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On the structure of the set of higher order spreading models

by

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Abstract. We generalize some results concerning the classical notion of a spreading model to spreading models of order ξ. Among other results, we prove that the set $SM^w_\xi(X)$ of ξ-order spreading models of a Banach space X generated by subordinated weakly null $\mathcal{F}$-sequences endowed with the pre-partial order of domination is a semilattice. Moreover, if $SM^w_\xi(X)$ contains an increasing sequence of length ω then it contains an increasing sequence of length ω_1. Finally, if $SM^w_\xi(X)$ is uncountable, then it contains an antichain of size continuum.

1. Introduction. In 1974, A. Brunel and L. Sucheston [BS] introduced the notion of a spreading model, which plays quite a central role in the asymptotic Banach space theory (see for example [AK, Kr, OS]). A higher order extension of this notion has been introduced and studied in [AKT2] and [AKT1]. In particular, for every countable ordinal ξ, the ξ-order spreading models of a Banach space X are defined. The order one spreading models coincide with the classical ones. In this note, we extend some of the results concerning the structure of the set of classical spreading models to the setting of ξ-order ones. Consider the set $SM_w(X)$ of (equivalence classes of) spreading models of a Banach space X generated by weakly null sequences endowed with the partial order given by domination of bases. This partially ordered set is proven to have interesting features:

(i) Every countable subset of $SM_w(X)$ admits an upper bound in this set, in particular, $SM_w(X)$ is an upper semilattice [AOST].
(ii) Existence of an increasing sequence (of length ω) in $SM_w(X)$ yields the existence of an increasing sequence of length ω_1 [Sa].
(iii) Suppose X is separable. If $SM_w(X)$ is uncountable, then it contains an antichain of size continuum. If $SM_w(X)$ contains a decreasing sequence of length ω_1 then it contains an increasing sequence of

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length ω_1. If $SM_w(X)$ does not contain any infinite increasing sequence then there exists $\zeta < \omega_1$ such that $SM_w(X)$ contains no decreasing sequence of length ζ [Do].

We show that for every $\xi < \omega_1$, these results extend to the set $SM_\xi^w(X)$ of ξ-order spreading models of X generated by subordinated weakly null $\mathcal{F}$-sequences. Subordinated $\mathcal{F}$-sequences are a higher order analogue of ordinary weakly convergent sequences.

A brief introduction to higher order spreading models and some new facts regarding subordinated $\mathcal{F}$-sequences are given in Sections 2 and 3. Sections 4, 5 and 6 are devoted to the proof of the generalization of the above results (i), (ii), and (iii), respectively. In particular, the main results of the paper are Corollary 5.2 and Theorems 4.10, 4.11, 6.2 and 6.4.

It is worth mentioning that in general $SM_\xi^w(X)$ does not coincide with $SM_1^w(X)$, and therefore the transfinite hierarchy $(SM_\xi^w(X))_{\xi<\omega_1}$ is not trivial. In fact, for every k there exists a Banach space X such that $SM_k^w(X)$ is a proper subset of $SM_{k+1}^w(X)$ [AKT1]. Moreover, there are reflexive Banach spaces X and Y which have, up to equivalence, the same set of spreading models of the first order but not of the second order. In Section 3 we also recall these known examples.

2. ξ-order spreading models of a Banach space. We will use capital letters $L, M, N, \dots$ to denote infinite subsets and lower case letters $s, t, u, \dots$ to denote finite subsets of $\mathbb{N} = \{1, 2, \dots\}$. For every infinite subset L of $\mathbb{N}$, $[L]^{<\infty}$ (resp. $[L]^\infty$) stands for the set of all finite (resp. infinite) subsets of L. For an infinite subset $L = \{l_1 < l_2 < \cdots\}$ of $\mathbb{N}$ and a positive integer $k \in \mathbb{N}$, we set $L(k) = l_k$. Similarly, for a finite subset $s = \{n_1 < \cdots < n_m\}$ of $\mathbb{N}$ and for $1 \le k \le m$ we set $s(k) = n_k$. For an infinite subset $L = \{l_1 < l_2 < \cdots\}$ of $\mathbb{N}$ and a finite subset $s = \{n_1 < \cdots < n_m\}$ (resp. for an infinite subset $N = \{n_1 < n_2 < \cdots\}$ of $\mathbb{N}$), we set $L(s) = \{l_{n_1}, \dots, l_{n_m}\} = \{L(s(1)), \dots, L(s(m))\}$ (resp. $L(N) = \{l_{n_1}, l_{n_2}, \dots\} = \{L(N(1)), L(N(2)), \dots\}$).

$\mathcal{F}$-spreading models are generated by sequences indexed by the elements of a regular thin family, which we are about to define. Let $\mathcal{R}$ be a family of finite subsets of $\mathbb{N}$. The family $\mathcal{R}$ is called *compact* if the set of all characteristic functions of the elements of $\mathcal{R}$ is a compact subset of the set of all functions from $\mathbb{N}$ into $\{0, 1\}$ endowed with the product topology. The family $\mathcal{R}$ is called *hereditary* if it contains every subset of its elements, and *spreading* if for every $s \in \mathcal{R}$ and $t \in [\mathbb{N}]^{<\infty}$ of the same cardinality, say n, such that $s(i) \le t(i)$ for all $1 \le i \le n$, the set t also belongs to $\mathcal{R}$.

A family of finite subsets of $\mathbb{N}$ is called *regular* if it is compact, hereditary and spreading. A family of finite subsets of $\mathbb{N}$ is called *regular thin* if it consists of the maximal elements, under inclusion, of some regular family. A family $\mathcal{H}$ of finite subsets of $\mathbb{N}$ is called *thin* if there are no s and t in $\mathcal{H}$ such that s is a proper initial segment of t. Clearly every regular thin family is also thin. A brief presentation of regular and regular thin families as well as relations between them can be found in Section 2 of [AKT2].

Finally, given a regular thin family $\mathcal{F}$, an $\mathcal{F}$-*sequence* in a Banach space is a sequence of the form $(x_s)_{s\in\mathcal{F}}$ indexed by $\mathcal{F}$, while an $\mathcal{F}$-*subsequence* is a sequence of the form $(x_s)_{s\in\mathcal{F}\restriction L}$ indexed by $\mathcal{F}\restriction L$, where L is an infinite subset of $\mathbb{N}$ and the restriction $\mathcal{F}\restriction L$ of $\mathcal{F}$ to L is defined by

$$\mathcal{F}\restriction L = \{s \in \mathcal{F} : s \subseteq L\}. \tag{2.1}$$

The connection between $\mathcal{F}$-spreading models and $\mathcal{F}$-sequences generating them is described by the notion of plegma families.

Definition 2.1. Let l be a positive integer and $s_1, \dots, s_l$ nonempty finite subsets of $\mathbb{N}$. The l-tuple $(s_j)_{j=1}^l$ is called a *plegma* family if:

(i) For every i, j in $\{1, \dots, l\}$ and k in $\mathbb{N}$ with $i<j$ and $k\le\min(|s_i|, |s_j|)$, we have $s_i(k) < s_j(k)$.
(ii) For every i, j in $\{1, \dots, l\}$ and k in $\mathbb{N}$ with $k \le \min(|s_i|, |s_j| - 1)$, we have $s_i(k) < s_j(k + 1)$.

For instance, a pair $(\{n_1, m_1\}, \{n_2, m_2\})$ of doubletons is plegma if and only if $n_1 < n_2 < m_1 < m_2$. More generally for two nonempty $s, t \in [\mathbb{N}]^{<\infty}$ with $|s| \le |t|$ the pair (s, t) is a plegma pair if and only if $s(1) < t(1) < s(2) < t(2) < \dots < s(|s|) < t(|s|)$.

The properties of plegma families are explored in Section 3 of [AKT2]. Moreover, for every regular thin family $\mathcal{F}$, every infinite subset L of $\mathbb{N}$ and every positive integer k we set

$$\mathrm{Plm}_k(\mathcal{F}\restriction L) = \{(s_i)_{i=1}^k : s_1, \dots, s_k \in \mathcal{F}\restriction L \text{ with } (s_i)_{i=1}^k \text{ plegma}\}. \tag{2.2}$$

Now we are ready to state the definition of $\mathcal{F}$-spreading models.

Definition 2.2. Let X be a Banach space, $\mathcal{F}$ a regular thin family and $(x_s)_{s\in\mathcal{F}}$ an $\mathcal{F}$-sequence in X. Also let $(E, \|\cdot\|_*)$ be an infinite-dimensional seminormed linear space with Hamel basis $(e_n)_n$. Finally, let M in $[\mathbb{N}]^\infty$ and $(\delta_n)_n$ be a null sequence of positive real numbers.

We say that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\restriction M}$ *generates* $(e_n)_n$ *as an* $\mathcal{F}$-*spreading model* (with respect to $(\delta_n)_n$) if for all l, k in $\mathbb{N}$ with $1 \le k \le l$, every finite sequence $(a_i)_{i=1}^k$ in $[-1, 1]$ and every $(s_j)_{j=1}^k$ in $\mathrm{Plm}_k(\mathcal{F}\restriction M)$ with $s_1(1) \ge M(l)$, we have

$$\Big|\Big\|\sum_{j=1}^k a_j x_{s_j}\Big\| - \Big\|\sum_{j=1}^k a_j e_j\Big\|_*\Big| \le \delta_l. \tag{2.3}$$

We also say that $(x_s)_{s\in\mathcal{F}}$ *admits* $(e_n)_n$ *as an* $\mathcal{F}$*-spreading model* if there exists M in $[\mathbb{N}]^\infty$ such that $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model.

Finally, for a subset A of X, $(e_n)_n$ is an $\mathcal{F}$*-spreading model* of A if there exists an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ in A which admits $(e_n)_n$ as an $\mathcal{F}$-spreading model.

The existence of $\mathcal{F}$-spreading models is established in [AKT2, Theorem 4.5]. For every regular family $\mathcal{R}$ its *order* $o(\mathcal{R})$ is defined to be the rank of the set $\mathcal{R}$ partially ordered by reverse inclusion. By the compactness of $\mathcal{R}$, its order is well defined and it is a countable ordinal number (see also [AKT2, Section 2]). The *order* $o(\mathcal{F})$ of a regular thin family $\mathcal{F}$ is defined to be the order of the regular family for which $\mathcal{F}$ is the set of maximal elements under inclusion. It turns out that the order of the regular thin family $\mathcal{F}$ involved in the definition of $\mathcal{F}$-spreading models is an important feature in the following sense. By Corollary 4.7 of [AKT2], for every subset A of a Banach space and every pair $\mathcal{F},\mathcal{G}$ of regular thin families of the same order, a sequence $(e_n)_n$ is an $\mathcal{F}$-spreading model of A if and only if it is a $\mathcal{G}$-spreading model of A. This fact gives rise to the following definition.

Definition 2.3. Let A be a subset of a Banach space X and $\xi\geq 1$ be a countable ordinal. We say that $(e_n)_n$ is a ξ*-order spreading model* of A if there exists a regular thin family $\mathcal{F}$ with $o(\mathcal{F})=\xi$ such that $(e_n)_n$ is an $\mathcal{F}$-spreading model of A.

An $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ in a Banach space X is defined to be *weakly null* if for every x^* in the dual of X and every $\varepsilon>0$ there exists some n_0 such that for every $s\in\mathcal{F}\upharpoonright L$ with $\min s\geq n_0$, we have $|x^*(x_s)|<\varepsilon$. For a regular thin family $\mathcal{F}$ its *closure* is defined by

$$\widehat{\mathcal{F}}=\{s\in[\mathbb{N}]^{<\infty}: \text{there exists } t\in\mathcal{F} \text{ such that } s\subseteq t\}. \tag{2.4}$$

If we identify each subset of $\mathbb{N}$ with its characteristic map and endow $\{0,1\}^{\mathbb{N}}$ with the product topology, then $\mathcal{F}$ (resp. $\mathcal{F}\upharpoonright L$) becomes a discrete subset of $\{0,1\}^{\mathbb{N}}$ and $\widehat{\mathcal{F}}$ (resp. $\widehat{\mathcal{F}}\upharpoonright L$) is its topological closure.

