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Remarks on the critical Besov space and its embedding into weighted Besov–Orlicz spaces

by

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Abstract. We present several continuous embeddings of the critical Besov space $B_p^{n/p,\rho}(\mathbb{R}^n)$. We first establish a Gagliardo-Nirenberg type estimate

$$\|u\|_{\dot{B}^{0,\nu}_{q,w_r}} \le C_n \left(\frac{1}{n-r}\right)^{\frac{1}{q}+\frac{1}{\nu}-\frac{1}{\rho}} \left(\frac{q}{r}\right)^{\frac{1}{\nu}-\frac{1}{\rho}} \|u\|_{\dot{B}^{0,\rho}_p}^{\frac{(n-r)p}{nq}} \|u\|_{\dot{B}^{n/p,\rho}_p}^{1-\frac{(n-r)p}{nq}},$$

for $1<p\le q<\infty$, $1\le \nu<\rho\le\infty$ and the weight function $w_r(x)=1/|x|^r$ with $0<r<n$. Next, we prove the corresponding Trudinger type estimate, and obtain it in terms of the embedding $B_p^{n/p,\rho}(\mathbb{R}^n)\hookrightarrow B^{0,\nu}_{\Phi_0,w_r}(\mathbb{R}^n)$, where the function Φ_0 of the weighted Besov–Orlicz space $B^{0,\nu}_{\Phi_0,w_r}(\mathbb{R}^n)$ is a Young function of the exponential type. Another point of interest is to embed $B_p^{n/p,\rho}(\mathbb{R}^n)$ into the weighted Besov space $B^{0,\rho}_{p,w_n}(\mathbb{R}^n)$ with the critical weight $w_n(x)=1/|x|^n$; more precisely, we prove $B_p^{n/p,\rho}(\mathbb{R}^n)\hookrightarrow B^{0,\rho}_{p,W_s}(\mathbb{R}^n)$ with the weight $W_s(x)=\frac{1}{|x|^n[\log(e+1/|x|)]^s}$ for any $s>1$.

1. Introduction and main results. The main purpose of this paper is to investigate the properties of the critical Besov space in terms of continuous embeddings into weighted Besov or Besov–Orlicz spaces. Firstly, we should recall the Gagliardo–Nirenberg type inequalities on the fractional Sobolev space with the critical differential order, $H_p^{n/p}(\mathbb{R}^n)$, where $n\in\mathbb{N}$ and $1<p<\infty$. The Sobolev embedding theorem states that $H_p^{n/p}(\mathbb{R}^n)$ can be embedded into $L_q(\mathbb{R}^n)$ for all q with $p\le q<\infty$, but not into $L_\infty(\mathbb{R}^n)$. Ozawa [Oz] gave a precise estimate for this embedding:

$$\|u\|_{L_q} \le C_{n,p} q^{1/p'} \|u\|_{L_p}^{p/q} \|(-\Delta)^{n/2p}u\|_{L_p}^{1-p/q} \tag{1.1}$$

holds for all $u\in H_p^{n/p}(\mathbb{R}^n)$ and $p\le q<\infty$, where $p':=p/(p-1)$ denotes the Hölder conjugate exponent of p. Furthermore, $C_{n,p}$ indicates that the constant depends only on n and p, a convention we shall adopt throughout this paper. The inequality (1.1) was originally obtained by Ogawa [Og]

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and Ogawa–Ozawa [OgOz] in the case $p = 2$, i.e., for $H_2^{n/2}(\mathbb{R}^n)$, and then (1.1) was generalized further in several ways. We refer to Nagayasu–Wadade [NW], who established a generalization of (1.1) with a weighted Lebesgue norm of the following type:

$$(1.2) \qquad \|u\|_{L_{q,w_r}} \le C_{n,p}\left(\frac{1}{n-r}\right)^{\frac{1}{q}+\frac{1}{p'}} q^{\frac{1}{p'}} \|u\|_{L_p}^{\frac{(n-r)p}{nq}} \|(-\Delta)^{\frac{n}{2p}} u\|_{L_p}^{1-\frac{(n-r)p}{nq}}$$

for all $u \in H_p^{n/p}(\mathbb{R}^n)$, $0 \le r < n$ and $\tilde{p} \le q < \infty$, where $\tilde{p} \in (p, \infty)$ only depends on n and p, and the weight function w_r is the homogeneous function

$$(1.3) \qquad w_r(x) := 1/|x|^r \quad \text{for } x \in \mathbb{R}^n \setminus \{0\}.$$

In [NW], the authors concentrated on the investigation of the growth order in q as $q \to \infty$, and they proved the inequality (1.2) for $q \ge \tilde{p}$, where the constant $\tilde{p}$ is chosen suitably.

Note that L_{q,w_r} on the left-hand side of (1.2) represents the weighted Lebesgue space, and in general, for a positive measurable function w, we define the weighted Lebesgue space $L_{p,w}(\mathbb{R}^n) = L_p(\mathbb{R}^n\,;\, w(x)dx)$ endowed with the norm

$$(1.4) \qquad \begin{aligned} \|u\|_{L_{p,w}} &:= \Big(\int_{\mathbb{R}^n} |u(x)|^p w(x)\,dx\Big)^{1/p} \text{ for } 1 \le p < \infty, \\ \|u\|_{L_{\infty,w}} &:= \|u\|_{L_\infty}. \end{aligned}$$

Recall that the particular weight w_r in (1.3) belongs to the class of Muckenhoupt weights, which was originally defined in Muckenhoupt [M]. Moreover, the special case $r = 0$ in (1.2) coincides with (1.1) with the same growth order $q^{1/p'}$ as $q \to \infty$.

For another way to generalize (1.1), we refer to Wadade [W]. In [W], the author obtained the following Gagliardo–Nirenberg type inequality on the critical Besov space:

$$(1.5) \qquad \|u\|_{L_q} \le C_{n,p} q^{1/\rho'} \|u\|_{L_p}^{p/q} \|u\|_{\dot{B}_p^{n/p,\rho}}^{1-p/q}$$

for all $u \in (L_p \cap \dot{B}_p^{n/p,\rho})(\mathbb{R}^n)$, $1 \le p \le q < \infty$ and $1 < \rho \le \infty$. The inequality (1.5) can also be regarded as a generalization of (1.1). Indeed, by noting the embedding $\dot{H}_p^{n/p}(\mathbb{R}^n) \hookrightarrow \dot{B}_p^{n/p,p}(\mathbb{R}^n)$ if $2 \le p < \infty$, and taking $p = \rho$ in (1.5), we have (1.1) immediately in the case $2 \le p < \infty$. We refer to Bergh–Löfström [BL] and Triebel [T1, T2, T3] for the relationship between Sobolev and Besov spaces, and their detailed properties as function spaces. Additionally, an estimate similar to (1.5) was also given in [W]:

$$(1.6) \qquad \|u\|_{\dot{B}_q^{0,\nu}} \le C_n q^{1/\nu - 1/\rho} \|u\|_{\dot{B}_p^{0,\nu}}^{p/q} \|u\|_{\dot{B}_p^{n/p,\rho}}^{1-p/q}$$

for all $u \in (\dot{B}_p^{0,\nu} \cap \dot{B}_p^{n/p,\rho})(\mathbb{R}^n)$, $1 \le p \le q < \infty$ and $1 \le \nu < \rho \le \infty$.

Keeping the above results in mind, we shall prove a Gagliardo–Nirenberg type interpolation inequality from the critical Besov space into a weighted Besov space, which is a generalization of the inequalities (1.2) and (1.6). The definition of weighted Besov spaces will be given in (2.2) and (2.3) in Section 2. We refer to Bui [B1, B2, B3] and Triebel [T3] for detailed information about weighted Besov and weighted Triebel–Lizorkin spaces. Our first theorem now reads:

THEOREM 1.1. *Let $n \in \mathbb{N}$. Then there exists $C_n > 0$ such that*

$$\|u\|_{\dot{B}^{0,\nu}_{q,w_r}} \le C_n \left(\frac{1}{r}\right)^{\frac{1}{\nu}-\frac{1}{\rho_1}} \left(\frac{1}{n-r}\right)^{\frac{1}{q}+\frac{1}{\nu}-\frac{1}{\rho_2}} q^{\frac{1}{\nu}-\frac{1}{\max\{\rho_1,\rho_2\}}} \|u\|_{\dot{B}^{0,\rho_1}_p}^{\frac{(n-r)p}{nq}} \|u\|_{\dot{B}^{n/p,\rho_2}_p}^{1-\frac{(n-r)p}{nq}} \tag{1.7}$$

for all $u \in (\dot{B}^{0,\rho_1}_p \cap \dot{B}^{n/p,\rho_2}_p)(\mathbb{R}^n)$, $1 < p \le q < \infty$, $0 < r < n$, $1 \le \nu \le \min\{\rho_1, \rho_2\} \le \infty$, where the weight function w_r is as in (1.3).

REMARK. (i) The inequality (1.7) cannot hold in the limiting cases $r = 0$ and $q = \infty$ in general. However, as expected from the powers in the constants of the right-hand side, the special cases $\nu = \rho_1$ and $\nu = \max\{\rho_1, \rho_2\}$ enable us to take $r = 0$ and $q = \infty$, respectively. On the other hand, we cannot put $r = n$ all the time when the weight function becomes $1/|x|^n$, and this limiting case will be the next target in our consideration.

(ii) The particular case with $\nu = \rho_1$ and $r = 0$ coincides with (1.6) where the growth order as $q \to \infty$ becomes $q^{1/\nu - 1/\rho_2}$. Furthermore, take $\nu = 1$ and $2 \le p = \rho_1 = \rho_2 < \infty$ in (1.7), and note the well-known embeddings

$$\begin{cases} \|u\|_{L_{p,w}} \le \|u\|_{\dot{B}^{0,1}_{p,w}} & \text{for all } 1 \le p \le \infty, \\ \|u\|_{\dot{B}^{s,p}_p} \le C_{n,s,p} \|u\|_{\dot{H}^s_p} & \text{for all } s \in \mathbb{R} \text{ and } 2 \le p < \infty. \end{cases} \tag{1.8}$$

Then the inequality (1.2) except for $r = 0$ follows from (1.7) and (1.8) with the same growth orders as $r \uparrow n$ and $q \to \infty$.

