

Involutions on the second duals of group algebras versus subamenable groups

by

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Abstract. Let $L^1(G)^{**}$ be the second dual of the group algebra $L^1(G)$ of a locally compact group G . We study the question of involutions on $L^1(G)^{**}$. A new class of subamenable groups is introduced which is universal for all groups. There is no involution on $L^1(G)^{**}$ for a subamenable group G .

1. Introduction. Let A be a complex Banach algebra, A^* its dual and A^{**} its second dual. We follow R. Arens ([A1], [A2]) and equip A^{**} with the first Arens product \square or the second Arens product \diamond defined as follows.

For $\varphi, \psi \in A$, $f \in A^*$, and $F, H \in A^{**}$,

$$f\varphi(\psi) = f(\varphi\psi), \quad Ff(\varphi) = F(f\varphi), \quad H \square F(f) = H(Ff);$$

whereas

$$\varphi f(\psi) = f(\psi\varphi), \quad fF(\varphi) = F(\varphi f), \quad F \diamond H(f) = H(fF).$$

The Banach algebra A is said to be *Arens regular* if $\square = \diamond$.

For a continuous conjugate linear map $T : A \rightarrow A$, the conjugate-adjoint $T^c : A^* \rightarrow A^*$ and the second conjugate-adjoint $T^{cc} : A^{**} \rightarrow A^{**}$ are defined via

$$T^c f = \overline{f \circ T}, \quad T^{cc} F = \overline{F \circ T^c} \quad \text{for } f \in A^*, F \in A^{**}.$$

Both T^c and T^{cc} are conjugate linear mappings and can very well be called the *first adjoint* and the *second adjoint* of T respectively. We will denote them by T^* and T^{**} respectively only if no confusion can arise.

P. Civin and B. Yood [CY] noted that a continuous (*algebra*) *involution* on A , i.e. a continuous conjugate linear anti-homomorphism of period two, can be extended to an involution on A^{**} if A is Arens regular. The converse is also true. On the other hand, M. Grosser [G] showed that a necessary

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condition for the bidual A^{**} of A with bounded approximate right identity to admit an involution with respect to the first Arens product is that $A^*A = A^*$. To get an idea for the case of $A = L^1(G)$, the group algebra of a locally compact (Hausdorff) group (with left Haar measure), we just mention that (i) A is Arens regular if and only if G is finite, as proved by Civin and Yood [CY] for the abelian case and by N. J. Young [Y] for the general case; and (ii) $A^*A = A^*$ if and only if G is discrete as indicated by Civin and Yood for the abelian case in the proof of Theorem 3.12 of [CY] and proved by Grosser [G] for the general case. We record the important consequence due to Grosser.

THEOREM 1.1 ([G]). *If $L^1(G)^{**}$ admits an involution then G is discrete.*

The background for $L^1(G)^{**}$ for discrete G is well set by M. M. Day [Da] and Civin and Yood [CY], and an important role is played by the Stone–Čech compactification βG of G identified as a semigroup under the multiplication \square of $L^1(G)^{**}$. See also R. G. Douglas [Do].

The question of involutions on the second duals of group algebras was initially given attention by J. Duncan and S. A. R. Hosseiniun [DH] and further studied by H. Farhadi and F. Ghahramani [FG]; see also the related paper by M. Neufang [N].

Our interest in this short paper is to further strengthen Theorem 1.1 above. We confine our attention to the first Arens product only. We define the notion of subamenability for a locally compact (Hausdorff) group G and prove

THEOREM 1.2. *Let G be a subamenable non-compact locally compact group. Then $L^1(G)^{**}$ has no involution.*

We also show that the class of subamenable groups properly contains that of amenable groups. Another interesting fact is that every discrete group G is a subgroup of a subamenable discrete group \tilde{G} with cardinality $\#\tilde{G} \leq 2\#G$.

We refer to E. Hewitt and K. A. Ross [HR], J.-P. Pier [Pi], T. W. Palmer [Pal], N. Hindman [H] and N. Hindman and D. Strauss ([HS1], [HS3]), M. Filali and A. I. Singh [FSi] and H. G. Dales, A. T.-M. Lau and D. Strauss [DLS], for instance, for basics and some further details. These sources permit us to concentrate on our new results utilizing the relevant formulations of known results rather than tracing their history.