Definition 2.4. Let $\mathcal{F}$ be a regular thin family and M an infinite subset of $\mathbb{N}$. Let $(x_s)_{s\in\mathcal{F}}$ be an $\mathcal{F}$-sequence in a Banach space X. We say that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is *subordinated* (with respect to the weak topology) if there exists a continuous map $\widehat{\varphi}:\widehat{\mathcal{F}}\upharpoonright L\to X$, where X is considered with the weak topology, such that $x_s=\widehat{\varphi}(s)$ for all $s\in\mathcal{F}\upharpoonright L$.

It is easy to see that a subordinated $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is weakly convergent to $\widehat{\varphi}(\emptyset)$, where $\widehat{\varphi}$ is the map witnessing the fact that $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is subordinated. Thus a subordinated $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is weakly null if and only if $\widehat{\varphi}(\emptyset)=0$.

Definition 2.5. Let X be a Banach space and ξ a countable ordinal. We denote by $SM_\xi^w(X)$ the set of all ξ-order spreading models generated by subordinated weakly null $\mathcal{F}$-subsequences in X for some regular thin family $\mathcal{F}$ of order ξ.

3. On subordinated weakly null $\mathcal{F}$-sequences. Our first goal is the following strengthening of Theorem 3.17 from [AKT2].

Theorem 3.1. *Let $\mathcal{F}$ and $\mathcal{G}$ be regular thin families with $o(\mathcal{F}) \le o(\mathcal{G})$. Then for all infinite subsets M and N of $\mathbb{N}$ there exist M' in $[M]^\infty$, N' in $[N]^\infty$ and a continuous map $\widehat{\varphi} : \widehat{\mathcal{G}}\restriction N' \to \widehat{\mathcal{F}}\restriction M'$ satisfying the following. Let φ be the restriction of $\widehat{\varphi}$ to $\mathcal{G}\restriction N'$. Then φ is a plegma preserving map onto $\mathcal{F}\restriction M'$ such that $\min\varphi(t) \ge M'(l)$ for every $l \in \mathbb{N}$ and $t \in \mathcal{G}\restriction N'$ with $\min t \ge N'(l)$.*

For the proof of the above theorem we will need Corollary 2.17 from [AKT2]. To state it we recall some notation. For two families $\mathcal{H}_1$ and $\mathcal{H}_2$ of finite subsets of $\mathbb{N}$, write $\mathcal{H}_1 \sqsubseteq \mathcal{H}_2$ if every element in $\mathcal{H}_1$ has an extension in $\mathcal{H}_2$ and every element in $\mathcal{H}_2$ has an initial segment in $\mathcal{H}_1$. Moreover, for every infinite subset L of $\mathbb{N}$ and every family $\mathcal{H}$ of finite subsets of $\mathbb{N}$, let

$$L(\mathcal{H}) = \{L(s) : s \in \mathcal{H}\}. \tag{3.1}$$

Proposition 3.2 ([AKT2, Corollary 2.17]). *Let $\mathcal{F}$ and $\mathcal{G}$ be regular thin families with $o(\mathcal{F}) \le o(\mathcal{G})$. Then there exists L_0 in $[\mathbb{N}]^\infty$ such that for every M in $[\mathbb{N}]^\infty$ there exists L in $[L_0(M)]^\infty$ satisfying $L_0(\mathcal{F})\restriction L \sqsubseteq \mathcal{G}\restriction L$.*

Proof of Theorem 3.1. Let M and N be two infinite subsets of $\mathbb{N}$. For every finite subset t of $\mathbb{N}$, there exists a unique finite subset s of $\mathbb{N}$ such that $N(s) = t$. Set $N^{-1}(t) = s$ and $\mathcal{G}' = \{N^{-1}(t) \in [\mathbb{N}]^{<\infty} : t \in \mathcal{G}\}$. It follows that $\mathcal{G}'$ is a regular thin family with $o(\mathcal{G}') = o(\mathcal{G})$. Moreover,

$$N(\mathcal{G}') = \mathcal{G}\restriction N. \tag{3.2}$$

By Proposition 3.2, there exist infinite subsets L_0 of $\mathbb{N}$ and L of $L_0(M)$ such that

$$L_0(\mathcal{F})\restriction L \sqsubseteq \mathcal{G}'\restriction L. \tag{3.3}$$

We essentially need to prove the following.

Claim. *There exists a continuous map $\widehat{\varphi}_1 : \widehat{\mathcal{G}'}\restriction L \to L_0(\widehat{\mathcal{F}})\restriction L$ whose restriction $\widehat{\varphi}_1$ to $\mathcal{G}'\restriction L$ is a plegma preserving map onto $L_0(\mathcal{F})\restriction L$ such that for every $t \in \mathcal{G}'\restriction L$, $\varphi_1(t)$ is an initial segment of t and so $\min t = \min\varphi_1(t)$.*

Proof of Claim. Since $\mathcal{F}$ is a thin family, so is $L_0(\mathcal{F})$. Hence, by (3.3), for every $t \in \mathcal{G}'\restriction L$ there exists a unique $s_t \in L_0(\mathcal{F})$ such that s_t is an initial segment of t. Let $\mathcal{A} = (\widehat{\mathcal{G}'}\restriction L) \setminus (L_0(\widehat{\mathcal{F}})\restriction L)$. Also observe that for every $\widehat{t} \in \mathcal{A}$

and any t, t' in $\mathcal{G}'\restriction L$ that both end-extend $\widehat{t}$ we have $s_t = s_{t'}$ and both $s_t, s_{t'}$ are initial segments of $\widehat{t}$. For every $\widehat{t} \in \mathcal{A}$ we set $s_{\widehat{t}} = s_t$ where t is any element from $\mathcal{G}'\restriction L$ that end-extends $\widehat{t}$. Finally, for every $\widehat{t} \in L_0(\widehat{\mathcal{F}})\restriction L$ we set $s_{\widehat{t}} = \widehat{t}$. Setting $\widehat{\varphi}_1(\widehat{t}) = s_{\widehat{t}}$ for all $\widehat{t} \in \mathcal{G}'\restriction L$, we find that $\widehat{\varphi}_1$ is as desired in the claim.

Since $L \in [L_0(M)]^\infty$, there exists M' in $[M]^\infty$ such that $L_0(M') = L$. Then $L_0(\widehat{\mathcal{F}}\restriction M') = L_0(\widehat{\mathcal{F}})\restriction L$. Moreover, let $N' = N(L)$. It is easy to check that $\widehat{\mathcal{G}}\restriction N' = N(\widehat{\mathcal{G}}'\restriction L)$. Define $\widehat{\varphi} : \widehat{\mathcal{G}}\restriction N' \to \widehat{\mathcal{F}}\restriction M'$ by setting $\widehat{\varphi}(\widehat{t}) = L_0^{-1}(\widehat{\varphi}_1(N^{-1}(\widehat{t})))$ for every $\widehat{t} \in \widehat{\mathcal{G}}\restriction N'$, where $L_0^{-1}(s)$ is defined similarly to $N^{-1}(s)$ for every $s \in [L_0]^{<\infty}$. It follows readily that $\widehat{\varphi}$ is as desired. ∎

Theorem 3.1 has the following immediate corollary.

Corollary 3.3. *Let X be a Banach space, ξ a countable ordinal and $(e_n)_n \in SM_\xi^w(X)$. Then for every regular thin family $\mathcal{F}$ of order at least ξ and every infinite subset M of $\mathbb{N}$ there exist a further infinite subset L of M and an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ in X such that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\restriction L}$ is subordinated, weakly null and generates $(e_n)_n$ as an $\mathcal{F}$-spreading model.*

Proof. Since $(e_n)_n$ belongs to $SM_\xi^w(X)$, there exist a regular thin family $\mathcal{G}$ of order ξ, an infinite subset N of $\mathbb{N}$ and a $\mathcal{G}$-sequence $(y_s)_{s\in\mathcal{G}}$ such that the $\mathcal{G}$-subsequence $(y_s)_{s\in\mathcal{G}\restriction N}$ is subordinated, weakly null and generates $(e_n)_n$ as a $\mathcal{G}$-spreading model. Let $\widehat{\varphi}_1 : \widehat{\mathcal{G}}\restriction N \to X$ be the continuous map witnessing that $(y_s)_{s\in\mathcal{G}\restriction N}$ is subordinated. Fix a regular thin family $\mathcal{F}$ of order at least ξ and an infinite subset M of $\mathbb{N}$. By Theorem 3.1 there exist an infinite subset L of M, an infinite subset N' of N and a continuous map $\widehat{\varphi} : \widehat{\mathcal{F}}\restriction L \to \widehat{\mathcal{G}}\restriction N'$ whose restriction φ to $\mathcal{F}\restriction L$ is a plegma preserving map onto $\mathcal{G}\restriction N'$ such that for every $l \in \mathbb{N}$ and $t \in \mathcal{F}\restriction L$,

$$\text{if } \min t \geq L(l) \text{ then } \min \varphi(t) \geq N'(l). \tag{3.4}$$

Set $\widehat{\varphi}_2 = \widehat{\varphi}_1 \circ \widehat{\varphi}$ and $x_s = y_{\widehat{\varphi}_2(s)}$ for all $s \in \widehat{\mathcal{F}}\restriction L$. Then $\widehat{\varphi}_2$ is continuous and therefore $(x_s)_{s\in\mathcal{F}\restriction L}$ is subordinated. Since $(y_s)_{s\in\mathcal{G}\restriction N}$ generates $(e_n)_n$ as a $\mathcal{G}$-spreading model, it follows that so does $(y_s)_{s\in\mathcal{G}\restriction N'}$, and therefore, invoking (3.4), we find that $(x_s)_{s\in\mathcal{F}\restriction L}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model too. The continuity of $\widehat{\varphi}$ and (3.4) yield $\widehat{\varphi}(\emptyset) = \emptyset$. Thus, since $(y_s)_{s\in\mathcal{G}\restriction N}$ is weakly null, so is $(x_s)_{s\in\mathcal{F}\restriction L}$. ∎

Corollary 3.3 implies that the transfinite hierarchy $(SM_\xi^w(X))_{\xi<\omega_1}$ is increasing. In general, this hierarchy is nontrivial and the class $SM_\xi^w(X)$ for $\xi > 1$ is richer than $SM_1^w(X)$. In fact, for every positive integer k there exists a reflexive space $\mathfrak{X}_{k+1}$ with an unconditional basis such that $\mathfrak{X}_{k+1}$ has no k-order spreading model equivalent to the standard basis of ℓ^1, while $\mathfrak{X}_{k+1}$ admits a $k+1$-order spreading model equivalent to that basis [AKT1, Section 12]. Since $\mathfrak{X}_{k+1}$ is reflexive, it follows that $SM_k^w(\mathfrak{X}_{k+1})$ is a proper subset of $SM_{k+1}^w(\mathfrak{X}_{k+1})$. This is due to the fact that for a reflexive Banach space X

the set of (Schauder basic) spreading models of order ξ coincides with the set of spreading models generated by subordinated weakly null $\mathcal{F}$-sequences of the same order. Indeed, if (e_n) is a spreading model of order ξ and $\mathcal{F}$ a regular thin family of order ξ, then by the reflexivity of X and Proposition 6.16 of [AKT2], (e_n) is generated by a subordinated $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright M}$. If (e_n) is not equivalent to the ℓ^1 basis, then by Theorem 6.14 of [AKT2], $(x_s)_{s\in\mathcal{F}\upharpoonright M}$ is weakly null. If (e_n) is equivalent to the ℓ^1 basis, then, again by Theorem 6.14 of [AKT2], (e_n) belongs to $SM^w_\xi(X)$.

It is important to point out that the higher order spreading models generated by subordinated weakly null $\mathcal{F}$-sequences do induce a new isomorphic invariance for Banach spaces. In particular, there exist two Banach spaces X and Y such that $SM^w_1(X) = SM^w_1(Y)$ and $SM^w_2(X) \neq SM^w_2(Y)$. This was known to the authors of [AKT3], though it was not explicitly stated. Indeed, let X be the space $X^2_{T,2,3}$ given in [AKT3, Theorem 12.11] and Y the direct sum of the Tsirelson space and ℓ^2. Both spaces are reflexive and therefore for every countable ordinal ξ and every Schauder basic ξ-order spreading model $(e_n)_n$ of X (resp. Y), $(e_n)_n$ belongs to $SM^w_\xi(X)$ (resp. $SM^w_\xi(Y)$). By Theorems 7.3 and 9.18 in [AKT3], it is easy to see that every Schauder basic spreading model of Y of any order ξ is equivalent to either the standard basis of ℓ^1 or the standard basis of ℓ^2, and, of course, both cases occur for every ξ. On the other hand, it is shown there that while every $(e_n)_n$ in $SM^w_1(X)$ is equivalent to either the standard basis of ℓ^1 or the standard basis of ℓ^2, $SM^w_2(X)$ contains a sequence equivalent to the standard basis of ℓ^3.

