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Essential norms of weighted composition operators on the space $\mathcal{H}^\infty$ of Dirichlet series

by

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Abstract. We estimate the essential norm of a weighted composition operator relative to the class of Dunford–Pettis operators or the class of weakly compact operators, on the space $\mathcal{H}^\infty$ of Dirichlet series. As particular cases, we obtain the precise value of the generalized essential norm of a composition operator and of a multiplication operator.

0. Introduction. The aim of this paper is to investigate the complete continuity and weak compactness of weighted composition operators on the space $\mathcal{H}^\infty$ of Dirichlet series. Composition operators have been investigated in many papers. The monographs [CmC] and [S] bring very good surveys of this topic. These operators are very often investigated on H^p spaces ($1 < p < \infty$), where their weak compactness and complete continuity are trivial problems (because of reflexivity). Investigations in the setting of Dirichlet series are more recent: see, for example, [B2], [GH] and [Q2].

Let us recall some terminology. We are going to work on half-planes

$$\mathbb{C}_\theta = \{s \in \mathbb{C};\ \mathrm{Re}(s) > \theta\}, \qquad \theta \geq 0.$$

In particular, $\mathcal{H}(\mathbb{C}_0)$ denotes the space of analytic functions on $\mathbb{C}_0$.

The space of Dirichlet series is

$$\mathcal{H}^\infty = \Big\{ f \in \mathcal{H}(\mathbb{C}_0);\ f \text{ bounded},\quad f(s) = \sum_{n\geq 1} a_n n^{-s} \text{ on some half-plane } \mathbb{C}_\varepsilon \text{ with } \varepsilon > 0 \Big\}.$$

(In fact, a result of Bohr [Bo] implies that any $\varepsilon > 0$ works.) The space $\mathcal{H}^\infty$ is the version of the classical Hardy space H^∞ in the setting of Dirichlet series.

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It is natural to introduce the counterpart of the disk algebra,

$$\mathcal{A} = \{f \in \mathcal{H}^\infty;\ f \text{ continuous on } \overline{\mathbb{C}}_0\}.$$

Both $\mathcal{H}^\infty$ and $\mathcal{A}$ are normed by $\|f\|_\infty = \sup\{|f(s)|;\ s \in \mathbb{C}_0\}$.

Before taking up some special properties of composition operators on $\mathcal{H}^\infty$, we have to know when they are defined. Actually, the case $\mathcal{H}^\infty$ is less complicated than the case of general $\mathcal{H}^p$ spaces: *An analytic function $\varphi : \mathbb{C}_0 \to \mathbb{C}_0$ defines a bounded composition operator $C_\varphi : f \mapsto f \circ \varphi$ on $\mathcal{H}^\infty$ if and only if $\varphi(s) = \alpha_0 s + \sum_{n\ge 1} \alpha_n n^{-s}$ with $\alpha_0 \in \mathbb{N}$* (see [B1, after Cor. 2, p. 217], or [B2, p. 65]). We shall always assume that φ satisfies this condition. We then have $\|C_\varphi\| = 1$.

A characterization of compact composition operators on $\mathcal{H}^\infty$ is due to Bayart [B1, Th. 18]. Actually, Bayart estimates the (classical) essential norm of a composition operator on $\mathcal{H}^\infty$. Let us recall his result:

Theorem ([B1], [B2]). *Let C_φ be a composition operator on $\mathcal{H}^\infty$. Then C_φ is compact if and only if $\varphi(\mathbb{C}_0) \subset \mathbb{C}_\varepsilon$ for some $\varepsilon > 0$.*

The compactness of weighted composition operators was studied in the classical frame of the disk algebra in [K]. Some extensions are studied in [L], where generalized essential norms are computed.

We are going to use rather elementary techniques, adapted from [L], to estimate the essential norm, relative to Dunford–Pettis operators and weakly compact operators, of weighted composition operators on $\mathcal{H}^\infty$.