2. Subamenable groups and the question of involutions on $L^1(G)^{}$.** We begin with a bit of prevalent notation and a variant of the notion of amenability for a locally compact (Hausdorff) group G with identity e and left Haar measure λ , simply denoted by dx at times.

Let $k(G)$ be the minimal number of compact sets required to cover G . Clearly, $k(G)$ is an infinite cardinal number except when G is compact, and in that case $k(G) = 1$. Further, for an infinite discrete group, $k(G) = k$, the cardinality $\#G$ of G .

For $x \in G$ and a function f on G , let $\ell_x f$ be the function on G given by $\ell_x f(y) = f(xy)$ for y in G . For F in $L^1(G)^{**} = L^\infty(G)^*$ let L_F be the bounded linear operator from $L^\infty(G)$ to itself given by $L_F f = Ff$. Let $\mathbf{1}$ be the constant function one on G . A mean M on G is a positive linear functional on $L^\infty(G)$ with $M(\mathbf{1}) = 1$. In case G is discrete, $L^1(G)$, $L^\infty(G)$, $L^1(G)^{**}$ etc. will be denoted, at times, by $\ell^1(G)$, $\ell^\infty(G)$, $\ell^1(G)^{**}$ etc. respectively. For a subset W of a vector space X , $\Gamma(W)$ will denote its linear span.

DEFINITION 2.1.

- (i) A non-empty subset \mathcal{F} of $L^1(G)^{**}$ will be called an *invoid* for G if $\Gamma(\mathcal{F})$ is a left ideal in $L^1(G)^{**}$.
- (ii) An invoid for G containing at least one mean M will be called a *meanoid* for G .
- (iii) For a cardinal number α , G will be called α -*amenable* if G has a meanoid with cardinality $\leq \alpha$.
- (iv) G will be called *subamenable* if G is α -amenable for some $1 \leq \alpha < 2^{2^{k(G)}}$, i.e., $L^1(G)^{**}$ has a left ideal which has dimension $< 2^{2^{k(G)}}$ and contains a mean.

The definition is designed to have the following result.

THEOREM 2.2. *Let G be a subamenable non-compact locally compact group. Then $L^1(G)^{**}$ has no involution.*

Proof. By the definition of subamenability, $L^1(G)^{**}$ has a γ -dimensional left ideal with $1 \leq \gamma < 2^{2^{k(G)}}$. By Filali and P. Salmi [FSa, Theorem 6], $L^1(G)^{**}$ has no non-trivial right ideal with dimension $< 2^{2^{k(G)}}$. So $L^1(G)^{**}$ has no involution. ■

We now come to the reason for the nomenclature in Definition 2.1.

REMARK 2.3. Let \mathcal{F} be a non-empty subset of $L^1(G)^{**}$.

(i) The set \mathcal{F} is an invoid for G if and only if for each F in \mathcal{F} and each non-zero H in $L^1(G)^{**}$ there exist $F_j \in \mathcal{F}$, $h_j \in L^\infty(G)$, $1 \leq j \leq n$, satisfying $H \circ (L_F - \sum_{j=1}^n F_j \otimes h_j) = 0$, i.e., for each f in $L^\infty(G)$, $H(Ff) = \sum_{j=1}^n H(h_j)F_j(f)$. To see this, we have only to note that $H \square F(f) = H(Ff)$ and for a scalar β there exists $h \in L^\infty(G)$ with $\beta = H(h)$. Hence the set \mathcal{F} is an invoid for G if and only if for each F in \mathcal{F} and each closed hyperplane Π in $L^\infty(G)$ there is a finite rank bounded operator T in $\Gamma(\mathcal{F}) \otimes L^\infty(G)$ from $L^\infty(G)$ to itself with the range of $L_F - T$ in Π .