4. The semilattice structure of $SM^w_\xi(X)$. Let X be a Banach space and ξ a countable ordinal. By Theorem 6.11 of [AKT2], every sequence $(e_n)_n$ in $SM^w_\xi(X)$ is either 1-suppression unconditional or $\|\sum_{i=1}^n a_ie_i\| = 0$ for all $n \in \mathbb{N}$ and $(a_i)_1^n \in \mathbb{R}$. We endow $SM^w_\xi(X)$ with the pre-partial order $\preccurlyeq$ of domination. That is, for two sequences $(e^1_n)_n$ and $(e^2_n)_n$ in $SM^w_\xi(X)$ and $C > 0$ we say that $(e^2_n)_n$ *C-dominates* $(e^1_n)_n$ if

$$\Big\|\sum_{i=1}^n a_ie^1_i\Big\| \le C\Big\|\sum_{i=1}^n a_ie^2_i\Big\| \tag{4.1}$$

for all $n \in \mathbb{N}$ and $(a_i)_{i=1}^n \in \mathbb{R}$. Write $(e^1_n)_n \preccurlyeq (e^2_n)_n$ if $(e^2_n)_n$ C-dominates $(e^1_n)_n$ for some $C > 0$. $(e^1_n)_n$ and $(e^2_n)_n$ are *equivalent*, denoted $(e^1_n)_n \sim (e^2_n)_n$, if $(e^1_n)_n \preccurlyeq (e^2_n)_n$ and $(e^2_n)_n \preccurlyeq (e^1_n)_n$. We write $(e^1_n)_n \prec (e^2_n)_n$, if $(e^1_n)_n \preccurlyeq (e^2_n)_n$ and $(e^1_n)_n \nsim (e^2_n)_n$. It is easy to see that $\sim$ is an equivalence relation on $SM^w_\xi(X)$. Let $\mathbf{SM}^w_\xi(X) = SM^w_\xi(X)/_\sim$ be endowed with the partial order induced by $\preccurlyeq$, still denoted by $\preccurlyeq$.

In this section we will generalize some results from [AOST]. The arguments are similar to the original ones. In particular, we will show that $\mathbf{SM}_\xi^w$ is a semilattice and that every countable subset of $\mathbf{SM}_\xi^w$ admits an upper bound in $\mathbf{SM}_\xi^w$. Towards achieving that we introduce a further generalization of $\mathcal{F}$-spreading models which we call joint $\mathcal{F}$-models.

First, we need some additional notation. For every k in $\mathbb{N}$, define a map $i_k : \mathbb{N} \to \mathbb{N}$ by setting, for every $j \in \mathbb{N}$,

$$i_k(j) = ((j-1) \bmod k) + 1. \tag{4.2}$$

Definition 4.1. Let X be a Banach space, $\mathcal{F}$ a regular thin family, $k \in \mathbb{N}$ and $((x_s^i)_{s\in\mathcal{F}})_{i=1}^k$ a k-tuple of $\mathcal{F}$-sequences in X. Let $(E, \|\cdot\|_*)$ be an infinite-dimensional seminormed linear space with Hamel basis $(e_n)_n$. Let $M \in [\mathbb{N}]^\infty$ and $\delta_n \searrow 0$.

We say that the k-tuple $((x_s^i)_{s\in\mathcal{F}\upharpoonright M})_{i=1}^k$ generates $(e_n)_n$ as a *joint $\mathcal{F}$-model* (with respect to $(\delta_n)_n$) if for all m,n in $\mathbb{N}$ with $1 \le m \le n$, every finite sequence $(a_i)_{i=1}^m$ in $[-1,1]$ and every $(s_j)_{j=1}^m$ in $\mathrm{Plm}_m(\mathcal{F}\upharpoonright M)$ with $s_1(1) \ge M(n)$, we have

$$\left| \Big\| \sum_{j=1}^m a_j x_{s_j}^{i_k(j)} \Big\| - \Big\| \sum_{j=1}^m a_j e_j \Big\|_* \right| \le \delta_n. \tag{4.3}$$

A k-tuple $((x_s^i)_{s\in\mathcal{F}})_{i=1}^k$ is said to admit $(e_n)_n$ as a joint $\mathcal{F}$-model if there exists $M \in [\mathbb{N}]^\infty$ such that the k-tuple $((x_s^i)_{s\in\mathcal{F}\upharpoonright M})_{i=1}^k$ generates $(e_n)_n$ as a joint $\mathcal{F}$-model.

Note that a joint $\mathcal{F}$-model is not necessarily spreading. The arguments establishing the existence of joint $\mathcal{F}$-models (see Theorem 4.4 below) are similar to the ones concerning $\mathcal{F}$-spreading models. We will need the following result (see [AKT2, Theorem 3.6]) which establishes the Ramsey property for plegma families.

Theorem 4.2. *Let $\mathcal{F}$ be a regular thin family, M an infinite subset of $\mathbb{N}$ and $l \in \mathbb{N}$. Then for every finite partition $\mathrm{Plm}_l(\mathcal{F}\upharpoonright M) = \bigcup_{i=1}^p \mathcal{P}_i$, there exist $L \in [M]^\infty$ and $1 \le i_0 \le p$ such that $\mathrm{Plm}_l(\mathcal{F}\upharpoonright L) \subseteq \mathcal{P}_{i_0}$.*

Lemma 4.3. *Let X be a Banach space and $\mathcal{F}$ a regular thin family. Also let $k \in \mathbb{N}$ and $(x_s^1)_{s\in\mathcal{F}}, \dots, (x_s^k)_{s\in\mathcal{F}}$ bounded $\mathcal{F}$-sequences in X. Then for every infinite subset M of $\mathbb{N}$, every $\varepsilon > 0$ and every $l \in \mathbb{N}$, there exists an infinite subset L of M such that for all $(s_j)_{j=1}^l, (t_j)_{j=1}^l$ in $\mathrm{Plm}_l(\mathcal{F}\upharpoonright L)$ and every choice of reals $(a_j)_{j=1}^l$ from $[-1,1]$ we have*

$$\left| \Big\| \sum_{j=1}^l a_j x_{s_j}^{i_k(j)} \Big\| - \Big\| \sum_{j=1}^l a_j x_{t_j}^{i_k(j)} \Big\| \right| < \varepsilon, \tag{4.4}$$

where i_k is as defined in (4.2).

Proof. Fix an infinite subset M of $\mathbb{N}$, a positive real ε and $l \in \mathbb{N}$. Assume $\|x_s^i\| \leq C$ for all $s \in \mathcal{F}$ and $i = 1, \ldots, k$. Let Λ be a finite $\frac{\varepsilon}{3lC}$-net of $[-1,1]$ and $((a_j^q)_{j=1}^l)_{q=1}^n$ an enumeration of all the l-tuples consisting of elements from Λ, where $n = |\Lambda|^l$.

Setting $L_0 = M$, we inductively construct a decreasing sequence $(L_q)_{q=0}^n$ of infinite subsets of $\mathbb{N}$ such that for every $q = 1, \ldots, n$ and all $(s_j)_{j=1}^l$ and $(t_j)_{j=1}^l$ from $\mathrm{Plm}_l(\mathcal{F}\restriction L_q)$, we have

$$\Big|\Big\|\sum_{j=1}^{l} a_j^q x_{s_j}^{i_k(j)}\Big\| - \Big\|\sum_{j=1}^{l} a_j^q x_{t_j}^{i_k(j)}\Big\|\Big| < \frac{\varepsilon}{3}. \tag{4.5}$$

The inductive step is an application of Theorem 4.2. Indeed, assume that for some $1 \leq q \leq n$ the set L_{q-1} has been chosen. Let $(A_r)_{r=1}^p$ be a partition of $[0, lC]$ such that A_r is of diameter at most $\varepsilon/3$ for all $r = 1, \ldots, p$. Observe that for every $(s_j)_{j=1}^l$ in $\mathrm{Plm}_l(\mathcal{F}\restriction L_q)$ the vector $\sum_{j=1}^l a_j^q x_{s_j}^{i_k(j)}$ is of norm at most lC. Thus if, for every $r = 1, \ldots, p$, $\mathcal{P}_r$ is the set of all $(s_j)_{j=1}^l$ from $\mathrm{Plm}_l(\mathcal{F}\restriction L_{q-1})$ such that the norm of the vector $\sum_{j=1}^l a_j^q x_{s_j}^{i_k(j)}$ belongs to A_r, then $(\mathcal{P}_r)_{r=1}^p$ forms a partition of $\mathrm{Plm}_l(\mathcal{F}\restriction L_{q-1})$. An application of Theorem 4.2 yields the desired L_q and the proof of the inductive step is complete.

We set $L = L_n$. Clearly, for every $q = 1, \ldots, n$ and all $(s_j)_{j=1}^l, (t_j)_{j=1}^l$ from $\mathrm{Plm}_l(\mathcal{F}\restriction L)$, we have

$$\Big|\Big\|\sum_{j=1}^{l} a_j^q x_{s_j}^{i_k(j)}\Big\| - \Big\|\sum_{j=1}^{l} a_j^q x_{t_j}^{i_k(j)}\Big\|\Big| < \frac{\varepsilon}{3}. \tag{4.6}$$

It remains to show that L is as desired. Indeed, choose $(s_j)_{j=1}^l, (t_j)_{j=1}^l$ from $\mathrm{Plm}_{kl}(\mathcal{F}\restriction L)$ and $(a_j)_{j=0}^l$ from $[-1,1]$. Pick $q_0 \in \{1, \ldots, n\}$ such that $|a_j - a_j^{q_0}| \leq \varepsilon/(lC)$ for all $j = 1, \ldots, l$. By the triangle inequality and the choice of C,

$$\Big|\Big\|\sum_{j=1}^{l} a_j x_{s_j}^{i_k(j)}\Big\| - \Big\|\sum_{j=1}^{l} a_j^{q_0} x_{s_j}^{i_k(j)}\Big\|\Big| < \frac{\varepsilon}{3}, \tag{4.7}$$

$$\Big|\Big\|\sum_{j=1}^{l} a_j x_{t_j}^{i_k(j)}\Big\| - \Big\|\sum_{j=1}^{l} a_j^{q_0} x_{t_j}^{i_k(j)}\Big\|\Big| < \frac{\varepsilon}{3}. \tag{4.8}$$

Inequalities (4.6)–(4.8) yield

$$\Big|\Big\|\sum_{j=1}^{l} a_j x_{s_j}^{i_k(j)}\Big\| - \Big\|\sum_{j=1}^{l} a_j x_{t_j}^{i_k(j)}\Big\|\Big| < \varepsilon, \tag{4.9}$$

and the proof is complete. ∎

By iterating the above lemma and diagonalizing, we obtain the following.

THEOREM 4.4. *Let X be a Banach space and $\mathcal{F}$ a regular thin family. Also let $k \in \mathbb{N}$ and $(x_s^1)_{s\in\mathcal{F}}, \dots, (x_s^k)_{s\in\mathcal{F}}$ be bounded $\mathcal{F}$-sequences in X. Then for every infinite subset M of $\mathbb{N}$ there exists a further infinite subset L of M such that the k-tuple $((x_s^i)_{s\in\mathcal{F}\restriction L})_{i=1}^k$ generates a joint $\mathcal{F}$-model.*

We proceed to the following analogue of Theorem 6.11 from [AKT2] for joint models.

