Duality, reflexivity and atomic decompositions in Banach spaces

by

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Abstract. We study atomic decompositions and their relationship with duality and reflexivity of Banach spaces. To this end, we extend the concepts of "shrinking" and "boundedly complete" Schauder basis to the atomic decomposition framework. This allows us to answer a basic duality question: when an atomic decomposition for a Banach space generates, by duality, an atomic decomposition for its dual space. We also characterize the reflexivity of a Banach space in terms of properties of its atomic decompositions.

Introduction. The concept of Schauder basis for a Banach space X is a natural way to introduce a coordinate system in the Banach space setting. This notion corresponds to orthogonal basis in Hilbert spaces. The biorthogonal dual sequence, in X', recovers the coordinates of the expansion of the elements of X in terms of the basis. In general, this dual sequence is not necessarily a basis for the whole space X', but only for the closed subspace the sequence spans. In the context of nonharmonic Fourier series, Duffin and Schaeffer [9] introduced frames for Hilbert spaces. These are sequences that allow reconstruction formulae that need not be unique, as opposed to what happens with bases. This redundancy makes frames useful in different applications such as sampling theory and noise reduction in image and signal processing (see, for instance, [1, 7, 16]).

In the eighties, Gröchenig [12] introduced atomic decompositions as an extension of the notion of frames from Hilbert space theory to a Banach space setting. Another concept extending the notion of Hilbert space frames is that of Banach frames. Both new concepts have much in common and in many situations are equivalent. Loosely speaking, atomic decompositions involve the existence of a reconstruction formula as a series expansion, while Banach frames give rise to a reconstruction operator.

We concentrate on some mathematical aspects of atomic decompositions from a functional analytic point of view, within the framework of abstract

approximation theory in Banach spaces. The results obtained can be applied to the study of approximation of functions on Banach spaces admitting this type of structure (see, for example, [2]).

In this article, we relate duality and reflexivity with atomic decompositions. In the Schauder basis setting, properties of a basis of a Banach space X allow us to determine if its dual basic sequence is actually a basis of X'. Also, the reflexivity of X can be characterized in terms of attributes of the basis [13, 14].

The study of duality for atomic decompositions has a natural approach. Indeed, if $((x_i'), (x_i))$ is an atomic decomposition for X, we investigate conditions ensuring that $((x_i), (x_i'))$ is an atomic decomposition for X'. In order to deal with this question we introduce the concept of shrinking and strongly shrinking atomic decomposition, which extend the notion of shrinking Schauder basis. We show that an atomic decomposition for X is shrinking if and only if the dual pair is an atomic decomposition for the dual space X'. Examples are presented in order to clarify the situation and to compare it with the Schauder basis context.

A Banach space X with Schauder basis is reflexive if and only if the basis is shrinking and boundedly complete. We introduce the notion of boundedly complete atomic decomposition and discuss the connection between these two properties of an atomic decomposition and the reflexivity of X.

For further information on atomic decompositions and Banach frames see, for example, [5, 6, 12] and the references therein. We refer to [8, 15] for background in Banach space theory.

1. Duality for atomic decompositions. By a *Banach sequence space* we mean a Banach space of scalar sequences for which the coordinate functionals are continuous. We say that the space is a *Schauder sequence space* if, in addition, the unit vectors $\{e_i\}$ given by $(e_i)_j = \delta_{i,j}$ form a basis for it. In this case, a sequence $a = (a_i)$ can be written as $a = \sum_i a_i e_i$.

DEFINITION 1.1. Let X be a Banach space and Z be a Banach sequence space. Let (x_i') and (x_i) be sequences in X' and X respectively. We say that $((x_i'), (x_i))$ is an *atomic decomposition* of X with respect to Z if for all $x \in X$:

(a) $(\langle x_i', x \rangle) \in Z$,
(b) $A\|x\| \le \|(\langle x_i', x \rangle)\|_Z \le B\|x\|$, with A and B positive constants,
(c) $x = \sum_i \langle x_i', x \rangle x_i$.

We will often refer to property (c) in the above definition as the *reconstruction formula* associated to the atomic decomposition.

A separable Banach space admits an atomic decomposition if and only if it has the bounded approximation property (this result is contained in [17]). Moreover, if $((x_i'), (x_i))$ is an atomic decomposition of X with respect

to some Banach sequence space Z, we can always find a Schauder sequence space X_d and an operator $S \colon X_d \to X$ such that $Se_i = x_i$ and $((x_i'), (x_i))$ is also an atomic decomposition of X with respect to X_d [17, 4]. In the following, we consider atomic decompositions of the form $((x_i'), (Se_i))$ associated to a Schauder sequence space X_d. See [3] for a complete treatment of the approximation property.