(ii) For a mean M on G , $\mathcal{F} = \{M\}$ is a meanoid for G if and only if $L_M = M \otimes \mathbf{1}$, i.e., $Mf = M(f)\mathbf{1}$ for each f in $L^\infty(G)$, i.e. M is topologically left invariant. To see this, let $H \in L^1(G)^{**} = L^\infty(G)^*$ be arbitrary. We note that $H \square M = \beta_H M$ for a scalar β_H if and only if $H(Mf) = \beta_H M(f)$ for each f in $L^\infty(G)$. The basic fact that $M\mathbf{1} = \mathbf{1}$ gives that this is so only if $\beta_H = H(\mathbf{1})$. Thus, $H \square M = \beta_H M$ for a scalar β_H if and only if $H(Mf) = H(\mathbf{1})M(f)$ for each f in $L^\infty(G)$, i.e., $H \circ (L_M - M \otimes \mathbf{1}) = 0$.

(iii) By (ii) above, G is 1-amenable if and only if G is amenable.

(iv) If G is α_1 -amenable and $\alpha_1 < \alpha_2$, then G is α_2 -amenable.

(v) G is $2^{2^{\#G}}$ -amenable. To see this it is enough to note that for G infinite, $\#L^\infty(G) \leq 2^{\#G}$ and, therefore, $\#L^1(G)^{**} \leq 2^{2^{\#G}}$.

(vi) We do not yet have any example of a non-subamenable group. A fortiori, we do not have an example of an infinite G with an involution on $L^1(G)^{**}$. But Theorem 2.9 below gives a large class of subamenable groups.

COROLLARY 2.4. *Let G be an amenable non-compact locally compact group. Then $L^1(G)^{**}$ has no involution.*

PROPOSITION 2.5. *If G is m -amenable for some $m \in \mathbb{N}$, the set of natural numbers, then G is amenable.*

Proof. Let \mathcal{F} be a meanoid for G with cardinality m . Then \mathcal{F} contains a mean M . Since $M \square M(\mathbf{1}) = 1$, M is not a right annihilator of $L^1(G)^{**}$.

So $\Gamma(\mathcal{F})$ is a finite-dimensional left ideal in $L^1(G)^{**}$ which is not generated by right annihilators of $L^1(G)^{**}$. By Filali [F2, Corollary (3) of Theorem 3], G is amenable. ■

We now prepare to show that there exist subamenable non-amenable groups. In fact, we will show that each discrete group G is a subgroup of some subamenable group G' with $k(G'_\sigma) = \#G'_\sigma \leq 2^{\#G}$. Hence the class of subamenable groups is universal for all groups.

REMARK 2.6. In view of Grosser's Theorem 1.1 in the first section, we can confine our attention to discrete groups. *From now onwards, G is an infinite discrete group with cardinality k unless otherwise stated.*

(i) The result on the dimension of non-trivial right ideals in $L^1(G)^{**}$ by Filali and Salmi used in the proof of Theorem 2.2 is needed only for discrete groups. In that case the theorem also follows from Filali and J. Pym [FP, Theorem 5] simply because for discrete G , $\mathcal{LUC}(G)^* = L^1(G)^{**}$.

(ii) An independent proof of Corollary 2.4 can be given using the relevant part and version of an earlier result of Filali [F1, Theorem 3], viz., for an infinite discrete group G , $L^1(G)^{**}$ has no non-trivial finite-dimensional right ideal, together with the basic fact that for amenable G , $L^1(G)^{**}$ has non-trivial finite-dimensional left ideals.

PROPOSITION 2.7. *Let G be a discrete group and α be a cardinal number.*

- (i) *If π is a homomorphism from G onto a discrete group S and G is α -amenable then S is α -amenable. Equivalently, the quotient group S of G by a normal subgroup N is α -amenable if G is so.*
- (ii) *If S is a subgroup of G and G is α -amenable then so is S .*
- (iii) *If N is an amenable normal subgroup of G with the quotient G/N α -amenable then G is α -amenable.*

Proof. Some basic ingredients for the proof are taken from the well-known amenable version. We may refer to Pier [Pi, §13], for instance.