THEOREM 4.5. *Let X be a Banach space, $\mathcal{F}$ a regular thin family and $k \in \mathbb{N}$. Also let M be an infinite subset of $\mathbb{N}$ and $(x_s^1)_{s\in\mathcal{F}}, \dots, (x_s^k)_{s\in\mathcal{F}}$ seminormalized $\mathcal{F}$-sequences in X such that the $\mathcal{F}$-subsequences $(x_s^1)_{s\in\mathcal{F}\restriction M}, \dots, (x_s^k)_{s\in\mathcal{F}\restriction M}$ are subordinated and weakly null. Also assume that the k-tuple $((x_s^i)_{s\in\mathcal{F}\restriction M})_{i=1}^k$ generates a joint $\mathcal{F}$-model $(e_n)_n$. Then $(e_n)_n$ is (suppression) 1-unconditional.*

The proof of Theorem 4.5 follows along similar lines to the proof of Theorem 6.11 from [AKT2]. Let $l \in \mathbb{N}$ and $F_1, \dots, F_l$ be subsets of $[\mathbb{N}]^{<\infty}$. Recall that $(F_j)_{j=1}^l$ is *completely plegma connected* if for every choice $s_j \in F_j$ for all $1 \le j \le l$, the l-tuple $(s_j)_{j=1}^l$ is a plegma family. Moreover, the convex hull of a subset A of a Banach space is denoted by $\operatorname{conv} A$. For the proof of Theorem 4.5 we need to recall Lemma 6.10 from [AKT2].

LEMMA 4.6. *Let X be a Banach space, l a positive integer, $\mathcal{F}_1, \dots, \mathcal{F}_l$ regular thin families and L an infinite subset of $\mathbb{N}$. Assume that for every $i = 1, \dots, l$, there exists a continuous map $\widehat{\varphi}_i : \widehat{\mathcal{F}_i}\restriction L \to X$, where X is considered with the weak topology. Then for every $\varepsilon > 0$ there exists a completely plegma connected family $(F_i)_{i=1}^l$ such that $F_i \subseteq [\mathcal{F}_i\restriction L]^{<\infty}$ and $\operatorname{dist}(\widehat{\varphi}_i(\emptyset), \operatorname{conv}\widehat{\varphi}_i(F_i)) < \varepsilon$ for every $i = 1, \dots, l$.*

Proof of Theorem 4.5. Fix $l \in \mathbb{N}$, $1 \le p \le l$ and $a_1, \dots, a_l$ in $[-1,1]$. It suffices to show that for every $\varepsilon > 0$ we have

$$\Big\|\sum_{\substack{j=1\\ j\neq p}}^{l} a_j e_j\Big\|_* < \Big\|\sum_{j=1}^{l} a_j e_j\Big\|_* + \varepsilon. \tag{4.10}$$

Fix $\varepsilon > 0$. Since $((x_s^i)_{s\in\mathcal{F}\restriction M})_{i=1}^k$ generates $(e_n)_n$ as a joint $\mathcal{F}$-model, by passing to a tail of M if necessary, we may assume that

$$\Big|\Big\|\sum_{\substack{j=1\\ j\neq p}}^{l} a_j x_{s_j}^{i_k(j)}\Big\| - \Big\|\sum_{\substack{j=1\\ j\neq p}}^{l} a_j e_j\Big\|_*\Big| < \frac{\varepsilon}{3}, \quad \Big|\Big\|\sum_{j=1}^{l} a_j x_{s_j}^{i_k(j)}\Big\| - \Big\|\sum_{j=1}^{l} a_j e_j\Big\|_*\Big| < \frac{\varepsilon}{3}, \tag{4.11}$$

for every plegma l-tuple $(s_j)_{j=1}^l$ in $\mathcal{F}\restriction M$, where i_k is as defined in (4.2). The first inequality in (4.11) is obtained by setting a_p to be 0. Since for every $1\le i\le k$ the $\mathcal{F}$-subsequence $(x_s^i)_{s\in\mathcal{F}\upharpoonright M}$ is subordinated, there exists a continuous map $\widehat{\varphi}_i:\widehat{\mathcal{F}}\upharpoonright M\to X$ such that $\widehat{\varphi}_i(s)=x_s^i$ for every $s\in\mathcal{F}\upharpoonright M$. Moreover, for every $1\le i\le k$, since $(x_s^i)_{s\in\mathcal{F}\upharpoonright M}$ is weakly convergent to $\widehat{\varphi}_i(\emptyset)$ and by assumption weakly null, we have $\widehat{\varphi}_i(\emptyset)=0$. Therefore by Lemma 4.6 (for "$\mathcal{F}_j=\mathcal{F}$" and "$\widehat{\varphi}_j=\widehat{\varphi}_{i_k(j)}$", for all $j=1,\dots,l$), there exist a completely plegma connected family $(F_j)_{j=1}^l$ and a sequence $(x_j)_{j=1}^l$ in X such that $F_j\subset[\mathcal{F}\upharpoonright M]^{<\infty}$, $x_j\in\operatorname{conv}\widehat{\varphi}_{i_k(j)}(F_j)$ and $\|x_j\|<\varepsilon/3$, for all $1\le j\le l$. Let $(\mu_s)_{s\in F_p}$ be a sequence in $[0,1]$ such that $\sum_{s\in F_p}\mu_s=1$ and $x_p=\sum_{s\in F_p}\mu_s\widehat{\varphi}_{i_k(p)}(s)$ and for each $j\neq p$ choose $s_j\in F_j$. Clearly $\|x_p\|=\|\sum_{s\in F_p}\mu_s x_s\|<\varepsilon/3$. Since $(F_j)_{j=1}^l$ is completely plegma connected, for every s in F_p the l-tuple $(s_1,\dots,s_{p-1},s,s_{p+1},\dots,s_l)$ is a plegma family. Therefore by (4.11) we have

$$\begin{aligned}\Big\|\sum_{\substack{j=1\\j\neq p}}^l a_je_j\Big\|_*&\le\Big\|\sum_{\substack{j=1\\j\neq p}}^l a_jx_{s_j}^{i_k(j)}\Big\|+\frac{\varepsilon}{3}\le\Big\|\sum_{\substack{j=1\\j\neq p}}^l a_jx_{s_j}^{i_k(j)}+a_px_p\Big\|+|a_p|\frac{\varepsilon}{3}+\frac{\varepsilon}{3}\\&\le\Big\|\sum_{\substack{j=1\\j\neq p}}^l a_jx_{s_j}^{i_k(j)}+a_p\sum_{s\in F_p}\mu_sx_s^{i_k(p)}\Big\|+\frac{2\varepsilon}{3}\\&\le\sum_{s\in F_p}\mu_s\Big\|\sum_{\substack{j=1\\j\neq p}}^l a_jx_{s_j}^{i_k(j)}+a_px_s^{i_k(p)}\Big\|+\frac{2\varepsilon}{3}\\&\le\sum_{s\in F_p}\mu_s\bigg(\Big\|\sum_{j=1}^l a_je_j\Big\|_*+\frac{\varepsilon}{3}\bigg)+\frac{2\varepsilon}{3}=\Big\|\sum_{j=1}^l a_je_j\Big\|_*+\varepsilon.\end{aligned}$$

The proof is complete. ∎

The following lemma allows us to establish the semilattice structure of $SM_\xi^w(X)$.

Lemma 4.7. *Let X be a Banach space and ξ a countable ordinal. Also let $(e_n^1)_n,\dots,(e_n^k)_n$ be elements of $SM_\xi^w(X)$. Then there exists $(e_n)_n$ in $SM_\xi^w(X)$ such that*

$$\max_{1\le i\le k}\Big\|\sum_{j=1}^l a_je_j^i\Big\|\le\Big\|\sum_{j=1}^l a_je_j\Big\|\le\sum_{i=1}^k\Big\|\sum_{j=1}^l a_je_j^i\Big\|\quad\Big(\le k\max_{1\le i\le k}\Big\|\sum_{j=1}^l a_je_j^i\Big\|\Big)\tag{4.12}$$

for every choice of $l\in\mathbb{N}$ and $a_1,\dots,a_l\in\mathbb{R}$.

Before we proceed to the proof of Lemma 4.7 let us recall some notation. A family $\mathcal{H}$ of finite subsets of $\mathbb{N}$ is called *large* (resp. *very large*) in an infinite subset M of $\mathbb{N}$ if every further infinite subset L of $\mathbb{N}$ contains an element (resp. has an initial segment) in $\mathcal{H}$. It is immediate that both the notions of large and very large are hereditary, i.e. if $\mathcal{H}$ is large (resp. very large) in some M then $\mathcal{H}$ is large (resp. very large) in any infinite subset L of M. We will need the following well known result due to F. Galvin and K. Prikry [GP] (it is actually a reformulation provided in [Go]).

Theorem 4.8. *Let $\mathcal{H}$ be a family of finite subsets of $\mathbb{N}$ and M an infinite subset of $\mathbb{N}$. If $\mathcal{H}$ is large in M then there exists an infinite subset L of M such that $\mathcal{H}$ is very large in L.*

It is easy to see that every regular thin family $\mathcal{F}$ is large in $\mathbb{N}$. Thus by the above theorem we have the following.

Corollary 4.9. *Let $\mathcal{F}$ be a regular thin family and M an infinite subset of $\mathbb{N}$. Then there exists an infinite subset L of M such that $\mathcal{F}$ is very large in L.*

Proof of Lemma 4.7. Let $\mathcal{F}$ be a regular thin family of order ξ. Applying Corollary 3.3, we obtain an infinite subset M_0 of $\mathbb{N}$ and $\mathcal{F}$-sequences $(x_s^1)_{s\in\mathcal{F}},\dots,(x_s^k)_{s\in\mathcal{F}}$ in X such that for every $1\le i\le k$ the $\mathcal{F}$-subsequence $(x_s^i)_{s\in\mathcal{F}\upharpoonright M_0}$ is subordinated, weakly null and generates $(e_n^i)_n$ as an $\mathcal{F}$-spreading model. By Corollary 4.9, we may assume that $\mathcal{F}$ is very large in M_0. Using Theorem 4.4, we pass to an infinite subset M of M_0 such that the k-tuple $((x_s^i)_{s\in\mathcal{F}\upharpoonright M})_{i=1}^k$ generates a joint $\mathcal{F}$-model $(v_n)_n$. Theorem 4.5 shows that $(v_n)_n$ is (suppression) 1-unconditional.

We pick a sequence $(F_n)_n$ of finite subsets of M such that $\max F_n<\min F_{n+1}$ and F_n is of cardinality k for all $n\in\mathbb{N}$. We set $N=\{\max F_n : n\in\mathbb{N}\}$. Clearly N is an infinite subset of M. For every $s\in\mathcal{F}\upharpoonright N$ and every $1\le i\le k$ we set t_s^i to be the unique element in $\mathcal{F}$ being an initial segment of $\{F_{s(q)}(i) : 1\le q\le |s|\}$. Observe that the existence of t_s^i is guaranteed by the spreading property of $\widehat{\mathcal{F}}$ and the fact that $\mathcal{F}$ is very large, while its uniqueness is a consequence of the fact that $\mathcal{F}$ is thin. Also observe that for every $l\in\mathbb{N}$ and every plegma family $\mathbf{s}=(s_j)_{j=1}^l$ in $\mathcal{F}\upharpoonright N$ of length l, the family $(t_q^{\mathbf{s}})_{q=1}^{kl}=(t_{s_1}^1,t_{s_1}^2,\dots,t_{s_1}^k,\dots,t_{s_l}^1,t_{s_l}^2,\dots,t_{s_l}^k)$ is a plegma family in $\mathcal{F}\upharpoonright M$. For every $s\in\mathcal{F}\upharpoonright N$ we set $z_s=\sum_{i=1}^k x_{t_s^i}^i$. Pass to an infinite subset L of N such that $(z_s)_{s\in\mathcal{F}\upharpoonright L}$ generates an $\mathcal{F}$-spreading model $(e_n)_n$. The following claim holds.