Whenever a sequence $(x_i') \subset X'$ satisfies conditions (a) and (b) of Definition 1.1 and there exists a linear operator $S \colon X_d \to X$ satisfying $S(\langle x_i', x \rangle) = x$ for all $x \in X$, the pair $((x_i'), S)$ is said to be a *Banach frame* for X with respect to X_d [4, 5]. Note that any atomic decomposition associated to an operator S defines a Banach frame, with the additional property of the reconstruction formula given by property (c) (which is stronger than having the identity $S(\langle x_i', x \rangle) = x$). In the case where X_d is a Schauder sequence space, $((x_i'), (Se_i))$ is an atomic decomposition if and only if $((x_i'), S)$ is a Banach frame. Therefore, all the results stated below for atomic decompositions can be reformulated for Banach frames with respect to Schauder sequence spaces.

If $((x_i'), (Se_i))$ is an atomic decomposition of X with respect to X_d, we define the natural inclusion $J \colon X \to X_d$ by

$$(1.1) \qquad J(x) = (\langle x_i', x \rangle) = \sum_i \langle x_i', x \rangle e_i.$$

If (e_i') is the dual basic sequence of (e_i), then $x_i' = J' e_i'$. Note that $SJ = I_X$ and so X is isomorphic to a complemented subspace of X_d. Conversely, we have the following

REMARK 1.2. Let X be a Banach space and X_d be a Schauder sequence space. Suppose there exist continuous operators $J \colon X \to X_d$ and $S \colon X_d \to X$ such that $SJ = I_X$. Then, if (e_i') is the dual basic sequence of (e_i), the pair $((J' e_i'), (Se_i))$ is an atomic decomposition for X with respect to X_d.

Indeed, condition (a) in the definition is satisfied since Jx belongs to X_d for all $x \in X$. Condition (c) holds by the continuity of S, and the constants A and B of condition (b) may be chosen to be $\|S\|^{-1}$ and $\|J\|$ respectively.

A natural question arises: if $((x_i'), (Se_i))$ is an atomic decomposition for X, is $((Se_i), (x_i'))$ an atomic decomposition for X'?

The notion of shrinking basis gives an answer to this question in the case of Schauder bases instead of atomic decompositions. Indeed, for a Schauder basis (x_i) of X, its dual sequence (x_i') is a Schauder basis of X' if and only if (x_i) is shrinking. Recall that (x_i) is said to be *shrinking* if for all $x' \in X'$,

$$\lim_{N \to \infty} \|x'|_{[x_i \,:\, i \geq N]}\| = 0,$$

where $[x_i : i \geq N]$ is the subspace spanned by $\{x_i\}_{i \geq N}$.

Our goal is then to extend the notion of shrinking basis to shrinking atomic decompositions. A proper definition for a shrinking atomic decomposition should satisfy the following: an atomic decomposition $((x_i'), (Se_i))$ of X with respect to X_d is shrinking if and only if $((Se_i), (x_i'))$ is an atomic decomposition of X' with respect to some sequence space. In order to establish the correct definition, we believe the following discussion is necessary.

First, note that if X_d is a Schauder sequence space, then the dual $(X_d)'$ is a Banach sequence space, provided we identify each linear functional $b' \in (X_d)'$ with the sequence $(\langle b', e_i \rangle)$.

One way to ensure that $((Se_i), (x_i'))$ is an atomic decomposition of X' is to require that (e_i) be a shrinking basis of X_d (see Proposition 1.9). In this case, (e_i') turns out to be a basis of $(X_d)'$ and this last space is therefore a Schauder sequence space.

Another natural attempt is to impose that

$$(1.2) \qquad \lim_{N \to \infty} \|x'|_{[Se_i : i \geq N]}\| = 0 \quad \text{for all } x' \in X'.$$

As we will see, condition (1.2) implies that $((Se_i), (x_i'))$ is an atomic decomposition of X'. This condition is the most direct extension of a shrinking basis. However, it does not take into account the linear functionals (x_i'). Although for a basis the dual linear functionals are uniquely determined, the sequence (Se_i) by itself does not characterize the atomic decomposition. Therefore, one should not expect (1.2) to be the appropriate definition for a shrinking atomic decomposition.

In fact, none of the previous two conditions is equivalent to $((Se_i), (x_i'))$ being an atomic decomposition of X'. Moreover, we will see that both conditions are strictly stronger than desired (see Proposition 1.9 and Examples 1.10, 1.11).

Although we state our results for atomic decompositions which are equivalent to Banach frames, we want to give the definition of shrinking atomic decompositions for the general case. Let, then, $((x_i'), (x_i))$ be an atomic decomposition of X with respect to a Banach sequence space Z. For each $N \in \mathbb{N}$, define the linear operator $T_N \colon X \to X$ by

$$(1.3) \qquad T_N(x) = \sum_{i \geq N} \langle x_i', x \rangle x_i.$$

Each T_N is a bounded linear operator as a consequence of the Banach–Steinhaus theorem. Also, again by that theorem, they are uniformly bounded. Now we define:

DEFINITION 1.3. Let X be a Banach space, Z be a Banach sequence space and $((x_i'), (x_i))$ be an atomic decomposition of X with respect to Z.

We say that $((x'_i), (x_i))$ is *shrinking* if for all $x' \in X'$,

$$(1.4) \qquad \|x' \circ T_N\| \xrightarrow[N \to \infty]{} 0,$$

where T_N is defined by (1.3).