Observe that in the limiting case $r = n$ the inequality (1.7) fails because the critical weight $1/|x|^n$ is not integrable near the origin. Keeping this in mind, the next goal is to establish the embedding of the critical Besov space into a weighted Besov space, where the weight function is almost critical, namely, we take the weight as $\frac{1}{|x|^n(\log(1/|x|))^s}$ near the origin with $s > 1$. Our second result now reads:

THEOREM 1.2. *Let $n \in \mathbb{N}$, $1 < p < \infty$ and $1 \le \rho \le \infty$.*

(i) (Subcritical weight case) *The following continuous embedding holds:*

$$B^{r/p,\rho}_p(\mathbb{R}^n) \hookrightarrow B^{0,\rho}_{p,w_r}(\mathbb{R}^n), \quad \textit{where} \quad w_r(x) := 1/|x|^r \textit{ with } 0 \le r < n.$$

Furthermore, we have the estimate

$$\|u\|_{B^{0,\rho}_{p,w_r}} \le C_n\left(\frac{1}{n-r}\right)^{1/p}\|u\|_{B^{r/p,\rho}_p}.$$

(ii) (Critical weight case) *The following continuous embedding holds:*

$$B^{n/p,\rho}_p(\mathbb{R}^n) \hookrightarrow B^{0,\rho}_{p,W_s}(\mathbb{R}^n),\ \textit{where } W_s(x) := \frac{1}{|x|^n[\log(e+1/|x|)]^s}\ \textit{with } s>1.$$

Furthermore, we have the estimate

$$\|u\|_{B^{0,\rho}_{p,W_s}} \le C_{n,p,s}\|u\|_{B^{n/p,\rho}_p}.$$

Remark. (i) There are more general results on such embeddings in the case of Besov and Triebel–Lizorkin spaces including the Sobolev scale (cf. Haroske–Skrzypczak [HS] and Kühn–Leopold–Sickel–Skrzypczak [KLSS1, KLSS2]), but restricted to Muckenhoupt weights or so-called admissible weights the former weights may have a local singularity, while the latter are a class of smooth functions. We emphasize that these classes of weight functions do not cover the limiting situation of Theorem 1.2(ii). Indeed, it is well known that the weight W_s is not even a Muckenhoupt weight.

(ii) The assertion of Theorem 1.2(ii) might inspire us to consider the continuous embedding from $B^{n/p,\rho}_p(\mathbb{R}^n)$ into $B^{0,\rho}_{p,\mathcal{W}_s}(\mathbb{R}^n)$, where $\mathcal{W}_s$ is the weight function of the double logarithmic type:

$$\mathcal{W}_s(x) \simeq \frac{1}{|x|^n\big(\log\frac{1}{|x|}\big)\big[\log\big(\log\frac{1}{|x|}\big)\big]^s} \quad \text{for } |x|\ll 1 \text{ with } s>1.$$

However, we do not explore the case of multiple logarithmic weights in this article; this will be studied in the forthcoming paper.

As another kind of a critical Sobolev embedding, we next consider a Trudinger type inequality. As already stated, the Gagliardo–Nirenberg type inequality (1.1) was obtained in [Oz] with the optimal growth order $q^{1/p'}$ as $q\to\infty$. In the same paper, the author also showed that (1.1) implies the following Trudinger type embedding:

$$\text{(1.9)}\qquad \begin{cases} H^{n/p}_p(\mathbb{R}^n) \hookrightarrow L_{\Phi_1}(\mathbb{R}^n), \\ \Phi_1(t) := \exp(t^{p'}) - \displaystyle\sum_{k=0}^{k_1-1}\frac{t^{p'k}}{k!}, \quad k_1 := \min\{k\in\mathbb{N} : p'k \ge p\}, \end{cases}$$

where $1<p<\infty$ and $L_{\Phi_1}(\mathbb{R}^n)$ denotes the usual Orlicz space with the Young function Φ_1. For the precise definition of the Orlicz space, see (2.5) and (2.6) in Section 2, where the weighted Orlicz space $L_{\Phi,w}(\mathbb{R}^n)$ will be introduced, and note that $L_\Phi(\mathbb{R}^n) = L_{\Phi,1}(\mathbb{R}^n)$. We refer to Rao–Ren [RaRe] for abundant information about Orlicz spaces defined on general measure spaces.

The same procedure which shows that the inequality (1.1) yields the embedding (1.9) can also be applied to the inequalities (1.2) and (1.5). Indeed, in Nagayasu–Wadade [NW] the authors derived from (1.2) the Trudinger type embedding

$$\begin{cases} H_p^{n/p}(\mathbb{R}^n) \hookrightarrow L_{\Phi_2,w_r}(\mathbb{R}^n), \\ \Phi_2(t) := \exp(t^{p'}) - \displaystyle\sum_{k=0}^{k_2-1} \frac{t^{p'k}}{k!}, \quad k_2 := \min\{k \in \mathbb{N} : p'k \geq \tilde{p}\}, \end{cases} \tag{1.10}$$

where $1 < p < \infty$, $0 \leq r < n$ and $\tilde{p}$ depends only on n and p. Note that the embedding (1.10) generalizes (1.9) since the special case $r = 0$ in (1.10) corresponds to (1.9). Furthermore, the inequality (1.5) obtained in Wadade [W] implies the embedding

$$\begin{cases} (L_p \cap \dot{B}_p^{n/p,\rho})(\mathbb{R}^n) \hookrightarrow L_{\Phi_3}(\mathbb{R}^n), \\ \Phi_3(t) := \exp(t^{\rho'}) - \displaystyle\sum_{k=0}^{k_3-1} \frac{t^{\rho'k}}{k!}, \quad k_3 := \min\{k \in \mathbb{N} : \rho'k \geq p\}, \end{cases} \tag{1.11}$$

for $1 \leq p < \infty$ and $1 < \rho \leq \infty$. Note that the embedding (1.11) is also a generalization of (1.9). Indeed, we can obtain (1.9) in the case $2 \leq p < \infty$ by taking $p = \rho$ in (1.11) and using the embedding $\dot{H}_p^{n/p}(\mathbb{R}^n) \hookrightarrow \dot{B}_p^{n/p,p}(\mathbb{R}^n)$ for $2 \leq p < \infty$.

As seen in the above observations, the Gagliardo–Nirenberg type inequality in the critical space provides the corresponding Trudinger type embedding in general. Keeping this in mind, our final goal in the present paper is to establish a Trudinger type estimate corresponding to the inequality (1.7) in Theorem 1.1. However, the method to get embeddings (1.9), (1.10) and (1.11) through the Gagliardo–Nirenberg type inequalities can no longer work for (1.7) since the norm on the left-hand side of (1.7) is a weighted Besov norm, while the inequalities (1.1), (1.2) and (1.5) have Lebesgue type norms on the left-hand side which are all formed by direct integration of functions. Therefore, we shall prove the expected Trudinger type estimate without making use of (1.7) by calculating the corresponding Besov–Orlicz norm directly. For simplicity, we restrict our considerations to the case $\rho_1 = \rho_2$ in (1.7), which then becomes

$$\|u\|_{\dot{B}^{0,\nu}_{q,w_r}} \leq C_{n,r} q^{\frac{1}{\nu}-\frac{1}{\rho}} \|u\|_{\dot{B}_p^{0,\rho}}^{\frac{(n-r)p}{nq}} \|u\|_{\dot{B}_p^{n/p,\rho}}^{1-\frac{(n-r)p}{nq}} \tag{1.12}$$

for $1 < p \leq q < \infty$, $0 < r < n$ and $1 \leq \nu < \rho \leq \infty$. To construct the Trudinger type estimate corresponding to (1.12), it is natural to introduce weighted Besov–Orlicz spaces with an exponential type Young function (see the definition (2.4) in Section 2). Thus our last theorem now reads:

Theorem 1.3. *Let* $n \in \mathbb{N}$, $1 < p < \infty$, $0 \le r < n$, $1 \le \nu < \rho \le \infty$ *with* $(\nu, \rho) \ne (1, \infty)$ *and* $0 < \delta \le \mu - 1$, *where*

$$\mu := \begin{cases} \dfrac{\rho\nu}{\rho - \nu} & \text{if } \rho < \infty, \\ \nu & \text{if } \rho = \infty. \end{cases}$$

Then the following continuous embedding holds:

$$\begin{cases} B_p^{n/p,\rho}(\mathbb{R}^n) \hookrightarrow B^{0,\nu}_{\Phi_0, w_r}(\mathbb{R}^n), \\ \Phi_0(t) := \exp(t^{\mu-\delta}) - \displaystyle\sum_{k=0}^{k_0-1} \frac{t^{(\mu-\delta)k}}{k!}, \quad k_0 := \min\{k \in \mathbb{N} : (\mu - \delta)k \ge p\}. \end{cases}$$

Furthermore, we have the estimate

$$\|u\|_{B^{0,\nu}_{\Phi_0, w_r}} \le C_{n,\delta} \left(\frac{1}{n - r} \right)^{2/(\mu-\delta)} \|u\|_{B_p^{n/p,\rho}}.$$

Remark. Since the growth order as $q \to \infty$ in (1.12) is $q^{1/\nu - 1/\rho}$, one would expect that the condition $\delta > 0$ can be removed. However, a technical reason forces us to assume $\delta > 0$ in the proof. Furthermore, the condition $\delta \le \mu - 1$, that is, $\mu - \delta \ge 1$ guarantees that Φ_0 is a Young function. Indeed, if $\mu - \delta < 1$, then Φ_0 is no longer a Young function, which implies the weighted Besov–Orlicz space is not necessarily a normed space.

Let us describe the organization of this article. Section 2 is devoted to defining weighted function spaces and preparing lemmas for the proof of the main theorems; the theorems are proved in Section 3.

