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A theorem of Gel'fand–Mazur type

by

Hung Le Pham (Edmonton)

Abstract. Denote by $\mathfrak{c}$ any set of cardinality continuum. It is proved that a Banach algebra A with the property that for every collection $\{a_\alpha : \alpha \in \mathfrak{c}\} \subset A$ there exist $\alpha \neq \beta \in \mathfrak{c}$ such that $a_\alpha \in a_\beta A^{\#}$ is isomorphic to

$$\bigoplus_{i=1}^{r}(\mathbb{C}[X]/X^{d_i}\mathbb{C}[X]) \oplus E,$$

where $d_1, \ldots, d_r \in \mathbb{N}$, and E is either $X\mathbb{C}[X]/X^{d_0}\mathbb{C}[X]$ for some $d_0 \in \mathbb{N}$ or a 1-dimensional $\bigoplus_{i=1}^{r} \mathbb{C}[X]/X^{d_i}\mathbb{C}[X]$-bimodule with trivial right module action. In particular, $\mathbb{C}$ is the unique non-zero prime Banach algebra satisfying the above condition.

The classical Gel'fand–Mazur theorem states that $\mathbb{C}$ is the unique (complex) normed division ring. A division ring has only $\{0\}$ and itself as the right [left] principal ideals, and thus, it is a special case of unital domains whose set of right [left] principal ideals forms a chain. It is obvious that the set of right principal ideals of a unital algebra A forms a chain if and only if, for any $a, b \in A$, either $a \in bA$ or $b \in aA$. In the commutative case, a *valuation ring* is defined as a unital integral domain whose set of principal ideals forms a chain.

In [3], J. Esterle proved that $\mathbb{C}$ is the unique commutative Banach algebra which is a valuation ring. This was then extended in [4] as follows.

Theorem (Esterle). *Let A be a commutative unital Banach algebra whose set of principal ideals forms a chain. Then A is isomorphic to the quotient $\mathbb{C}[X]/X^d\mathbb{C}[X]$ for some $d \in \mathbb{N}$.*

In this paper, we shall remove the commutativity hypothesis from this result. Thus, in particular, $\mathbb{C}$ is the unique unital prime Banach algebra whose set of right principal ideals forms a chain. For this purpose, we only need to prove that a unital Fréchet algebra whose set of right principal ideals forms a chain must be commutative; the above theorems of Esterle can then

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be applied. This will also remove the commutativity assumption from the following theorem of Bouloussa [1]; here, a Fréchet algebra A is an *algebra of power series* if there exists a continuous monomorphism from A into $\mathbb{C}[[X]]$ whose image contains $\mathbb{C}[X]$.

THEOREM (Bouloussa). *A commutative unital Fréchet algebra A of dimension at least 2 which is a valuation ring must be an algebra of power series. If, in addition, A has no continuous norm then A is topologically isomorphic to $\mathbb{C}[[X]]$.*

In fact, we shall consider Banach algebras A having a weaker property that for every collection $\{a_\lambda : \lambda \in \mathfrak{c}\} \subset A$ there exist $\alpha \neq \beta \in \mathfrak{c}$ such that $a_\alpha \in a_\beta A^\#$; where $A^\#$ is the *conditional unitization* of A. A consequence of Theorem 3.4 is that an infinite-dimensional unital Banach algebra must contain right [left] principal ideals I_λ $(\lambda \in \mathfrak{c})$ such that $I_\alpha \not\subset I_\beta$ $(\alpha \neq \beta \in \mathfrak{c})$.

Note that there are infinite-dimensional prime Banach algebras A where A and $\{0\}$ are the only two-sided ideals, for example $A = \mathcal{B}(\ell^2)/\mathcal{K}(\ell^2)$.

1. Preliminaries. More details of most of the following can be found, for example, in [2].

Let A be an algebra. We denote by either $\mathbf{e}_A$ or $\mathbf{e}$ the identity of the conditional unitization $A^\#$ of A.

