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On the first sign change in Mertens' theorem

by

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1. Introduction. Mertens' Theorem states that

$$\mathit{\Delta}_M(x) := \sum_{p\le x} \frac{1}{p} - \log\log(x) - M = O(\log(x)^{-1})$$

for $x \to \infty$, where $M = 0.26149\ldots$ denotes the Mertens constant [8]. Rosser and Schoenfeld observed that $\mathit{\Delta}_M(x)$ is positive for $1 \le x \le 10^8$ and posed the question whether this would always be the case [12, p. 72f]. This has been answered by Robin who showed that $\mathit{\Delta}_M(x)$ changes sign infinitely often [10].

In this paper we show that the first sign change occurs before $\exp(495.702833165) = 1.909875\ldots \times 10^{215}$. More specifically, we prove

THEOREM 1.1. *There exists an*

$$x_0 \in [\exp(495.702833109), \exp(495.702833165)]$$

such that $\mathit{\Delta}_M(x) < 0$ *for all* $x \in [x_0 - \exp(239.046541), x_0]$.

This problem is similar to bounding the Skewes number, the number in $[2,\infty)$ where the first sign change of $\mathit{\Delta}(x) = \pi(x) - \mathrm{li}(x)$ occurs [14]; this number is by now known to lie between 10^{19} (see [2]) and $\exp(727.951335792)$ (see [13]). The functions $\mathit{\Delta}(x)$ and $\mathit{\Delta}_M(x)$ are closely related and the Prime Number Theorem, $\mathit{\Delta}(x) = o(\mathrm{li}(x))$ for $x \to \infty$, is in fact equivalent to $\mathit{\Delta}_M(x) = o(\log(x)^{-1})$ for $x \to \infty$. But since $\mathit{\Delta}(x)$ and $\mathit{\Delta}_M(x)$ are biased in opposite directions, there is no correlation between the sign changes of the two functions. On the Riemann Hypothesis, sign changes of $\mathit{\Delta}_M(x)$ rather occur at points where $\mathit{\Delta}(x) \approx -2\sqrt{x}/\log(x)$.

Theorem 1.1 is proven by an adaption of the Lehman method for bounding the Skewes number [6], using explicit formulas and numerical approximations to part of the zeros of the Riemann zeta function from [4]. In doing so, the kernel function in Lehman's method is replaced by the Logan function [7],

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which appears to be more suitable for this problem. This is done in such generality that it can easily be reapplied to the original Lehman method.

2. Notation. As usual, $\zeta(s)$ denotes the Riemann zeta function and zeros of $\zeta(s)$ are denoted by $\rho = \beta + i\gamma$ with $\beta, \gamma \in \mathbb{R}$. The Euler constant is denoted by $C_0 = 0.57721\ldots$ and the Mertens constant by

$$M = C_0 - \sum_p \sum_{m=2}^{\infty} \frac{1}{mp^m} = 0.26149\ldots. \tag{2.1}$$

We use the symbol $\sum'$ to define normalized summatory functions, i.e. we define

$$\sum_{x<n<y}{}' a_n := \frac{1}{2} \sum_{x<n<y} a_n + \frac{1}{2} \sum_{x\le n\le y} a_n.$$

Moreover, we define the Mertens prime-counting functions

$$\pi_M(x) = \sum_{p<x}{}' \frac{1}{p} \quad \text{and} \quad \pi_M^*(x) = \sum_{m=1}^{\infty} \frac{\pi_M(x^{1/m})}{m}.$$

The Fourier transform of a function f is denoted by $\hat{f}$ and defined by

$$\hat{f}(x) = \int_{-\infty}^{\infty} f(t)e^{-itx}\,dt.$$

Finally, we will use Turing's big theta notation for explicit estimates and write $f(x) = \Theta(g(x))$ for $|f(x)| \le g(x)$.

3. Description of the method. The method we use is similar to the Lehman method for finding regions where $\pi(x) - \mathrm{li}(x)$ is positive [6]. We aim to calculate upper bounds for a weighted mean value

$$\int_{\omega-\varepsilon}^{\omega+\varepsilon} K(y-\omega) y e^{y/2} [\pi_M(e^y) - \log(y) - M]\,dy, \tag{3.1}$$

where $K(y)$ is a non-negative kernel function. By using explicit formulas this mean value can be expressed as a sum over the non-trivial zeros of $\zeta(s)$, which can be approximated numerically. Then, if an ω can be found for which the value in (3.1) is negative, there must exist an $x \in [\exp(\omega-\varepsilon), \exp(\omega+\varepsilon)]$ such that $\pi_M(x) - \log\log(x) - M$ is negative.