2. Preliminaries. In this section, we first give the definition of weighted Besov spaces, and then prove several lemmas. Let φ be a non-negative function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ such that $\operatorname{supp} \varphi = \{1/2 \le |x| \le 2\}$, $\varphi(x) > 0$ for all x with $1/2 < |x| < 2$ and $\sum_{j=-\infty}^{\infty} \varphi(2^{-j}x) = 1$ if $x \ne 0$. It is well known that such a function exists (see, for instance, Bergh–Löfström [BL]). Moreover, we define $\mathcal{S}(\mathbb{R}^n)$ functions φ_j for $j \in \mathbb{Z}$ and ψ as follows:

$$\hat{\varphi}_j(x) = \varphi(2^{-j}x) \quad \text{and} \quad \hat{\psi}(x) = 1 - \sum_{j=1}^{\infty} \hat{\varphi}_j(x), \tag{2.1}$$

where $\hat{f}$ denotes the Fourier transform of f, i.e., $\hat{f}(x) := \int_{\mathbb{R}^n} e^{-2\pi i x \cdot \xi} f(\xi)\, d\xi$. Then for a positive measurable function w, the *inhomogeneous Besov space* $B^{s,q}_{p,w}(\mathbb{R}^n)$ and the *homogeneous Besov space* $\dot{B}^{s,q}_{p,w}(\mathbb{R}^n)$ are respectively defined by

$$\begin{cases} B^{s,q}_{p,w}(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{B^{s,q}_{p,w}} < \infty\}, \\ \|u\|_{B^{s,q}_{p,w}} := \|\psi * u\|_{L_{p,w}} + \Big(\displaystyle\sum_{j=1}^{\infty} (2^{sj} \|\varphi_j * u\|_{L_{p,w}})^q \Big)^{1/q}, \end{cases} \tag{2.2}$$

and

$$\left\{\begin{array}{l}\dot{B}^{s,q}_{p,w}(\mathbb{R}^n) := \{u \in (\mathcal{S}'/\mathcal{P})(\mathbb{R}^n) : \|u\|_{\dot{B}^{s,q}_{p,w}} < \infty\},\\ \|u\|_{\dot{B}^{s,q}_{p,w}} := \Big(\sum_{j=-\infty}^{\infty} (2^{sj}\|\varphi_j * u\|_{L_{p,w}})^q\Big)^{1/q},\end{array}\right. \tag{2.3}$$

for $s \in \mathbb{R}$ and $1 \le p, q \le \infty$. In the above definitions, $\mathcal{S}'(\mathbb{R}^n)$ and $\mathcal{P}(\mathbb{R}^n)$ represent the classes of tempered distributions and of polynomials in $\mathbb{R}^n$, respectively. In addition, we make the usual modification if $q = \infty$, and $L_{p,w}(\mathbb{R}^n)$ denotes the weighted Lebesgue space as in (1.4).

Next, we introduce inhomogeneous weighted Besov–Orlicz spaces, which are naturally defined by replacing weighted Lebesgue spaces with weighted Orlicz spaces in the definition (2.2). Let w be a positive measurable function, and let Φ be a *Young function*, that is, Φ is a continuous increasing convex function on $[0, \infty)$ satisfying $\Phi(0) = 0$ and $\lim_{t\to\infty} \Phi(t) = \infty$. Then for $s \in \mathbb{R}$ and $1 \le q \le \infty$, the *inhomogeneous weighted Besov–Orlicz space* is defined by

$$\left\{\begin{array}{l}B^{s,q}_{\Phi,w}(\mathbb{R}^n) := \{u \in \mathcal{S}'(\mathbb{R}^n) : \|u\|_{B^{s,q}_{\Phi,w}} < \infty\},\\ \|u\|_{B^{s,q}_{\Phi,w}} := \|\psi * u\|_{L_{\Phi,w}} + \Big(\sum_{j=1}^{\infty}(2^{sj}\|\varphi_j * u\|_{L_{\Phi,w}})^q\Big)^{1/q},\end{array}\right. \tag{2.4}$$

where we make the usual modification if $q = \infty$, and $L_{\Phi,w}(\mathbb{R}^n)$ denotes the weighted Orlicz space, which is a class of measurable functions defined by

$$L_{\Phi,w}(\mathbb{R}^n) := \Big\{f : \int_{\mathbb{R}^n} \Phi(\varepsilon|f|)w\,dx < \infty \text{ for some } \varepsilon > 0\Big\} \tag{2.5}$$

equipped with the Luxemburg norm

$$\|f\|_{L_{\Phi,w}} := \inf\Big\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi(|f|/\lambda)w\,dx \le 1\Big\}. \tag{2.6}$$

Clearly, $B^{s,q}_{\Phi,w}(\mathbb{R}^n)$ coincides with $B^{s,q}_{p,w}(\mathbb{R}^n)$ for $\Phi(t) = t^p$ and $1 \le p < \infty$.

In this section, we shall show two key lemmas, inspired by Rakotondratsimba [R1, R2], who proved weighted Young inequalities for convolutions with kernel functions behaving like the Riesz potential. However, for the purpose of the precise investigation of constants, we need to reconstruct this procedure with kernel functions belonging to the Schwartz class.

First, we recall the following n-dimensional Hardy type inequality:

Theorem A. (i) (n-dimensional Hardy type inequality) *Let U_1 and V_1 be positive measurable weight functions in $\mathbb{R}^n$, $n \in \mathbb{N}$ and $1 < p \le q < \infty$. Then the inequality*

$$\Big(\int_{\mathbb{R}^n}\Big(\int_{\{2|y|<|x|\}} f(y)\,dy\Big)^q U_1(x)\,dx\Big)^{1/q} \le C_1\Big(\int_{\mathbb{R}^n} f(x)^p V_1(x)\,dx\Big)^{1/p}$$

holds for all $f \in L_{p,V_1}(\mathbb{R}^n)$ *with* $f \geq 0$ *a.e. in* $\mathbb{R}^n$ *if and only if*

$$A_1 := \sup_{R>0}\Big(\int_{\{|x|>2R\}} U_1(x)\,dx\Big)^{1/q}\Big(\int_{\{|x|<R\}} V_1(x)^{-(p'-1)}\,dx\Big)^{1/p'} < \infty.$$

Moreover, the constant C_1 *can be taken as*

$$C_1 = (p')^{1/p'}p^{1/q}A_1.$$

(ii) (n-dimensional dual-Hardy type inequality) *Let* U_2 *and* V_2 *be positive weight functions in* $\mathbb{R}^n$, $n \in \mathbb{N}$ *and* $1 < p \leq q < \infty$. *Then the inequality*

$$\Big(\int_{\mathbb{R}^n}\Big(\int_{\{|y|>2|x|\}} f(y)\,dy\Big)^q U_2(x)\,dx\Big)^{1/q} \leq C_2\Big(\int_{\mathbb{R}^n} f(x)^p V_2(x)\,dx\Big)^{1/p}$$

holds for all $f \in L_{p,V_2}(\mathbb{R}^n)$ *with* $f \geq 0$ *a.e. in* $\mathbb{R}^n$ *if and only if*

$$A_2 := \sup_{R>0}\Big(\int_{\{|x|<R\}} U_2(x)\,dx\Big)^{1/q}\Big(\int_{\{|x|>2R\}} V_2(x)^{-(p'-1)}\,dx\Big)^{1/p'} < \infty.$$

Moreover, the constant C_2 *can be taken as*

$$C_2 = (p')^{1/p'}p^{1/q}A_2.$$

Indeed, Theorem A can be proved immediately from the n-dimensional Hardy and dual-Hardy inequalities shown by Drábek–Heinig–Kufner [DHK, Theorem 2.1, p. 4] through scaling and changing variables.

We first prove a norm estimate with a subcritical homogeneous weight:

LEMMA 2.1. (i) *Let* $\Psi \in \mathcal{S}(\mathbb{R}^n)$ *be any fixed kernel function with* $n \in \mathbb{N}$. *Then*

$$\|\Psi * f\|_{L_{q,w_r}} \leq C_n\Big(\frac{1}{n-r}\Big)^{1/q}\|f\|_{L_p} \quad \text{for all } f \in L_p(\mathbb{R}^n),$$

$1 < p \leq q < \infty$ *and* $0 \leq r < n$, *where* w_r *is the homogeneous weight as in* (1.3).

(ii) *Let* $n \in \mathbb{N}$. *Then*

$$\|\varphi_j * f\|_{L_{q,w_r}} \leq C_n\Big(\frac{1}{n-r}\Big)^{1/q}2^{(\frac{n}{p}-\frac{n-r}{q})j}\|f\|_{L_p} \quad \text{for all } f \in L_p(\mathbb{R}^n),$$

$1 < p \leq q < \infty$, $0 \leq r < n$ *and* $j \in \mathbb{Z}$, *where* $\{\varphi_j\}$ *are the Schwartz functions as in* (2.1).

Proof. Note that (ii) is an immediate consequence of (i). Indeed, since $(\varphi_j * f)(x/2^j) = (\varphi_0 * [f(\cdot/2^j)])(x)$, by applying (i) with $\Psi = \varphi_0$ we obtain

$$\begin{aligned}\|\varphi_j * f\|_{L_{q,w_r}} &= 2^{-\frac{n-r}{q}j}\|\varphi_0 * [f(\cdot/2^j)]\|_{L_{q,w_r}} \\ &\leq C_n 2^{-\frac{n-r}{q}j}\Big(\frac{1}{n-r}\Big)^{1/q}\|f(\cdot/2^j)\|_{L_p} = C_n\Big(\frac{1}{n-r}\Big)^{1/q}2^{(\frac{n}{p}-\frac{n-r}{q})j}\|f\|_{L_p}.\end{aligned}$$

We turn to the proof of (i). Without loss of generality, we may assume $f \geq 0$ a.e. in $\mathbb{R}^n$, and that Ψ is positive, since otherwise we can replace Ψ by $|\Psi| + \eta$ with any fixed positive function $\eta \in \mathcal{S}(\mathbb{R}^n)$. Note that Ψ might no longer be smooth after such a replacement. However, the only property we will need is its decay estimate, i.e., for any $\alpha \geq 0$, there exists a positive constant $C_{n,\alpha}$ such that

$$0 < |\Psi(x)| + \eta(x) \leq C_{n,\alpha}(1 + |x|)^{-\alpha} \quad \text{for all } x \in \mathbb{R}^n. \tag{2.7}$$

We first decompose the integral into three parts:

$$\begin{aligned} \|\Psi * f\|_{L_{q,w_r}}^q \leq 3^q \Bigg[& \int_{\mathbb{R}^n} \bigg(\int_{\{|y|<|x|/2\}} \Psi(x-y) f(y)\, dy \bigg)^q \frac{dx}{|x|^r} \\ & + \int_{\mathbb{R}^n} \bigg(\int_{\{|x|/2 \leq |y| \leq 2|x|\}} \Psi(x-y) f(y)\, dy \bigg)^q \frac{dx}{|x|^r} \\ & + \int_{\mathbb{R}^n} \bigg(\int_{\{|y|>2|x|\}} \Psi(x-y) f(y)\, dy \bigg)^q \frac{dx}{|x|^r} \Bigg] \\ =: {} & 3^q (I_1 + I_2 + I_3). \end{aligned} \tag{2.8}$$

First, we estimate I_1. Note that $|y| < |x|/2$ implies $|x|/2 < |x-y|$. Hence,

$$I_1 \leq \int_{\mathbb{R}^n} \bigg(\int_{\{|y|<|x|/2\}} f(y)\, dy \bigg)^q \tilde{\Psi}(x)^q \frac{dx}{|x|^r}, \quad \text{where} \quad \tilde{\Psi}(x) := \sup_{\{|z|>|x|/2\}} \Psi(z).$$

To apply Theorem A(i), we need to check the condition

$$\bigg(\int_{\{2R<|x|\}} \tilde{\Psi}(x)^q \frac{dx}{|x|^r} \bigg)^{1/q} \bigg(\int_{\{|x|<R\}} dx \bigg)^{1/p'} \leq A_1 \quad \text{for all } R > 0. \tag{2.9}$$