(i) Let \mathcal{F} be a meanoid for G with $\#\mathcal{F} \leq \alpha$ and M a mean present in \mathcal{F} .

For $\xi \in \ell^\infty(S)$, let $\hat{\xi} = \xi \circ \pi$. Then $\hat{\xi}$ is in $\ell^\infty(G)$ and is constant on $\pi^{-1}\{b\}$ for each b in S .

We put $\ell_\pi^\infty(G) = \{f \in \ell^\infty(G) : f \text{ is constant on } \pi^{-1}\{b\} \text{ for each } b \text{ in } S\} = \{\hat{\xi} : \xi \in \ell^\infty(S)\} = \widehat{\ell^\infty(S)}$. In fact, $\xi \mapsto \hat{\xi}$ is a 1-1 correspondence between $\ell^\infty(S)$ and $\ell_\pi^\infty(G)$. For $\Phi \in \ell^1(S)^{**} = \ell^\infty(S)^*$, let $\hat{\Phi}$ be any Hahn-Banach extension to $\ell^\infty(G)$ of Φ_1 on $\ell_\pi^\infty(G)$ given by $\Phi_1(\hat{\xi}) = \Phi(\xi)$. For $H \in \ell^1(G)^{**} = \ell^\infty(G)^*$, we define $\check{H} \in \ell^1(S)^{**}$ by $\check{H}(\xi) = H(\hat{\xi})$. Then $(\Phi)^{\wedge\vee} = \Phi$ for $\Phi \in \ell^1(S)^{**}$. Next, for $\xi \in \ell^\infty(S)$, $a \in G$, $b = \pi(a)$, we have $\ell_a \hat{\xi} = (\ell_b \xi)^\wedge$, and therefore, for $H \in \ell^1(G)^{**}$,

$$(H\hat{\xi})(a) = H(\ell_a \hat{\xi}) = H((\ell_b \xi)^\wedge) = \check{H}(\ell_b \xi) = (\check{H}\xi)(b) = (\check{H}\xi)^\wedge(a).$$

So, $H\hat{\xi} = (\check{H}\xi)^\wedge$, and further, $H\hat{\xi}$ is in $\ell_\pi^\infty(G)$. Moreover,

$$\widetilde{M}(\mathbf{1}_s) = M(\mathbf{1}) = 1 = \|M\| = \|\widetilde{M}\|.$$

Hence \widetilde{M} is a mean on S . Now let $\check{\mathcal{F}} = \{\check{F} : F \in \mathcal{F}\}$. Let $F \in \mathcal{F}$ and $\Phi \in \ell^1(S)^{**}$. Put $H = \hat{\Phi}$. Since \mathcal{F} is an invoid for G , there exist $F_j \in \mathcal{F}$, $\beta_j \in \mathbb{C}$, $1 \leq j \leq n$, satisfying

$$H \square F = \sum_{j=1}^n \beta_j F_j.$$

So for $\xi \in \ell^\infty(S)$,

$$H(F\hat{\xi}) = \sum_{j=1}^n \beta_j F_j(\hat{\xi}),$$

which in turn gives

$$H((\check{F}\xi)^\wedge) = \sum_{j=1}^n \beta_j (F_j)^\vee(\xi), \quad \text{i.e.,} \quad \check{H}(\check{F}\xi) = \left(\sum_{j=1}^n \beta_j (F_j)^\vee \right)(\xi).$$

It follows that

$$\Phi(\check{F}\xi) = \left(\sum_{j=1}^n \beta_j(F_j)^\vee \right)(\xi).$$

Therefore,

$$\Phi \square \check{F} = \sum_{j=1}^n \beta_j(F_j)^\vee.$$

Thus \mathcal{F} is an invoid for S .

As $\check{M} \in \check{\mathcal{F}}$ we conclude that $\check{\mathcal{F}}$ is a meanoid for S . The result follows on noting that $\#\check{\mathcal{F}} \leq \alpha$.

The second part follows from the first just by applying it to the canonical homomorphism of G onto G/N .