An algebra A is *prime* if $\{0\}$ is a prime ideal in A, i.e. if either $a = 0$ or $b = 0$ whenever $a, b \in A$ and $aAb = \{0\}$. A domain is obviously prime.

Let E be a left A-module. Denote by $A \cdot E$ the set $\{ax : a \in A,\ x \in E\}$. Let $(a_n) \subset A$. Define

$$\varprojlim a_1 \cdots a_n \cdot E = \{x \in E : \text{there exists a sequence } (x_n) \text{ in } E \text{ such that } x = x_1 \text{ and } x_n = a_n x_{n+1}\ (n \in \mathbb{N})\}.$$

Let E be an A-bimodule. Then $A \oplus E$ becomes an algebra with pointwise addition and the following standard multiplication:

$$(a \oplus x)(b \oplus y) = ab \oplus (ay + xb) \quad (a \oplus x, b \oplus y \in A \oplus E).$$

A *Fréchet algebra* is a topological algebra A whose topology is determined by a sequence (p_n) of algebra seminorms such that

$$d(a,b) = \sum_{n=1}^{\infty} \frac{\min\{p_n(a-b), 1\}}{2^n} \qquad (a, b \in A)$$

is a complete metric.

We shall need the following result, which is [4, Corollary 3.5]; it was proved for Banach algebras, but the proof works for Fréchet algebras.

THEOREM 1.1 (Esterle). *Let R be a radical Fréchet algebra. Then the following conditions are equivalent*:

(i) *There exists a non-zero element $a \in R$ such that $a \in \overline{Ra}$.*
(ii) *There exists a sequence (a_n) in R such that $\varprojlim a_1 \cdots a_n \cdot R \neq \{0\}$.*
(iii) *There exists a strictly increasing sequence of left principal ideals in $R^{\#}$.*

2. Right principal submodules. In this section, we shall prove a result for modules similar to results in [1], [3], and [4] (and without the commutativity assumption). The proof follows the same ideas of using Baire's category theorem and Liouville's theorem.

LEMMA 2.1. *Let R be a radical Fréchet algebra. Let E be a topological vector space and a left R-module such that $a \mapsto ax$, $R \to E$, is continuous for all $x \in E$, and $E = \overline{R \cdot E}$. Then, for each $\phi : E \to \mathbb{R}^+$ and each non-zero continuous seminorm q on E, the set*

$$U = \{(x, y) \in E \times E : \phi(vx) < q(vy) \text{ for some } v \in R^{\#}\}$$

is dense in $E \times E$.

Proof. First, let (x, y) be arbitrary in $E \times E$ with $x \in R \cdot E$ and $q(y) \neq 0$. Let $x' \in E$ and $c \in R$ be such that $cx' = x$. We see that, for each $r \in \mathbb{N}$, there exists $\lambda_r \in \mathbb{C}$ such that

$$0 < |\lambda_r| < 1/r \quad \text{and} \quad q((\lambda_r + c)^{-1}y) > \phi(x');$$

for otherwise the Hahn–Banach and Liouville theorems imply $q((\lambda + c)^{-1}y) = 0$ $(\lambda \in \mathbb{C})$ (cf. [1, Lemme 1.1]), and so

$$q(y) = \lim_{\lambda \to \infty} q(\lambda(\lambda + c)^{-1}y) = 0.$$

Set $v_r = \lambda_r + c \in R^{\#}$ and $x_r = v_r x' \in E$. Then

$$\phi(v_r^{-1}x_r) = \phi(x') < q(v_r^{-1}y),$$

so that $(x_r, y) \in U$ $(r \in \mathbb{N})$. We have $\lim x_r = x$, so $(x, y) \in \overline{U}$. The set $\{y \in E : q(y) \neq 0\}$ is dense in E. Therefore, U is dense in $E \times E$ as claimed. ■