Indeed, once (2.9) has been verified, Theorem A(i) yields

$$I_1^{1/q} \leq (p')^{1/p'} p^{1/q} A_1 \|f\|_{L_p} \leq e^{2/e} A_1 \|f\|_{L_p}.$$

Since Ψ decays rapidly as $|x| \to \infty$, $\tilde{\Psi}$ also satisfies the same estimate as in (2.7), namely, for any $\alpha \geq 0$, there exists a positive constant $C_{n,\alpha}$ such that

$$\tilde{\Psi}(x) \leq C_{n,\alpha}(1 + |x|)^{-\alpha} \quad \text{for all } x \in \mathbb{R}^n. \tag{2.10}$$

We now distinguish two cases:

Case 1: $R \geq 1$. Then by making use of the decay estimate (2.10) with $\alpha = 2n$, we see that

$$
\tag{2.11}
\begin{aligned}
&\Big(\int\limits_{\{2R<|x|\}} \tilde{\Psi}(x)^q \,\frac{dx}{|x|^r}\Big)^{1/q}\Big(\int\limits_{\{|x|<R\}} dx\Big)^{1/p'}\\
&\qquad\le C_n\Big(\int\limits_{\{2R<|x|\}} |x|^{-\alpha q-r}\,dx\Big)^{1/q}\Big(\int\limits_{\{|x|<R\}} dx\Big)^{1/p'}\\
&\qquad\le C_n\Big(\frac{1}{\alpha q-(n-r)}\Big)^{1/q} R^{-\alpha+\frac{n-r}{q}+\frac{n}{p'}} \le C_n,
\end{aligned}
$$

where we have used the facts that $R \ge 1$ and

$$
\begin{cases}
\alpha q-(n-r) \ge \alpha-(n-r) = n+r > 0,\\
\Big(\dfrac{1}{\alpha q-(n-r)}\Big)^{1/q} \le \Big(\dfrac{1}{n+r}\Big)^{1/q} \le 1,\\
-\alpha+\dfrac{n-r}{q}+\dfrac{n}{p'} \le -2n+(n-r)+n = -r \le 0.
\end{cases}
$$

Case 2: $0 < R < 1$. In this case,

$$
\begin{aligned}
&\Big(\int\limits_{\{2R<|x|\}} \tilde{\Psi}(x)^q \,\frac{dx}{|x|^r}\Big)^{1/q}\\
&\qquad\le \Big(\int\limits_{\{2R<|x|<2\}} \tilde{\Psi}(x)^q \,\frac{dx}{|x|^r}\Big)^{1/q} + \Big(\int\limits_{\{|x|\ge 2\}} \tilde{\Psi}(x)^q \,\frac{dx}{|x|^r}\Big)^{1/q}\\
&\qquad=: J_1 + J_2.
\end{aligned}
$$

First, by using (2.10) with $\alpha = 0$, we obtain

$$
J_1 \le C_n\Big(\int\limits_{\{|x|<2\}} \frac{dx}{|x|^r}\Big)^{1/q} \le C_n\Big(\frac{1}{n-r}\Big)^{1/q}.
$$

Moreover, by (2.10) with $\alpha = n+1$,

$$
J_2 \le C_n\Big(\int\limits_{\{|x|\ge 2\}} |x|^{-\alpha q-r}\,dx\Big)^{1/q} \le C_n\Big(\frac{1}{\alpha q-(n-r)}\Big)^{1/q} \le C_n,
$$

where we have used the facts that

$$
\begin{cases}
\alpha q-(n-r) \ge \alpha-(n-r) = 1+r > 0,\\
\Big(\dfrac{1}{\alpha q-(n-r)}\Big)^{1/q} \le \Big(\dfrac{1}{\alpha-(n-r)}\Big)^{1/q} = \Big(\dfrac{1}{1+r}\Big)^{1/q} \le 1.
\end{cases}
$$

To sum up, we get

$$\left(\int_{\{2R<|x|\}} \tilde{\Psi}(x)^q \frac{dx}{|x|^r}\right)^{1/q} \left(\int_{\{|x|<R\}} dx\right)^{1/p'} \le C_n \left(\frac{1}{n-r}\right)^{1/q} R^{n/p'} \le C_n \left(\frac{1}{n-r}\right)^{1/q} \tag{2.12}$$

since $R < 1$. Thus by (2.11), (2.12) and Theorem A(i), we obtain

$$I_1^{1/q} \le C_n \left(\frac{1}{n-r}\right)^{1/q} \|f\|_{L_p}. \tag{2.13}$$

Next, we estimate I_3. Note that $2|x| < |y|$ implies $|y|/2 < |x-y|$. Thus

$$I_3 \le \int_{\mathbb{R}^n} \Big(\int_{\{|y|>2|x|\}} \tilde{\Psi}(y) f(y)\, dy\Big)^q \frac{dx}{|x|^r}.$$

To apply Theorem A(ii), we need to check the condition

$$\left(\int_{\{|x|<R\}} \frac{dx}{|x|^r}\right)^{1/q} \Big(\int_{\{2R<|x|\}} (\tilde{\Psi}(x)^{-p})^{-(p'-1)}\, dx\Big)^{1/p'} \le A_2 \quad \text{for all } R > 0. \tag{2.14}$$

Indeed, once (2.14) has been verified, Theorem A(ii) yields

$$I_3^{1/q} \le (p')^{1/p'} p^{1/q} A_2 \Big(\int_{\mathbb{R}^n} (\tilde{\Psi}(x) f(x))^p \tilde{\Psi}(x)^{-p}\, dx\Big)^{1/p} \le e^{2/e} A_2 \|f\|_{L_p}.$$

Just as for I_1, we distinguish two cases:

Case 1: $R \ge 1$. Then by (2.10) with $\alpha = 2n$,

$$\Big(\int_{\{2R<|x|\}} \tilde{\Psi}(x)^{p'}\, dx\Big)^{1/p'} \le C_n \Big(\int_{\{2R<|x|\}} |x|^{-\alpha p'}\, dx\Big)^{1/p'} \le C_n \left(\frac{1}{\alpha p' - n}\right)^{1/p'} R^{-(\alpha - n/p')} \le C_n R^{-(\alpha - n/p')},$$

where we have used the facts that

$$\alpha p' - n \ge \alpha - n = n > 0, \qquad \left(\frac{1}{\alpha p' - n}\right)^{1/p'} \le \left(\frac{1}{\alpha - n}\right)^{1/p'} = \left(\frac{1}{n}\right)^{1/p'} \le 1.$$

Thus we obtain

$$\left(\int_{\{|x|<R\}} \frac{dx}{|x|^r}\right)^{1/q} \Big(\int_{\{2R<|x|\}} \tilde{\Psi}(x)^{p'}\, dx\Big)^{1/p'} \le C_n \left(\frac{1}{n-r}\right)^{1/q} R^{-\alpha + \frac{n-r}{q} + \frac{n}{p'}} \le C_n \left(\frac{1}{n-r}\right)^{1/q} \tag{2.15}$$

since $R \ge 1$ and $-\alpha + \frac{n-r}{q} + \frac{n}{p'} \le -2n + (n-r) + n = -r \le 0$.

CASE 2: $0 < R < 1$. In this case,

$$
\begin{aligned}
&\Big(\int_{\{2R<|x|\}} \tilde{\Psi}(x)^{p'}\,dx\Big)^{1/p'} \\
&\qquad\qquad \le \Big(\int_{\{2R<|x|<2\}} \tilde{\Psi}(x)^{p'}\,dx\Big)^{1/p'} + \Big(\int_{\{|x|\ge 2\}} \tilde{\Psi}(x)^{p'}\,dx\Big)^{1/p'} \\
&\qquad\qquad =: K_1 + K_2.
\end{aligned}
$$

First, by (2.10) with $\alpha = 0$, we obtain

$$K_1 \le C_n\Big(\int_{\{|x|<2\}} dx\Big)^{1/p'} \le C_n.$$

Moreover, by (2.10) with $\alpha = 2n$,

$$K_2 \le C_n\Big(\int_{\{|x|\ge 2\}} |x|^{-\alpha p'}\,dx\Big)^{1/p'} \le C_n\Big(\frac{1}{\alpha p' - n}\Big)^{1/p'} 2^{-(\alpha - n/p')} \le C_n.$$

To sum up, we get

$$
\begin{aligned}
(2.16)\qquad &\Big(\int_{\{|x|<R\}} \frac{dx}{|x|^r}\Big)^{1/q}\Big(\int_{\{2R<|x|\}} \tilde{\Psi}(x)^{p'}\,dx\Big)^{1/p'} \\
&\qquad\qquad \le C_n\Big(\frac{1}{n-r}\Big)^{1/q} R^{(n-r)/q} \le C_n\Big(\frac{1}{n-r}\Big)^{1/q}
\end{aligned}
$$

since $R < 1$. Thus by (2.15), (2.16) and Theorem A(ii), we obtain

$$I_3^{1/q} \le C_n\Big(\frac{1}{n-r}\Big)^{1/q}\|f\|_{L_p}. \tag{2.17}$$

Finally, we estimate I_2. Since $|x|/2 \le |y| \le 2|x|$ and $2^k \le |x| < 2^{k+1}$ imply $2^{k-1} \le |y| < 2^{k+2}$, we have

$$
\begin{aligned}
(2.18)\qquad I_2 &= \sum_{k\in\mathbb{Z}} \int_{\{2^k\le|x|<2^{k+1}\}} \Big(\int_{\{|x|/2\le|y|\le 2|x|\}} \Psi(x-y)f(y)\,dy\Big)^q \frac{dx}{|x|^r} \\
&\le \sum_{k\in\mathbb{Z}} 2^{-kr} \int_{\{2^k\le|x|<2^{k+1}\}} (\Psi * [f\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}])(x)^q\,dx.
\end{aligned}
$$

Here, take $t := nq/(n-r) \in [q,\infty)$. Then (2.18), Hölder's inequality and Young's inequality imply

$$
\begin{aligned}
(2.19)\qquad I_2 &\le C_n \sum_{k\in\mathbb{Z}} 2^{-kr+kn(1-q/t)} \\
&\quad \times \Big(\int_{\{2^k\le|x|<2^{k+1}\}} (\Psi * [f\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}])(x)^t\,dx \Big)^{q/t} \\
&\le C_n \sum_{k\in\mathbb{Z}} \|\Psi * [f\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}]\|_{L_t}^q \\
&\le C_n \sum_{k\in\mathbb{Z}} \|\Psi\|_{L_{\tilde t}}^q \|f\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}\|_{L_p}^q \\
&\le C_n^q \sum_{k\in\mathbb{Z}} \|f\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}\|_{L_p}^q \le C_n^q \|f\|_{L_p}^q,
\end{aligned}
$$

where $\tilde t \in [1,p')$ is the exponent satisfying $1/t+1 = 1/\tilde t + 1/p$. Moreover, the fact $\max_{1\le\tau\le\infty}\|\Psi\|_{L_\tau} < \infty$ was used above. By (2.8), (2.13), (2.17) and (2.19), we obtain the desired estimate. ∎