(ii) Let G/S be the set of left cosets $[x] = xS$. Let h be any function from G/S to G with $h([x]) \in [x]$ for each $[x]$ in G/S and $h([e]) = e$, the identity of G . Let C be the range of h . Then $h_0 = \chi_C$ is a Bruhat function attached to the pair (G, S) . For $\xi \in \ell^\infty(S)$, let $\hat{\xi}(x) = \xi(xh[x^{-1}])$ for x in G . Then $\hat{\xi} \in \ell^\infty(G)$. Also, for $a \in S$, $\hat{\xi}(a) = \xi(ah([a^{-1}])) = \xi(ae) = \xi(a)$. So $\hat{\xi}|S = \xi$. For $H \in \ell^1(G)^{**}$, let $\check{H} \in \ell^1(S)^{**} = \ell^\infty(S)^*$ be defined by $\check{H}(\xi) = H(\hat{\xi})$. Consider any $H \in \ell^1(G)^{**}$, $\xi \in \ell^\infty(S)$ and $a \in S$. Then for $x \in G$,

$$(\ell_a \xi)^\wedge(x) = \ell_a \xi(xh([x^{-1}])) = \xi(axh([(ax)^{-1}])) = \hat{\xi}(ax) = \ell_a \hat{\xi}(x),$$

and therefore $(\ell_a \xi)^\wedge = \ell_a \hat{\xi}$. So

$$(H\hat{\xi})(a) = H(\ell_a \hat{\xi}) = H((\ell_a \xi)^\wedge) = \check{H}(\ell_a \xi) = \check{H}\xi(a).$$

Hence $\check{H}\xi = (H\hat{\xi})|S$. For $\Phi \in \ell^1(S)^{**}$, let $\hat{\Phi} : \ell^\infty(G) \rightarrow \mathbb{C}$ be given by $\hat{\Phi}(f) = \Phi(f|S)$. Then $\hat{\Phi} \in \ell^\infty(G)^* = \ell^1(G)^{**}$. Also

$$(\hat{\Phi})^\vee(\xi) = \hat{\Phi}(\hat{\xi}) = \Phi(\hat{\xi}|S) = \Phi(\xi).$$

So $(\hat{\Phi})^\vee = \Phi$.

Let \mathcal{F} be a meanoid for G with $\#\mathcal{F} \leq \alpha$ and M a mean present in \mathcal{F} . Let $F \in \mathcal{F}$ and $\Phi \in \ell^1(S)^{**}$. Since \mathcal{F} is a meanoid, there exist $F_j \in \mathcal{F}$, $\beta_j \in \mathbb{C}$, $1 \leq j \leq n$, such that $\hat{\Phi} \square F = \sum_{j=1}^n \beta_j F_j$. So, for $\xi \in \ell^\infty(S)$,

$$\hat{\Phi}(F\hat{\xi}) = \sum_{j=1}^n \beta_j F_j(\hat{\xi}), \quad \text{i.e.,} \quad \Phi(F\hat{\xi}|S) = \sum_{j=1}^n \beta_j (F_j)^\vee(\xi),$$

which in turn gives

$$\Phi(\check{F}\xi) = \sum_{j=1}^n \beta_j (F_j)^\vee(\xi).$$

So

$$\Phi \square \check{F} = \sum_{j=1}^n \beta_j (F_j)^\vee.$$

Thus $\check{F} = \{\check{F} : F \in \mathcal{F}\}$ is an invoid for S . Also $\check{M}(\mathbf{1}_s) = M(\hat{\mathbf{1}}_s) = M(\mathbf{1}) = 1$. Therefore, $\check{\mathcal{F}}$ is a meanoid for S and $\#\check{\mathcal{F}} \leq \alpha$. Hence S is α -amenable.