THEOREM 1.4. *Let X be a Banach space and X_d be a Schauder sequence space. Let $S \colon X_d \to X$ be a continuous operator and $(x'_i) \subset X'$ be a sequence so that $((x'_i), (Se_i))$ is an atomic decomposition for X with respect to X_d. Then $((Se_i), (x'_i))$ is an atomic decomposition for X' with respect to $(X_d)'$ if and only if $((x'_i), (Se_i))$ is shrinking.*

Proof. Assume that $((x'_i), (Se_i))$ is a shrinking atomic decomposition for X with respect to X_d. Since $\langle x', Se_i \rangle = \langle S'x', e_i \rangle$ for all i, the sequence $(\langle x', Se_i \rangle)$ belongs to $(X_d)'$ considered as a Banach sequence space, and condition (a) is satisfied.

To show that the norms $\|x'\|$ and $\|(\langle x', Se_i \rangle)\|$ are equivalent, note that S is surjective, therefore S' is an isomorphism onto its image.

Finally, we have to show the validity of the reconstruction formula, i.e., $x' = \sum_i \langle x', x_i \rangle x'_i$, where $x_i = Se_i$. It is clear that $\langle x', x \rangle = \sum_i \langle x', x_i \rangle \langle x'_i, x \rangle$ for all $x \in X$, so it only remains to show that the series $\sum_i \langle x', x_i \rangle x'_i$ converges. Let us see that it is a Cauchy series:

$$\sup_{\|x\| \leq 1} \Big| \sum_{i=N}^{M-1} \langle x', x_i \rangle \langle x'_i, x \rangle \Big| = \sup_{\|x\| \leq 1} |\langle x', (T_M - T_N)x \rangle|$$

$$\leq \|x' \circ T_M\| + \|x' \circ T_N\| \xrightarrow[N,M \to \infty]{} 0.$$

Therefore, we have shown that $((Se_i), (x'_i))$ is an atomic decomposition for X' with respect to $(X_d)'$.

The converse follows immediately from

$$\|x' \circ T_N\| = \sup_{\|x\| \leq 1} |\langle x', T_N x \rangle| = \Big\| \sum_{i \geq N} \langle x', Se_i \rangle x'_i \Big\|$$

and the fact that $((Se_i), (x'_i))$ is an atomic decomposition for X' with respect to $(X_d)'$. ∎

If X_d is a Schauder sequence space, it is easy to see that the pair $((Se_i), J')$ defines a Banach frame for X' with respect to $(X_d)'$, even if the atomic decomposition $((x'_i), (Se_i))$ is not shrinking. Indeed, since $J'S' = \mathrm{Id}_{X'}$ and $S'x' = (\langle x', Se_i \rangle)_i$, we have $J'(\langle x', Se_i \rangle) = x'$. The continuity of S and J gives the frame bounds. However, in this case the reconstruction formula does not necessarily hold.

On the other hand, if $((x'_i), (Se_i))$ is shrinking and the range of S' is contained in the closed subspace of $(X_d)'$ spanned by the dual sequence (e'_i), then

we can obtain an atomic decomposition for X' with respect to a Schauder sequence space. This fact motivates the definition of strongly shrinking atomic decompositions. First we state the following corollary, which is an immediate consequence of the relationship between atomic decompositions and the bounded approximation property:

COROLLARY 1.5. *Suppose that X admits a shrinking atomic decomposition $((x_i'), (Se_i))$. Then X' is separable and has the bounded approximation property.*

The existence of the operator S provides us with a sequence (S_N) of continuous operators which is useful to avoid the obstacle arising from the fact that we work with nonunique representations.

Fix N, and consider the operator $S_N \colon X_d \to X$ defined by

$$S_N(a) = \sum_{i \geq N} a_i x_i.$$

Then (S_N) is bounded by $2K\|S\|$, where K is the basis constant of (e_i).

DEFINITION 1.6. Let X be a Banach space, X_d be a Schauder sequence space, and $S \colon X_d \to X$ be a continuous operator such that $((x_i'), (Se_i))$ is an atomic decomposition of X with respect to X_d. We say that $((x_i'), (Se_i))$ is *strongly shrinking* if for all $x' \in X'$,

$$\|x' \circ S_N\| \to 0 \quad \text{as } N \to \infty,$$

where $S_N \colon X_d \to X$ is given by $S_N(a) = \sum_{i \geq N} a_i x_i$.

It is clear that any strongly shrinking atomic decomposition is shrinking, since $T_N = S_N \circ J$. On the other hand, at the end of this section we present examples showing that the converse is not true.