Lehman's method uses the Gaussian function as a kernel function but we prefer to use dilatations of the function

$$K_c(y) := \begin{cases} \dfrac{c}{2\sinh(c)} I_0(c\sqrt{1-y^2}), & |y| < 1, \\ 0, & \text{otherwise,} \end{cases}$$

where $I_0(t) := \sum_{n=0}^{\infty}(t/2)^{2n}/(n!)^2$ denotes the 0th modified Bessel function. The Fourier transform of K_c is given by the Logan function (see [4, Proposition 4.1])

$$\hat{K}_c(x) = \ell_c(x) := \frac{c}{\sinh(c)} \frac{\sin(\sqrt{t^2-c^2})}{\sqrt{t^2-c^2}},$$

which satisfies an optimality property well-suited for this problem [7], and which outperforms the Gaussian function in the similar context of calculating the prime-counting function analytically [4].

We define

$$K_{c,\varepsilon}(y) := \frac{1}{\varepsilon}K_c(y/\varepsilon) \quad \text{and} \quad \ell_{c,\varepsilon}(x) := \hat{K}_{c,\varepsilon}(x) = \ell_c(\varepsilon x).$$

Then our main result is

THEOREM 3.1. *Let $0 < \varepsilon < 10^{-3}$, $c \geq 3$, $\omega - \varepsilon > 200$, and let $H \geq c/\varepsilon$ be a number such that $\beta = 1/2$ holds for all zeros $\rho = \beta + i\gamma$ of the Riemann zeta function with $0 < \gamma \leq H$. Furthermore, let $h = 0$ if the Riemann Hypothesis holds and $h = 1$ otherwise. Then*

$$\int_{\omega-\varepsilon}^{\omega+\varepsilon} K_{c,\varepsilon}(y-\omega) y e^{y/2}[\pi_M(e^y) - \log(y) - M]\,dy \tag{3.2}$$

$$\leq \sum_{|\gamma|\leq c/\varepsilon} e^{-i\gamma\omega}\ell_{c,\varepsilon}(\gamma)\left(\frac{1}{\rho} - \frac{1}{\omega\rho^2}\right) + 1 + 5.4\times 10^{-10} + \mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3,$$

where

$$\mathcal{E}_1 \leq 0.33 e^{h\omega/2}\frac{e^{0.71\sqrt{c\varepsilon}}}{\sinh c}\log(3c)\log\left(\frac{c}{\varepsilon}\right), \tag{3.3}$$

$$\mathcal{E}_2 \leq \frac{3.36 + 126\,\varepsilon}{1000\omega^2} + 2.8\left(\frac{e}{2H}\right)^{\omega/2-1}\log(H), \tag{3.4}$$

$$\mathcal{E}_3 \leq \frac{e^{\omega/2}}{1.99H}\log(H)\left(\frac{ce^{3.12\sqrt{c\varepsilon}}}{\omega\sinh(c)} + \left(\frac{e\varepsilon}{\omega}\right)^{\omega/2}\right). \tag{3.5}$$

Moreover, if $a \in (0,1)$ satisfies $ac/\varepsilon \geq 10^3$ in addition to the previous conditions, then

$$\sum_{ac/\varepsilon<|\gamma|\leq c/\varepsilon}\left|e^{-i\gamma\omega}\ell_{c,\varepsilon}(\gamma)\left(\frac{1}{\rho} - \frac{1}{\omega\rho^2}\right)\right| \tag{3.6}$$

$$\leq \frac{0.32 + 3.51c\varepsilon}{ca^2}\log\left(\frac{c}{\varepsilon}\right)\frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)}.$$

The proof needs some preparation.

4. The explicit formula for $\pi_M^*(x)$. The first ingredient is the explicit formula for $\pi_M^*(x)$. We define the auxiliary function

$$\widetilde{\mathrm{Ei}}(z) = \int_0^\infty \frac{e^{z-t}}{z-t}\,dt,$$

which coincides with the exponential integral $\mathrm{Ei}(z)$ in $\mathbb{R}\setminus\{0\}$, and which occurs naturally in explicit formulas for prime-counting functions.

LEMMA 4.1. *Let $x > 1$. Then*

$$\pi_M^*(x) = \log\log(x) + C_0 - \sum_\rho{}^{*}\, \widetilde{\mathrm{Ei}}(-\rho\log(x)) + \int_x^\infty \frac{dt}{t^2\log(t)(t^2-1)}, \tag{4.1}$$

where $\sum^*$ means that the sum over zeros is calculated as

$$\lim_{T\to\infty} \sum_{|\gamma|<T} \widetilde{\mathrm{Ei}}(-\rho\log(x)).$$

Proof. The argument is similar to the original proof of the Riemann explicit formula [15]. Let

$$\psi(x,r) = \sum_{p^m<x}{}' \frac{\log(p)}{p^{mr}}. \tag{4.2}$$

Then we have

$$\pi_M^*(x) = \int_1^\infty \psi(x,r)\,dr.$$

From [5, (39)] we get the explicit formula

$$\psi(x,r) = \frac{x^{1-r}}{1-r} - \sum_\rho{}^{*} \frac{x^{\rho-r}}{\rho-r} - \sum_{n=1}^\infty \frac{x^{-2n-r}}{-2n-r} - \frac{\zeta'}{\zeta}(r).$$