We first specify some terminology:

Definition 0.1. Let X, Y be Banach spaces and $\mathcal{I}$ a closed subspace of the space $B(X,Y)$ of bounded operators from X to Y. The *essential norm* of $T \in B(X,Y)$ *relative to* $\mathcal{I}$ is the distance from T to $\mathcal{I}$:

$$\|T\|_{\mathrm{e},\mathcal{I}} = \inf\{\|T + S\|;\ S \in \mathcal{I}\}.$$

This is the canonical norm on the quotient space $B(X,Y)/\mathcal{I}$.

If moreover $\mathcal{I}$ is an ideal of $B(X)$ then $B(X)/\mathcal{I}$ is an algebra.

The classical case is that of compact operators, $\mathcal{I} = \mathcal{K}(X,Y)$ (in this case, the preceding quotient space is the Calkin algebra). Below, we are interested in the case of weakly compact operators: $\mathcal{I} = \mathcal{W}(X,Y)$, and in the case of completely continuous operators (= Dunford–Pettis operators): $\mathcal{I} = \mathcal{DP}(X,Y)$. Compact operators are both weakly compact and completely continuous.

Recall that a Banach space X has the *Dunford–Pettis property* if, for every Banach space Y and every operator $T : X \to Y$ which is weakly compact, T maps any weakly Cauchy sequence in X into a norm Cauchy sequence. A good survey on the subject (until the early eighties) is the paper of Diestel [D]. A Banach space X has the *property* (V) *of Pełczyński* if, for every Banach space Y and every operator $T:\ X \to Y$ which is not weakly compact, there exists a subspace X_0 of X isomorphic to c_0 such that $T_{|X_0}$ is an isomorphic embedding.

If the space $\mathcal{H}^\infty$ of Dirichlet series had both property (V) and the Dunford–Pettis property, then the ideals $\mathcal{W}(\mathcal{H}^\infty, Y)$ and $\mathcal{DP}(\mathcal{H}^\infty, Y)$ would coincide for every Banach space Y. It turns out that $\mathcal{H}^\infty$ does not have property (V) and it is unknown whether it has the Dunford–Pettis property.

Claim. *$\mathcal{H}^\infty$ does not have property* (V) (we have no reference for this remark).

This is a consequence of the Bohr inequality (see [Q1]):

$$\sum_{p\in\mathcal{P}} |a_p| \le \|f\|_\infty \quad \text{for every } f \in \mathcal{H}^\infty,$$

where $\mathcal{P}$ stands for the set of prime numbers. The inequality implies that $\{f \in \mathcal{H}^\infty;\ f(s) = \sum_{p\in\mathcal{P}} a_p p^{-s}\}$ is a complemented subspace of $\mathcal{H}^\infty$, isomorphic to ℓ^1. Thus, the corresponding projection can neither be weakly compact, nor fix a copy of c_0. This proves the claim.

Let us point out too that the same argument implies that the space $\mathcal{H}^\infty$ does not satisfy the Grothendieck theorem: the projection (given by the Bohr inequality) from $\mathcal{H}^\infty$ to ℓ^1 is bounded and cannot be 2-summing.

Given $u \in \mathcal{H}^\infty$ and an analytic function φ from $\mathbb{C}_0$ to $\mathbb{C}_0$ defining a composition operator, we shall study the (generalized) essential norm of the weighted composition operator $T_{u,\varphi}$:

$$T_{u,\varphi}(f) = u \cdot (f \circ \varphi) \quad \text{where } f \in \mathcal{H}^\infty.$$

Of course, when $u = \mathbb{1}$, this operator is the classical composition operator, simply denoted by C_φ. When $\varphi = \mathrm{Id}_{\mathbb{C}_0}$, it is the multiplication operator M_u by u.

Observe that $T_{u,\varphi}$ is always bounded from $\mathcal{H}^\infty$ to $\mathcal{H}^\infty$, with $\|T_{u,\varphi}\| = \|u\|_\infty$, where $\|u\|_\infty = \sup\{|u(s)|;\ s \in \mathbb{C}_0\}$.