Before stating the duality property of strongly shrinking atomic decompositions we need the following:

LEMMA 1.7. *Let X be a Banach space and X_d be a Schauder sequence space. Let $S \colon X_d \to X$ be a continuous operator. If $((x_i'), (Se_i))$ is a strongly shrinking atomic decomposition then $S'(X')$ is contained in the closure of $[e_i' : i \geq 1]$ in $(X_d)'$ and $S'x' = \sum_i \langle x', Se_i \rangle e_i'$ for all $x' \in X'$.*

Proof. Put $x_i = Se_i$. Since $\langle S'x', e_i \rangle = \langle x', x_i \rangle$, it is enough to see that $\sum_{i=1}^{M} \langle x', x_i \rangle e_i'$ is a Cauchy sequence in $(X_d)'$. Now

$$\left\| \sum_{i=N}^{M-1} \langle x', x_i \rangle e_i' \right\| = \sup_{\|a\| \leq 1} \left| \sum_{i=N}^{M-1} \langle x', x_i \rangle \langle e_i', a \rangle \right| = \sup_{\|a\| \leq 1} \left| \left\langle x', \sum_{i=N}^{M-1} a_i x_i \right\rangle \right|$$

$$= \sup_{\|a\| \leq 1} |\langle x', (S_N - S_M)a \rangle| \leq \|x' \circ S_N\| + \|x' \circ S_M\|.$$

Since $((x_i'), (Se_i))$ is strongly shrinking, the result follows. ∎

If X is a Banach space and X_d is an associated Schauder sequence space with basis (e_i), we denote by X_d' the closed subspace spanned by (e_i') in $(X_d)'$, where (e_i') is the dual sequence of (e_i). Since (e_i') is a basic sequence, X_d' is a Schauder sequence space.

Now we can establish:

THEOREM 1.8. *Let X be a Banach space and X_d be a Schauder sequence space. Let $S: X_d \to X$ be a continuous operator and $(x_i') \subset X'$ be a sequence such that $((x_i'), (Se_i))$ is an atomic decomposition for X with respect to X_d. Then $((Se_i), (x_i'))$ is an atomic decomposition for X' with respect to X_d' if and only if $((x_i'), (Se_i))$ is strongly shrinking.*

Proof. If $((x_i'), (Se_i))$ is a strongly shrinking atomic decomposition for X with respect to X_d, it is shrinking. Then a combination of Lemma 1.7 and Theorem 1.4 shows that $((Se_i), (x_i'))$ is an atomic decomposition for X' with respect to X_d'. The converse follows as in the proof of Theorem 1.4. ∎

Now we see that our first two *candidates* for the definition of shrinking atomic decomposition are strictly stronger than the proper definition. Moreover, they are even stronger than the definition of strongly shrinking atomic decomposition. First we have

PROPOSITION 1.9. *Let X be a Banach space, X_d be a sequence space and $S: X_d \to X$ be a continuous operator such that $((x_i'), (Se_i))$ is an atomic decomposition of X with respect to X_d. Consider the following statements*:

(a) *the basis (e_i) of X_d is shrinking*,
(b) *(Se_i) satisfies condition (1.2), i.e. $\lim_{N \to \infty} \|x'|_{[Se_i : i \geq N]}\| = 0$ for all $x' \in X'$.*

Then either (a) *or* (b) *implies that $((x_i'), (Se_i))$ is a strongly shrinking atomic decomposition.*

Proof. If (a) holds, let K be the basis constant of the shrinking basis (e_i) and $F_N = [e_i : i \geq N]$. We have

$$\|x' \circ S_N\| = \sup_{\|a\| \leq 1} \left| \left\langle S'x', \sum_{i \geq N} a_i e_i \right\rangle \right| \leq \sup_{\substack{a \in F_N \\ \|a\| \leq 1+K}} \left| \left\langle S'x', \sum_{i \geq N} a_i e_i \right\rangle \right|$$

$$= (1 + K)\|S'x'|_{F_N}\|,$$

which goes to zero since (e_i) is shrinking. The proof for (b) is straightforward. ∎

Now, we present counterexamples for the converses.

EXAMPLE 1.10 (A strongly shrinking atomic decomposition for which statement (a) of Proposition 1.9 does not hold). Take $X = \ell_2$, $X_d = \ell_1 \oplus \ell_2$ with the basis (f_k) that alternates $(e_i, 0)$ and $(0, e_i)$. Consider the projection

$S\colon X_d \to X$ on the second coordinate. If we define $x'_{2j-1} = 0$ and $x'_{2j} = e'_j$ on X for $j = 1, 2, \ldots$, then $((x'_k), (Sf_k))$ is a strongly shrinking atomic decomposition of X with respect to X_d, since condition (b) of Proposition 1.9 is satisfied. However, the basis for X_d is not shrinking. Indeed, $a' = (1, 1, \ldots) \times (0, 0, \ldots) \in \ell_\infty \times \ell_2$ satisfies $\|a'|_{[f_k\colon k \geq N]}\| = 1$ for all N.