Since $\mathrm{Ei}(-x) = \log(x) + C_0 + o(x)$ for $x \searrow 0$ [9, p. 40], and since $\log(\zeta(1+\varepsilon)) = -\log(\varepsilon) + o(1)$ for $\varepsilon \searrow 0$, we have

$$\int_1^\infty \left(\frac{x^{1-r}}{1-r} - \frac{\zeta'}{\zeta}(r)\right) dr = \lim_{\varepsilon\searrow 0}[\mathrm{Ei}(-\varepsilon\log(x)) + \log(\zeta(1+\varepsilon))] = \log\log(x) + C_0.$$

The sum over zeros takes the form

$$\int_1^\infty \sum_\rho{}^{*} \frac{x^{\rho-r}}{\rho-r}\,dr = \sum_\rho{}^{*}\, \widetilde{\mathrm{Ei}}((\rho-1)\log(x)) = \sum_\rho{}^{*}\, \widetilde{\mathrm{Ei}}(-\rho\log(x)),$$

and for the sum over the trivial zeros we find

$$\int_1^\infty \sum_{n=1}^\infty \frac{x^{-2n-r}}{2n+r}\,dr = \int_1^\infty \sum_{n=1}^\infty x^{-(2n+1)r}\,\frac{dr}{r} = \int_1^\infty \frac{x^{-3r}}{1-x^{-2r}}\,dr$$
$$= \int_x^\infty \frac{dt}{t^2\log(t)(t^2-1)}. \blacksquare$$

5. The difference $\pi_M^*(x) - \pi_M(x)$. By definition of the Mertens constant (2.1) we have

$$\pi_M(x) = \pi_M^*(x) + M - C_0 + r_M(x), \quad \text{where} \quad r_M(x) = \sideset{}{'}\sum_{\substack{p^m>x\\ m\ge 2}} \frac{1}{mp^m}.$$

The term $r_M(x)$ is responsible for the positive bias in Mertens' Theorem and needs to be bounded from above.

LEMMA 5.1. *Let* $\log(x) > 200$. *Then*

$$r_M(x) \le \frac{1+5.3\times 10^{-10}}{\sqrt{x}\log(x)}.$$

Proof. First we consider the contribution of the squares of prime numbers which yield the main term. Let $r(t) = \psi(t) - t$, where $\psi(t) := \psi(t,0)$ in the sense of (4.2) denotes the normalized Chebyshev function, and assume $|r(t)| < \varepsilon t$ for $t \ge \sqrt{x}$ and some $\varepsilon > 0$. Then partial summation gives

$$\sideset{}{'}\sum_{p>\sqrt{x}} \frac{1}{p^2} < \left[\frac{-r(t)}{t^2\log(t)}\right]_{\sqrt{x}}^\infty + \int_{\sqrt{x}}^\infty \frac{dt}{t^2\log(t)} - \int_{\sqrt{x}}^\infty r(t)\,\frac{d}{dt}\left(\frac{1}{t^2\log(t)}\right)dt \tag{5.1}$$
$$< 2\frac{1+3\varepsilon}{\sqrt{x}\log(x)}.$$

For $3 \le m \le \log(x)$ we use

$$\sum_{p\ge x^{1/m}} \frac{1}{p^m} \le \frac{1}{x} + \int_{x^{1/m}}^\infty \frac{dt}{t^m} = \frac{1}{x} + \frac{1}{m-1}x^{1/m-1},$$

which gives

$$\sum_{\substack{p^m\ge x\\ 3\le m\le \log(x)}} \frac{1}{mp^m} \le \frac{\log(x)}{x} + (\zeta(2)-1)x^{-2/3} < \frac{10^{-12}}{\sqrt{x}\log(x)}.$$

For $m > \log(x)$ we estimate trivially:

$$\sum_p \frac{1}{p^m} \le \sum_{n=3}^\infty n^{-m} + 2^{-m} \le 2^{-m} + \int_2^\infty \frac{dt}{t^m} = 2^{-m}\left(1 + \frac{2}{m-1}\right).$$

Therefore, we get

$$\sum_{\substack{p^m \geq x \\ m > \log x}} \frac{1}{mp^m} \leq \frac{1.01}{\log(x)} \sum_{m \geq \log(x)} 2^{-m} \leq \frac{2.02 \times 2^{-\log(x)}}{\log(x)} < \frac{10^{-16}}{\sqrt{x}\log(x)}.$$

By [3, Table 1], (5.1) holds with $\varepsilon = 1.752 \times 10^{-10}$ and so the assertion follows. ∎

6. Evaluating the sum over zeros. The next problem is to approximate the following integral of the sum over zeros:

$$\int_{-\varepsilon}^{\varepsilon} K_{c,\varepsilon}(y-\omega) y e^{y/2} \sum_{\rho}{}^{*} \widetilde{\mathrm{Ei}}(-\rho y)\, dy.$$

Here, integral and sum may be interchanged, since the sum converges locally in L^1. Therefore, we may treat each summand individually.