(iii) Let M be a left invariant mean on N , \mathcal{F} a meanoid for G/N with $\#\mathcal{F} \leq \alpha$, and Φ a mean present in \mathcal{F} . Consider any $f \in \ell^\infty(G)$. For $x \in G$, the function $f_1^{(x)} = \ell_x f$ restricted to N is in $\ell^\infty(N)$. Also, for $a \in N$, $\ell_a f_1^{(x)} = f_1^{(xa)}$. We define $\tilde{f} \in \ell^\infty(G)$ by $\tilde{f}(x) = M(f_1^{(x)})$ for x in G . Then, for $x \in G$ and $a \in N$, $\tilde{f}(xa) = M(f_1^{(xa)}) = M(\ell_a f_1^{(x)}) = M(f_1^{(x)}) = \tilde{f}(x)$. So \tilde{f} is constant on the left coset xN . Thus we may define $\hat{f} \in \ell^\infty(G/N)$ by putting $\hat{f}(xN) = \tilde{f}(x)$.

For $x, y \in G$ and $f \in \ell^\infty(G)$,

$$\begin{aligned} (\ell_y f)^\wedge(xN) &= (\ell_y f)^\sim(x) = M((\ell_y f)_1^{(x)}) = M(f_1^{(yx)}) \\ &= \tilde{f}(yx) = \hat{f}(yxN) = \hat{f}((yN)(xN)) = \ell_{yN} \hat{f}(xN). \end{aligned}$$

Now for $\Psi \in \ell^1(G/N)^{**}$, let $\check{\Psi} \in \ell^1(G)^{**} = \ell_\infty(G)^*$ be given by $\check{\Psi}(f) = \Psi(\hat{f})$. Then for $f \in \ell^\infty(G)$, $y \in G$ we have

$$\check{\Psi} f(y) = \check{\Psi}(\ell_y f) = \Psi((\ell_y f)^\wedge) = \Psi(\ell_{yN} \hat{f}) = (\Psi \hat{f})(yN).$$

Thus, $\check{\Psi} f$ is constant on each left coset xN for $x \in G$. Let $Y = \{f \in \ell^\infty(G) : f \text{ is constant on left cosets of } N\} = \{\xi \circ \pi : \xi \in \ell^\infty(G/N)\}$, where π is the canonical map given by $\pi(x) = xN$. Then $\ell^\infty(G/N)$ can be identified with Y by $\xi \mapsto \hat{\xi} = \xi \circ \pi$. For $H \in \ell^1(G)^{**}$ let $\hat{H} \in \ell^1(G/N)^{**} = \ell^\infty(G/N)^*$ be given by $\hat{H}(\xi) = H(\hat{\xi})$.

Let $\check{\mathcal{F}} = \{\check{\Psi} : \Psi \in \mathcal{F}\}$. Then for $\Psi \in \mathcal{F}$, $H \in \ell^1(G)^{**}$, there exist $\Psi_j \in \mathcal{F}$, $\beta_j \in \mathbb{C}$, $1 \leq j \leq n$, with $\hat{H} \square \Psi = \sum_{j=1}^n \beta_j \Psi_j$.

Now for $f \in \ell^\infty(G)$,

$$\begin{aligned} (H \square \check{\Psi})(f) &= H(\check{\Psi} f) = H((\Psi \hat{f})^\wedge) = \hat{H}(\Psi \hat{f}) = (\hat{H} \square \Psi)(\hat{f}) \\ &= \left(\sum_{j=1}^n \beta_j \Psi_j \right) (\hat{f}) = \sum_{j=1}^n \beta_j \Psi_j(\hat{f}) = \sum_{j=1}^n \beta_j (\Psi_j)^\vee(f). \end{aligned}$$

So $H \square \check{\Psi} = \sum_{j=1}^n \beta_j (\Psi_j)^\vee$.

Therefore, $\check{\mathcal{F}}$ is an invoid for G . Also $\check{\Phi}$ is a mean on G , and as a consequence, $\check{\mathcal{F}}$ is a meanoid for G . As $\#\check{\mathcal{F}} \leq \#\mathcal{F} \leq \alpha$ we have the desired result. ■

COROLLARY 2.8. *Let G be an infinite discrete group with $\#G = k$.*

- (i) *Let G be subamenable. Let S be a discrete group with $\#S = k$. If S is a homomorphic image of G (equivalently, a quotient group of G by a normal subgroup N) or a subgroup of G , then S is subamenable.*
- (ii) *Let N be an amenable normal subgroup of G such that G/N is subamenable. Then G is subamenable. In particular, it is so if N is taken to be G_{FC} , the subgroup of G consisting of elements in G with finite conjugacy classes.*

Proof. We have only to note that subamenability of G or S can be replaced by α -amenability for some $\alpha < 2^{2^k}$ and then we can apply Proposition 2.7 above. For the second part of (ii) we use the well-known fact that G_{FC} is amenable (cf. [Pi, Proposition 12.9]). ■

THEOREM 2.9.