The following quantity plays a crucial role in the estimate of the essential norm:

$$n_\varphi(u) = \lim_{r\to 0^+} \sup\{|u(s)|;\ s \in \mathbb{C}_0,\ \mathrm{Re}(\varphi(s)) \le r\},$$

which is finite since u is bounded.

If $\inf \mathrm{Re}(\varphi) > 0$ then $n_\varphi(u) = 0$ (i.e. the supremum over the empty set is taken as 0).

1. Characterization of weak compactness and complete continuity. We first need the following lemma.

Lemma 1.1. *Let* $(h_n)_{n\ge 0}$ *be a sequence in the disk algebra* $A(\mathbb{D})$, *to which we associate the sequence in* $\mathcal{A}$ *defined by* $H_n(s) = h_n(2^{-s})$. *If* $(h_n)_{n\ge 0}$ *is weakly Cauchy in* $A(\mathbb{D})$, *then* $(H_n)_{n\ge 0}$ *is weakly Cauchy in* $\mathcal{A}$. *Moreover*,

(i) $(H_n)_{n\ge 0}$ *is weakly null if and only if* $H_n(ix) \to 0$ *for every* $x \in \mathbb{R}$.
(ii) $(H_n)_{n\ge 0}$ *is weakly Cauchy if and only if* $(H_n(ix))$ *is convergent for every* $x \in \mathbb{R}$.

Proof. First notice that in (i) and (ii) the "only if" part is obvious since $H \mapsto H(ix)$ clearly defines a linear functional on $\mathcal{A}$ for each $x \in \mathbb{R}$.

Observe that, for every $h \in A(\mathbb{D})$, $H(s) = h(2^{-s})$ defines a function in $\mathcal{A}$. Indeed, if $h(z) = \sum c_j z^j$ for z in the open unit disk $\mathbb{D}$, then $H(s) = \sum c_j 2^{-js}$ is convergent for $s \in \mathbb{C}_0$. Moreover, H is continuous on $i\mathbb{R}$.

Now, let ξ be a linear functional on $\mathcal{A}$. We can define a linear functional on $A(\mathbb{D})$ in the following way: $\chi(h) = \xi(H)$, with $H(s) = h(2^{-s})$. The first part of the lemma easily follows: $\xi(H_n)$ converges.

Thus, there is a Borel measure μ on $\mathbb{T}$ such that $\xi(H) = \int_{\mathbb{T}} h\,d\mu$. We can deduce the "if" part in (i) and (ii) because $H_n(ix) = h_n(2^{-ix})$ and the dominated convergence theorem applies. ∎

Now, we can establish the following characterization, which is a generalization of [B1, Th. 8].

Theorem 1.2. *With the previous notations*, *the following assertions are equivalent*:

(1) $T_{u,\varphi} : \mathcal{A} \to \mathcal{H}^\infty$ *is completely continuous.*
(2) $T_{u,\varphi} : \mathcal{A} \to \mathcal{H}^\infty$ *is weakly compact.*
(3) $n_\varphi(u) = 0$.
(4) $T_{u,\varphi} : \mathcal{H}^\infty \to \mathcal{H}^\infty$ *is compact.*

Proof. Obviously (4) implies (1) and (2).

(1)⇒(3). Assume that $\inf \mathrm{Re}(\varphi) = 0$ and $n_\varphi(u) > \varepsilon_0 > 0$.

Choose any sequence $s_j \in \mathbb{C}_0$ such that $\mathrm{Re}(\varphi(s_j))$ converges to 0 and $|u(s_j)| \ge \varepsilon_0$. Extracting a subsequence if necessary, we may suppose that $2^{-\varphi(s_j)}$ converges to some a belonging to the unit circle. We shall write $a = 2^{-i\alpha}$ where $\alpha \in \mathbb{R}$.

Now, we consider the sequence of functions $F_n(s) = f_n(2^{-s})$ where $f_n(z) = 2^{-n}(\bar{a}z+1)^n$ lies in the unit ball of the disk algebra. (F_n) is clearly a weakly Cauchy sequence in $\mathcal{A}$ thanks to Lemma 1.1(ii). Actually $F_n(s) \to 0$ for every $s \in \overline{\mathbb{C}}_0 \setminus \{i\alpha\}$ and $F_n(i\alpha) = 1$.