6.1. Asymptotic expansion of the summands. Since the Logan kernel should also be of interest for the question of finding regions where $\pi(x) - \mathrm{li}(x)$ is positive, the following lemma is presented in a more general version, which also covers the classical case.

Lemma 6.1. *Let $0 < \varepsilon < \omega$, and let $K \in L^1([-\varepsilon, \varepsilon])$ satisfy $\|K\|_{L^1} = 1$. Let $a \in [0,1]$, let $\rho = \beta + i\gamma$, where $0 \leq \beta \leq 1$ and $\gamma \in \mathbb{R} \setminus \{0\}$, and let*

$$\Phi_{\omega,\rho,a} = \int_{\omega-\varepsilon}^{\omega+\varepsilon} K(y-\omega) y e^{(1/2-a)y}\, \widetilde{\mathrm{Ei}}((a-\rho)y)\, dy.$$

Then

$$\Phi_{\omega,\rho,a} = \sum_{j=1}^{k} (j-1)! \frac{F_{\omega,\rho}^{(-j)}(0)}{(\rho-a)^j} + \Theta\left(\frac{k! e^{\varepsilon/2} e^{(1/2-\beta)\omega}}{(\omega-\varepsilon)^k |\gamma|^{k+1}}\right), \tag{6.1}$$

where $F_{\omega,\rho}^{(-1)}(0) = -e^{(1/2-\rho)\omega} \hat{K}\left(\frac{\rho}{i} - \frac{1}{2i}\right)$ and for $j \geq 2$ and any $m \geq 0$,

$$F_{\omega,\rho}^{(-j)}(0) = (-1)^j e^{(1/2-\rho)\,\omega} \sum_{n=0}^{m} \binom{n+j-2}{n} \frac{(-i)^n \hat{K}^{(n)}\left(\frac{\rho}{i} - \frac{1}{2i}\right)}{\omega^{n+j-1}} + \Theta\left(\frac{e^{j-2+\varepsilon/2} e^{(1/2-\beta)\omega}}{\omega^{j-1}} \frac{(e\varepsilon/\omega)^{m+1}}{1 - e\varepsilon/\omega}\right). \tag{6.2}$$

Proof. By definition of $\widetilde{\mathrm{Ei}}$ we have

$$\Phi_{\omega,\rho,a} = \int_{\omega-\varepsilon}^{\omega+\varepsilon} K(y-\omega) y e^{(1/2-a)y} \int_0^\infty \frac{e^{(a-\rho-r)y}}{a-\rho-r}\, dr\, dy \tag{6.3}$$
$$= \int_0^\infty \frac{1}{a-\rho-r} \int_{\omega-\varepsilon}^{\omega+\varepsilon} K(y-\omega) y e^{(1/2-\rho-r)y}\, dy\, dr.$$

Now let

$$F_{\omega,\rho}^{(-j)}(r) := (-1)^j \int_{\omega-\varepsilon}^{\omega+\varepsilon} y^{1-j} K(y-\omega) e^{(1/2-\rho-r)y}\, dy,$$

which is well defined since $\omega > \varepsilon$, and satisfies $\frac{d}{dr} F_{\omega,\rho}^{(-j)} = F_{\omega,\rho}^{(1-j)}$. Then partial summation gives

$$\Phi_{\omega,\rho,a} = -\int_0^\infty \frac{F_{\omega,\rho}^{(0)}(r)}{r+\rho-a}\, dr \quad = \sum_{j=1}^{k} (j-1)! \frac{F_{\omega,\rho}^{(-j)}(0)}{(\rho-a)^j} - k! \int_0^\infty \frac{F_{\omega,\rho}^{(-k)}(r)}{(r+\rho-a)^{k+1}}\, dr.$$