The following lemma yields a norm inequality involving the critical weight function W_s defined in the statement of Theorem 1.2(ii):

Lemma 2.2. (i) *Let $\Psi \in \mathcal{S}(\mathbb{R}^n)$ and $1 < p, s < \infty$. Then*

$$\|\Psi * f\|_{L_{p,W_{s,j}}} \le C_{n,p,s}\|f\|_{L_p} \quad \textit{for all } f \in L_p(\mathbb{R}^n) \textit{ and } j \in \mathbb{N}\cup\{0\},$$

where the weight function $W_{s,j}$ is defined by

$$W_{s,j}(x) := \frac{1}{|x|^n[\log(e+2^j/|x|)]^s} \quad \textit{for } x \in \mathbb{R}^n \setminus \{0\}.$$

(ii) *Let $n \in \mathbb{N}$ and $1 < p, s < \infty$. Then*

$$\|\varphi_j * f\|_{L_{p,W_s}} \le C_{n,p,s}\, 2^{\frac{n}{p}j}\|f\|_{L_p} \quad \textit{for all } f \in L_p(\mathbb{R}^n) \textit{ and } j \in \mathbb{N}\cup\{0\},$$

where $\{\varphi_j\}$ are the Schwartz functions as in (2.1), *and the weight function W_s is defined by*

$$W_s(x) := \frac{1}{|x|^n[\log(e+1/|x|)]^s} \quad \textit{for } x \in \mathbb{R}^n \setminus \{0\}.$$

Proof. Note that (ii) is an immediate consequence of (i). Indeed, by noting $(\varphi_j * f)(x/2^j) = (\varphi_0 * [f(\cdot/2^j)])(x)$ and applying (i) with $\Psi = \varphi_0$, we see

$$
\begin{aligned}
\|\varphi_j * f\|_{L_{p,W_s}} &= \|\varphi_0 * [f(\cdot/2^j)]\|_{L_{p,W_{s,j}}} \le C_{n,p,s}\|f(\cdot/2^j)\|_{L_p} \\
&= C_{n,p,s}\, 2^{\frac{n}{p}j}\|f\|_{L_p}.
\end{aligned}
$$

To prove (i), just as in the proof of Lemma 2.1, we may assume that $f \ge 0$ a.e. in $\mathbb{R}^n$, Ψ is positive, and for any $\alpha \ge 0$, there exists $C_{n,\alpha} > 0$ such that

$$(2.20)\qquad 0 < \Psi(x) \le C_{n,\alpha}(1+|x|)^{-\alpha} \quad \text{for all } x \in \mathbb{R}^n.$$

Let us decompose the integral into three parts:

$$\begin{aligned}(2.21)\qquad \|\varPsi * f\|^p_{L_{p,W_{s,j}}} &\le 3^p\Big[\int_{\mathbb{R}^n}\Big(\int_{\{|y|<|x|/2\}}\varPsi(x-y)f(y)\,dy\Big)^p W_{s,j}(x)\,dx \\ &\quad + \int_{\mathbb{R}^n}\Big(\int_{\{|x|/2\le|y|\le 2|x|\}}\varPsi(x-y)f(y)\,dy\Big)^p W_{s,j}(x)\,dx \\ &\quad + \int_{\mathbb{R}^n}\Big(\int_{\{|y|>2|x|\}}\varPsi(x-y)f(y)\,dy\Big)^p W_{s,j}(x)\,dx\Big] \\ &=: 3^p(L_1+L_2+L_3).\end{aligned}$$

First, we estimate L_1. As $|y|<|x|/2$ implies $|x|/2<|x-y|$, we have

$$L_1 \le \int_{\mathbb{R}^n}\Big(\int_{\{|y|<|x|/2\}} f(y)\,dy\Big)^p \tilde{\varPsi}(x)^p W_{s,j}(x)\,dx, \quad \text{where } \tilde{\varPsi}(x) := \sup_{\{|z|>|x|/2\}}\varPsi(z).$$

To apply Theorem A(i), we need to check the condition

$$\Big(\int_{\{2R<|x|\}}\tilde{\varPsi}(x)^p W_{s,j}(x)\,dx\Big)^{1/p}\Big(\int_{\{|x|<R\}}dx\Big)^{1/p'} \le A_1 \quad \text{for all } R>0.$$

By (2.20) with any $\alpha>0$,

$$\begin{aligned}(2.22)\qquad &\Big(\int_{\{2R<|x|\}}\tilde{\varPsi}(x)^p W_{s,j}(x)\,dx\Big)^{1/p} \\ &\le C_{n,\alpha}\Big(\int_{\{2R<|x|\}}|x|^{-\alpha p-n}\frac{dx}{[\log(e+2^j/|x|)]^s}\Big)^{1/p} \\ &\le C_{n,\alpha}\Big(\int_{\{2R<|x|\}}|x|^{-\alpha p-n}\,dx\Big)^{1/p} \le C_{n,\alpha}R^{-\alpha} \quad \text{for all } R>0.\end{aligned}$$

Thus by taking $\alpha = n/p'$ in (2.22), we obtain

$$\Big(\int_{\{2R<|x|\}}\tilde{\varPsi}(x)^p W_{s,j}(x)\,dx\Big)^{1/p}\Big(\int_{\{|x|<R\}}dx\Big)^{1/p'} \le C_{n,p} \quad \text{for all } R>0,$$

and then Theorem A(i) yields

$$(2.23)\qquad L_1^{1/p} \le C_{n,p}\|f\|_{L_p}.$$

Next, we estimate L_3. Since $2|x|<|y|$ implies $|y|/2<|x-y|$, we have

$$L_3 \le \int_{\mathbb{R}^n}\Big(\int_{\{|y|>2|x|\}}\tilde{\varPsi}(y)f(y)\,dy\Big)^p W_{s,j}(x)\,dx.$$

To apply Theorem A(ii), we need to check the condition

$$\Big(\int_{\{|x|<R\}} W_{s,j}(x)\,dx\Big)^{1/p}\Big(\int_{\{2R<|x|\}} (\tilde{\Psi}(x)^{-p})^{-(p'-1)}\,dx\Big)^{1/p'} \le A_2 \quad \text{for all } R>0.$$

We distinguish two cases:

CASE 1: $R > 2^{j-1}$. In this case,

$$\int_{\{|x|<R\}} W_{s,j}(x)\,dx = \int_{\{|x|<2^{j-1}\}} W_{s,j}(x)\,dx + \int_{\{2^{j-1}\le|x|<R\}} W_{s,j}(x)\,dx =: M_1+M_2.$$

By changing variables $x = 2^j y$, it follows that

$$\begin{aligned}(2.24)\qquad M_1 &\le \int_{\{|x|<2^{j-1}\}} \frac{dx}{|x|^n[\log(2^j/|x|)]^s} = \int_{\{|y|<1/2\}} \frac{dy}{|y|^n[\log(1/|y|)]^s}\\ &= C_n\int_{\log 2}^{\infty} \frac{d\tau}{\tau^s} = C_{n,s},\end{aligned}$$

where the last integral in (2.24) is finite because of $s>1$. Next,

$$M_2 \le \int_{\{2^{j-1}\le|x|<R\}} \frac{dx}{|x|^n} = C_n \log(R/2^{j-1}).$$

Moreover, by (2.20) with $\alpha = n/p'+1$,

$$\Big(\int_{\{2R<|x|\}} \tilde{\Psi}(x)^{p'}\,dx\Big)^{1/p'} \le C_{n,\alpha}\Big(\int_{\{2R<|x|\}} |x|^{-\alpha p'}\,dx\Big)^{1/p'} = C_{n,p}R^{-1}.$$

To sum up, we obtain

$$\begin{aligned}(2.25)\qquad &\Big(\int_{\{|x|<R\}} W_{s,j}(x)\,dx\Big)^{1/p}\Big(\int_{\{2R<|x|\}} \tilde{\Psi}(x)^{p'}\,dx\Big)^{1/p'}\\ &\le C_{n,p,s}[1+\log(R/2^{j-1})]^{1/p}R^{-1}\\ &= C_{n,p,s}[1+\log(R/2^{j-1})]^{1/p}(R/2^{j-1})^{-1}2^{-(j-1)} \le C_{n,p,s},\end{aligned}$$

where we have used $j\in\mathbb{N}\cup\{0\}$, $R/2^{j-1}>1$ and $\max_{1\le\tau<\infty}(1+\log\tau)^{1/p}\tau^{-1} < \infty$.

CASE 2: $R\le 2^{j-1}$. In this case, by using (2.24) again, we have

$$\Big(\int_{\{|x|<R\}} W_{s,j}(x)\,dx\Big)^{1/p} \le \Big(\int_{\{|x|<2^{j-1}\}} W_{s,j}(x)\,dx\Big)^{1/p} = M_1^{1/p} \le C_{n,s}.$$

Moreover, by (2.20) with $\alpha = n/p' + 1$,

$$\Big(\int_{\{2R<|x|\}} \tilde{\Psi}(x)^{p'}\,dx \Big)^{1/p'} \le C_{n,\alpha} \Big(\int_{\{2R<|x|\}} (1+|x|)^{-\alpha p'}\,dx \Big)^{1/p'} \le C_{n,\alpha} \Big(\int_{\mathbb{R}^n} (1+|x|)^{-\alpha p'}\,dx \Big)^{1/p'} = C_{n,p}.$$

Thus we obtain

$$\Big(\int_{\{|x|<R\}} W_{s,j}(x)\,dx \Big)^{1/p} \Big(\int_{\{2R<|x|\}} \tilde{\Psi}(x)^{p'}\,dx \Big)^{1/p'} \le C_{n,p,s}. \tag{2.26}$$

Therefore, (2.25), (2.26) and Theorem A(ii) yield

$$L_3^{1/p} \le C_{n,p,s} \|f\|_{L_p}. \tag{2.27}$$

Finally, we estimate L_2. Since $|x|/2 \le |y| \le 2|x|$ and $2^k \le |x| < 2^{k+1}$ imply $2^{k-1} \le |y| < 2^{k+2}$, by Young's inequality we have

$$\begin{aligned} L_2 &= \sum_{k\in\mathbb{Z}} \int_{\{2^k\le|x|<2^{k+1}\}} \Big(\int_{\{|x|/2\le|y|\le2|x|\}} \Psi(x-y) f(y)\,dy \Big)^p W_{s,j}(x)\,dx \\ &\le \sum_{k\in\mathbb{Z}} \int_{\{2^k\le|x|<2^{k+1}\}} \Big(\int_{\{|x|/2\le|y|\le2|x|\}} \Psi(x-y) f(y)\,dy \Big)^p \frac{dx}{|x|^n} \\ &\le \sum_{k\in\mathbb{Z}} 2^{-kn} \int_{\{2^k\le|x|<2^{k+1}\}} (\Psi * [f\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}])(x)^p\,dx \\ &\le C_n \sum_{k\in\mathbb{Z}} \|\Psi * [f\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}]\|_{L_\infty}^p \\ &\le C_n \sum_{k\in\mathbb{Z}} \|\Psi\|_{L_{p'}}^p \|f\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}\|_{L_p}^p \\ &\le C_n^p \sum_{k\in\mathbb{Z}} \|f\chi_{\{2^{k-1}\le|\cdot|<2^{k+2}\}}\|_{L_p}^p \le C_n^p \|f\|_{L_p}^p, \end{aligned} \tag{2.28}$$

where we have used $\max_{1\le\tau\le\infty} \|\Psi\|_{L_\tau} < \infty$. Hence, by (2.21), (2.23), (2.27) and (2.28), we obtain the desired estimate. ∎

3. Proofs of theorems

Proof of Theorem 1.1. We consider only the case $\nu < \min\{\rho_1,\rho_2\} \le \max\{\rho_1,\rho_2\} < \infty$ since the limiting cases can be obtained quite similarly.