PROPOSITION 2.2. *Let R be a radical Fréchet algebra. Let E be a Fréchet left R-module having a non-zero element $x \in E$ such that $x \in \overline{R \cdot x}$. Then there exists a sequence (x_n) in $\overline{R \cdot x}$ with the properties that $x_j \notin x_i \cdot A$ $(i \neq j \in \mathbb{N})$ for each algebra A such that A acts continuously on the right on E and $E \in R$-mod-A.*

Proof. Set $F = \overline{R \cdot x}$. We see that $0 \neq x \in F$ and $F = \overline{R \cdot F}$. Denote by Ω the product space $F^{\mathbb{N}}$; its topology is defined by a complete metric. Let (q_k) be an increasing sequence of non-zero seminorms defining the topology of E.

For each triple $s = (l, i, j)$ of natural numbers with $i \neq j$, set

$$U_s = \{(x_r) \in \Omega : l^2 q_l(vx_i) < q_1(vx_j) \text{ for some } v \in R^\#\}.$$

By Lemma 2.1, this is a dense (open) subset of Ω. By the Baire category theorem, there exists a sequence (x_r) contained in all U_s.

Let $i \neq j \in \mathbb{N}$. We need to show that

$$x_j \notin x_i \cdot A.$$

Indeed, assume the contrary that $x_j = x_i c$ for some $c \in A$. The previous paragraph shows that there exist (v_l) in $R^\#$ such that

$$q_l(v_l x_i) < 1/l \quad \text{and} \quad q_1(v_l x_j) > l \quad (l \in \mathbb{N}).$$

This shows that $\lim_{l\to\infty} v_l x_i c = 0$; however, $(v_l x_j : l \in \mathbb{N})$ can never converge, a contradiction. ∎

3. Main results

Lemma 3.1. *Let R be an algebra and let $a \in R$. Suppose that $\varprojlim a^n \cdot R = \{0\}$ and $R = aR^\#$. Then R is commutative.*

Proof. Denote by $\mathbf{e}$ the identity of $R^\#$. We also set $[x, b, y] = xby - ybx$ and $[x, y] = [x, \mathbf{e}, y] = xy - yx$ $(b, x, y \in R^\#)$.

Let $x, y \in R$ be arbitrary. Then $R = aR^\#$ implies that there exist sequences (α_n), (β_n) in $\mathbb{C}$ and (x_n), (y_n) in R such that $x = x_0$, $y = y_0$, and $x_{n-1} = a(\alpha_n \mathbf{e} + x_n)$ and $y_{n-1} = a(\beta_n \mathbf{e} + y_n)$ $(n \in \mathbb{N})$. Then we see that

$$[x_{n-1}, a^{n-1}, y_{n-1}] = a(\alpha_n[a^n, y_n] + \beta_n[x_n, a^n] + [x_n, a^n, y_n]) \qquad (n \in \mathbb{N});$$

our convention here is $a^0 = \mathbf{e}$. For each $n \in \mathbb{N}$, set

$$w_n = \Big[\sum_{k=1}^{n} \alpha_k a^k, y_n\Big] + \Big[x_n, \sum_{k=1}^{n} \beta_k a^k\Big] + [x_n, a^n, y_n].$$

Then $[x, y] = aw_1$ and, for $n > 1$,

$$\begin{aligned} w_{n-1} &= \Big[\sum_{k=1}^{n-1} \alpha_k a^k, y_{n-1}\Big] + \Big[x_{n-1}, \sum_{k=1}^{n-1} \beta_k a^k\Big] + [x_{n-1}, a^{n-1}, y_{n-1}] \\ &= \Big[\sum_{k=1}^{n-1} \alpha_k a^k, \beta_n a + a y_n\Big] + \Big[\alpha_n a + a x_n, \sum_{k=1}^{n-1} \beta_k a^k\Big] \\ &\quad + a(\alpha_n[a^n, y_n] + \beta_n[x_n, a^n] + [x_n, a^n, y_n]) \\ &= a w_n. \end{aligned}$$

Thus, we see that $[x, y] \in \varprojlim a^n \cdot R = \{0\}$. Hence, R is commutative. ∎

Recall that a *local algebra* is a unital algebra whose radical has codimension 1.