Here, the trivial bound

$$|F_{\omega,\rho}^{(-k)}(r)| \le \int_{-\varepsilon}^{\varepsilon} \frac{|K(y)|}{(\omega+y)^{k-1}} e^{(1/2-\beta-r)(y+\omega)}\, dy \le \frac{e^{\varepsilon/2}}{(\omega-\varepsilon)^{k-1}} e^{(1/2-\beta)\omega} e^{r(\varepsilon-\omega)}$$

yields

$$\int_0^\infty \left| \frac{F_{\omega,\rho}^{(-k)}(r)}{(r+\rho-a)^{k+1}} \right| dr \le \frac{e^{\varepsilon/2} e^{(1/2-\beta)\omega}}{(\omega-\varepsilon)^k |\gamma|^{k+1}},$$

which confirms (6.1). It remains to evaluate $F_{\omega,\rho}^{(-j)}(0)$. For $j=1$ we find

$$F_{\omega,\rho}^{(-1)}(0) = -e^{(1/2-\rho)\omega} \int_{-\varepsilon}^{\varepsilon} K(y) e^{-i(\frac{\rho}{i}-\frac{1}{2i})y}\, dy = -e^{(1/2-\rho)\omega} \hat{K}\left(\frac{\rho}{i} - \frac{1}{2i}\right).$$

For larger values of j we use the Taylor series expansion

$$\frac{1}{(\omega+y)^u} = \sum_{n=0}^{\infty} \binom{u+n-1}{n} \frac{(-y)^n}{\omega^{u+n}}$$

and

$$\int_{-\varepsilon}^{\varepsilon} K(y) y^n e^{-i(\frac{\rho}{i}-\frac{1}{2i})y}\, dy = i^n \hat{K}^{(n)}\left(\frac{\rho}{i} - \frac{1}{2i}\right), \tag{6.4}$$

which gives

$$F_{\omega,\rho}^{(-j)}(0) = (-1)^j e^{(1/2-\rho)\omega} \sum_{n=0}^{\infty} \binom{j+n-2}{n} \frac{(-i)^n \hat{K}^{(n)}\left(\frac{\rho}{i} - \frac{1}{2i}\right)}{\omega^{n+j-1}}.$$

From (6.4) we also get

$$\left|\hat{K}^{(n)}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right| \le e^{\varepsilon/2}\varepsilon^n;$$

moreover the inequality $\binom{a}{b} \le \left(\frac{ea}{b}\right)^b$, which follows from Stirling's lower bound for $b!$, implies

$$\binom{j+n-2}{n} \le e^n\left(1+\frac{j-2}{n}\right)^n \le e^{n+j-2}.$$

Thus, we have

$$\begin{aligned}\sum_{n=m+1}^{\infty}\binom{j+n-2}{n}\frac{\left|\hat{K}^{(n)}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{\omega^{n+j-1}} &\le \frac{e^{j-2+\varepsilon/2}}{\omega^{j-1}}\sum_{n=m+1}^{\infty}\left(\frac{e\varepsilon}{\omega}\right)^n\\ &= \frac{e^{j-2+\varepsilon/2}}{\omega^{j-1}}\,\frac{(e\varepsilon/\omega)^{m+1}}{1-e\varepsilon/\omega},\end{aligned}$$

which confirms the bound in (6.2). ∎

6.2. Bounds for the kernel function. We need some bounds to estimate the tails of the sum over zeros. These are provided by the following two lemmas from [1] and [3]:

Lemma 6.2 ([3, Lemma 2]). *Let $0<\varepsilon<10^{-3}$ and $c\ge 3$. Then*

$$\sum_{|\gamma|>c/\varepsilon}\frac{\left|\ell_{c,\varepsilon}\left(\frac{\rho}{i}-\frac{1}{2i}\right)\right|}{|\gamma|} \le 0.32\frac{e^{0.71\sqrt{c\varepsilon}}}{\sinh(c)}\log(3c)\log\left(\frac{c}{\varepsilon}\right). \tag{6.5}$$

Lemma 6.3 ([1, Lemma 5.5]). *Let $0<\varepsilon<10^{-3}$ and $c\ge 3$, and let $a\in(0,1)$ satisfy $ac/\varepsilon>10^3$. Then*

$$\sum_{ac/\varepsilon<|\gamma|\le c/\varepsilon}\frac{|\ell_{c,\varepsilon}(\gamma)|}{|\gamma|} \le \frac{1+11c\varepsilon}{\pi c a^2}\log\left(\frac{c}{\varepsilon}\right)\frac{\cosh(c\sqrt{1-a^2})}{\sinh(c)}. \tag{6.6}$$

We also need bounds for the derivatives $\ell_{c,\varepsilon}^{(n)}\left(\frac{\rho}{i}-\frac{1}{2i}\right)$ occurring in (6.2), for calculations not assuming the Riemann Hypothesis.