Claim. *Let $l\in\mathbb{N}$, $a_1,\dots,a_l$ in $[-1,1]$ and $\varepsilon>0$. Then*

$$\max_{1\le i\le k}\Big\|\sum_{j=1}^{l}a_je_j^i\Big\|-\varepsilon\le\Big\|\sum_{j=1}^{l}a_je_j\Big\|\le\sum_{i=1}^{k}\Big\|\sum_{j=1}^{l}a_je_j^i\Big\|+\frac{\varepsilon}{2}. \tag{4.13}$$

Proof of Claim. Let $(b_q)_{q=1}^{kl}$ be defined by $b_{(j-1)k+i} = a_j$ for all $1 \le j \le l$ and $1 \le i \le k$. We pass to a final segment L' of L such that for every plegma family $\mathbf{s} = (s_j)_{j=1}^{l}$ in $\mathcal{F}\upharpoonright L'$ we have

$$\Big|\Big\|\sum_{j=1}^{l} a_j e_j\Big\| - \Big\|\sum_{j=1}^{l} a_j z_{s_j}\Big\|\Big| < \varepsilon/4, \tag{4.14}$$

$$\Big|\Big\|\sum_{j=1}^{l} a_j e_j^i\Big\| - \Big\|\sum_{j=1}^{l} a_j x^i_{t^i_{s_j}}\Big\|\Big| < \varepsilon/(4k) \quad \text{for all } 1 \le i \le k, \tag{4.15}$$

$$\Big|\Big\|\sum_{q\in F} b_q v_q\Big\| - \Big\|\sum_{q\in F} b_q x^{i_k(q)}_{t^{\mathbf{s}}_q}\Big\|\Big| < \varepsilon/4 \qquad \text{for all } F \subseteq \{1,\dots,kl\}. \tag{4.16}$$

We set $F_i = \{(j-1)k + i : j = 1,\dots,l\}$ for all $1 \le i \le k$. Let us also fix some plegma family $\mathbf{s} = (s_j)_{j=1}^{l}$ in $\mathcal{F}\upharpoonright L'$. Recall that by Theorem 4.5, $(v_n)_n$ is (suppression) 1-unconditional. Hence for every $1 \le i \le k$, we have

(4.17)

$$\begin{aligned}
\Big\|\sum_{j=1}^{l} a_j e_j^i\Big\| - \varepsilon &\overset{(4.15)}{\le} \Big\|\sum_{j=1}^{l} a_j x^i_{t^i_{s_j}}\Big\| - \frac{3\varepsilon}{4} = \Big\|\sum_{q\in F_i} b_q x^{i_k(q)}_{t^{\mathbf{s}}_q}\Big\| - \frac{3\varepsilon}{4}\\
&\overset{(4.16)}{\le} \Big\|\sum_{q\in F_i} b_q v_q\Big\| - \frac{\varepsilon}{2} \le \Big\|\sum_{q=1}^{kl} b_q v_q\Big\| - \frac{\varepsilon}{2}\\
&\overset{(4.16)}{\le} \Big\|\sum_{q=1}^{kl} b_q x^{i_k(q)}_{t^{\mathbf{s}}_q}\Big\| - \frac{\varepsilon}{4} = \Big\|\sum_{j=1}^{l} a_j z_{s_j}\Big\| - \frac{\varepsilon}{4} \overset{(4.14)}{\le} \Big\|\sum_{j=1}^{l} a_j e_j\Big\|.
\end{aligned}$$

Since (4.17) holds for all $1 \le i \le k$ we get

$$\max_{1\le i\le k}\Big\|\sum_{j=1}^{l} a_j e_j^i\Big\| - \varepsilon \le \Big\|\sum_{j=1}^{l} a_j e_j\Big\|. \tag{4.18}$$

Making use of the triangle inequality we have

$$\begin{aligned}
\Big\|\sum_{j=1}^{l} a_j e_j\Big\| &\overset{(4.14)}{\le} \Big\|\sum_{j=1}^{l} a_j z_{s_j}\Big\| + \frac{\varepsilon}{4} = \Big\|\sum_{q=1}^{kl} b_q x^{i_k(q)}_{t^{\mathbf{s}}_q}\Big\| + \frac{\varepsilon}{4}\\
&\le \sum_{i=1}^{k}\Big\|\sum_{q\in F_i} b_q x^{i_k(q)}_{t^{\mathbf{s}}_q}\Big\| + \frac{\varepsilon}{4} = \sum_{i=1}^{k}\Big\|\sum_{j=1}^{l} a_j x^{i_k(F_i(j))}_{t^{\mathbf{s}}_{F_i(j)}}\Big\| + \frac{\varepsilon}{4}\\
&= \sum_{i=1}^{k}\Big\|\sum_{j=1}^{l} a_j x^{i}_{t^{\mathbf{s}}_{F_i(j)}}\Big\| + \frac{\varepsilon}{4} \overset{(4.15)}{\le} \sum_{i=1}^{k}\Big\|\sum_{j=1}^{l} a_j e^i_j\Big\| + \frac{\varepsilon}{2}.
\end{aligned} \tag{4.19}$$

Clearly (4.13) follows from (4.18) and (4.19). ∎

By the Claim above the conclusion of Lemma 4.7 is immediate. ■

The above lemma has the following immediate consequence.

THEOREM 4.10. *Let X be a Banach space and ξ a countable ordinal. Then $\mathbf{SM}^w_\xi(X)$ is an upper semilattice.*

THEOREM 4.11. *Let X be a Banach space and ξ a countable ordinal. Let $(c_k)_k$ be a sequence of positive reals satisfying $\sum_{k=1}^\infty c_k^{-1} < \infty$ and for every $k \in \mathbb{N}$ let $(e_n^k)_n$ be a normalized sequence that belongs to $SM^w_\xi(X)$. Then there exist $(e_n)_n$ in $SM^w_\xi(X)$ and a real K with $\max_{k\in\mathbb{N}} c_k^{-1} \le K \le \sum_{k=1}^\infty c_k^{-1}$ such that:*

(i) *The sequence $(e_n)_n$ is normalized.*
(ii) *The sequence $(e_n)_n$ $(c_k K)$-dominates $(e_n^k)_n$ for all $k \in \mathbb{N}$.*
(iii) *For every $l \in \mathbb{N}$ and every choice of reals $a_1, \ldots, a_l$ in $[-1,1]$ we have $\|\sum_{j=1}^l a_j e_j\| \le K^{-1}\sum_{k=1}^\infty c_k^{-1}\|\sum_{j=1}^l a_j e_j^k\|$.*

Proof. Let $\mathcal{F}$ be a regular thin family of order ξ. Applying Corollary 4.9 we obtain an infinite subset M_0 of $\mathbb{N}$ such that $\mathcal{F}$ is very large in M_0. Using Corollary 3.3, we inductively obtain a decreasing sequence $(M'_k)_k$ of infinite subsets of M_0 and for every $k \in \mathbb{N}$ a seminormalized $\mathcal{F}$-sequence $(x_s^k)_{s\in\mathcal{F}}$ such that for every $k \in \mathbb{N}$ the $\mathcal{F}$-subsequence $(x_s^k)_{s\in\mathcal{F}\upharpoonright M'_k}$ is subordinated, weakly null and generates $(e_n^k)_n$ as an $\mathcal{F}$-spreading model. Moreover, setting $B_k = \sup\{\|x_s^k\| : s \in \mathcal{F}\upharpoonright M'_k\}$, we may assume that $B_k \le 2$ for all $k \in \mathbb{N}$. Hence $\sum_{k=1}^\infty c_k^{-1}B_k < \infty$. For every $k \in \mathbb{N}$ and $s \in \mathcal{F}$ we set $y_s^k = c_k^{-1}x_s^k$. Clearly for every $k \in \mathbb{N}$ the $\mathcal{F}$-sequence $(y_s^k)_{s\in\mathcal{F}}$ is seminormalized, while $(y_s^k)_{s\in\mathcal{F}\upharpoonright M'_k}$ is subordinated, weakly null and generates $(c_k^{-1}e_n^k)_n$ as an $\mathcal{F}$-spreading model.

Let M' be an infinite subset of M_0 such that $M'(k) \in M'_k$ for all $k \in \mathbb{N}$. Using Theorem 4.4 we obtain a decreasing sequence $(M_k)_k$ of infinite subsets of M' such that for every $k \in \mathbb{N}$ the k-tuple $((y_s^i)_{s\in\mathcal{F}\upharpoonright M_k})_{i=1}^k$ generates a joint $\mathcal{F}$-model $(v_n^k)_n$. Observe that, by Theorem 4.5, the sequence $(v_n^k)_n$ is (suppression) 1-unconditional for all $k \in \mathbb{N}$.

Let M be an infinite subset of M' such that $M(k) \in M_k$ for all $k \in \mathbb{N}$. Fix a sequence $(F_n)_n$ of finite subsets of M such that for every $n \in \mathbb{N}$ $\max F_n < \min F_{n+1}$ and F_n is of cardinality n. Set $L = \{\max F_n : n \in \mathbb{N}\}$. For every $s \in \mathcal{F}\upharpoonright L$ let n_s be such that $\min s = \max F_{n_s}$, and t_s^i be the unique initial segment of $\{F_{s(j)}(i) : j = 1, \ldots, |s|\}$ belonging to $\mathcal{F}$, for all $1 \le i \le n_s$. The existence of t_s^i follows from the fact that $\mathcal{F}$ is very large in L and $\widehat{\mathcal{F}}$ is spreading, while the uniqueness of t_s^i follows from $\mathcal{F}$ being thin. Observe that for every $s \in \mathcal{F}\upharpoonright L$ and every $1 \le i \le n_s$, $\min t_s^i \ge M(i) \ge M'(i) \in M'_i$ and therefore

$$\|x^i_{t_s^i}\| \le B_i. \tag{4.20}$$

For every s in $\mathcal{F}\upharpoonright L$ set $z_s = \sum_{i=1}^{n_s} y^i_{t^i_s}$ and pass to an infinite subset L' of L such that $(z_s)_{s\in\mathcal{F}\upharpoonright L'}$ generates an $\mathcal{F}$-spreading model $(e'_n)_n$. Since $(c_k^{-1}B_k)_k$ is summable and for each $k\in\mathbb{N}$ the $\mathcal{F}$-subsequence $(x^k_s)_{s\in\mathcal{F}\upharpoonright L'}$ is subordinated and weakly null, by (4.20), one can easily derive that $(z_s)_{s\in\mathcal{F}\upharpoonright L'}$ is also subordinated and weakly null. Thus $(e'_n)_n$ belongs to $SM^w_\xi(X)$. First we prove the following claim.

Claim. *The sequence $(e'_n)_n$ c_k-dominates $(e^k_n)_n$ for all $k\in\mathbb{N}$.*

Proof of Claim. Fix some $k\in\mathbb{N}$. Pick $l\in\mathbb{N}$, real numbers $a_1,\dots,a_l$ in $[-1,1]$ and $\varepsilon>0$. We will show that

$$\Big\|\sum_{j=1}^{l} a_j e^k_j\Big\| \le c_k\Big\|\sum_{j=1}^{l} a_j e'_j\Big\| + \varepsilon. \tag{4.21}$$

Pick $k'\ge k$ such that

$$\sum_{q=k'+1}^{\infty} c_q^{-1}B_q < \frac{\varepsilon}{5c_k\sum_{j=1}^{l}|a_j|}. \tag{4.22}$$

