y_1}(I - P_{TY_1})$ is a projection onto $Y_1 + TY_1$, so that $Z := Y_1$ satisfies the conclusions of the lemma.

We now turn to the proof of the main step. Let P be a projection from X onto Y. Since $d(Y, TY) > 0$ and $T|_Y$ is an isomorphism, it follows that $T' := (I - P)TP$ is an isomorphism on Y, where I is the identity operator on X. Let W be a subspace of $T'(Y)$ that is isomorphic to X and complemented in X. Then $Y_1 := (T'|_Y)^{-1}(W)$ is also isomorphic to X and complemented in X. Let P_W be a projection from X onto W. Let S be the inverse of the isomorphism $T(y) \mapsto T'(y)$, $y \in Y$. Then it is straightforward to verify that $P_{TY_1} := SP_W(I - P)$ is a projection from X onto TY_1 such that $P_{TY_1}Y_1 = \{0\}$. ∎

The root of the proof of Theorem 2.15 goes back to [A1]; see also [DJ, Lemma 4.1].

Theorem 2.15. *Let X be a complementably homogeneous Banach space isomorphic to $(\sum X)_p$, $p \in [1,\infty] \cup \{0\}$, and suppose the set of all X-strictly singular operators on X form an ideal in $L(X)$. Let $T : X \to X$ be a bounded linear operator such that $T - \lambda' I$ is not X-strictly singular for any $\lambda' \in \mathbb{C}$. If there is a $\lambda \in \mathbb{C}$ and a subspace Y of X isomorphic to X and such that $(T - \lambda I)|_Y$ is X-strictly singular then T is a commutator.*

Proof. Since X is complementably homogeneous, by passing to a subspace of Y, we may assume that Y is complemented in X. Let $I - P$ be a bounded projection from X onto Y. By passing to a further subspace of Y, we can assume additionally that PX contains a complemented subspace isomorphic to X and hence PX is isomorphic to X by Pełczyński's decomposition method [LT1, p. 54]. To simplify the notation, set $T_\lambda := T - \lambda I$. Then $T_\lambda(I - P)$ is X-strictly singular. Now we consider the operator $(I - P)T_\lambda P$. If it is not X-strictly singular, then there is a subspace Z of PX that is isomorphic to X and such that $(I - P)T_\lambda P|_Z$ is an isomorphism. By passing to a subspace of Z, we may assume that Z is complemented in X. By the construction, we immediately get $d(Z, (I - P)T_\lambda PZ) > 0$ and hence $d(Z, T_\lambda Z) > 0$. By Proposition 2.1 in [DJ], $d(Z, TZ) > 0$. By Lemma 2.14, we may assume $Z + TZ$ is complemented in X and hence T is a commutator in virtue of Lemma 2.13. If $(I - P)T_\lambda P$ is X-strictly singular, we write

$$T_\lambda = T_\lambda(I - P) + (I - P)T_\lambda P + PT_\lambda P.$$

Since T_λ is not X-strictly singular, $PT_\lambda P$ is not X-strictly singular by the hypothesis that the X-strictly singular operators are closed under addition. Let A be an isomorphism from PX onto $(I - P)X$ and let $B : (I - P)X \to PX$ be its inverse, so that $BAP = P$ and $AB(I - P) = I - P$. We define an operator S on X by

$$\sqrt{2}S = P + AP - (I - P) + B(I - P).$$

A direct computation shows that $S^2 = I$.

Now we consider the operator $R := 2(I - P)ST_\lambda SP$. We claim that R is not X-strictly singular. To see this, substitute for S in the expression for R and compute

$$R = APT_\lambda P + [AP - (I - P)]T_\lambda AP - (I - P)T_\lambda P =: APT_\lambda P + \alpha - \beta.$$

The operator β is X-strictly singular by assumption and α is X-strictly singular because AP has range $(I - P)X$ and $T_\lambda(I - P)$ is X-strictly singular. We mentioned above that $PT_\lambda P$ is not X-strictly singular, so also $APT_\lambda P$ is not X-strictly singular because A is an isomorphism on PX, and hence $R = 2(I - P)ST_\lambda SP$ is also not X-strictly singular.

Therefore there is a complemented subspace Z of PX such that Z is isomorphic to X and $d(Z, ST_\lambda S(Z)) > 0$. That is, $d(S(Z), T_\lambda S(Z)) > 0$. By

Proposition 2.1 in [DJ] again, $d(S(Z), TS(Z)) > 0$. Hence T is a commutator by Lemmas 2.14 and 2.13. ∎

COROLLARY 2.16. *Let X be a complementably homogeneous Banach space such that X is isomorphic to $(\sum X)_p$, $p \in [1,\infty] \cup \{0\}$, and the set of all X-strictly singular operators on X form an ideal in $\mathcal{L}(X)$ and are commutators in $\mathcal{L}(X)$. Assume that for every operator T on X either there is a $\lambda \in \mathbb{C}$ and a subspace Y of X that is isomorphic to X and such that $(T - \lambda I)|_Y$ is X-strictly singular, or there is a $\lambda \in \mathbb{C}$ and a subspace Y of X that is isomorphic to X and such that $(T - \lambda I)|_Y$ is an isomorphism and $d((T - \lambda I)(Y), Y) > 0$. Then $T \in \mathcal{L}(X)$ is a commutator if and only if $T - \lambda' I$ is not X-strictly singular for every non-zero $\lambda' \in \mathbb{C}$. Consequently, X is a Wintner space.*