EXAMPLE 1.11 (A strongly shrinking atomic decomposition for which statement (b) of Proposition 1.9 does not hold). We show a family of examples of strongly shrinking atomic decompositions not satisfying condition (b). Take $X = X_d = \ell_2$, fix $b \in \ell_2$ with $b_j \neq 0$ for all j and $b_1 = 1$ and consider the operator $S\colon X_d \to X$ given by the infinite matrix

$$\begin{pmatrix} b_1 & b_2 & 0 & b_3 & 0 & 0 & b_4 & 0 & 0 & 0 & b_5 & 0 & 0 & 0 & 0 & b_6 & \cdots \\ 0 & 0 & b_1 & 0 & b_2 & 0 & 0 & b_3 & 0 & 0 & 0 & b_4 & 0 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & b_2 & 0 & 0 & 0 & b_3 & 0 & 0 & 0 & \cdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & b_1 & 0 & 0 & 0 & b_2 & 0 & 0 & \cdots \\ \vdots & \vdots & \vdots & & & & & & & & & & & & & & \ddots \end{pmatrix}$$

Note that each column of the matrix has a unique nonzero coefficient. Then consider the subsequences

$$N_i = \{j \in \mathbb{N}\colon \text{the element in column } j \text{ and row } i \text{ is nonzero}\}$$

obtaining the disjoint partition $\mathbb{N} = \bigcup_i N_i$. If we write $N_i = \{(n_{ij})_j\}$, for any $a \in \ell_2$ we obtain

$$Sa = \sum_{i=1}^{\infty} \Big(\sum_{j=1}^{\infty} b_j a_{n_{ij}} \Big) e_i.$$

Also, note that for each i, the element in row i and column $i(i+1)/2$ is $b_1 = 1$. This means that $Se_{i(i+1)/2} = e_i$ for all i.

Hölder's inequality gives the continuity of S:

$$\|Sa\|^2 = \sum_{i=1}^{\infty} \Big| \sum_{j=1}^{\infty} b_j a_{n_{ij}} \Big|^2 \leq \sum_{i=1}^{\infty} \Big(\sum_{j=1}^{\infty} |b_j|^2 \Big) \Big(\sum_{j=1}^{\infty} |a_{n_{ij}}|^2 \Big) = \|b\|^2 \|a\|^2.$$

Let $x'_k = e'_i$ if $k = i(i+1)/2$ and $x'_k = 0$ otherwise. It is clear that $(\langle x'_k, x \rangle)_k \in X_d$ for all $x \in \ell_2$. Also,

$$\sum_{k=1}^{\infty} \langle x'_k, x \rangle Se_k = \sum_{i=1}^{\infty} \langle x'_{i(i+1)/2}, x \rangle Se_{i(i+1)/2} = \sum_{i=1}^{\infty} \langle e'_i, x \rangle e_i = x.$$

If $J\colon X \to X_d$ is the canonical inclusion, $\|Jx\| \leq \|x\|$ and $\|x\| = \|SJx\| \leq \|b\| \, \|(\langle x'_i, x \rangle)\|$, and then $((x'_i), (Se_i))$ is an atomic decomposition.

To see it is strongly shrinking, it is enough to observe that condition (a) of Proposition 1.9 holds. On the other hand, for any N, the subspace

$[Se_i : i \geq N]$ is in fact all ℓ_2. Therefore, condition (b) of Proposition 1.9 is not satisfied.

It is clear that different examples can be obtained for any disjoint partition $\mathbb{N} = \bigcup_i N_i$.

Note that in Example 1.10 the atomic decomposition satisfies condition (b) but not (a) while in Example 1.11 condition (a) is satisfied but (b) is not. A direct combination of both examples provides a strongly shrinking atomic decomposition for which neither (a) nor (b) of Remark 1.9 is satisfied.

The following examples show that shrinking and strongly shrinking are not equivalent notions for atomic decompositions. Note that these two concepts coincide for Schauder bases. Also, a Schauder basis of a reflexive Banach space is always shrinking, and consequently strongly shrinking. The examples show that a reflexive space can admit an atomic decomposition that is not strongly shrinking. The first example is a modification of Example 1.10. We thank the referee for kindly suggesting it to us.

EXAMPLE 1.12 (An atomic decomposition for a reflexive Banach space X which is shrinking but not strongly shrinking). Take $X = \ell_2$, $X_d = \ell_1 \oplus \ell_2$ with the basis (f_k) that alternates $(e_i, 0)$ and $(0, e_i)$. We define $S \colon X_d \to X$ by $S(e_i, 0) = e_1$, $S(0, e_i) = e_i$, and $x'_{2j-1} = 0$ and $x'_{2j} = e'_j$ on X for $j = 1, 2, \ldots$. Since $\|(e'_1, 0) \circ S_N\| = 1$ for all N, we see that $((x'_k), (Sf_k))$ is not strongly shrinking. By a direct calculation or using Theorem 1.4 we see that it is shrinking.