The operator $T_{u,\varphi}$ being a Dunford–Pettis operator, the sequence $(u \cdot F_n \circ \varphi)_{n\in\mathbb{N}}$ is norm-Cauchy, hence converging to some $\sigma \in \mathcal{H}^\infty$. But for every fixed $s \in \mathbb{C}_0$, $u(s) \cdot F_n \circ \varphi(s)$ converges both to 0 and $\sigma(s)$, so that $\sigma = 0$.

For any fixed $\varepsilon > 0$, there exists n_0 such that $\sup_{s\in\mathbb{C}_0} |u(s)F_{n_0}\circ\varphi(s)| \le \varepsilon$. Choosing $s = s_{j_0}$ with j_0 so large that $|F_{n_0} \circ \varphi(s_{j_0})| \ge 1-\varepsilon$, we have

$$\varepsilon \ge |u(s_{j_0})|(1-\varepsilon) \ge (1-\varepsilon)\varepsilon_0.$$

As ε is arbitrary, this gives a contradiction.

(2)⇒(3). Assume that $\inf \mathrm{Re}(\varphi) = 0$ and $n_\varphi(u) > \varepsilon_0 > 0$. As above, choose any sequence $s_j \in \mathbb{C}_0$ such that $\mathrm{Re}(\varphi(s_j)) \to 0$ and $|u(s_j)| \ge \varepsilon_0$. We may assume that $2^{-\varphi(s_j)}$ converges to some $a = 2^{-i\alpha} \in \mathbb{T}$ and we consider the same sequence of functions F_n. The operator $T_{u,\varphi}$ being weakly compact, there exists a sequence (n_k) of integers such that $(u\cdot F_{n_k}\circ\varphi)_{k\in\mathbb{N}}$ is weakly convergent to some $\sigma \in \mathcal{H}^\infty$. Testing the weak convergence on the point evaluation $\delta_s \in (\mathcal{H}^\infty)^*$, for each $s \in \mathbb{C}_0$, we conclude that $\sigma = 0$.

By the Mazur theorem, there exists a sequence of convex combinations of these functions which is norm convergent to 0:

$$\sum_{k\in I_m} c_k^{(m)} u\cdot(F_{n_k}\circ\varphi) \to 0$$

where $c_k^{(m)} \ge 0$ and $\sum_{k\in I_m} c_k^{(m)} = 1$. Now, fixing $\varepsilon \in (0, \varepsilon_0/2)$, we have, for a suitable m_0,

$$\sup_{s\in\mathbb{C}_0}\Big|\sum_{k\in I_{m_0}} c_k^{(m_0)}\, u(s)\cdot F_{n_k}(\varphi(s))\Big| \le \varepsilon.$$

So, for every j,

$$\varepsilon_0\Big|\sum_{k\in I_{m_0}} c_k^{(m_0)}\cdot F_{n_k}(\varphi(s_j))\Big| \le \Big|\sum_{k\in I_{m_0}} c_k^{(m_0)}\, u(s_j)\cdot F_{n_k}(\varphi(s_j))\Big| \le \varepsilon.$$

Letting j tend to infinity, we have $F_{n_k}(\varphi(s_j)) \to F_{n_k}(i\alpha) = 1$, for each $k \in I_{m_0}$, so that

$$\varepsilon_0 = \varepsilon_0\Big|\sum_{k\in I_{m_0}} c_k^{(m_0)}\Big| \le \varepsilon.$$

This gives a contradiction.

(3)⇒(4). Note that $T_{u,\varphi} = M_u \circ C_\varphi$.

If $\inf \mathrm{Re}(\varphi) > 0$ then $\varphi(\mathbb{C}_0) \subset \mathbb{C}_\varepsilon$ for some $\varepsilon > 0$ and C_φ is compact thanks to Bayart's theorem, recalled in the introduction.