LEMMA 3.2. *Let A be a local Fréchet algebra with the property that for every collection $\{a_\alpha \in \operatorname{rad} A : \alpha \in \mathfrak{c}\}$ there exist $\alpha \neq \beta \in \mathfrak{c}$ such that $a_\alpha \in a_\beta A$. Then A is either an algebra of power series or isomorphic to $\mathbb{C}[X]/X^d\mathbb{C}[X]$ for some $d \in \mathbb{N}$. If, in addition, A is a Banach algebra, it must be of the latter form.*

Proof. We have $A = R^\#$ where R is the radical of A. Suppose that $R \neq \{0\}$. Let $a, b \in R$. We *claim* that either $a \in bR^\#$ or $b \in aR^\#$. Indeed, set $a_\alpha = \alpha a + b$ $(\alpha \in \mathbb{C})$. The assumption implies that there exist $\alpha \neq \beta$ such that $\alpha a + b \in (\beta a + b)R^\#$; say $\alpha a + b = (\beta a + b)(\lambda \mathbf{e} + x)$ for some $\lambda \in \mathbb{C}$ and $x \in R$. Then we see that

$$a[(\alpha - \lambda\beta)\mathbf{e} - \beta x] = b[(\lambda - 1)\mathbf{e} + x],$$

and that $\alpha - \lambda\beta$ and $\lambda - 1$ cannot be 0 simultaneously. The claim then follows since $R^\# \setminus R = \operatorname{Inv} R^\#$.

Proposition 2.2 implies that R does not satisfy any of the conditions in Theorem 1.1. In particular, Theorem 1.1(ii) shows that there exists $a \in R \setminus R \cdot R$. Let $x \in R$ be arbitrary. Then we have shown that either $x \in aR^\#$ or $a \in xR^\#$. The latter implies that $a \in x \cdot (R^\# \setminus R)$, and so $x \in a \cdot (R^\# \setminus R)$. Thus in both cases we have $x \in aR^\#$. Hence, R and a satisfy the hypothesis of Lemma 3.1, and so R is commutative.

Since for any $x \in R \setminus \{0\}$ we have $x = a^n u$ for some $n \in \mathbb{N}$ and $u \in R^\# \setminus R$, either R is an integral domain or a is nilpotent. In the former case, by [1, Théorème 2.5], A is an algebra of power series. In the latter case, it can be proved as in the last part of the proof of [4, Theorem 8.4] that A is isomorphic to $\mathbb{C}[X]/X^d\mathbb{C}[X]$ for some $d \in \mathbb{N}$. The last assertion also follows from [4, Theorem 8.4]. ■

The above is sufficient to remove the commutativity assumption from the above mentioned results.

THEOREM 3.3. *Let A be a unital Fréchet algebra A whose set of right principal ideals forms a chain. Then*:

(i) *A is either an algebra of power series or isomorphic to $\mathbb{C}[X]/X^d\mathbb{C}[X]$ for some $d \in \mathbb{N}$.*
(ii) *If A is a Banach algebra then A is isomorphic to $\mathbb{C}[X]/X^d\mathbb{C}[X]$.*
(iii) *If A has no continuous norm then A is topologically isomorphic to $\mathbb{C}[[X]]$.*

Proof. The assumption implies that the set of right ideals in A forms a chain. Let R be the unique maximal right ideal. Then R is the radical of A, and it can be seen that R is closed. We *claim* that $A/R = \mathbb{C}$. Let p be any seminorm on A and set $N = \ker p$. Then $N \subset R$, by the maximality of R. Now, A/N is a normed algebra and R/N is the unique maximal right ideal for A/N. We can then deduce from Rickart–Jacobson's density theorem for A/N that $A/R \cong (A/N)/(R/N)$ must be of dimension 1. Thus $A = R^{\#}$. Lemma 3.2 then completes the proof; (iii) follows from the last assertion of [1, Théorème 2.5]. ∎

For the class of Banach algebras, it is possible to weaken the hypothesis further.