Lemma 6.4. *Let $0<\varepsilon\le\delta<c/100$, and let $z\in\mathbb{C}$ satisfy $|\Re(z)|\ge c/\varepsilon$ and $|\Im(z)|\le 1/2$. Then*

$$|\ell_{c,\varepsilon}^{(n)}(z)| \le n!\frac{ce^{1.56\sqrt{\delta c}}}{\sinh(c)}\left(\frac{2\varepsilon}{\delta}\right)^n.$$

Proof. The bound follows from the Cauchy formula

$$\ell_{c,\varepsilon}^{(n)}(z) = \frac{n!}{2\pi i}\oint_{|z-\xi|=\delta/(2\varepsilon)}\frac{\ell_{c,\varepsilon}(\xi)}{(z-\xi)^{n+1}}\,d\xi$$

if we show that

$$(6.7)\qquad |\ell_{c,\varepsilon}(\xi)| \le \frac{ce^{1.56\sqrt{\delta c}}}{\sinh(c)}$$

in the range of integration. By basic properties of $\ell_{c,\varepsilon}$ it suffices to prove this bound for $\varepsilon = 1$ under the conditions $\Re(\xi) \ge c - \delta$, $0 \le \Im(\xi) \le \delta$, and we may also assume $\delta < c/100$. Since we have

$$\begin{aligned}|\Im(\sqrt{\xi^2 - c^2})| &\le |\Im(\sqrt{(c-\delta+i\delta)^2 - c^2})| \\ &\le \sqrt{2|1+i|\delta c}\sin\left(\frac{\pi}{4} + \frac{1}{2}\arctan\left(\frac{\delta c - \delta^2}{\delta c}\right)\right) \\ &\le 2^{3/4}\sin(1.181)\sqrt{\delta c} \le 1.56\sqrt{\delta c}\end{aligned}$$

under these conditions, the desired bound follows from

$$\left|\frac{\sin(z)}{z}\right| \le e^{|\Im(z)|}. \ \blacksquare$$

7. Proof of Theorem 3.1. By Lemmas 4.1 and 5.1 we have

$$\begin{aligned}\pi_M(e^y) - \log(y) - M &= \pi_M^*(e^y) - \log(y) - C_0 + r_M(e^y) \\ &\le -\sum_\rho{}^{*} \widetilde{\mathrm{Ei}}(-\rho y) + \frac{1 + 5.4\times 10^{-10}}{y} e^{-y/2}\end{aligned}$$

for $y > 200$, where we have estimated the integral in (4.1) trivially by e^{-3y}. Therefore

$$\begin{aligned}\int\limits_{\omega-\varepsilon}^{\omega+\varepsilon} K_{c,\varepsilon}(y-\omega) y e^{y/2}[\pi_M(e^y) - \log(y) - M]\,dy \\ \le -\sum_\rho \Phi_{\omega,\rho,0} + 1 + 5.4\times 10^{-10},\end{aligned}$$

with $\Phi_{\omega,\rho,0}$ as defined in Lemma 6.1 with $K = K_{c,\varepsilon}$ and $\hat{K} = \ell_{c,\varepsilon}$. We subdivide the sum over zeros into two parts. For $0 < \gamma \le H$ we choose $k = 2$ and $m = 0$ in Lemma 6.1, which gives

$$(7.1)\qquad \begin{aligned}-\sum_{|\gamma|\le H}\Phi_{\omega,\rho,0} &\le \sum_{|\gamma|\le c/\varepsilon} e^{-i\gamma\omega}\ell_{c,\varepsilon}(\gamma)\left(\frac{1}{\rho} - \frac{1}{\omega\rho^2}\right) \\ &+ \sum_{c/\varepsilon<|\gamma|\le H}\left|\frac{\ell_{c,\varepsilon}(\gamma)}{\gamma}\right|\left(1+\frac{\varepsilon}{c\omega}\right) + \frac{1}{\omega^2}\sum_{|\gamma|<H}\left(\frac{2.72\varepsilon}{\gamma^2} + \frac{2.01}{|\gamma|^3}\right),\end{aligned}$$

where we have used $\varepsilon \le 10^{-3}$. For $\gamma > H$ we have

$$\sum_{|\gamma|>H} |\Phi_{\omega,\rho,0}| \le e^{h\omega/2} \sum_{|\gamma|>H} \left| \frac{\ell_{c,\varepsilon}(\frac{\rho}{i} - \frac{1}{2i})}{\gamma} \right| \sum_{j=1}^{k} \frac{(j-1)!}{\omega^{j-1}} H^{1-j} \tag{7.2}$$
$$+ e^{h\omega/2} \sum_{|\gamma|>H} \sum_{j=2}^{k} \frac{(j-1)!}{|\gamma|^j} \left(\sum_{n=1}^{m} \binom{n+j-2}{n} \frac{\left|\ell_{c,\varepsilon}^{(n)}(\frac{\rho}{i} - \frac{1}{2i})\right|}{\omega^{n+j-1}} + \frac{e^{j-2+\varepsilon/2}(e\varepsilon)^{m+1}}{\omega^{j+m-1}(\omega - e\varepsilon)} \right)$$
$$+ e^{h\omega/2} \sum_{|\gamma|>H} \frac{k! e^{\varepsilon/2}}{(\omega-\varepsilon)^k |\gamma|^{k+1}}$$