Let $(b_q)_{q=1}^{k'l}$ be defined by $b_{(j-1)k'+i}=a_j$ for all $1\le j\le l$ and $1\le i\le k'$. Moreover, for every plegma family $\mathbf{s}=(s_j)_{j=1}^{l}$ in $\mathcal{F}\upharpoonright L$ with $n_{s_1}\ge L(k')$ we set $(t^{\mathbf{s}}_q)_{q=1}^{k'l}=(t^1_{s_1},\dots,t^{k'}_{s_1},\dots,t^1_{s_l},\dots,t^{k'}_{s_l})$. Observe that for every plegma family $\mathbf{s}=(s_j)_{j=1}^{l}$ in $\mathcal{F}\upharpoonright L$ with $n_{s_1}\ge L(k')$ both the $k'l$-tuple $(t^{\mathbf{s}}_q)_{q=1}^{k'l}$ and the $(\sum_{j=1}^{l} n_{s_j})$-tuple $(t^1_{s_1},\dots,t^{n_{s_1}}_{s_1},\dots,t^1_{s_l},\dots,t^{n_{s_l}}_{s_l})$ are plegma families. We pass to an infinite subset L' of L such that $\min L'\ge L(k')$ and for every plegma family $\mathbf{s}=(s_j)_{j=1}^{l}$ in $\mathcal{F}\upharpoonright L'$ we have

$$\bigg|\Big\|\sum_{j=1}^{l} a_j e'_j\Big\| - \Big\|\sum_{j=1}^{l} a_j z_{s_j}\Big\|\bigg| < \frac{\varepsilon}{5c_k}, \tag{4.23}$$

$$\bigg|c_k^{-1}\Big\|\sum_{j=1}^{l} a_j e^k_j\Big\| - \Big\|\sum_{j=1}^{l} a_j y^k_{t^k_{s_j}}\Big\|\bigg| < \frac{\varepsilon}{5c_k}, \tag{4.24}$$

$$\bigg|\Big\|\sum_{q\in F} b_q y^{i_{k'}(q)}_{t^{\mathbf{s}}_q}\Big\| - \Big\|\sum_{q\in F} b_q v^{k'}_q\Big\|\bigg| < \frac{\varepsilon}{5c_k} \quad \text{for all } F\subseteq\{1,\dots,k'l\}, \tag{4.25}$$

where $i_{k'}$ is as defined in (4.2). Fix a plegma family $\mathbf{s}=(s_j)_{j=1}^{l}$ in $\mathcal{F}\upharpoonright L'$ and set

$$F_k=\{(j-1)k'+k : j=1,\dots,l\}.$$

By the unconditionality of $(v_n^{k'})_n$ we get

$$\tag{4.26}\begin{aligned}\Big\|\sum_{q=1}^{k'l} b_q y^{i_{k'}(q)}_{t^{\mathbf{s}}_q}\Big\| &\overset{(4.25)}{\geq} \Big\|\sum_{q=1}^{k'l} b_q v_q^{k'}\Big\| - \frac{\varepsilon}{5c_k} \geq \Big\|\sum_{q\in F_k} b_q v_q^{k'}\Big\| - \frac{\varepsilon}{5c_k}\\ &\overset{(4.25)}{\geq} \Big\|\sum_{q\in F_k} b_q y^{i_{k'}(q)}_{t^{\mathbf{s}}_q}\Big\| - \frac{2\varepsilon}{5c_k} = \Big\|\sum_{j=1}^{l} a_j y^k_{t^k_{s_j}}\Big\| - \frac{2\varepsilon}{5c_k}\\ &\overset{(4.24)}{\geq} c_k^{-1}\Big\|\sum_{j=1}^{l} a_j e_j^k\Big\| - \frac{3\varepsilon}{5c_k}.\end{aligned}$$

Moreover,

(4.27)

$$\begin{aligned}c_k\Big\|\sum_{j=1}^{l} a_j e'_j\Big\| + \varepsilon &\overset{(4.23)}{\geq} c_k\Big\|\sum_{j=1}^{l} a_j z_{s_j}\Big\| + \frac{4\varepsilon}{5} = c_k\Big\|\sum_{j=1}^{l} a_j \sum_{i=1}^{n_{s_j}} y^i_{t^i_{s_j}}\Big\| + \frac{4\varepsilon}{5}\\ &\geq c_k\Big\|\sum_{j=1}^{l} a_j \sum_{i=1}^{k'} y^i_{t^i_{s_j}}\Big\| - c_k\sum_{j=1}^{l}|a_j|\cdot\Big\|\sum_{i=k'+1}^{n_{s_j}} y^i_{t^i_{s_j}}\Big\| + \frac{4\varepsilon}{5}\\ &\geq c_k\Big\|\sum_{j=1}^{l} a_j \sum_{i=1}^{k'} y^i_{t^i_{s_j}}\Big\| - c_k\sum_{j=1}^{l}|a_j|\sum_{i=k'+1}^{n_{s_j}} c_i^{-1}\|x^i_{t^i_{s_j}}\| + \frac{4\varepsilon}{5}\\ &\overset{(4.20),(4.22)}{\geq} c_k\Big\|\sum_{j=1}^{l} a_j \sum_{i=1}^{k'} y^i_{t^i_{s_j}}\Big\| + \frac{3\varepsilon}{5}\\ &= c_k\Big\|\sum_{q=1}^{k'l} b_q y^{i_{k'}(q)}_{t^{\mathbf{s}}_q}\Big\| + \frac{3\varepsilon}{5}.\end{aligned}$$

Clearly (4.21) follows from (4.26) and (4.27). Since (4.21) holds for every choice of natural numbers k, l, real numbers $a_1, \ldots, a_l$ in $[-1,1]$ and $\varepsilon > 0$, the claim follows. ∎

Let $K = \|e'_1\|$ and $(e_n)_n = (K^{-1}e'_n)_n$. Then $(e_n)_n$ is a normalized sequence belonging to $SM^w_\xi(X)$, i.e. assertion (i) of Theorem 4.11 is satisfied. By the Claim, assertion (ii) is immediate and $K = \|e_1\| \geq c_k^{-1}\|e_1^k\| = c_k^{-1}$ for all $k \in \mathbb{N}$. Thus $K \geq \max_{k\in\mathbb{N}} c_k^{-1}$. Finally, by the definition of $(e'_n)_n$ and $(e_n)_n$ it is easy to check that assertion (iii) is also true and $K \leq \sum_{k=1}^\infty c_k^{-1}$. The proof of Theorem 4.11 is complete. ∎

5. From countable to uncountable increasing sequences of spreading models. Using identical arguments to the ones used in [Sa, proof of Theorem 2.2] one can prove the following.

THEOREM 5.1. *Let $\mathcal{C}$ be a family of normalized Schauder basic sequences satisfying:*

(i) *For every $(x_n)_n$ and $(y_n)_n$ in $\mathcal{C}$ there exists $(z_n)_n$ in C such that $(z_n)_n$ 2-dominates both $(x_n)_n$ and $(y_n)_n$.*
(ii) *For every sequence $(c_k)_k$ of positive reals satisfying $\sum_{k=1}^{\infty} c_k^{-1} < \infty$ and every infinite sequence $(x_n^1)_n, (x_n^2)_n, \ldots$ in $\mathcal{C}$ there exist $(x_n)_n$ in $\mathcal{C}$ and a constant K with $\max_{k\in\mathbb{N}} c_k^{-1} \le K \le \sum_{k=1}^{\infty} c_k^{-1}$ such that:*
   (a) *The sequence $(x_n)_n$ $(c_k K)$-dominates $(x_n^k)_n$ for all $k \in \mathbb{N}$.*
   (b) *For every $l \in \mathbb{N}$ and every choice of reals $a_1, \ldots, a_l$ in $[-1, 1]$ we have $\|\sum_{j=1}^{l} a_j x_j\| \le K^{-1} \sum_{k=1}^{\infty} c_k^{-1} \|\sum_{j=1}^{l} a_j x_j^k\|$.*

If $\mathcal{C}$ contains a strictly increasing (with respect to domination) sequence of length ω, then $\mathcal{C}$ contains a strictly increasing sequence of length ω_1.

By Lemma 4.7 and Theorem 4.11 the collection $\mathcal{C}$ of all normalized elements of $SM_\xi^w(X)$ satisfies the assumptions of Theorem 5.1. Hence we have the following.

COROLLARY 5.2. *Let X be a Banach space and ξ a countable ordinal. If $SM_\xi^w(X)$ contains a strictly increasing sequence of length ω, then $\mathcal{C}$ contains a strictly increasing sequence of length ω_1.*

6. On the richness of $SM_\xi^w(X)$. In this section we will generalize some results from [Do]. We will code the set $SM_\xi^w(X)$, for X separable, as an analytic subset of $[\mathbb{N}]^\infty$. Recall that a binary relation $\preccurlyeq$ on some set A is called a *pre-partial order* if:

(i) $a \preccurlyeq a$ for every $a \in A$, and
(ii) for every $a, b, c \in A$, if $a \preccurlyeq b$ and $b \preccurlyeq c$, then $a \preccurlyeq c$.

As usual, every binary relation on a set A can be viewed as a subset of the Cartesian product $A \times A$. If A is a topological space, we endow the Cartesian product with the product topology.

PROPOSITION 6.1. *Let $\preccurlyeq$ be an F_σ pre-partial order on $[\mathbb{N}]^\infty$ and $\approx$ the equivalence relation defined by $a \approx b$ iff $a \preccurlyeq b$ and $b \preccurlyeq a$. Let A be an analytic subset of $[\mathbb{N}]^\infty$ such that either A does not contain any strictly increasing sequence of length ω, or A contains a strictly increasing sequence of length ω_1. Then:*

(i) *If $A/_\approx$ is uncountable then A contains an antichain of size continuum, i.e. there exists a subset $P \subset A$ of cardinality $\mathfrak{c}$ such that for every $a \ne b$ in P we have $a \not\preccurlyeq b$ and $b \not\preccurlyeq a$.*
(ii) *If A contains a strictly decreasing sequence of length ω_1, then A contains a strictly increasing sequence of length ω_1.*

(iii) *If A does not contain a strictly increasing sequence of length ω, then there exists a countable ordinal ζ such that A does not contain any decreasing sequence of length ζ.*

Assertion (i) is a consequence of a result due to J. H. Silver [Si] (see also [Do, Lemma 5] for a simplified version adapted to our needs). Actually, by Silver's Theorem the set P can be chosen to be a perfect subset of A. Assertions (ii) and (iii) follow by similar arguments to the ones developed in [Do, proof of Theorem 3].

THEOREM 6.2. *Let X be a Banach space with separable dual and ξ a countable ordinal. Then:*

(i) *If $\mathbf{SM}_\xi^w(X)$ is uncountable then there exist continuum many pairwise incomparable elements of $SM_\xi^w(X)$.*
(ii) *If $SM_\xi^w(X)$ contains a strictly decreasing sequence of length ω_1, then $SM_\xi^w(X)$ contains a strictly increasing sequence of length ω_1.*
(iii) *If $SM_\xi^w(X)$ does not contain a strictly increasing sequence of length ω, then there exists a countable ordinal ζ such that $SM_\xi^w(X)$ does not contain any decreasing sequence of length ζ.*

The above theorem follows from Proposition 6.1 and the following analogue of Lemma 7 from [Do] which provides the desired coding of $SM_\xi^w(X)$. In order to state it we need some additional notation. Let $(u_n)_n$ be the standard unconditional Schauder basis of Pełczyński's universal space for unconditional Schauder basic sequences (see [Pe]). We define the following pre-partial ordering $\preccurlyeq$ on $[\mathbb{N}]^\infty$: for every L and M in $[\mathbb{N}]^\infty$ we set $L \preccurlyeq M$ iff $(u_n)_{n\in M}$ dominates $(u_n)_{n\in L}$.

LEMMA 6.3. *Let X be a Banach space with separable dual and ξ be a countable ordinal. Then there exists an analytic subset A of $[\mathbb{N}]^\infty$ satisfying:*

(i) *For every $(e_n)_n$ in $SM_\xi^w(X)$ there exists M in A such that the sequences $(e_n)_n$ and $(u_n)_{n\in M}$ are equivalent.*
(ii) *For every M in A there exists $(e_n)_n$ in $SM_\xi^w(X)$ such that the sequences $(e_n)_n$ and $(u_n)_{n\in M}$ are equivalent.*

Proof. Fix a regular thin family $\mathcal{F}$ of order ξ. We consider the following subset G of $[\mathbb{N}]^\infty \times [\mathbb{N}]^\infty \times X^{\mathcal{F}} \times X^{\widehat{\mathcal{F}}}$. We write $(M, L, (x_s)_{s\in\mathcal{F}}, (y_t)_{t\in\widehat{\mathcal{F}}}) \in G$ if:

(a) There exists $C > 0$ such that for every $k \in \mathbb{N}$, every $(s_j)_{j=1}^k$ in $\mathrm{Plm}_k(\mathcal{F}\restriction L)$ with $s_1(1) \geq L(k)$ and every $a_1, \ldots, a_k$ reals we have

$$C^{-1}\Big\|\sum_{j=1}^k a_j x_{s_j}\Big\| \leq \Big\|\sum_{j=1}^k a_j u_{M(j)}\Big\| \leq C\Big\|\sum_{j=1}^k a_j x_{s_j}\Big\| \tag{6.1}$$

where $(u_n)_n$ is the standard unconditional Schauder basis of Pełczyński's universal space for unconditional Schauder basic sequences.

(b) The $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is subordinated and weakly null. Moreover, if $\widehat{\varphi}:\widehat{\mathcal{F}}\upharpoonright L\to X$ witnesses $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ being subordinated, then $\widehat{\varphi}(t)=y_t$ for all $t\in\widehat{\mathcal{F}}\upharpoonright L$.