THEOREM 3.4. *Let A be a Banach algebra with the property that for every collection $\{a_\alpha : \alpha \in \mathfrak{c}\} \subset A$ there exist $\alpha \neq \beta \in \mathfrak{c}$ such that $a_\alpha \in a_\beta A^{\#}$. Then there exist $d_1, \ldots, d_r \in \mathbb{N}$ such that A is isomorphic to either*

$$\bigoplus_{i=1}^{r} (\mathbb{C}[X]/X^{d_i}\mathbb{C}[X]) \oplus (X\mathbb{C}[X]/X^{d_0}\mathbb{C}[X])$$

for some $d_0 \in \mathbb{N}$, or

$$\bigoplus_{i=1}^{r} (\mathbb{C}[X]/X^{d_i}\mathbb{C}[X]) \oplus E$$

where E is a 1-dimensional $\bigoplus_{i=1}^{r} \mathbb{C}[X]/X^{d_i}\mathbb{C}[X]$-bimodule with trivial right module action. If, in addition, A is unital, then A is isomorphic to

$$\bigoplus_{i=1}^{r} \mathbb{C}[X]/X^{d_i}\mathbb{C}[X].$$

Proof. Assume toward a contradiction that $\sigma(a)$ is uncountable for some $a \in A$. Then we see that the boundary $\partial\sigma(a)$ of $\sigma(a)$ is uncountable (and compact) and hence has cardinality $\mathfrak{c}$. Set $S = \partial\sigma(a) \setminus \{0\}$. The hypothesis then implies that there exist $\alpha \neq \beta \in S$ such that

$$(a - \beta)a \in (a - \alpha)aA^{\#} \quad \text{and so} \quad a \in (a - \alpha)aA^{\#} \subset (a - \alpha)A^{\#}.$$

It follows that $\alpha \in (a-\alpha)A^{\#}$, and thus $a-\alpha$ cannot be a topological divisor of zero, contradicting the fact that $\alpha \in \partial\sigma(a)$.

Assume toward a contradiction that $\sigma(a)$ is infinitely countable for some $a \in A$. Then there exists an infinite sequence of disjoint subsets of $\sigma(a) \setminus \{0\}$ which are both open and closed in $\sigma(a)$. The functional calculus for a then implies the existence of an orthogonal sequence (e_n) of non-zero idempotents in A. For each subset E of $\mathbb{N}$, define

$$a_E = \sum_{k \in E} \frac{1}{2^k \|e_k\|} e_k.$$

Then we see that $a_E \notin a_F A^{\#}$ whenever $E \not\subset F$. Combining this with Sierpiński's family $\{E_\alpha : \alpha \in \mathfrak{c}\}$ of infinite subsets of $\mathbb{N}$ with the property that $E_\alpha \cap E_\beta$ is finite for each $\alpha \neq \beta \in \mathfrak{c}$ (for more details see [5]) will give us a contradiction to the assumption.