for arbitrary $k \ge 2$ and $m \ge 1$, where $h = 0$ if the Riemann Hypothesis holds and $h = 1$ otherwise. So the inequality in (3.2) holds with

$$\mathcal{E}_1 = \sum_{c/\varepsilon<|\gamma|\le H} \left| \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} \right| \left(1 + \frac{\varepsilon}{c\omega} \right) \tag{7.3}$$
$$+ e^{h\omega/2} \sum_{|\gamma|>H} \left| \frac{\ell_{c,\varepsilon}(\frac{\rho}{i} - \frac{1}{2i})}{\gamma} \right| \sum_{j=1}^{k} \frac{(j-1)!}{\omega^{j-1}} H^{1-j},$$

$$\mathcal{E}_2 = \frac{1}{\omega^2} \sum_{\rho} \left(\frac{2.72\varepsilon}{\gamma^2} + \frac{2.01}{|\gamma|^3} \right) + e^{h\omega/2} \sum_{|\gamma|>H} \frac{k! e^{\varepsilon/2}}{(\omega-\varepsilon)^k |\gamma|^{k+1}}, \tag{7.4}$$

$$\mathcal{E}_3 = e^{\omega/2} \sum_{|\gamma|>H} \sum_{j=2}^{k} \frac{(j-1)!}{|\gamma|^j} \tag{7.5}$$
$$\times \left(\sum_{n=1}^{m} \binom{n+j-2}{n} \frac{|\ell_{c,\varepsilon}^{(n)}(\frac{\rho}{i} - \frac{1}{2i})|}{\omega^{n+j-1}} + \frac{e^{j-2+\varepsilon/2}(e\varepsilon)^{m+1}}{\omega^{j+m-1}(\omega - e\varepsilon)} \right).$$

We proceed by bounding $\mathcal{E}_k$. To this end we choose $k = m = \lfloor \omega/2 \rfloor$. In (7.3) we take $H = c/\varepsilon$, which gives

$$\mathcal{E}_1 \le e^{h\omega/2} \sum_{c/\varepsilon<|\gamma|} \left| \frac{\ell_{c,\varepsilon}(\gamma)}{\gamma} \right| \sum_{j=0}^{k-1} \frac{j!}{\omega^j} \left(\frac{\varepsilon}{c} \right)^j, \tag{7.6}$$

where the inner sum is bounded by

$$\sum_{j=0}^{\infty} \left(\frac{\varepsilon}{2c} \right)^j \le \left(1 - \frac{1}{6000} \right)^{-1} \le 1.0002,$$

since $c \ge 3$. Using this and (6.5) in (7.6) gives (3.3).

In (7.4) we use the bounds $\sum_\gamma \gamma^{-2} < 0.0463$ and $\sum_\gamma |\gamma|^{-3} < 0.00167$ from [11, Lemma 17], the bound

$$\sum_{|\gamma|>T} |\gamma|^{-k} \le T^{1-k}\log(T) \tag{7.7}$$

for $T \ge 2\pi e$ and $k \ge 2$ from [6, Lemma 2], and the inequality $(\omega-\varepsilon)^k \ge e^{-\varepsilon}\omega^k$, which follows from $k \le \omega/2$, and get

$$\begin{aligned}\mathcal{E}_2 &\le \frac{0.00336+0.126\varepsilon}{\omega^2} + e^{\omega/2}\frac{e^{2\varepsilon}k!}{(\omega H)^k}\log(H)\\ &\le \frac{3.36+126\varepsilon}{1000\omega^2} + 2.8\left(\frac{e}{2H}\right)^{\omega/2-1}\log(H).\end{aligned}$$

In (7.5) we use (7.7) again and the bound from Lemma 6.4, where we choose $\delta = 4\varepsilon$, which gives

$$\begin{aligned}\mathcal{E}_3 \le e^{\omega/2}\sum_{j=2}^{k} H^{1-j}\log(H)\biggl(&\frac{ce^{3.12\sqrt{c\varepsilon}}}{\sinh(c)}\sum_{n=1}^{m}\frac{j-1}{\omega}\frac{(n+j-2)!}{\omega^{n+j-2}}2^{-n}\\ &+\frac{1.002e^{j-1}}{e}\frac{(j-1)!}{\omega^{j-1}}\left(\frac{e\varepsilon}{\omega}\right)^{m+1}\biggr).\end{aligned} \tag{7.8}$$

Since $n+j-2 \le \omega$ we have $(n+j-2)!/\omega^{n+j-2} \le 1/\omega$, so the inner sum is bounded by $1/(2\omega)$. In the second summand, we use $(j-1)!/\omega^{j-1} \le 2^{1-j}$. Since $\sum_{j=1}^{\infty} H^{-j} \le 1.001/H$, $\sum_{j=1}^{\infty}(2H/e)^{-j} \le 1.001e/(2H)$, and $m+1 \ge \omega/2$, we obtain the bound in (3.5).