If $\inf \mathrm{Re}(\varphi) = 0$ and $\lim_{r\to 0^+} \sup\{|u(s)|;\ s \in \mathbb{C}_0,\ \mathrm{Re}(\varphi(s)) \le r\} = 0$ then $T_{u,\varphi}$ is compact. Indeed, given a sequence in the unit ball of $\mathcal{H}^\infty$, we can extract a subsequence $(f_n)_n$ uniformly converging on every half-plane $\mathbb{C}_\theta$ with $\theta > 0$. This is due to a version for Dirichlet series of the classical Montel theorem, proved by Bayart (see [B1, Lemma 18] or [B2, Lemme 5.2]). Hence, given $\varepsilon > 0$, we choose $\theta > 0$ such that $|u(s)| \le \varepsilon$ when $\mathrm{Re}(\varphi(s)) \le \theta$.

Then we have

$$\|u\cdot(f_n-f_m)\circ\varphi\|_\infty \le \max\{\|u\|_\infty \sup_{\varphi(s)\in\mathbb{C}_\theta} |(f_n-f_m)\circ\varphi(s)|, 2\varepsilon\},$$

which is less than 2ε when n, m are large enough. ∎

Corollary 1.3. *Let C_φ be a composition operator on $\mathcal{H}^\infty$. The following assertions are equivalent*:

(i) *C_φ is completely continuous.*
(ii) *C_φ is weakly compact.*
(iii) *C_φ is compact.*
(iv) $\inf \mathrm{Re}(\varphi) > 0$.

Proof. If $\inf \mathrm{Re}(\varphi) > 0$, then C_φ is indeed compact. If C_φ is completely continuous (resp. weakly compact) on $\mathcal{H}^\infty$ then its restriction to $\mathcal{A}$ is as well. The result follows from the preceding theorem in the case $u = \mathbb{1}$. ∎

Remark. We have the same results when the operators act from $\mathcal{A}$ into itself (under the extra assumption that $\varphi \in \mathcal{A}$).

From Theorem 1.2, we can deduce

Corollary 1.4. *Let $u \in \mathcal{H}^\infty$.*

(1) *Assume that $E = \{y \in \mathbb{R}; \inf_{x>0} \mathrm{Re}(\varphi(x+iy)) = 0\}$ has positive Lebesgue measure. Then $T_{u,\varphi}$ is weakly compact or completely continuous if and only if $u = 0$.*
(2) *$M_u : \mathcal{A} \to \mathcal{H}^\infty$ is weakly compact or completely continuous if and only if $u = 0$.*

Remark. Actually, the hypothesis on E means that the (nontangential) boundary values of φ, defined almost everywhere on the imaginary axis, vanish on a set of positive Lebesgue measure.

Proof. Under the hypothesis of weak compactness or complete continuity of $T_{u,\varphi}$, we have $n_\varphi(u) = 0$, due to Theorem 1.2. Let us fix $\varepsilon > 0$ and take $r > 0$ such that for every $s \in \mathbb{C}_0$,

$$\mathrm{Re}(\varphi(s)) < r \;\Rightarrow\; |u(s)| \le \varepsilon.$$

The hypothesis on φ implies that, for every $y \in E$, there is a sequence $(x_n)_n$ in $(0,\infty)$ with

$$\mathrm{Re}(\varphi(s_n)) \to 0$$

where $s_n = x_n + iy$. Moreover, we may suppose that $x_n \to 0^+$, since for every $a > 0$, $\varphi(\mathbb{C}_a) \subset \mathbb{C}_b$ for some $b > 0$ (see [GH, Prop. 4.2]). Actually, we could replace s_n by any sequence in $\mathbb{C}_0$ nontangentially converging to iy.

But for almost every $y \in E$ (say for $y \in E_0$ where $E_0 \subset E$ has positive Lebesgue measure), $u(s_n) \to u^*(iy)$, the boundary value of u, defined almost everywhere on the imaginary axis.

Therefore, for every $y \in E_0$ and n large enough, we have $\mathrm{Re}(\varphi(s_n)) < r$, hence $|u^*(iy)| \leq \varepsilon$. Since $\varepsilon > 0$ is arbitrary the boundary value of u vanishes on a set of positive Lebesgue measure, so $u = 0$ everywhere on $\mathbb{C}_0$.