Invoking the separability of X^*, it is easy to check that G is a Borel subset of $[\mathbb{N}]^\infty\times[\mathbb{N}]^\infty\times X^{\mathcal{F}}\times X^{\widehat{\mathcal{F}}}$. We let A be the projection of G onto the first coordinate, that is,

$$A=\{M\in[\mathbb{N}]^\infty : \text{there is } (L,(x_s)_{s\in\mathcal{F}},(y_t)_{t\in\widehat{\mathcal{F}}})\in[\mathbb{N}]^\infty\times X^{\mathcal{F}}\times X^{\widehat{\mathcal{F}}} \text{ such that } (M,L,(x_s)_{s\in\mathcal{F}},(y_t)_{t\in\widehat{\mathcal{F}}})\in G\}. \tag{6.2}$$

Since G is Borel, A is analytic. It remains to check that A satisfies (i) and (ii) of the lemma. Indeed, let $(e_n)_n\in SM^w_\xi(X)$. By Corollary 3.3, there exist an infinite subset L of $\mathbb{N}$ and an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ in X such that $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is subordinated, weakly null and generates $(e_n)_n$ as an $\mathcal{F}$-spreading model. Moreover, by the universality of Pełczyński's space, there exists an infinite subset M of $\mathbb{N}$ such that $(u_n)_{n\in M}$ and $(e_n)_n$ are equivalent. Finally, if $\widehat{\varphi}:\widehat{\mathcal{F}}\to X$ is the continuous map witnessing that $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ is subordinated, then we set $y_t=\widehat{\varphi}(t)$ for all $t\in\widehat{\mathcal{F}}\upharpoonright L$ and $y_t=0$ otherwise. It follows readily that $(M,L,(x_s)_{s\in\mathcal{F}},(y_t)_{t\in\widehat{\mathcal{F}}})$ belongs to G and therefore M belongs to A. Since $(u_n)_{n\in M}$ and $(e_n)_n$ are equivalent, conclusion (i) is satisfied.

Conversely, let $M\in A$. By the definition of A, there exist an infinite subset L of $\mathbb{N}$, an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}})$ in X and a family $(y_t)_{t\in\widehat{\mathcal{F}}}$ of elements in X such that $(M,L,(x_s)_{s\in\mathcal{F}},(y_t)_{t\in\widehat{\mathcal{F}}})$ belongs to G. We pass to an infinite subset L' of L such that $(x_s)_{s\in\mathcal{F}\upharpoonright L'}$ generates an $\mathcal{F}$-spreading model $(e_n)_n$. By (6.1), it is easy to see that $(e_n)_n$ and $(u_n)_{n\in M}$ are equivalent, while by (b) above, $(e_n)_n$ belongs to $SM^w_\xi(X)$. That is, (ii) also holds true. ∎

A question of interest is whether one can drop the separable dual assumption in Theorem 6.2. In that direction we have the following.

Theorem 6.4. *Let X be a separable Banach space admitting no spreading model of order 1 equivalent to the standard basis of ℓ^1, and ξ a countable ordinal. Then:*

(i) *If $\mathbf{SM}^w_\xi(X)$ is uncountable then there exist continuum many pairwise incomparable elements of $SM^w_\xi(X)$.*

(ii) *If $SM^w_\xi(X)$ contains a strictly decreasing sequence of length ω_1, then $SM^w_\xi(X)$ contains a strictly increasing sequence of length ω_1.*

(iii) *If $SM^w_\xi(X)$ does not contain any strictly increasing sequence of length ω, then there exists a countable ordinal ζ such that $SM^w_\xi(X)$ does not contain any decreasing sequence of length ζ.*

Theorem 6.4 follows from Proposition 6.1 and the following analogue of Lemma 6.3 which provides us with the desired analytic coding of $SM_\xi^w(X)$.

LEMMA 6.5. *Let X be a separable Banach space admitting no spreading model of order 1 equivalent to the standard basis of ℓ^1 and ξ a countable ordinal. Then there exists an analytic subset A of $[\mathbb{N}]^\infty$ satisfying:*

(i) *For every $(e_n)_n$ in $SM_\xi^w(X)$ there exists M in A such that the sequences $(e_n)_n$ and $(u_n)_{n\in M}$ are equivalent.*
(ii) *For every M in A there exists $(e_n)_n$ in $SM_\xi^w(X)$ such that the sequences $(e_n)_n$ and $(u_n)_{n\in M}$ are equivalent.*

Let us recall that $(u_n)_n$ is the standard unconditional Schauder basis of Pełczyński's universal space for unconditional Schauder basic sequences (see [Pe]). For the proof of Lemma 6.5 we will need the following lemma.

LEMMA 6.6. *Let $\mathcal{F}$ be a regular thin family and L an infinite subset of $\mathbb{N}$. Also let X be a Banach space and $\widehat{\varphi}:\widehat{\mathcal{F}}\restriction L\to X$ a map such that for every t in $(\widehat{\mathcal{F}}\restriction L)\setminus\mathcal{F}$ and every sequence $(s_n)_n$ in $\mathcal{F}\restriction L$ convergent to t, we have $\widehat{\varphi}(s_n)\xrightarrow{w}\widehat{\varphi}(t)$. Then $\widehat{\varphi}$ is subordinated.*

Proof. Since the topology on $\widehat{\mathcal{F}}\restriction L$ is metrizable, it suffices to check that $\widehat{\varphi}$ is sequentially continuous. Let $(t_n)_n$ be a sequence in $\widehat{\mathcal{F}}\restriction L$ converging to some t. Clearly t belongs to $\widehat{\mathcal{F}}\restriction L$. Moreover, without loss of generality, we may assume that $\min(t_n\setminus t)\to\infty$. We need to show that $\widehat{\varphi}(t_n)\xrightarrow{w}\widehat{\varphi}(t)$. Fix $x^*\in X^*$. For every $n\in\mathbb{N}$, we pick $s_n\in\mathcal{F}\restriction L$ such that $t_n\sqsubseteq s_n$ and $|x^*(\widehat{\varphi}(t_n))-x^*(\widehat{\varphi}(s_n))|<1/n$. Since $\min(t_n\setminus t)\to\infty$, we see that $(s_n)_n$ converges to t. By the assumptions on $\widehat{\varphi}$, we find that $x^*(\widehat{\varphi}(s_n))\to x^*(\widehat{\varphi}(t))$. Since $|x^*(\widehat{\varphi}(t_n))-x^*(\widehat{\varphi}(s_n))|<1/n$ for all $n\in\mathbb{N}$, it follows that $x^*(\widehat{\varphi}(t_n))\to x^*(\widehat{\varphi}(t))$, and the proof is complete. ∎

Before we proceed to the proof of Lemma 6.5, we need to introduce some additional notation. Let $\mathcal{F}$ be a regular thin family, L an infinite subset of $\mathbb{N}$ and k a positive integer. We set

$$\mathrm{Bl}_k(\mathcal{F}\restriction L)=\{(s_i)_{i=1}^k:(s_i)_{i=1}^k \text{ is a block sequence in } \mathcal{F}\restriction L\}. \tag{6.3}$$

Let us observe that the families $\mathrm{Bl}_k(\mathcal{F}\restriction L)$ have the Ramsey property. In particular, we have the following.

PROPOSITION 6.7. *Let $\mathcal{F}$ be a regular thin family, L an infinite subset of $\mathbb{N}$ and k a positive integer. Then for every finite coloring of $\mathrm{Bl}_k(\mathcal{F}\restriction L)$ there exists an infinite subset L' of L such that $\mathrm{Bl}_k(\mathcal{F}\restriction L')$ is monochromatic.*

Proof. By passing to an infinite subset of L if necessary, we may assume that $\mathcal{F}$ is very large in L. For every infinite subset L' of L, we set

$$\mathrm{UBl}_k(\mathcal{F}\restriction L') = \Big\{\bigcup_{i=1}^{k} s_i : (s_i)_{i=1}^k \in \mathrm{Bl}_k(\mathcal{F}\restriction L')\Big\}. \tag{6.4}$$

It is easy to observe that $\mathrm{UBl}_k(\mathcal{F}\restriction L)$ is a family of finite subsets of $\mathbb{N}$ which is very large in L and thin. Moreover, for every infinite subset L' of L, the operator sending each $(s_i)_{i=1}^k$ from $\mathrm{Bl}_k(\mathcal{F}\restriction L')$ to $\bigcup_{i=1}^k s_i$ is 1-1 and onto $\mathrm{UBl}_k(\mathcal{F}\restriction L')$. So fixing a finite coloring on $\mathrm{Bl}_k(\mathcal{F}\restriction L)$ we induce a finite coloring on $\mathrm{UBl}_k(\mathcal{F}\restriction L)$. By the Ramsey property for thin families (see [NW], [PR] or [AKT2, Proposition 2.6]) there is an infinite subset L' of L such that $\mathrm{UBl}_k(\mathcal{F}\restriction L')$ is monochromatic. Clearly, $\mathrm{Bl}_k(\mathcal{F}\restriction L')$ is then also monochromatic. ∎

Finally, let $\mathcal{F}$ be a regular thin family and t an element of $\widehat{\mathcal{F}}$. We set

$$\mathcal{F}_{[t]} = \{s \in [\mathbb{N}]^{<\infty} : \min s > \max t \text{ and } t \cup s \in \mathcal{F}\}. \tag{6.5}$$

It follows easily that $\mathcal{F}_{[t]}$ is regular thin. We are ready to proceed to the proof of Lemma 6.5.

Proof of Lemma 6.5. Let $\mathcal{F}$ be a regular thin family of order ξ. We define a subset G of the product $[\mathbb{N}]^\infty \times [\mathbb{N}]^\infty \times X^{\widehat{\mathcal{F}}} \times X^{\mathcal{F}}$ as follows. We say that $(M, L, (y_t)_{t\in\widehat{\mathcal{F}}}, (x_s)_{s\in\mathcal{F}})$ is in G if:

(I) There exists $C > 0$ such that for every $k \in \mathbb{N}$, every $(s_j)_{j=1}^k$ in $\mathrm{Plm}_k(\mathcal{F}\restriction L)$ with $s_1(1) \geq L(k)$ and any real $a_1, \ldots, a_k$ we have

$$C^{-1}\Big\|\sum_{j=1}^{k} a_j x_{s_j}\Big\| \leq \Big\|\sum_{j=1}^{k} a_j u_{M(j)}\Big\| \leq C\Big\|\sum_{j=1}^{k} a_j x_{s_j}\Big\| \tag{6.6}$$

where $(u_n)_n$ is the standard unconditional Schauder basis of Pełczyński's universal space for unconditional Schauder basic sequences.

(II) For every $s \in \mathcal{F}\restriction L$ we have $y_s = x_s$ and $y_\emptyset = 0$.

(III) For every t in $(\widehat{\mathcal{F}}\restriction L) \setminus \mathcal{F}$, set $L' = \{q \in L : q > \max t\}$; then for every $\varepsilon > 0$ there exists $n_0 \in \mathbb{N}$ such that for every $n \geq n_0$ and every block sequence $(t'_j)_{j=1}^n$ in $\mathcal{F}_{[t]}\restriction L$ with $\min t'_1 \geq L'(n)$ we have

$$\Big\|\frac{1}{n}\sum_{j=1}^{n} y_{t\cup t'_j} - y_t\Big\| \leq \varepsilon. \tag{6.7}$$

It is easy to check that G is a Borel subset of $[\mathbb{N}]^\infty \times [\mathbb{N}]^\infty \times X^{\widehat{\mathcal{F}}_k} \times X^{\mathcal{F}_k}$. We have the following claim.

Claim 1. *Let $(M, L, (y_t)_{t\in\widehat{\mathcal{F}}}, (x_s)_{s\in\mathcal{F}}) \in G$. Then every $\mathcal{F}$-spreading model generated by an $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\restriction L}$ belongs to $SM_\xi^w(X)$ and it is equivalent to $(u_n)_{n\in M}$.*

Proof of Claim 1. First observe that the second part of the conclusion is immediate by property (I) above. Next, let $\widehat{\varphi} : \widehat{\mathcal{F}}\restriction L \to X$ with $\widehat{\varphi}(t) = y_t$ for all $t \in \widehat{\mathcal{F}}\restriction L$. We need to show that $\widehat{\varphi}$ is continuous, where X is considered with the weak topology. By Lemma 6.6, it suffices to show that for every $t \in \widehat{\mathcal{F}}\restriction L$ and every $(s_n)_n$ in $\mathcal{F}\restriction L$ convergent to t, we have $y_{s_n} \xrightarrow{w} y_t$.