Finally, the estimate in (3.6) follows from (6.6) since

$$\sum_{ac/\varepsilon<|\gamma|\le c/\varepsilon}\left|\frac{\ell_{c,\varepsilon}(\gamma)}{\rho}\left(1-\frac{1}{\omega\rho}\right)\right| \le \left(1+\frac{1}{200\times 1000}\right)\sum_{ac/\varepsilon<|\gamma|\le c/\varepsilon}\left|\frac{\ell_{c,\varepsilon}(\gamma)}{\gamma}\right|. \blacksquare$$

8. Numerical results. To locate potential regions where the left hand side of (3.2) should be small, the function

$$\sigma_T(y) = \sum_{|\gamma|\le T}\frac{e^{i\gamma y}}{1/2-i\gamma}.$$

has been evaluated for $T = 10^6$ at all points in $10^{-7}\mathbb{Z}\cap[1,2500]$. Since $\ell_{c,\varepsilon}(\gamma) = 1+O((\varepsilon\gamma)^2/c)$ for $\gamma\to 0$, this gives a reasonably good approximation to the first part of the sum in (3.2), and the objective is thus to find regions where $\sigma_T(y)$ is smaller than -1.

The evaluation has been done using the method for fast multiple evaluation of trigonometric sums from [4]. A more detailed search with $T = 10^8$ around 495.7028078, the first point where $\sigma_{10^6}(y)$ turned out to be promisingly small, revealed a short region of length $\approx 2.8\times 10^{-8}$ about 495.702833137 where $\sigma_{10^8}(y) < -1$.

Proof of Theorem 1.1. The assertion now follows by an application of Theorem 3.1 with $\omega = 495.702833137$, $c = 280$, $\varepsilon = 2.8 \times 10^{-8}$, $H = 10^{11}$ (which has been reported in [4]) and $a = 0.4$.

Table 1. Values of $y \in [1, 2500]$ for which $\sigma_{10^6}(y) < -0.95$

y	$\sigma_{10^6}(y)$
495.7028078	$-0.9972\ldots$
1423.957207	$-0.9740\ldots$
1623.9204309	$-0.9807\ldots$
1859.1291846	$-1.0511\ldots$
2107.5263606	$-1.0214\ldots$
2285.3917834	$-1.0454\ldots$
2430.3039554	$-1.0172\ldots$
2447.6661764	$-1.0028\ldots$

The sum over zeros was calculated using approximations to the zeros with imaginary part up to 4×10^9 which were given within an absolute accuracy of 2^{-64}. The sum was evaluated using multiple precision arithmetic, which gave the bound

$$\sum_{|\gamma| \le 4\times 10^9} e^{-i\gamma\omega} \ell_{c,\varepsilon}(\gamma)\left(\frac{1}{\rho} - \frac{1}{\omega\rho^2}\right) \le -1.00015419. \tag{8.1}$$

The sum in (3.6) is then bounded by 1.2×10^{-11} and we have

$$\mathcal{E}_1 + \mathcal{E}_2 + \mathcal{E}_3 \le 1.2 \times 10^{-12} + 1.37 \times 10^{-8} + 1.6 \times 10^{-24} \le 1.38 \times 10^{-8}.$$

Thus, the left hand side of (3.2) is bounded by

$$-1.00015419 + 1.2 \times 10^{-11} + 1 + 5.4 \times 10^{-10} + 1.38 \times 10^{-8} < -0.000154.$$

Consequently, there exists an $x \in [\exp(w - \varepsilon), \exp(w + \varepsilon)]$ such that

$$\pi_M(x) - \log\log(x) - M < -0.000154/(\sqrt{x}\log(x)).$$

Obviously, we have

$$\begin{aligned} \pi_M(x-y) - \log\log(x-y) - M &\le \pi_M(x) - \log\log(x) - M + \int_{x-y}^{x} \frac{dt}{t\log t} \\ &\le -\frac{0.000154}{\sqrt{x}\log(x)} + \frac{y}{(x-y)\log(x-y)}, \end{aligned}$$

which is negative for $y \le 0.00015\sqrt{x}$. Since $0.00015\sqrt{x} > \exp(239.046541)$, the theorem follows. ∎

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