EXAMPLE 1.13 (Another atomic decomposition for a reflexive Banach space X which is shrinking but not strongly shrinking). Let $X = \ell_2$ and let X_d be the full 3-fold projective tensor product $X_d = \bigotimes_\pi^3 \ell_2$. This is a sequence space with basis $(e_\alpha)_{\alpha \in \mathbb{N}^3}$, where $e_\alpha = e_{\alpha_1} \otimes e_{\alpha_2} \otimes e_{\alpha_3}$ and $\mathbb{N}^3$ is considered with the square ordering [10] (see also [11]). Let Φ be the bilinear form on $\ell_2 \times \ell_2$ given by $\Phi(y, z) = \sum_{k \geq 1} y_k z_k$. Consider the continuous operators $S \colon X_d \to X$, $S(x \otimes y \otimes z) = \Phi(y, z)x$, and $J \colon X \to X_d$, $J(x) = x \otimes e_1 \otimes e_1$, and set $x'_\alpha = J'(e'_{\alpha_1} \otimes e'_{\alpha_2} \otimes e'_{\alpha_3})$. Note that $S(e_{\alpha_1} \otimes e_{\alpha_2} \otimes e_{\alpha_3}) = e_{\alpha_1}$. We claim that the pair $((x'_\alpha)_{\alpha \in \mathbb{N}^3}, (e_{\alpha_1})_{\alpha \in \mathbb{N}^3})$ is a shrinking atomic decomposition for ℓ_2 with respect to $\bigotimes_\pi^3 \ell_2$ which is not strongly shrinking. By Remark 1.2, to see that it is an atomic decomposition we only need to show that $SJ = I_{\ell_2}$, which is clear since $\Phi(e_1, e_1) = 1$.

To see that $((x'_\alpha), (e_{\alpha_1}))$ is shrinking, by Theorem 1.4, it is enough to check the reconstruction formula for each $x' \in (\ell_2)' = \ell_2$ with respect to the dual atomic decomposition $((e_{\alpha_1}), (x'_\alpha))$. Since $x'_\alpha = \delta_{\alpha_2,1} \cdot \delta_{\alpha_3,1} \cdot e'_{\alpha_1}$, we have

$$x' = \sum_{\alpha_1 \in \mathbb{N}} \langle x', e_{\alpha_1} \rangle e'_{\alpha_1} = \sum_{\alpha \in \mathbb{N}^3} \langle x', e_{\alpha_1} \rangle \delta_{\alpha_2,1} \cdot \delta_{\alpha_3,1} \cdot e'_{\alpha_1} = \sum_{\alpha \in \mathbb{N}^3} \langle x', e_{\alpha_1} \rangle x'_\alpha.$$

Now, suppose it is a strongly shrinking atomic decomposition. By Lemma 1.7, $S'(X')$ should be contained in $X'_d = \overline{[(e_\alpha)']}$. Identifying $(\bigotimes^3_\pi \ell_2)'$ with the space of 3-linear forms on ℓ_2, we conclude that $S'(X')$ must consist of trilinear forms which are separately weakly continuous on bounded sets.

Given $x' \in X'$ choose a unit vector $x \in \ell_2$ so that $\langle x', x \rangle = \|x'\|$. Then, as a trilinear form,

$$S'(x')(x, y, z) = \langle x', S(x \otimes y \otimes z) \rangle = \Phi(y, z) \langle x', x \rangle.$$

Now, $(e_N)_N$ is a weakly null sequence in ℓ_2 but $S'(x')(x, e_N, e_N) = \|x'\|$. Therefore, $S'(x')$ does not belong to X'_d for any nonzero $x' \in X'$ and the atomic decomposition is not strongly shrinking.

2. Atomic decompositions and reflexivity. Banach spaces with Schauder basis admit a characterization of reflexivity in terms of the properties of the basis. To be precise, a Banach space X with Schauder basis (e_i) is reflexive if and only if (e_i) is shrinking and boundedly complete.

Recall that a Schauder basis (e_i) of X is said to be *boundedly complete* if for each scalar sequence (a_i) such that

$$\sup_N \left\| \sum_{i=1}^N a_i e_i \right\| < \infty,$$

the series $\sum_{i=1}^\infty a_i e_i$ converges in X. The following proposition establishes an equivalent formulation of this definition. This result can be found, for instance, in [18, Theorem II.6.2]. Nevertheless, we prefer to include a proof for the sake of completeness.

PROPOSITION 2.1. *Let X be a Schauder sequence space with basis (e_i). Then (e_i) is boundedly complete if and only if for all $x'' \in X''$, the series $\sum_{i=1}^\infty \langle x'', e'_i \rangle e_i$ converges.*

Proof. Suppose (e_i) is boundedly complete with basis constant K. Take $x'' \in X''$. To show that $\sum_{i=1}^N \langle x'', e'_i \rangle e_i$ is uniformly bounded we consider its norm as the supremum over $x' \in B_{X'}$; then

$$\left| \left\langle x', \sum_{i=1}^N \langle x'', e'_i \rangle e_i \right\rangle \right| = \left| \left\langle x'', \sum_{i=1}^N \langle x', e_i \rangle e'_i \right\rangle \right| \leq \|x''\| \left\| \sum_{i=1}^N \langle x', e_i \rangle e'_i \right\|.$$

Since

$$\left\| \sum_{i=1}^N \langle x', e_i \rangle e'_i \right\| = \sup_{\|x\| \leq 1} \left| \left\langle x', \sum_{i=1}^N \langle e'_i, x \rangle e_i \right\rangle \right| \leq K \|x'\|,$$

we see that $K \|x''\|$ is a bound for $\| \sum_{i=1}^N \langle x'', e'_i \rangle e_i \|$, for all $N \in \mathbb{N}$.