By using the partition of unity $\sum_{j\in\mathbb{Z}} \hat{\varphi}_j \equiv 1$, we decompose

$$u = \sum_{j=-\infty}^{Z-1} \varphi_j * u + \sum_{j=Z}^{\infty} \varphi_j * u =: u_1 + u_2 \quad \text{for any fixed } Z \in \mathbb{Z}.$$

We first estimate $\|u_1\|_{\dot{B}^{0,\nu}_{q,w_r}}$. Since $\operatorname{supp}\hat{\varphi}_j = \{2^{j-1} \le |x| \le 2^{j+1}\}$, we see that $\varphi_j * \varphi_l \equiv 0$ if $|j-l| > 1$. Therefore,

$$\|u_1\|_{\dot{B}^{0,\nu}_{q,w_r}} = \Big(\sum_{l=-\infty}^{\infty} \|\varphi_l * u_1\|^{\nu}_{L_{q,w_r}}\Big)^{1/\nu} = \Big(\sum_{l=-\infty}^{Z} \|\varphi_l * u_1\|^{\nu}_{L_{q,w_r}}\Big)^{1/\nu}$$
$$= \Big(\sum_{l=-\infty}^{Z} \|\tilde{\varphi}_l * \varphi_l * u_1\|^{\nu}_{L_{q,w_r}}\Big)^{1/\nu},$$

where $\tilde{\varphi}_l := \varphi_{l-1} + \varphi_l + \varphi_{l+1}$. By Lemma 2.1(ii) and Hölder's inequality,

$$\text{(3.1)} \qquad \|u_1\|_{\dot{B}^{0,\nu}_{q,w_r}} \le C_n\Big(\frac{1}{n-r}\Big)^{1/q}\Big(\sum_{l=-\infty}^{Z} 2^{(\frac{n}{p}-\frac{n-r}{q})l\nu}\|\varphi_l * u_1\|^{\nu}_{L_p}\Big)^{1/\nu}$$
$$\le C_n\Big(\frac{1}{n-r}\Big)^{1/q}\Big(\sum_{l=-\infty}^{Z} 2^{(\frac{n}{p}-\frac{n-r}{q})\cdot\frac{\rho_1\nu}{\rho_1-\nu}l}\Big)^{\frac{\rho_1-\nu}{\rho_1\nu}}\Big(\sum_{l=-\infty}^{Z}\|\varphi_l * u_1\|^{\rho_1}_{L_p}\Big)^{1/\rho_1}$$
$$\le C_n\Big(\frac{1}{n-r}\Big)^{1/q}\Big(\sum_{l=-\infty}^{Z} 2^{(\frac{n}{p}-\frac{n-r}{q})\cdot\frac{\rho_1\nu}{\rho_1-\nu}l}\Big)^{\frac{\rho_1-\nu}{\rho_1\nu}}\|u_1\|_{\dot{B}^{0,\rho_1}_{p}}.$$

Moreover, by Young's inequality,

$$\text{(3.2)} \qquad \|u_1\|_{\dot{B}^{0,\rho_1}_{p}} = \Big(\sum_{l=-\infty}^{\infty}\Big\|\varphi_l * \Big(\sum_{j=-\infty}^{Z-1}\varphi_j * u\Big)\Big\|^{\rho_1}_{L_p}\Big)^{1/\rho_1}$$
$$\le \Big(\sum_{l=-\infty}^{\infty}\Big(\sum_{j=l-1}^{l+1}\|\varphi_l * \varphi_j * u\|_{L_p}\Big)^{\rho_1}\Big)^{1/\rho_1}$$
$$\le \Big(\sum_{l=-\infty}^{\infty}\Big(\sum_{j=l-1}^{l+1}\|\varphi_j\|_{L_1}\|\varphi_l * u\|_{L_p}\Big)^{\rho_1}\Big)^{1/\rho_1}$$
$$= 3\|\varphi_0\|_{L_1}\Big(\sum_{l=-\infty}^{\infty}\|\varphi_l * u\|^{\rho_1}_{L_p}\Big)^{1/\rho_1} = 3\|\varphi_0\|_{L_1}\|u\|_{\dot{B}^{0,\rho_1}_{p}},$$

where we have used the fact that $\|\varphi_j\|_{L_1} = \|\varphi_0\|_{L_1}$ for all $j \in \mathbb{Z}$. Next, we investigate the geometric series which appears in (3.1). Since $t/(2^t - 1) \le 1/\log 2$ for all $t > 0$, we see that

$$\text{(3.3)} \qquad \Big(\sum_{l=-\infty}^{Z} 2^{(\frac{n}{p}-\frac{n-r}{q})\cdot\frac{\rho_1\nu}{\rho_1-\nu}l}\Big)^{\frac{\rho_1-\nu}{\rho_1\nu}}$$
$$= 2^{(\frac{n}{p}-\frac{n-r}{q})(Z+1)}\Bigg(\frac{1}{2^{(\frac{n}{p}-\frac{n-r}{q})\cdot\frac{\rho_1\nu}{\rho_1-\nu}} - 1}\Bigg)^{\frac{\rho_1-\nu}{\rho_1\nu}}$$

$$\begin{aligned}
&\le C_n 2^{(\frac{n}{p}-\frac{n-r}{q})Z}\left(\frac{1}{(\frac{n}{p}-\frac{n-r}{q})\cdot\frac{\rho_1\nu}{\rho_1-\nu}}\right)^{\frac{\rho_1-\nu}{\rho_1\nu}}\\
&\le C_n\, 2^{(\frac{n}{p}-\frac{n-r}{q})Z}\left(\frac{p\cdot\frac{q}{p}}{n\cdot\frac{q}{p}-(n-r)}\right)^{\frac{\rho_1-\nu}{\rho_1\nu}}\\
&\le C_n 2^{(\frac{n}{p}-\frac{n-r}{q})Z} p^{\frac{\rho_1-\nu}{\rho_1\nu}}\left[\sup_{t\ge 1}\frac{t}{n\,t-(n-r)}\right]^{\frac{\rho_1-\nu}{\rho_1\nu}}\\
&\le C_n 2^{(\frac{n}{p}-\frac{n-r}{q})Z}\left(\frac{q}{r}\right)^{\frac{\rho_1-\nu}{\rho_1\nu}},
\end{aligned}$$

where we have used the fact that $\sup_{t\ge 1} t/(nt-(n-r)) = 1/r$. Summing up, by (3.1)–(3.3), we get

$$\|u_1\|_{\dot B^{0,\nu}_{q,w_r}} \le C_n\left(\frac{1}{n-r}\right)^{1/q}\left(\frac{q}{r}\right)^{\frac{1}{\nu}-\frac{1}{\rho_1}} 2^{(\frac{n}{p}-\frac{n-r}{q})Z}\|u\|_{\dot B^{0,\rho_1}_p} \quad \text{for all } Z\in\mathbb{Z}. \tag{3.4}$$

Next,

$$\begin{aligned}
\|u_2\|_{\dot B^{0,\nu}_{q,w_r}} &= \Big(\sum_{l=-\infty}^{\infty}\|\varphi_l * u_2\|^{\nu}_{L_{q,w_r}}\Big)^{1/\nu} = \Big(\sum_{l=Z-1}^{\infty}\|\varphi_l * u_2\|^{\nu}_{L_{q,w_r}}\Big)^{1/\nu}\\
&= \Big(\sum_{l=Z-1}^{\infty}\|\tilde\varphi_l * \varphi_l * u_2\|^{\nu}_{L_{q,w_r}}\Big)^{1/\nu}.
\end{aligned}$$

By Lemma 2.1(ii) and Hölder's inequality,

$$\begin{aligned}
\|u_2\|_{\dot B^{0,\nu}_{q,w_r}} &\le C_n\left(\frac{1}{n-r}\right)^{1/q}\Big(\sum_{l=Z-1}^{\infty}2^{(\frac{n}{p}-\frac{n-r}{q})l\nu}\|\varphi_l * u_2\|^{\nu}_{L_p}\Big)^{1/\nu}\\
&= C_n\left(\frac{1}{n-r}\right)^{1/q}\Big(\sum_{l=Z-1}^{\infty}2^{-\frac{n-r}{q}l\nu}(2^{\frac{n}{p}l}\|\varphi_k * u_2\|_{L_p})^{\nu}\Big)^{1/\nu}\\
&\le C_n\left(\frac{1}{n-r}\right)^{1/q}\Big(\sum_{l=Z-1}^{\infty}2^{-\frac{n-r}{q}\cdot\frac{\rho_2\nu}{\rho_2-\nu}l}\Big)^{\frac{\rho_2-\nu}{\rho_2\nu}}\\
&\quad\times\Big(\sum_{l=Z-1}^{\infty}(2^{\frac{n}{p}l}\|\varphi_l * u_2\|_{L_p})^{\rho_2}\Big)^{1/\rho_2}\\
&\le C_n\left(\frac{1}{n-r}\right)^{1/q}\Big(\sum_{l=Z-1}^{\infty}2^{-\frac{n-r}{q}\cdot\frac{\rho_2\nu}{\rho_2-\nu}l}\Big)^{\frac{\rho_2-\nu}{\rho_2\nu}}\|u_2\|_{\dot B^{n/p,\rho_2}_p}.
\end{aligned}\tag{3.5}$$