The second point is an immediate consequence of the first one. ■

2. Essential norms. In the following, X denotes either $\mathcal{A}$ or $\mathcal{H}^\infty$. We shall adapt techniques of Section 1 to compute essential norms. We get a generalization of the theorem of Bayart in several directions. We first need the following lower estimate:

LEMMA 2.1. *Let $u \in \mathcal{H}^\infty$ and $\varphi : \mathbb{C}_0 \to \mathbb{C}_0$ defining a composition operator. Assume that $\mathcal{I} \subset \mathcal{W}(X, \mathcal{H}^\infty) \oplus \mathcal{DP}(X, \mathcal{H}^\infty)$. Then*

$$n_\varphi(u) \leq \|T_{u,\varphi}\|_{\mathrm{e},\mathcal{I}}.$$

Proof. The proof combines the one of Theorem 1.2 with that of [B1] (relying on an idea due to Zheng [Z]) and is very similar to the one given in [L] in the framework of classical Hardy spaces. For completeness, we give the details. We already know that $\|T_{u,\varphi}\|_{\mathrm{e},\mathcal{I}} = 0$ if and only if $T_{u,\varphi}$ is completely continuous if and only if $n_\varphi(u) = 0$ if and only if $T_{u,\varphi}$ is compact. We now assume that $T_{u,\varphi}$ is not compact; this implies that $\inf \mathrm{Re}(\varphi) = 0$.

We choose a sequence $s_j \in \mathbb{C}_0$ such that $\mathrm{Re}(\varphi(s_j)) \to 0$ and $|u(s_j)| \to n_\varphi(u)$. We may assume that $2^{-\varphi(s_j)}$ converges to some $a = 2^{-i\alpha}$.

We introduce the sequence of functions (where $n \geq 2$)

$$H_n(s) = \frac{n\bar{a}2^{-s} - (n-1)}{n - (n-1)\bar{a}2^{-s}},$$

which lies in the unit ball of $\mathcal{A}$.

Obviously, $H_n(s) = h_n(2^{-s})$ where h_n lies in the unit ball of the disk algebra, with $h_n(z) \to -1$ for every $z \in \overline{\mathbb{D}} \setminus \{a\}$ and $h_n(a) = 1$. So, $H_n(s) \to -1$ for every $s \in \overline{\mathbb{C}}_0 \setminus \{i\alpha\}$ and $H_n(i\alpha) = 1$.

Now, let $S \in \mathcal{I}$. Write $S = D + W$, where W is weakly compact and D is Dunford–Pettis.

As $D \in \mathcal{DP}(X, \mathcal{H}^\infty)$ and $(H_n)_n$ is a weakly Cauchy sequence by Lemma 1.1, $(D(H_n))_n$ is a Cauchy sequence, hence convergent to some $\Delta \in \mathcal{H}^\infty$.

As $W \in \mathcal{W}(X, \mathcal{H}^\infty)$, up to extracting a subsequence, $(W(H_n))_n$ is weakly convergent to some $w \in \mathcal{H}^\infty$. By the Mazur theorem, we can find some $c_k^{(m)} \geq 0$ with $\sum_{k\in I_m} c_k^{(m)} = 1$, where $I_m \subset \mathbb{N}$, and $\sum_{k\in I_m} c_k^{(m)} W(H_k) \to w$. Moreover, we can assume that $\sup I_m < \inf I_{m+1}$.

Introducing $\widetilde{H}_m = \sum_{k\in I_m} c_k^{(m)} H_k$, we have $\widetilde{H}_m(s) \to -1$ for every $s \in \mathbb{C}_0$, and $\widetilde{H}_m(\varphi(s_j)) \to 1$ for every m. Clearly, $(D(\widetilde{H}_n))_n$ is norm convergent to Δ, so $(S(\widetilde{H}_n))_n$ is norm convergent to $\sigma = \Delta + w$.