- (i) *Every discrete group G is a subgroup of a subamenable discrete group G_σ with $\#G_\sigma \leq 2^{\#G}$.*
- (ii) *There exist non-amenable subamenable groups of any infinite cardinality $\geq c$, the cardinality of the continuum.*
- (iii) *Every discrete group G is a subgroup of a non-amenable subamenable discrete group G_Σ with $\#G_\Sigma \leq 2^{\#G^{\alpha_0}}$. Here α_0 is the cardinality of the set \mathbb{N} of natural numbers.*

Proof. (i) Let G be any discrete group. If G is finite then it is amenable (and therefore subamenable). Now let G be infinite with $k = \#G \geq \alpha_0$. Then by Remark 2.3(v), G is α -amenable for some $\alpha \leq 2^{2^k}$. Let $\ell > k$ be any cardinal number. Let N be any discrete amenable group with $\#N = \ell$. We may take N to be any abelian group with cardinality ℓ , the free abelian group with ℓ generators, for instance. Let G_ℓ be the product group $N \times G$. By Proposition 2.7(iii) above, G_ℓ is α -amenable. Now $k_\ell = \#G_\ell = \ell \cdot k = \ell > k$. So $2^{2^{k_\ell}} = 2^{2^\ell} > 2^{2^k} \geq \alpha$. So G_ℓ is subamenable. We put $G_\sigma = G_\ell$ with $\ell = 2^k = 2^{\#G}$.

(ii) We take G to be any non-amenable countable group, say the free group \mathbb{F}_2 with two generators, and then take ℓ to be any infinite cardinal number $\geq c$ in the proof of (i) above. Then G_ℓ is not amenable simply because it has a non-amenable subgroup G , whereas every subgroup of an amenable discrete group is amenable.

(iii) Given any group G , we let $\Gamma = G \times \mathbb{F}_2$ and then apply part (i) to Γ (in place of G) and put $G_\Sigma = \Gamma_\sigma$. ■

REMARK 2.10. (i) Obviously, in Theorem 2.9(i) above, if G is subamenable we may take $G_\sigma = G$; but we have divided the proof of Theorem 2.9(i) above into easily checkable cases of G finite and G infinite rather

than G subamenable and G non-subamenable. If G is a non-subamenable group and G_1 is a group containing G as a subgroup then, in view of Corollary 2.8(i), $\#G_1 = \#G$ forces G_1 to be non-subamenable, and therefore if G_1 is a subamenable group we must have $\#G_1 > \#G$. In the statement and the last step (sentence) of the proof of Theorem 2.9(i) we have used the cardinal number $2^{\#G}$ to avoid any questions regarding the truth of the *generalized continuum hypothesis* (GCH) or otherwise.

(ii) Theorem 2.9(i) has no amenable counterpart because of the well-known fact that a subgroup of an amenable discrete group is amenable (already used in the proof of part (ii) of Theorem 2.9 above) and the fact that there exist non-amenable groups.

(iii) On the opposite side, if G has an element x of infinite order then it contains the infinite cyclic group generated by x , which is abelian, and therefore amenable; and, on the other hand, if G is a torsion group then also it contains a finite (and therefore) amenable subgroup.

(iv) There is a vast standard literature on torsion groups (and other groups) vis-à-vis amenability created by mathematicians such as S. I. Adian, R. I. Grigorchuk, A. Y. Ol'shanskiĭ, D. Osin, M. V. Sapir and J. Tits. We have yet to properly utilize it for our purposes.

REMARK 2.11. (i) Let us call a group G *Arens-** if $\ell^1(G)^{**}$ has an involution. Every finite group is Arens-* and we have not been able to give an example of an Arens-* infinite group.