We have shown that $\sigma(a)$ is finite for all $a \in A$. A theorem due to Kaplansky (see, for example, [2, 2.6.29]) then shows that $A/\operatorname{rad} A$ has finite dimension. Now, let P be any primitive ideal of A. Then P is the kernel of a homomorphism π from A onto $\mathbb{M}_d(\mathbb{C})$ for some $d \in \mathbb{N}$. Assume toward a contradiction that $d \geq 2$. For each $\lambda \in \mathbb{C}$, define N_λ to be the right ideal of $\mathbb{M}_d(\mathbb{C})$ consisting of those matrices whose first row is λ times the second row. It is obvious that we can find $a_\lambda \in A$ such that $\pi(a_\lambda) \in N_\lambda$ but $\pi(a_\lambda) \notin N_\alpha$ for $\alpha \neq \lambda$. This again gives us a contradiction. Thus, we have $A/\operatorname{rad} A \cong \mathbb{C}^r$ for some $r \in \mathbb{N}$. It then follows from a Wedderburn decomposition theorem due to Feldman that there exists an orthogonal set $\{e_1, \ldots, e_r\}$ of idempotents of A such that $A = \bigoplus_{i=1}^r \mathbb{C}e_i \oplus \operatorname{rad} A$ (see, for example, [2, 1.5.18 and 2.4.2]). We also set $e_0 = \mathbf{e} - \sum_{i=1}^r e_r$.

Fix $i \in \{1, \ldots, r\}$. Set $A_i = e_iAe_i$. Then A_i is a local Banach algebra whose radical is $e_i(\operatorname{rad} A)e_i$. Let $\{a_\lambda : \lambda \in \mathfrak{c}\}$ be a family in A_i. Then there exist $\alpha \neq \beta \in \mathfrak{c}$ such that $a_\alpha \in a_\beta A^\#$. This implies that $a_\alpha \in a_\beta e_i A^\# e_i = a_\beta A_i$. Thus A_i satisfies the hypothesis of Lemma 3.2, and so A_i is isomorphic to $\mathbb{C}[X]/X^{d_i}\mathbb{C}[X]$ for some $d_i \in \mathbb{N}$. Similarly, we see that $A_0 = e_0Ae_0$ is a radical Banach algebra whose unitization, which is $e_0A^\#e_0$ in the case where $e_0 \neq 0$, satisfies the hypothesis of Lemma 3.2, and so e_0Ae_0 is isomorphic to $X\mathbb{C}[X]/X^{d_0}\mathbb{C}[X]$ for some $d_0 \in \mathbb{N}$.

Fix $j \in \{0, \ldots, r\}$. Let $u, v \in Ae_j \setminus \{0\}$ be arbitrary. The hypothesis then implies that there exist $\alpha \neq \beta \in \mathbb{C}$ such that $\alpha u + v \in (\beta u + v)A^\#$, and so $\alpha u + v \in (\beta u + v)e_jA^\#e_j$. Then, similar to Lemma 3.2, we see that either $u \in ve_jA^\#e_j$ or $v \in ue_jA^\#e_j$. Thus, we can deduce that at most one of the subspaces e_iAe_j $(0 \leq i \leq r)$ is non-zero. In particular, $e_iAe_j = \{0\}$ if both $0 \leq i \neq j \leq r$ and $j \geq 1$.

Suppose that $e_iAe_0 \neq \{0\}$ for some $1 \leq i \leq r$ (this forces $e_0Ae_0 = \{0\}$). Let $u, v \in e_iAe_0 \setminus \{0\}$. It follows from the previous paragraph and $e_0Ae_0 = \{0\}$ that u and v are linearly dependent. Thus e_iAe_0 has dimension 1. The result then follows. ∎

Conversely, it can be seen that if A is an algebra of either form in the previous theorem, then for every $a_1, \ldots, a_N \in A$, there exist $i \neq j \in \{1, \ldots, N\}$ such that $a_i \in a_jA^\#$; here N is a natural number depending on $d_0, d_1, \ldots, d_r$.

Corollary 3.5. *Let A be a non-zero prime Banach algebra with the property that for every collection $\{a_\alpha \in A : \alpha \in \mathfrak{c}\}$ there exist $\alpha \neq \beta \in \mathfrak{c}$ such that $a_\alpha \in a_\beta A^\#$. Then A is isomorphic to $\mathbb{C}$.*

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Department of Mathematical and Statistical Sciences
University of Alberta
Edmonton, Alberta T6G 2G1, Canada
E-mail: hlpham@math.ualberta.ca

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