Quite similarly as in (3.2), we obtain $\|u_2\|_{\dot B^{n/p,\rho_2}_p} \le 3\|\varphi_0\|_{L_1}\|u\|_{\dot B^{n/p,\rho_2}_p}$. Since

$t/(2^t-1) \le 1/\log 2$ for all $t>0$, the geometric series in (3.5) is estimated as follows:

$$\begin{aligned}
&\Big(\sum_{l=Z-1}^{\infty} 2^{-\frac{n-r}{q}\cdot\frac{\rho_2\nu}{\rho_2-\nu}l}\Big)^{\frac{\rho_2-\nu}{\rho_2\nu}}\\
&\quad = 2^{-\frac{n-r}{q}(Z-1)}\bigg(\frac{1}{1-2^{-\frac{n-r}{q}\cdot\frac{\rho_2\nu}{\rho_2-\nu}}}\bigg)^{\frac{\rho_2-\nu}{\rho_2\nu}} \le C_n 2^{-\frac{n-r}{q}Z}\bigg(\frac{1}{2^{\frac{n-r}{q}\cdot\frac{\rho_2\nu}{\rho_2-\nu}}-1}\bigg)^{\frac{\rho_2-\nu}{\rho_2\nu}}\\
&\quad \le C_n 2^{-\frac{n-r}{q}Z}\bigg(\frac{q}{n-r}\cdot\frac{\rho_2-\nu}{\rho_2\nu}\bigg)^{\frac{\rho_2-\nu}{\rho_2\nu}} \le C_n 2^{-\frac{n-r}{q}Z}\bigg(\frac{q}{n-r}\bigg)^{\frac{\rho_2-\nu}{\rho_2\nu}}.
\end{aligned}$$

To sum up, we get

$$\|u_2\|_{\dot B^{0,\nu}_{q,w_r}} \le C_n 2^{-\frac{n-r}{q}Z}\bigg(\frac{1}{n-r}\bigg)^{\frac1q+\frac1\nu-\frac1{\rho_2}} q^{\frac1\nu-\frac1{\rho_2}}\|u\|_{\dot B^{n/p,\rho_2}_p} \quad \text{for all } Z\in\mathbb{Z}. \tag{3.6}$$

Combining (3.4) with (3.6), we have, for any $Z\in\mathbb{Z}$,

$$\begin{aligned}
\|u\|_{\dot B^{0,\nu}_{q,w_r}} &\le C_n\bigg(\frac1r\bigg)^{\frac1\nu-\frac1{\rho_1}}\bigg(\frac{1}{n-r}\bigg)^{\frac1q+\frac1\nu-\frac1{\rho_2}} q^{\frac1\nu-\frac{1}{\max\{\rho_1,\rho_2\}}}\\
&\quad\times\big(2^{(\frac np-\frac{n-r}{q})Z}\|u\|_{\dot B^{0,\rho_1}_p} + 2^{-\frac{n-r}{q}Z}\|u\|_{\dot B^{n/p,\rho_2}_p}\big).
\end{aligned} \tag{3.7}$$

Furthermore, as $[t]\le t<[t]+1$ for all $t\in\mathbb{R}$ where $[t]$ denotes the integer part, it follows that

$$\begin{aligned}
\|u\|_{\dot B^{0,\nu}_{q,w_r}} &\le C_n\bigg(\frac1r\bigg)^{\frac1\nu-\frac1{\rho_1}}\bigg(\frac{1}{n-r}\bigg)^{\frac1q+\frac1\nu-\frac1{\rho_2}} q^{\frac1\nu-\frac{1}{\max\{\rho_1,\rho_2\}}}\\
&\quad\times\big(2^{(\frac np-\frac{n-r}{q})t}\|u\|_{\dot B^{0,\rho_1}_p} + 2^{-\frac{n-r}{q}t}\|u\|_{\dot B^{n/p,\rho_2}_p}\big)
\end{aligned} \tag{3.8}$$

for all $t\in\mathbb{R}$. In the end, we take $t=t_0$ in (3.8) with

$$t_0 := \frac{p}{n\log 2}\log\bigg(\frac{\|u\|_{\dot B^{n/p,\rho_2}_p}}{\|u\|_{\dot B^{0,\rho_1}_p}}\bigg)$$

to obtain

$$2^{(\frac np-\frac{n-r}{q})t_0}\|u\|_{\dot B^{0,\rho_1}_p} = 2^{-\frac{n-r}{q}t_0}\|u\|_{\dot B^{n/p,\rho_2}_p} = \|u\|_{\dot B^{0,\rho_1}_p}^{\frac{(n-r)p}{nq}}\|u\|_{\dot B^{n/p,\rho_2}_p}^{1-\frac{(n-r)p}{nq}},$$

which finishes the proof. ∎

Proof of Theorem 1.2. We deal only with the case $\rho<\infty$ since the limiting case $\rho=\infty$ requires no modification. First, we prove (i). By the definition of the inhomogeneous weighted Besov space, we have

$$\|u\|_{B^{0,\rho}_{p,w_r}} = \|\psi * u\|_{L_{p,w_r}} + \Big(\sum_{j=1}^{\infty} \|\varphi_j * u\|^{\rho}_{L_{p,w_r}}\Big)^{1/\rho}.$$

By noting $\psi = \sum_{j=-\infty}^{0} \varphi_j$, and applying Lemma 2.1(i) with $p = q$ and $\Psi = \sum_{l=-\infty}^{1} \varphi_l = \psi + \varphi_1$, we see that

$$\|\psi * u\|_{L_{p,w_r}} = \Big\|\Big(\sum_{l=-\infty}^{1} \varphi_l\Big) * \psi * u\Big\|_{L_{p,w_r}} \leq C_n \Big(\frac{1}{n-r}\Big)^{1/p} \|\psi * u\|_{L_p}. \tag{3.9}$$

Furthermore, by making use of Lemma 2.1(ii) with $p = q$, we obtain

$$\Big(\sum_{j=1}^{\infty} \|\varphi_j * u\|^{\rho}_{L_{p,w_r}}\Big)^{1/\rho} = \Big(\sum_{j=1}^{\infty} \|\tilde{\varphi}_j * \varphi_j * u\|^{\rho}_{L_{p,w_r}}\Big)^{1/\rho} \leq C_n \Big(\frac{1}{n-r}\Big)^{1/p} \Big(\sum_{j=1}^{\infty} (2^{\frac{r}{p}j} \|\varphi_j * u\|_{L_p})^{\rho}\Big)^{1/\rho}. \tag{3.10}$$

Hence, combining (3.9) with (3.10), we get the desired estimate.

Next, we prove (ii). First, by Lemma 2.2(i) with $j = 0$ and $\Psi = \sum_{l=-\infty}^{1} \varphi_l$,

$$\|\psi * u\|_{L_{p,W_s}} = \Big\|\Big(\sum_{l=-\infty}^{1} \varphi_l\Big) * \psi * u\Big\|_{L_{p,W_s}} \leq C_{n,p,s} \|\psi * u\|_{L_p}. \tag{3.11}$$

Furthermore, by Lemma 2.2(ii),

$$\Big(\sum_{j=1}^{\infty} \|\varphi_j * u\|^{\rho}_{L_{p,W_s}}\Big)^{1/\rho} = \Big(\sum_{j=1}^{\infty} \|\tilde{\varphi}_j * \varphi_j * u\|^{\rho}_{L_{p,W_s}}\Big)^{1/\rho} \leq C_{n,p,s} \Big(\sum_{j=1}^{\infty} (2^{\frac{n}{p}j} \|\varphi_j * u\|_{L_p})^{\rho}\Big)^{1/\rho}. \tag{3.12}$$

Hence, combining (3.11) with (3.12), we get the desired estimate. ∎

Proof of Theorem 1.3. We deal only with the case $\rho < \infty$ since the limiting case $\rho = \infty$ is similar. We first compute the weighted Orlicz norm $\|\psi * u\|_{L_{\Phi_0,w_r}}$. Let $\varepsilon > 0$, to be determined later. By Lemma 2.1(i) with $\Psi = \sum_{l=-\infty}^{1} \varphi_l = \psi + \varphi_1$,

$$\int_{\mathbb{R}^n} \Phi_0(\varepsilon|\psi * u|)w_r\,dx = \sum_{k=k_0}^{\infty} \frac{\varepsilon^{(\mu-\delta)k}}{k!}\|\psi * u\|_{L_{(\mu-\delta)k,w_r}}^{(\mu-\delta)k}$$
$$= \sum_{k=k_0}^{\infty} \frac{\varepsilon^{(\mu-\delta)k}}{k!}\Big\|\Big(\sum_{l=-\infty}^{1}\varphi_l\Big) * \psi * u\Big\|_{L_{(\mu-\delta)k,w_r}}^{(\mu-\delta)k}$$
$$\le \frac{1}{n-r}\sum_{k=k_0}^{\infty} \frac{(\varepsilon C_n\|\psi * u\|_{L_p})^{(\mu-\delta)k}}{k!}$$
$$\le \frac{1}{n-r}\sum_{k=1}^{\infty} \frac{(\varepsilon C_n\|\psi * u\|_{L_p})^{(\mu-\delta)k}}{k!}$$
$$= \frac{1}{n-r}[\exp((\varepsilon\, C_n\|\psi * u\|_{L_p})^{\mu-\delta}) - 1].$$

We now take $\varepsilon = \varepsilon_0 > 0$ satisfying

$$\frac{1}{n-r}[\exp((\varepsilon_0 C_n\|\psi * u\|_{L_p})^{\mu-\delta}) - 1] = 1,$$

so that

$$\varepsilon_0 = \frac{[\log(1+n-r)]^{1/(\mu-\delta)}}{C_n\|\psi * u\|_{L_p}}.$$

Thus by the definition of the Luxemburg norm, we have

$$\|\psi * u\|_{L_{\Phi_0,w_r}} \le \frac{1}{\varepsilon_0} = \frac{C_n\|\psi * u\|_{L_p}}{[\log(1+n-r)]^{1/(\mu-\delta)}}. \tag{3.13}$$

Here, we use the following elementary inequality: for any $a > 0$,

$$\frac{t}{\log(1+t)} \le \frac{a}{\log(1+a)} \quad \text{for all } 0 < t \le a. \tag{3.14}$$

Hence, by using (3.13), (3.14) and $\mu - \delta \ge 1$, we obtain

$$\|\psi * u\|_{L_{\Phi_0,w_r}} \le C_n\Big(\frac{1}{n-r}\Big)^{1/(\mu-\delta)}\|\psi * u\|_{L_p}. \tag{3.15}$$