For every integer n,

$$\|(T_{u,\varphi} - S)(\widetilde{H}_n)\|_\infty \ge \|T_{u,\varphi}(\widetilde{H}_n) - \sigma\|_\infty - \|S(\widetilde{H}_n) - \sigma\|_\infty$$

and we already know that $\|S(\widetilde{H}_n) - \sigma\|_\infty \to 0$.

For every $s \in \overline{\mathbb{C}}_0 \setminus \{i\alpha\}$, we have $|u(s) \cdot \widetilde{H}_n \circ \varphi(s) - \sigma(s)| \to |\sigma(s) + u(s)|$. If $|\sigma(s_0) + u(s_0)| > n_\varphi(u)$ for some $s_0 \in \mathbb{C}_0$, then

$$\begin{aligned}\|T_{u,\varphi} - S\| &\ge \overline{\lim}\, \|(T_{u,\varphi} - S)(\widetilde{H}_n)\|_\infty \ge \overline{\lim}\, |u(s_0) \cdot \widetilde{H}_n \circ \varphi(s_0) - \sigma(s_0)| \\ &= |\sigma(s_0) + u(s_0)| \ge n_\varphi(u).\end{aligned}$$

If not, then $\|\sigma + u\|_\infty \le n_\varphi(u)$ and $|\sigma(s) - u(s)| \ge 2|u(s)| - n_\varphi(u)$ for every $s \in \mathbb{C}_0$. Then, for every $n \ge 2$ and every integer j,

$$\begin{aligned}\|T_{u,\varphi} - S\| &\ge |u(s_j) \cdot \widetilde{H}_n \circ \varphi(s_j) - \sigma(s_j)| - \|S(\widetilde{H}_n) - \sigma\|_\infty \\ &\ge 2|u(s_j)| - n_\varphi(u) - |u(s_j)| \cdot |\widetilde{H}_n \circ \varphi(s_j) - 1| - \|S(\widetilde{H}_n) - \sigma\|_\infty.\end{aligned}$$

Letting first j tend to infinity, we obtain $\|T_{u,\varphi} - S\| \ge n_\varphi(u) - \|S(\widetilde{H}_n) - \sigma\|_\infty$. Finally, letting $n \to \infty$ yields $\|T_{u,\varphi} - S\| \ge n_\varphi(u)$, and the conclusion follows. ∎

For the upper estimate, we have

Lemma 2.2. *Let $u \in \mathcal{H}^\infty$ and $\varphi : \mathbb{C}_0 \to \mathbb{C}_0$ defining a composition operator. Then*

$$\|T_{u,\varphi}\|_{\mathrm{e}} \le \inf\{2n_\varphi(u), \|u\|_\infty\}.$$

Proof. Fix $\varepsilon > 0$. There exists $r \in (0,1)$ such that for every $s \in \mathbb{C}_0$,

$$\mathrm{Re}(\varphi(s)) \le r \;\Rightarrow\; |u(s)| \le n_\varphi(u) + \varepsilon.$$

Now, fixing $\varrho > 0$ for a while, we introduce the operator defined for $s \in \mathbb{C}_0$ by

$$S(f)(s) = u(s) \cdot f(\varphi(s) + \varrho).$$

In other words, $S = T_{u,\varphi_\varrho}$ with $\varphi_\varrho = \varphi + \varrho$. By the theorem of Bayart, S is compact since $\varphi_\varrho(\mathbb{C}_0) \subset \mathbb{C}_\varrho$. We have

$$\|T_{u,\varphi} - S\| = \sup_{\substack{f \in \mathcal{H}^\infty \\ \|f\|_\infty \le 1}} \sup_{\mathrm{Re}(\varphi(s)) > 0} |u(s)| \cdot |f \circ \varphi(s) - f \circ (\varphi(s) + \varrho)|.$$

First observe that

$$\sup_{\substack{f \in \mathcal{H}^\infty \\ \|f\|_\infty \le 1}} \sup_{\mathrm{Re}(\varphi(s)) \le r} |u(s)| \cdot |f \circ \varphi(s) - f \circ (\varphi(s) + \varrho)| \le 2(n_\varphi(u) + \varepsilon).$$