(ii) Theorems 2.2 and 2.9 put together give that

- (a) every infinite group G is a subgroup of some non-Arens-* group G_σ with $\#G_\sigma \leq 2^{\#G}$,
- (b) there are non-amenable non-Arens-* groups of all cardinalities $\geq c$,
- (c) every infinite amenable group is non-Arens-* and is a subgroup of a non-amenable non-Arens-* group,
- (d) every group G is a subgroup of a non-amenable non-Arens-* group G_Σ with $\#G_\Sigma \leq 2^{\#G^{\aleph_0}}$.

(iii) Every non-torsion group G is sandwiched between some pair of non-Arens-* groups. To see this we only have to combine (ii)(a) above with Remark 2.10(iii). In fact, there are infinitely many such pairs in view of the observation that every infinite cyclic group has infinitely many infinite cyclic subgroups, and on the other hand, (ii)(a) above indicates how to build up a chain of non-Arens-* groups containing G as a subgroup.

(iv) An infinite torsion group G is sandwiched between some pair consisting of a finite Arens-* group and a non-Arens-* group. Again, there are infinitely many such pairs.

REMARK 2.12. (i) It is well-known (see [CY], [Do], [DLS] for instance) that $\ell^1(G)^{**}$ can be identified with the space of finitely additive measures on G , or equivalently, with the space $M(\beta G)$ of complex regular Borel measures on the Stone–Čech compactification βG of G . Further, under the Arens product \square of $\ell^1(G)^{**}$, βG (identified with unit point masses) is a compact right topological semigroup equipped with the weak* topology coming from $\ell^\infty(G)$.

(ii) [DLS, Proposition 7.5(i)] implies that if L is a closed left ideal in βG then $M(L)$ is a weak*-closed left ideal in $M(\beta G)$.

(iii) There is an abundance of closed left ideals in βG : this follows as a special case of discrete G from Proposition 2 of Lau and A. L. T. Paterson [LPa]. Keeping in mind the proof of that proposition, particularly the proof of Lemma 3 there, we record it as: If G is an infinite discrete group, then βG contains at least $2^{2^{\#G}}$ minimal closed left ideals which are mutually disjoint.

(iv) A combination of (ii) and (iii) above gives that $M(\beta G)$ contains at least $2^{2^{\#G}}$ weak*-closed left ideals whose pairwise intersection is $\{0\}$. Hence, if $M(\beta G)$ has an involution then it must have at least $2^{2^{\#G}}$ right ideals whose pairwise intersection is $\{0\}$. We do not know the situation in this regard.

(v) In the absence of involution on $M(\beta G)$ for many groups (as seen above), we may consider involutions on subalgebras of $\ell^1(G)^{**}$. The algebra $\ell^1(G)$ and its (algebraic) centre (which, in fact, turns out to be the centre of $\ell^1(G)^{**}$ as well, using [L], [LL], and is therefore invariant under an involution, if any, on $\ell^1(G)^{**}$) have their own involutions (see [Pat]). There is also an abundance of such subalgebras of $\ell^1(G)^{**}$ which do not intersect $\ell^1(G)$ except for $\{0\}$. In fact, apart from the group G , the semigroup βG very often contains several groups (considered as subsemigroups of βG) contained in $\beta G \setminus G$. Work of T. Budak, N. Işık and J. Pym [BIP], N. Hindman and D. Strauss ([HS2], [HS4]), I. V. Protasov [Pr] and E. G. Zelenyuk [Z], for instance, gives concrete examples and some general results (see also [DLS]).

(vi) This brings us to the question of involutions on subalgebras of $L^1(G)^{**}$ for a locally compact group G as well (see [FG], [LPy], [S]).

(vii) Another idea is to consider a trivolution on $L^1(G)^{**}$, viz. a conjugate-linear anti-homomorphism ρ satisfying $\rho^3 = \rho$. Then ρ is its own generalized inverse instead of being inverse. The name is motivated by a notion studied by J. W. Degen [De]. One can find details in [FMS].

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