Assume to the contrary that there exist $t \in \widehat{\mathcal{F}}\restriction L$, a sequence $(s_n)_n$ in $\mathcal{F}\restriction L$ convergent to t, an element x^* in X^* of norm 1 and some $\varepsilon > 0$ such that $x^*(y_{s_n} - y_t) \geq 2\varepsilon$ for all n. Passing to a subsequence of $(s_n)_n$ if necessary, we may assume that each s_n end-extends t and $(s_n \setminus t)_n$ is a block sequence. We set $t_n = s_n \setminus t$ for all n. Then for every $n \in \mathbb{N}$,

$$\Big\|\frac{1}{n}\sum_{j=1}^{n} y_{s_{n+j}} - y_t\Big\| \geq x^*\Big(\frac{1}{n}\sum_{j=1}^{n} y_{s_{n+j}} - y_t\Big) \geq 2\varepsilon, \tag{6.8}$$

which contradicts (III).

Hence $(x_s)_{s\in\mathcal{F}\restriction L}$ is subordinated. Moreover, by (II), it follows that $\widehat{\varphi}(\emptyset) = 0$ and therefore $(x_s)_{s\in\mathcal{F}\restriction L}$ is weakly null. The proof of the claim is complete. ∎

The converse of Claim 1 holds as well.

Claim 2. *For every element $(e_n)_n$ of $SM_\xi^w(X)$ there exists an element $(M, L, (y_t)_{t\in\widehat{\mathcal{F}}}, (x_s)_{s\in\mathcal{F}})$ of G such that the $\mathcal{F}$-subsequence $(x_s)_{s\in\mathcal{F}\restriction L}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model.*

Proof of Claim 2. Fix $(e_n)_n$ in $SM_\xi^w(X)$. By the universality property of Pełczyński's space, there exists an infinite subset M of $\mathbb{N}$ such that $(e_n)_n$ and $(u_n)_{n\in M}$ are equivalent. By Corollary 3.3, there exist an infinite subset P of $\mathbb{N}$ and an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ such that $(x_s)_{s\in\mathcal{F}\restriction P}$ is subordinated, weakly null and generates $(e_n)_n$ as an $\mathcal{F}$-spreading model. For every $t \in \widehat{\mathcal{F}}\restriction P$ we set $y_t = \widehat{\varphi}(t)$ and we pick an arbitrary y_t for every $t \in \widehat{\mathcal{F}} \setminus (\widehat{\mathcal{F}}\restriction P)$. Clearly for every infinite subset P' of P, $(M, P', (y_t)_{t\in\widehat{\mathcal{F}}}, (x_s)_{s\in\mathcal{F}})$ satisfies (I) and (II). It suffices to choose an infinite subset L of P such that $(M, L, (y_t)_{t\in\widehat{\mathcal{F}}}, (x_s)_{s\in\mathcal{F}})$ satisfies (III).

First, let us observe that the following property holds true:

($\mathcal{P}$) For every infinite subset P' of P and every t in $(\widehat{\mathcal{F}}\restriction M) \setminus \mathcal{F}$, there exists an infinite subset P'' of P' such that $\min P'' > \max t$ and for every $n \in \mathbb{N}$ and any block sequences $(t_j)_{j=1}^n, (t'_j)_{j=1}^n$ in $\mathcal{F}_{[t]}\restriction P''$ with $\min t_1, \min t'_1 \geq P''(n)$ we have

$$\left| \left\| \frac{1}{n} \sum_{j=1}^{n} y_{t \cup t_j} - y_t \right\| - \left\| \frac{1}{n} \sum_{j=1}^{n} y_{t \cup t'_j} - y_t \right\| \right| \le \frac{1}{n}.$$

Indeed, let P' and t be as above. We set $Q_0 = P'$ and we inductively construct a decreasing sequence $(Q_n)_n$ of infinite subsets of Q_0 such that for every $n \in \mathbb{N}$ we have

$$\left| \left\| \frac{1}{n} \sum_{j=1}^{n} y_{t \cup t_j} - y_t \right\| - \left\| \frac{1}{n} \sum_{j=1}^{n} y_{t \cup t'_j} - y_t \right\| \right| \le \frac{1}{n}, \tag{6.9}$$

for every choice of block sequences $(t_j)_{j=1}^n, (t'_j)_{j=1}^n$ in $\mathcal{F}_{[t]} \upharpoonright Q_n$. Assume that for some $n \in \mathbb{N}$ the sets $Q_0, \ldots, Q_{n-1}$ have been chosen. Since $\widehat{\varphi}$ is continuous and $\widehat{\mathcal{F}} \upharpoonright Q_{n-1}$ is compact, the set $\{y_s : s \in \widehat{\mathcal{F}} \upharpoonright Q_{n-1}\}$ is weakly compact and therefore bounded. Let $C > 0$ be such that $\|y_s\| \le C$ for all $s \in \widehat{\mathcal{F}} \upharpoonright Q_{n-1}$. Let $(A_i)_{i=1}^{i_0}$ be a partition of $[0, C]$ into sets of diameter at most $1/n$. We define a finite coloring on $\mathrm{Bl}_n(\mathcal{F}_{[t]} \upharpoonright Q_{n-1})$ as follows. We assign to a block sequence $(t_j)_{j=1}^n$ in $\mathcal{F}_{[t]} \upharpoonright Q_{n-1}$ the color $i \in \{1, \ldots, i_0\}$ if

$$\left\| \frac{1}{n} \sum_{j=1}^{n} y_{t \cup t_j} - y_t \right\| \in A_i. \tag{6.10}$$

Applying Proposition 6.7 we obtain an infinite subset Q_n of Q_{n-1} such that the set $\mathrm{Bl}_n(\mathcal{F}_{[t]} \upharpoonright Q_n)$ is monochromatic. It is easy to check that Q_n is as desired and the inductive step is complete. Pick an infinite subset P'' of P' such that $P''(n) \in Q_n$ for all n. Thus $(\mathcal{P})$ holds.

Set $L_0 = P$ and inductively construct a decreasing sequence $(L_q)_q$ of infinite subsets of P and a strictly increasing sequence $(l_q)_q$ in P for every $q \in \mathbb{N}$:

(a) $l_q = \min L_q$.

(b) For every t subset of $\{l_p : 1 \le p \le q-1\}$ belonging to $\widehat{\mathcal{F}} \setminus \mathcal{F}$, every $n \in \mathbb{N}$ and any block sequences $(t_j)_{j=1}^n, (t'_j)_{j=1}^n$ in $\mathcal{F}_l \upharpoonright L_q$ with $\min t_1, \min t'_1 \ge L_q(n)$ we have

$$\left| \left\| \frac{1}{n} \sum_{j=1}^{n} y_{t \cup t_j} - y_t \right\| - \left\| \frac{1}{n} \sum_{j=1}^{n} y_{t \cup t'_j} - y_t \right\| \right| \le \frac{1}{n}. \tag{6.11}$$

The inductive step of the construction is carried out as follows. Assume that for some $q \in \mathbb{N}$ the sets $L_0, \ldots, L_{q-1}$ are constructed. Let $\{t^r\}_{r=1}^N$ be an enumeration of the set $\{t \in \widehat{\mathcal{F}} \setminus \mathcal{F} : t \subseteq \{l_p : 1 \le p \le q-1\}\}$. Applying property $(\mathcal{P})$, we construct a decreasing sequence $(Q_r)_{r=1}^N$ of infinite subsets of $L_{q-1} \setminus \{l_{q-1}\}$ such that for every $n \in \mathbb{N}$ and any block sequences $(t_j)_{j=1}^n, (t'_j)_{j=1}^n$ in $\mathcal{F}_{[t]} \upharpoonright Q_r$ with $\min t_1, \min t'_1 \ge Q_r(n)$ we have

$$\left|\left\|\frac{1}{n}\sum_{j=1}^{n} y_{t^r\cup t_j} - y_{t^r}\right\| - \left\|\frac{1}{n}\sum_{j=1}^{n} y_{t^r\cup t'_j} - y_{t^r}\right\|\right| \le \frac{1}{n}. \tag{6.12}$$

Setting $L_q = Q_N$ completes the inductive step of the construction.

We set $L = \{l_q : q \in \mathbb{N}\}$. Then for every t in $(\widehat{\mathcal{F}}\upharpoonright L) \setminus \mathcal{F}$, every $n \in \mathbb{N}$ and any block sequences $(t_j)_{j=1}^n, (t'_j)_{j=1}^n$ in $\mathcal{F}_{[t]}\upharpoonright L'$, where $L' = \{q \in L : q \ge \max t\}$, with $\min t_1, \min t'_1 \ge L'(n)$, it follows that

$$\left|\left\|\frac{1}{n}\sum_{j=1}^{n} y_{t\cup t_j} - y_t\right\| - \left\|\frac{1}{n}\sum_{j=1}^{n} y_{t\cup t'_j} - y_t\right\|\right| \le \frac{1}{n}. \tag{6.13}$$

In order to check that property (III) is satisfied we fix some t from $(\widehat{\mathcal{F}}\upharpoonright L)\setminus\mathcal{F}$. Let $L' = \{q \in L : q > \max t\}$. Let $(t_n)_n$ be a block sequence in $\mathcal{F}_{[t]}\upharpoonright L'$. By the lemma's assumptions, the sequence $(y_{t\cup t_n} - y_t)_n$ admits no spreading model equivalent to the standard basis of ℓ^1. Moreover, by the continuity of $\widehat{\varphi}$, $(y_{t\cup t_n} - y_t)_n$ is weakly null. Hence, by a well known dichotomy of H. P. Rosenthal concerning Cesàro summability and ℓ^1 spreading models, there exists a subsequence $(y_{t\cup t_{m_n}} - y_t)_n$ of $(y_{t\cup t_n} - y_t)_n$ which is Cesàro summable to zero. Hence

$$\begin{aligned} \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{j=1}^{n} y_{t\cup t_{m_{n+j}}} - y_t\right\| \\ \le 2\lim_{n\to\infty}\left\|\frac{1}{2n}\sum_{j=1}^{2n} y_{t\cup t_{m_j}} - y_t\right\| - \lim_{n\to\infty}\left\|\frac{1}{n}\sum_{j=1}^{n} y_{t\cup t_{m_j}} - y_t\right\| = 0. \end{aligned} \tag{6.14}$$

Clearly, (III) follows from (6.13) and (6.14). The proof of the claim is complete. ■

Let A be the projection of G onto the first coordinate, that is,

$$\begin{aligned} A = \{M \in [\mathbb{N}]^\infty : \text{there exists } (L, (y_t)_{t\in\widehat{\mathcal{F}}}, (x_s)_{s\in\mathcal{F}}) \in [\mathbb{N}]^\infty \times X^{\widehat{\mathcal{F}}} \times X^{\mathcal{F}} \\ \text{such that } (M, L, (y_t)_{t\in\widehat{\mathcal{F}}}, (x_s)_{s\in\mathcal{F}}) \in G\}. \end{aligned} \tag{6.15}$$

Since G is Borel, A is analytic. It remains to check that A satisfies (i) and (ii) of Lemma 6.5. Indeed, fix $(e_n)_n$ in $SM_\xi^w(X)$. By Claim 2, there is $(M, L, (y_t)_{t\in\widehat{\mathcal{F}}}, (x_s)_{s\in\mathcal{F}})$ in G such that $(x_s)_{s\in\mathcal{F}\upharpoonright L}$ generates $(e_n)_n$ as an $\mathcal{F}$-spreading model. By the definition of A, M belongs to A and by property (I) the sequences $(e_n)_n$ and $(u_n)_{n\in M}$ are equivalent. Conversely, let $M \in A$. By the definition of A, there exist an infinite subset L of $\mathbb{N}$, an $\mathcal{F}$-sequence $(x_s)_{s\in\mathcal{F}}$ in X and a family $(y_t)_{t\in\widehat{\mathcal{F}}}$ of elements in X such that $(M, L, (y_t)_{t\in\widehat{\mathcal{F}}}, (x_s)_{s\in\mathcal{F}})$ belongs to G. We pass to an infinite subset L' of L such that $(x_s)_{s\in\mathcal{F}\upharpoonright L'}$ generates an $\mathcal{F}$-spreading model $(e_n)_n$. By Claim 1 the sequences $(e_n)_n$ and $(u_n)_{n\in M}$ are equivalent and $(e_n)_n$ belongs to $SM_\xi^w(X)$. The proof of Lemma 6.5 is complete. ■

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