On the other hand, we claim that

$$\sup_{\substack{f \in \mathcal{H}^\infty \\ \|f\|_\infty \le 1}} \sup_{\mathrm{Re}(\varphi(s)) > r} |f \circ \varphi(s) - f \circ (\varphi(s) + \varrho)| \xrightarrow{\varrho \to 0^+} 0.$$

Indeed,

$$\sup_{\substack{f\in\mathcal{H}^\infty\\ \|f\|_\infty\le 1}} \sup_{\operatorname{Re}(\varphi(s))>r} |f\circ\varphi(s)-f\circ(\varphi(s)+\varrho)| \le \sup_{\substack{f\in\mathcal{H}^\infty\\ \|f\|_\infty\le 1}} \sup_{\operatorname{Re}(w)>r} |f(w)-f(w+\varrho)|$$

and using the analogue for Dirichlet series of the Montel theorem (cited above), it is easy to see that

$$\lim_{\varrho\to 0^+} \sup_{\operatorname{Re}(w)>r} \sup_{\substack{f\in\mathcal{H}^\infty\\ \|f\|_\infty\le 1}} |f(w)-f(w+\varrho)| = 0.$$

So we can choose $\varrho > 0$ such that

$$\sup_{\substack{f\in\mathcal{H}^\infty\\ \|f\|_\infty\le 1}} \sup_{\operatorname{Re}(\varphi(s))>r} |f\circ\varphi(s)-f\circ(\varphi(s)+\varrho)| \le \varepsilon.$$

Finally,

$$\|T_{u,\varphi}-S\| \le \max\{\varepsilon\|u\|_\infty, 2(n_\varphi(u)+\varepsilon)\}.$$

As $\varepsilon > 0$ is arbitrary, we conclude that $\|T_{u,\varphi}\|_{\mathrm{e}} \le 2n_\varphi(u)$. This gives the result. ∎

We summarize our results in the following theorem.

THEOREM 2.3. *Let $u \in \mathcal{H}^\infty$ and $\varphi : \mathbb{C}_0 \to \mathbb{C}_0$ defining a composition operator. Assume that $\mathcal{K}(X,\mathcal{H}^\infty) \subset \mathcal{I} \subset \mathcal{W}(X,\mathcal{H}^\infty) \oplus \mathcal{DP}(X,\mathcal{H}^\infty)$. Then*

$$\|T_{u,\varphi}\|_{\mathrm{e},\mathcal{I}} \approx n_\varphi(u).$$

More precisely,

$$n_\varphi(u) \le \|T_{u,\varphi}\|_{\mathrm{e},\mathcal{I}} \le \inf\{2n_\varphi(u), \|u\|_\infty\}.$$

As a particular case, *when $n_\varphi(u) = \|u\|_\infty$*, *we have the equality $\|T_{u,\varphi}\|_{\mathrm{e},\mathcal{I}} = \|T_{u,\varphi}\|_{\mathrm{e}} = \|u\|_\infty$.*

We specify two particular cases.

COROLLARY 2.4. *Let $u \in \mathcal{H}^\infty$ and $\varphi : \mathbb{C}_0 \to \mathbb{C}_0$ defining a composition operator. Assume that $\mathcal{K}(X,\mathcal{H}^\infty) \subset \mathcal{I} \subset \mathcal{W}(X,\mathcal{H}^\infty) \oplus \mathcal{DP}(X,\mathcal{H}^\infty)$. Then*

(1) $\|M_u\|_{\mathrm{e},\mathcal{I}} = \|M_u\|_{\mathrm{e}} = \|u\|_\infty$.
(2) $\|C_\varphi\|_{\mathrm{e},\mathcal{I}} = 1$ *if* $\inf \operatorname{Re}(\varphi) = 0$, *and* $\|C_\varphi\|_{\mathrm{e},\mathcal{I}} = 0$ *if* $\inf \operatorname{Re}(\varphi) > 0$.

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