Next, we handle $\|\varphi_j * u\|_{L_{\Phi_0,w_r}}$ for each fixed $j \in \mathbb{Z}$ in the same way as when estimating of $\|\psi * u\|_{L_{\Phi_0,w_r}}$. Let $\varepsilon > 0$, to be chosen later. Then by Lemma 2.1(ii),

$$\int_{\mathbb{R}^n} \Phi_0(\varepsilon|\varphi_j * u|)w_r\,dx = \sum_{k=k_0}^{\infty} \frac{\varepsilon^{(\mu-\delta)k}}{k!}\|\varphi_j * u\|_{L_{(\mu-\delta)k,w_r}}^{(\mu-\delta)k}$$
$$= \sum_{k=k_0}^{\infty} \frac{\varepsilon^{(\mu-\delta)k}}{k!}\|\tilde{\varphi}_j * \varphi_j * u\|_{L_{(\mu-\delta)k,w_r}}^{(\mu-\delta)k}$$

$$\leq \frac{1}{(n-r)2^{(n-r)j}} \sum_{k=k_0}^{\infty} \frac{(\varepsilon C_n 2^{\frac{n}{p}j} \|\varphi_j * u\|_{L_p})^{(\mu-\delta)k}}{k!}$$
$$\leq \frac{1}{(n-r)2^{(n-r)j}} \sum_{k=1}^{\infty} \frac{(\varepsilon C_n 2^{\frac{n}{p}j} \|\varphi_j * u\|_{L_p})^{(\mu-\delta)k}}{k!}$$
$$= \frac{1}{(n-r)2^{(n-r)j}} [\exp((\varepsilon C_n 2^{\frac{n}{p}j} \|\varphi_j * u\|_{L_p})^{\mu-\delta}) - 1].$$

In particular, we take $\varepsilon = \varepsilon_0 > 0$ satisfying

$$\frac{1}{(n-r)2^{(n-r)j}} [\exp((\varepsilon_0 C_n 2^{\frac{n}{p}j} \|\varphi_j * u\|_{L_p})^{\mu-\delta}) - 1] = 1,$$

so that

$$\varepsilon_0 = \frac{[\log(1+(n-r)2^{(n-r)j})]^{1/(\mu-\delta)}}{C_n 2^{\frac{n}{p}j} \|\varphi_j * u\|_{L_p}}.$$

Thus by the definition of the Luxemburg norm, we have

$$\|\varphi_j * u\|_{L_{\Phi_0,w_r}} \leq \frac{1}{\varepsilon_0} = \frac{C_n 2^{\frac{n}{p}j} \|\varphi_j * u\|_{L_p}}{[\log(1+(n-r)2^{(n-r)j})]^{1/(\mu-\delta)}} \quad \text{for all } j \in \mathbb{Z}. \tag{3.16}$$

Therefore, by (3.16) and Hölder's inequality,

$$\begin{aligned}
&\Big(\sum_{j=1}^{\infty} \|\varphi_j * u\|_{L_{\Phi_0,w_r}}^{\nu}\Big)^{1/\nu} \\
&\leq C_n \Big[\sum_{j=1}^{\infty} [\log(1+(n-r)2^{(n-r)j})]^{-\frac{\nu}{\mu-\delta}} (2^{\frac{n}{p}j} \|\varphi_j * u\|_{L_p})^{\nu}\Big]^{1/\nu} \\
&\leq C_n \Big[\sum_{j=1}^{\infty} [\log(1+(n-r)2^{(n-r)j})]^{-\frac{\mu}{\mu-\delta}}\Big]^{1/\mu} \Big[\sum_{j=1}^{\infty} (2^{\frac{n}{p}j} \|\varphi_j * u\|_{L_p})^{\rho}\Big]^{1/\rho}.
\end{aligned} \tag{3.17}$$

In what follows, we investigate the non-negative term series on the right-hand side of (3.17). For a technical reason, take $t_0 \in \mathbb{R}$ defined by the equation

$$\log(n-r) + t_0(n-r)\log 2 = \frac{t_0(n-r)\log 2}{1+\mu/\delta},$$

so that

$$t_0 = \frac{\mu+\delta}{\mu(\log 2)(n-r)} \log\Big(\frac{1}{n-r}\Big).$$

We distinguish two cases:

Case 1: $t_0 \geq 1$. Noting that $\mu > 1$, we decompose the sum into two parts:

$$\Big[\sum_{j=1}^{\infty}[\log(1+(n-r)2^{(n-r)j})]^{-\frac{\mu}{\mu-\delta}}\Big]^{1/\mu}$$
$$\le \Big[\sum_{j=1}^{[t_0]}[\log(1+(n-r)2^{(n-r)j})]^{-\frac{\mu}{\mu-\delta}}\Big]^{1/\mu}$$
$$+\Big[\sum_{j=[t_0]+1}^{\infty}[\log(1+(n-r)2^{(n-r)j})]^{-\frac{\mu}{\mu-\delta}}\Big]^{1/\mu} =: N_1+N_2.$$

We first estimate N_1. By (3.14), we have

$$\frac{(n-r)2^{(n-r)j}}{\log(1+(n-r)2^{(n-r)j})} \le \frac{(n-r)2^{(n-r)[t_0]}}{\log(1+(n-r)2^{(n-r)[t_0]})} \quad \text{for all } 1\le j\le [t_0]. \tag{3.18}$$

Moreover, direct computation yields

$$(n-r)2^{(n-r)[t_0]} \le (n-r)2^{(n-r)t_0} = \Big(\frac{1}{n-r}\Big)^{\delta/\mu}, \tag{3.19}$$

$$\begin{aligned}\frac{1}{\log(1+(n-r)2^{(n-r)[t_0]})} &\le \frac{1}{\log(1+(n-r)2^{(n-r)(t_0-1)})}\\ &= \frac{1}{\log(1+2^{-(n-r)}(n-r)^{-\delta/\mu})} \le 1/\log(1+2^{-n}n^{-1}),\end{aligned} \tag{3.20}$$

where we have used the facts that $[t_0]\le t_0<[t_0]+1$ and $\delta/\mu\le 1-1/\mu<1$. Hence, by (3.18)–(3.20),

$$\log(1+(n-r)2^{(n-r)j}) \ge C_n(n-r)^{(\mu+\delta)/\mu}\,2^{(n-r)j} \quad \text{for all } 1\le j\le [t_0]. \tag{3.21}$$

Therefore, by making use of (3.21) and $t/(2^t-1)\le 1/\log 2$ for all $t>0$, and using the condition $0<\delta\le\mu-1$ with $\mu>1$, we see that

$$\begin{aligned} N_1 &\le C_n(n-r)^{-\frac{\mu+\delta}{\mu(\mu-\delta)}}\Big(\sum_{j=1}^{\infty}2^{-\frac{\mu(n-r)}{\mu-\delta}j}\Big)^{1/\mu}\\ &\le C_n(n-r)^{-\frac{\mu+\delta}{\mu(\mu-\delta)}}\Big(\sum_{j=1}^{\infty}2^{-(n-r)j}\Big)^{1/\mu}\\ &= C_n(n-r)^{-\frac{\mu+\delta}{\mu(\mu-\delta)}}\Big(\frac{1}{2^{n-r}-1}\Big)^{1/\mu}\\ &\le C_n(n-r)^{-\frac{\mu+\delta}{\mu(\mu-\delta)}-\frac{1}{\mu}} = C_n(n-r)^{-\frac{2}{\mu-\delta}}.\end{aligned} \tag{3.22}$$

We next estimate N_2. Observe that the definition of t_0 implies

$$\log(n-r)+t(n-r)\log 2 \ge \frac{t(n-r)\log 2}{1+\mu/\delta} \quad \text{for all } t\ge t_0. \tag{3.23}$$

Since $t_0 < [t_0]+1$, by using (3.23) and $0 < \delta \le \mu - 1$, N_2 can be estimated as follows:

$$N_2 \le \Big[\sum_{j=[t_0]+1}^{\infty} [\log((n-r)2^{(n-r)j})]^{-\frac{\mu}{\mu-\delta}} \Big]^{1/\mu} \tag{3.24}$$

$$\le \Big[\sum_{j=[t_0]+1}^{\infty} \Big[\frac{j(n-r)\log 2}{1+\mu/\delta}\Big]^{-\frac{\mu}{\mu-\delta}} \Big]^{1/\mu}$$

$$\le C_\delta (n-r)^{-\frac{1}{\mu-\delta}} \Big(\sum_{j=[t_0]+1}^{\infty} j^{-\frac{\mu}{\mu-\delta}} \Big)^{1/\mu}$$

$$\le C_\delta (n-r)^{\frac{1}{\mu-\delta}} \Big(\int_1^\infty t^{-\frac{\mu}{\mu-\delta}}\, dt \Big)^{1/\mu} \le C_\delta (n-r)^{-\frac{1}{\mu-\delta}}.$$

Thus combining (3.22) with (3.24), we obtain

$$\Big[\sum_{j=1}^{\infty} [\log(1+(n-r)2^{(n-r)j})]^{-\frac{\mu}{\mu-\delta}}\Big]^{1/\mu} \le C_{n,\delta}(n-r)^{-\frac{2}{\mu-\delta}}. \tag{3.25}$$

Case 2: $t_0 < 1$. In this case, the inequality (3.23) is available for all $t = j \in \mathbb{N}$, and then, in the quite same manner as for N_2,

$$\Big[\sum_{j=1}^{\infty} [\log(1+(n-r)2^{(n-r)j})]^{-\frac{\mu}{\mu-\delta}}\Big]^{1/\mu} \le C_\delta\,(n-r)^{-\frac{1}{\mu-\delta}}. \tag{3.26}$$

To sum up, by (3.15), (3.17), (3.25) and (3.26), we have

$$\begin{aligned} \|u\|_{B^{0,\nu}_{\Phi_0,w_r}} &= \|\psi * u\|_{L_{\Phi_0,w_r}} + \Big(\sum_{j=1}^{\infty} \|\varphi_j * u\|^{\nu}_{L_{\Phi_0,w_r}} \Big)^{1/\nu} \\ &\le C_{n,\delta} \Big(\frac{1}{n-r} \Big)^{\frac{2}{\mu-\delta}} \Big[\|\psi * u\|_{L_p} + \Big(\sum_{j=1}^{\infty} (2^{\frac{n}{p}j} \|\varphi_j * u\|_{L_p})^{\rho} \Big)^{1/\rho} \Big] \\ &= C_{n,\delta} \Big(\frac{1}{n-r} \Big)^{\frac{2}{\mu-\delta}} \|u\|_{B^{n/p,\rho}_p}, \end{aligned}$$

which is the desired estimate. ∎

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