For the converse, take a scalar sequence (a_i) such that $\sup_N \| \sum_{i=1}^N a_i e_i \|$ is bounded and consider $\varphi_N = \sum_{i=1}^N a_i e_i \in X''$. Then there exists a subnet

$(\varphi_{n(\alpha)})$ of (φ_N) and $x'' \in X''$ so that $x'' = w^*\text{-}\lim \varphi_{n(\alpha)}$. Then $\langle x'', e'_k \rangle = a_k$ for each $k \in \mathbb{N}$, and therefore $\sum_{i=1}^{\infty} a_i e_i = \sum_{i=1}^{\infty} \langle x'', e'_i \rangle e_i$ converges. ∎

Note that the equivalent condition involves both the basis $(e_i)_i$ and its dual basic sequence $(e'_i)_i$. Therefore, this formulation is more suitable for extension to atomic decompositions:

DEFINITION 2.2. Let X be a Banach space and let Z be a Banach sequence space. Let $((x'_i), (x_i))$ be an atomic decomposition of X with respect to Z. The atomic decomposition is said to be *boundedly complete* if for each $x'' \in X''$, the series $\sum_{i=1}^{\infty} \langle x'', x'_i \rangle x_i$ converges in X.

It is immediate that whenever X is reflexive, any atomic decomposition is boundedly complete. Moreover, we have:

PROPOSITION 2.3. *Let X be a Banach space and X_d be a Schauder sequence space. Let $S \colon X_d \to X$ be a continuous operator and $(x'_i) \subset X'$ be a sequence such that $((x'_i), (Se_i))$ is a shrinking atomic decomposition for X with respect to X_d. Then $((Se_i), (x'_i))$ is a boundedly complete atomic decomposition for X' with respect to $(X_d)'$.*

Proof. Let $\varrho \colon X''' \to X'$ be the restriction mapping. Since $((x'_i), (Se_i))$ is shrinking, $((Se_i), (x'_i))$ is an atomic decomposition for X' with respect to $(X_d)'$. For any x''' in X''', we can apply the reconstruction formula to $x' = \rho(x''')$ to deduce that $\sum_{i=1}^{N} \langle x''', Se_i \rangle x'_i$ is convergent. Then $((Se_i), (x'_i))$ is boundedly complete. ∎

The analogous proposition for Schauder bases has a dual result: if a Banach space X has a boundedly complete basis (e_i), then X is isomorphic to the dual of a Banach space with shrinking basis (namely, the closed subspace of X' spanned by the dual basic sequence (e'_i)). The fact that the biorthogonal sequence (e'_i) is always a basic sequence in X' is crucial for the proof of this result.

For an atomic decomposition $((x'_i), (x_i))$, we do not know if $((x_i), (x'_i))$ is in general an atomic decomposition of the closed subspace of X' spanned by $(x'_i)_i$. If this is the case and $((x'_i), (x_i))$ is boundedly complete, it is not difficult to see that X is isomorphic to the dual of $Y = [(x'_i)_i]$, and that $((x_i), (x'_i))$ is a shrinking atomic decomposition of Y.

Also, not knowing if $((x_i), (x'_i))$ is an atomic decomposition (of the closure of $[(x'_i)_i]$) prevents us from giving, with full generality, a characterization of reflexivity in terms of properties of the atomic decomposition of X.

We have:

PROPOSITION 2.4. *Let X be a Banach space admitting an atomic decomposition $((x'_i), (x_i))$ (with respect to some sequence space Z). If $((x'_i), (x_i))$ is shrinking and boundedly complete, then X is reflexive.*

Proof. Since $((x_i'), (x_i))$ is boundedly complete, for each $x'' \in X''$ there exists $x \in X$ such that $x = \sum_{i=1}^{\infty} \langle x'', x_i' \rangle x_i$ is convergent. As the decomposition is shrinking, for every $x' \in X'$ we have the reconstruction formula

$$x' = \sum_{i=1}^{\infty} \langle x', x_i \rangle x_i'.$$

Then

$$\langle x', x \rangle = \sum_{i=1}^{\infty} \langle x'', x_i' \rangle \langle x', x_i \rangle = \langle x'', x' \rangle.$$

Therefore, $x'' = x \in X$, which shows that X is reflexive. ■

We do not know if the converse of the above proposition holds in general. However, we do have a characterization of reflexivity analogous to the one for Schauder bases if we impose some unconditionality assumptions on the atomic decomposition. Let us say that an atomic decomposition $((x_i'), (x_i))$ is *unconditional* if for any $x \in X$, its series expansion $\sum_{i=1}^{\infty} \langle x_i', x \rangle x_i$ converges unconditionally. Note that this happens, for example, when the atomic decomposition is associated to an operator $S \colon X_d \to X$ and the canonical basis of X_d is unconditional. With this definition, we have:

THEOREM 2.5. *Let X be a Banach space and X_d be a Schauder sequence space. Let $S \colon X_d \to X$ be a continuous operator and $(x_i') \subset X'$ be a sequence such that $((x_i'), (Se_i))$ is an unconditional atomic decomposition for X with respect to X_d. Then X is reflexive if and only if $((x_i'), (Se_i))$ is shrinking and boundedly complete.*

Proof. One implication is given by Proposition 2.4, so suppose that X is a reflexive space. Since any atomic decomposition for X is boundedly complete we only have to show that $((x_i'), (Se_i))$ is shrinking. By Theorem 1.4, this is equivalent to $((Se_i), (x_i'))$ being an atomic decomposition for X' with respect to $(X_d)'$. Since $\langle x', Se_i \rangle = \langle S'x', e_i \rangle$ and $SJ = I_X$, it only remains to show that the reconstruction formula holds for any $x' \in X'$, which means that

$$x' = \sum_i \langle x', Se_i \rangle x_i'$$

holds for all $x' \in X'$. Moreover, since the series weakly converges to x', we only need to show that it is norm convergent.

The Orlicz–Pettis theorem states that weak subseries convergence implies norm subseries convergence. Select an increasing sequence (i_k) of positive integers and check that $\sum \langle x', Se_{i_k} \rangle x_{i_k}'$ is weakly convergent. Since X is reflexive, it is enough to show that, for any $x \in X$, the sequence

$$(2.1) \qquad \left(\left\langle \sum_{k=1}^{N} \langle x', Se_{i_k} \rangle x_{i_k}', x \right\rangle \right)_N = \left(\left\langle x', \sum_{k=1}^{N} \langle x_{i_k}', x \rangle Se_{i_k} \right\rangle \right)_N$$

is Cauchy. Since $\sum_i \langle x'_i, x \rangle Se_i$ is unconditionally convergent in X, any subseries is convergent and, in particular, $\sum_{k=1}^{N} \langle x'_{i_k}, x \rangle Se_{i_k}$ converges. As a result, the sequence in (2.1) is a Cauchy sequence. ∎

References

[1] J. J. Benedetto and S. D. Li, *Subband coding and noise reduction in multiresolution analysis frames*, Proc. of SPIE Conf. on Mathematical Imaging, San Diego, 1994.

[2] D. Carando and S. Lassalle, *Atomic decompositions for tensor products and polynomial spaces*, preprint.

[3] P. Casazza, *Approximation properties*, in: Handbook of the Geometry of Banach spaces, Vol. I, W. B. Johnson and J. Lindenstrauss (eds.), Elsevier, 2001, 271–316.

[4] P. Casazza, O. Christensen and D. T. Stoeva, *Frame expansions in separable Banach spaces*, J. Math. Anal. Appl. 307 (2005), 710–723.

[5] P. Casazza, D. G. Han and D. Larson, *Frames for Banach spaces*, in: The Functional and Harmonic Analysis of Wavelets and Frames (San Antonio, TX, 1999), Contemp. Math. 247, Amer. Math. Soc., Providence, RI, 1999, 149–182.

[6] O. Christensen and Ch. Heil, *Perturbations of Banach frames and atomic decompositions*, Math. Nachr. 185 (1997), 33–47.

[7] I. Daubechies, *Ten Lectures on Wavelets*, SIAM, Philadelphia, PA, 1992.

[8] J. Diestel, *Sequences and Series in Banach Spaces*, Grad. Texts in Math. 92, Springer, New York, 1984.

[9] R. J. Duffin and A. C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc. 72 (1952), 341–366.

[10] B. R. Gelbaum and J. Gil de Lamadrid, *Bases of tensor products of Banach spaces*, Pacific J. Math. 11 (1961), 1281–1286.

[11] B. Grecu and R. Ryan, *Schauder bases for symmetric tensor products*, Publ. Res. Inst. Math. Sci. 41 (2005), 459–469.

[12] K. Gröchenig, *Describing functions: Atomic decompositions versus frames*, Monatsh. Math. 112 (1991), 1–42.

[13] R. C. James, *Bases and reflexivity of Banach spaces*, Ann. of Math. (2) 52 (1950), 518–527.

[14] —, *A non-reflexive Banach space isometric with its second conjugate space*, Proc. Nat. Acad. Sci. U.S.A. 37 (1951), 174–177.

[15] J. Lindenstrauss and L. Tzafriri, *Classical Banach Spaces I, II*, Springer, 1977.

[16] N. J. Munch, *Noise reduction in tight Weyl–Heisenberg frames*, IEEE Trans. Inform. Theory 38 (1992), 608–616.

[17]   A. Pełczyński, *Any separable Banach space with the bounded approximation property is a complemented subspace of a Banach space with a basis*, Studia Math. 40 (1971), 239–243.

[18]   I. Singer, *Bases in Banach Spaces I*, Springer, New York, 1970.

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