Proof. Let T be an operator on X such that $T-\lambda I$ is not X-strictly singular for all $\lambda \in \mathbb{C}$. If there is a $\lambda \in \mathbb{C}$ and a subspace Y of X that is isomorphic to X and such that $(T-\lambda I)|_Y$ is an isomorphism and $d((T - \lambda I)(Y), Y) > 0$, by Lemmas 2.14 and 2.13, T is a commutator. If there is a $\lambda \in \mathbb{C}$ and a subspace Y of X that is isomorphic to X and such that $(T-\lambda I)|_Y$ is X-strictly singular, then by Lemma 2.15, T is a commutator. ∎

Finally we complete the proof of our Main Theorem:

COROLLARY 2.17. *The space $X := Z_{p,q}$ with $1 \le q < \infty$ and $1 < p < \infty$ is a Wintner space. In fact, if T is an operator on X, then T is a commutator if and only if for all non-zero $\lambda \in \mathbb{C}$, the operator $T - \lambda I$ is not in $\mathcal{M}_X$ (i.e., the identity on X factors through $T - \lambda I$).*

Proof. For $1 \le q < p < \infty$, Corollary 2.16 applies to give the desired conclusions. The other cases follow from the obvious facts that if X is reflexive and T is an operator on X, then T is in $\mathcal{M}_X$ (respectively, is a commutator) if and only if T^* is in $\mathcal{M}_{X^*}$ (respectively, is a commutator). ∎

3. Open problems. Our first question concerns general classes of Banach spaces.

QUESTION 1. *Is every infinite-dimensional Banach space a Wintner space?*

As we mentioned in the introduction, there may be an infinite-dimensional Banach space on which every finite rank commutator has zero trace. A less striking negative example would be an infinite-dimensional Banach space on which every operator is of the form $\lambda I + T$ with T nuclear.

QUESTION 1.1. *If X admits a Pełczyński decomposition, is X a Wintner space?*

QUESTION 1.2. *What if also $\mathcal{M}_X$ is an ideal in $\mathcal{L}(X)$?*

Question 1.3. *What if also X is complementably homogeneous?*

We next turn to special spaces.

Question 2. *Is every $C(K)$ space a Wintner space when K is a compact Hausdorff space? What if also K is metrizable?*

Question 2.1. *Is $C[0,1]$ a Wintner space?*

An affirmative answer to Question 1.3 gives an affirmative answer to Question 2.1, but we suspect that Question 2.1, while difficult, is much easier than Question 1.3.

Question 3. *Is every complemented subspace of $L_p := L_p(0,1)$, $1 \le p < \infty$, a Wintner space?*

In regard to Question 3, it is open whether every infinite-dimensional complemented subspace of L_1 is isomorphic to either L_1 or to ℓ_1. There are, on the other hand, uncountably many different (up to isomorphism) complemented subspaces of L_p for $1 < p \neq 2 < \infty$ [BRS], including the Wintner spaces L_p [DJS], ℓ_p [A1], $\ell_p \oplus \ell_2$ [D], and $Z_{p,2}$. All of these are complementably homogeneous and have $\mathcal{M}_X$ as an ideal, but $\ell_p \oplus \ell_2$, while the direct sum of two spaces that admit Pełczyński decompositions, does not itself admit a Pełczyński decomposition.

Question 3.1.2. *Is Rosenthal's space X_p (see [R]) a Wintner space?*

The space X_p, $2 < p < \infty$, was the first "non-obvious" complemented subspace of L_p and it has played a central role in the modern development of the structure theory of L_p. It is small in the sense that it embeds isomorphically into $\ell_p \oplus \ell_2$ (ℓ_p is the only complemented subspace of L_p that does not contain any subspace isomorphic to $\ell_p \oplus \ell_2$), but not as a complemented subspace. The space X_p does not admit a Pełczyński decomposition, but it does admit something analogous (a "$p, 2$" decomposition) which might serve as a substitute. Not every operator in $\mathcal{M}_{X_p}$ is X_p-strictly singular because you can map X_p isomorphically into a subspace of X_p that is isomorphic to $\ell_p \oplus \ell_2$ and no isomorphic copy of X_p in $\ell_p \oplus \ell_2$ can be complemented. Probably the ideas in [JO] can be used to show that $\mathcal{M}_{X_p}$ is an ideal in $\mathcal{L}(X_p)$, but we have not yet tried to check this.

The famous problem, due to Brown and Pearcy, whether every compact operator on ℓ_2 is a commutator of compact operators, is still open. In fact, nothing is known in a more general setting, so we ask:

Question 4. *For what Banach spaces X is every compact operator a commutator of compact operators? Is this true for every infinite-dimensional X? Is it true for some infinite-dimensional X?*

QUESTION 5. *Assume that X is a complementably homogeneous Wintner space that has a Pełczyński decomposition and that $\mathcal{M}_X$ is an ideal in $\mathcal{L}_X$. If T is not in $\mathcal{M}_X$ and T is a commutator, does there exist a complemented subspace X_1 of X that is isomorphic to X and such that $(I - P_{X_1})T|_{X_1}$ is an isomorphism? Here P_{X_1} is a projection onto X_1.*

For $X = \ell_p$, $1 \le p \le \infty$, or c_0 or L_p, $1 \le p \le \infty$, or $Z_{p,q}$, $1 \le q < \infty$ and $1 < p < \infty$, the proofs that X is a Wintner space show that Question 5 has an affirmative answer.

Finally there is the obvious problem to give an affirmative answer to.

QUESTION 6. *Is $Z_{p,q}$ a Wintner space for values of p and q not covered by the Main Theorem?*

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